

Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science

Assignment 3 for Block III
of the course “Algorithms for NP-Hard Problems.”

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🏆 Difficulty of the problem is indicated by the total number of points next to it. Each problem consists of several subtasks. Points for each subtask are stated next to them. Collect at least 10 points.

📎 Please submit a PDF/Python notebook (code is not required) with illustrations of the required simulations and explanation of how you obtained them, e.g., what was the experimental design. For instance, if you are simulating Bernoulli trials, you are asked to explain that you simulated a random variable which takes values 1 and 0 according to certain conditions on the outcomes of independent uniform samples. There are no requirements on how long the explanation should be as long as it explains your experiments.

1 Bernoulli generator (1 point)

- (0.2 points) Based on the Bernoulli generator scheme, construct a generator for binomial distribution and negative binomial distribution. Define them analytically and illustrate the distributions for different values of parameters (e.g., probability, number of trials).
- (0.3 points) On a sample of 5000 check empirically the law of large numbers for frequencies. Provide illustration.
- (0.5 points) Consider the game of flipping a coin, an endless sequence of independent trials with the flip of the fair coin. Let us define by X_1, X_2, \dots a sequence of independent identically distributed random variables, each of which takes on values 1 if in the corresponding trial the tail is drawn, and -1 otherwise (with probability $= 1/2$). Denote the total gain by $S_n = X_1 + X_2 + \dots + X_n$. Perform $N = 1000$ trials and plot the trajectory of the process $X(t)$, $t \in [0, 1]$, which equals $X(n/N) = \frac{1}{\sqrt{N}}S_n$ at points n/N , where $n = 0, 1, \dots, N$, and in the other cases is determined by piecewise linear interpolation (i.e., in the form of a broken line).

1.1 Analytical solution

Provide an analytical solution and theoretical background.

1.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

2 Symmetry property (4 points)

- (1 point) Implement a Bernoulli scheme generator with a given probability of success p .
- (1 point) Construct a generator for a singular distribution that corresponds to the Cantor distribution function (“staircase” function).
- (1 point) Verify the symmetry property with respect to $1/2$ (X is equally distributed with $1 - X$), and the property of self-similarity (the conditional distribution of Y under the condition that Y belongs to the interval $[0, 1/3]$ coincides with the distribution $Y/3$).

- (1 point) Using the Cantor generator, check empirically the distribution function (on a sample volume of 1000).

See Appendix for hints.

2.1 Analytical solution

Provide an analytical solution and theoretical background.

2.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

3 From one distribution to another (3 points)

- (1 point) Construct an exponential distribution generator. Use it to verify the Poisson distribution generator.
- (1 point) Build a standard normal distribution generator by simulating random variables in pairs with transition to polar coordinates.
- (1 point) Construct a χ^2 distribution generator with n degrees of freedom.

See Appendix for hints.

3.1 Analytical solution

Provide an analytical solution and theoretical background.

3.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

4 LLN and CLT (2 points)

Let $X(i) \sim N(a, b^2)$. Verify empirically the law of large numbers (1 point) and the central limit theorem (1 point), i.e., investigate the behavior of the sum $S(n)/n$ and the empirical distributions of $\sqrt{n}(S(n)/n - a)$.

4.1 Analytical solution

Provide an analytical solution and theoretical background.

4.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

5 Integral (2 points)

Calculate the integral

$$I = \int_0^\infty \dots \int_0^\infty \exp \left\{ - \left(x_1 + \dots + x_{10} + \frac{1}{2^{20} x_1 \dots x_{10}} \right) \right\} x_1^{\frac{1}{11}-1} x_2^{\frac{2}{11}-1} \dots x_{10}^{\frac{10}{11}-1} dx_1 \dots dx_{10}$$

(1 point) by Monte Carlo method;

(1 point) by the quadrature method (reducing the problem to the calculation of the Riemann integral) evaluating the accuracy of calculations.

Hint: try representing the integrand as a sum of random variables following the same well-known distribution. Make use of LLN and CLT.

5.1 Analytical and numerical solution

Provide an analytical solution and theoretical background. Implement the solution and present numerical results.

6 Unit Simplex (2 points)

Sample from $X = \{(x_1, x_2, x_3) | 0 \leq x_i \leq 1, x_1 + x_2 + x_3 = 1\}$ (1 point) in a way that most of the samples are concentrated in the center and (1 point) uniformly. Visualize in two dimensions.

Hint: for uniform sampling, use approach from the lectures or your own.

6.1 Analytical solution

Provide an analytical solution and theoretical background.

6.2 Illustration

Implement the solution and provide plots illustrating implementation for different values of parameters.

References

- [Buslenko et al.(1966)] Nikolai Panteleimonovich Buslenko et al. *The Monte Carlo method: the method of statistical trials*, volume 87. Pergamon Press, 1966.
- [Shiryaev(2016)] Albert N Shiryaev. *Probability-1*, volume 95. Springer, 2016.
- [Shiryaev(2019)] Albert N Shiryaev. *Probability-2*, volume 95. Springer, 2019.
- [Shreider(2014)] Yu A Shreider. *The Monte Carlo method: the method of statistical trials*, volume 87. Elsevier, 2014.
- [Sobol(2018)] Ilya M Sobol. *A primer for the Monte Carlo method*. CRC press, 2018.
- [Sobol(1974)] IM Sobol. *The monte carlo method. popular lectures in mathematics*. 1974.

Appendix

Problem 2. Page 190 in Shiryaev "Probability-1".

Problem 3. Page 189-190 in Shiryaev "Probability-1". Page 315 in Buslenko "Monte-Carlo Method". For standard normal distribution, prove the following: Let r, φ be such that $r^2 \sim \text{Exp}(1/2)$ and $\varphi \sim U[0, 2\pi)$. Then $X = r \sin \phi$ and $Y = r \cos \phi$ are normally distributed. Chi-distribution can be expressed as a sum of normally distributed variables. Use Box-Muller method to generate them.

Problem 6. Distribution on a unit simplex is also known as Dirichlet distribution.