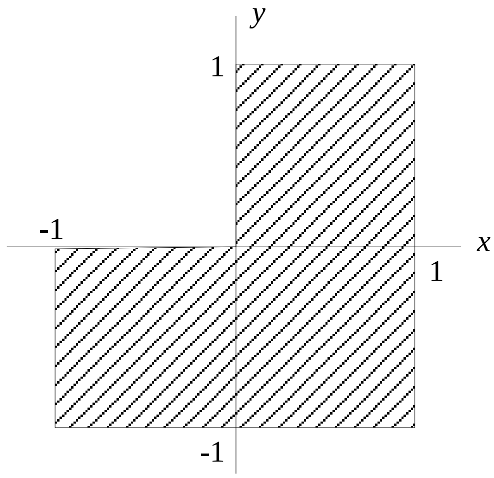
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.437 Inference and Information
Spring 2015

Problem Set 3

Issued: Tuesday, February 24, 2015 **Due:** Tuesday, March 3, 2015

Problem 3.1

Suppose x and y are random variables. Their joint density, depicted below, is constant in the shaded area and zero elsewhere.



(a) Determine $\hat{x}_{\mathrm{BLS}}(y)$, the Bayes least-squares estimate of x given y.

(b) Determine $\lambda_{\mathrm{BLS}} = \mathbb{E}\left[(\hat{x}_{\mathrm{BLS}}(y) - x)^2\right]$, the error variance associated with your estimator in (a).

(c) Consider the following "modified least-squares" cost function (corresponding to the cost of estimating x as $\hat{x}$):

$$C(x, \hat{x}) = \begin{cases} (x - \hat{x})^2, & x < 0 \\ K(x - \hat{x})^2, & x \geq 0 \end{cases}$$

where $K > 1$ is a constant.

Determine $\hat{x}_{\mathrm{MLS}}(y)$, the associated Bayes estimate of x for this cost criterion.

(d) Give a brief intuitive explanation for why your answers to (a) and (c) are either the same or different.

Problem 3.2

Let x and y be random variables such that the random variable x is exponential, and conditioned on knowledge of x, y is exponentially distributed with parameter x, i.e.,

$$\begin{aligned} p_x(x) &= \frac{1}{a} e^{-x/a}, \quad x > 0, \\ p_{y|x}(y|x) &= x e^{-xy}, \quad y > 0. \end{aligned}$$

(a) Determine $\hat{x}_{\mathrm{BLS}}(y)$, $\lambda_{x|y}(y) = \mathbb{E}\left[(\hat{x}_{\mathrm{BLS}}(y) - x)^2 \,|y = y\right]$, and $\lambda_{\mathrm{BLS}} = \mathbb{E}\left[(\hat{x}_{\mathrm{BLS}}(y) - x)^2\right]$.

(b) Determine $\hat{x}_{\mathrm{MAP}}(y)$, the MAP estimate of x based on observation y. Determine the bias and error variance for this estimator. (Recall that the bias is $\mathbb{E}\left[\hat{x}_{\mathrm{MAP}}(y) - x\right]$, and the error variance is $\mathbb{E}\left[(\hat{x}_{\mathrm{MAP}}(y) - x - \mathbb{E}\left[\hat{x}_{\mathrm{MAP}}(y) - x\right])^2\right]$.)

(c) Find $\hat{x}_{\mathrm{LLS}}(y)$, the linear least-squares estimate of x based on observation of y, and $\mathbb{E}\left[(\hat{x}_{\mathrm{LLS}}(y) - x)^2\right]$, the resulting mean-square estimation error.

Problem 3.3

You are given a coin and are allowed to toss it until you see the first head. The probability of heads for this particular coin is modeled as a random variable x, and its prior distribution $p_x(x)$ is uniform in the region $[0, 1]$. Let n be the number of times you see tails before the first head. You are then asked to estimate x from n. (We note that the likelihood for n is the geometric distribution, $p_n(n|x) = x(1-x)^n$ for $n = 0, 1, \ldots$, with $\mathbb{E}[n|x] = (1-x)/x$ and $\mathrm{var}(n|x) = (1-x)/x^2$.)

(a) Determine the MAP estimator of x. Is it unique?

(b) Find the Bayes least-squares estimate for x.
Hint: You might find the following formula useful:

$$\int_0^1 x^k (1-x)^n dx = \frac{n!k!}{(n+k+1)!}.$$

(c) We define relative bias to be the expected *ratio* of the true parameter to the estimated value, i.e., $r_{\hat{x}} = \mathbb{E}\left[x/\hat{x}\right]$. An estimator with a relative bias of 1 is called "relatively unbiased".

  (i) Is the MAP estimator you computed in part (a) relatively unbiased?

  (ii) Is the BLS estimator you computed in part (b) relatively unbiased?

Problem 3.4

Let y be an exponentially distributed random variable with parameter x. x is in turn an exponentially distributed random variable with parameter μ.

As a reminder, an exponential random variable r with parameter λ is distributed according to

$$p_{\mathsf{r}}(r;\lambda) = \begin{cases} \lambda e^{-\lambda r}, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

with expectation $\mathbb{E}[\mathsf{r}] = 1/\lambda$ and variance $\sigma_{\mathsf{r}}^2 = 1/\lambda^2$.

(a) Find the maximum likelihood (ML) estimator of μ given observation $\mathsf{y} = y$.

(b) Is the ML estimator unique?

(c) Is the ML estimator unbiased?

(d) Is the ML estimator efficient?

Problem 3.5

Suppose, for i=1,2

$$\mathsf{y}_i = x + \mathsf{w}_i$$

where x is an unknown but non-zero constant, and where w_1 and w_2 are statistically independent, zero-mean Gaussian random variables with

$$\begin{aligned} \operatorname{var}(\mathsf{w}_1) &= 1 \\ \operatorname{var}(\mathsf{w}_2) &= \begin{cases} 1 & x > 0 \\ 2 & x < 0 \end{cases}. \end{aligned}$$

(a) Calculate the Cramér-Rao bound for unbiased estimators of x based on observation of

$$\mathbf{y} = \begin{bmatrix} \mathsf{y}_1 \\ \mathsf{y}_2 \end{bmatrix}.$$

(b) Show that a minimum variance unbiased estimator $\hat{x}_{\mathrm{MVU}}(\mathbf{y})$ does not exist.

Hint: Consider the estimators

$$\begin{aligned} \hat{\mathsf{x}}_1(\mathbf{y}) &= \frac{1}{2}\mathsf{y}_1 + \frac{1}{2}\mathsf{y}_2, \\ \hat{\mathsf{x}}_2(\mathbf{y}) &= \frac{2}{3}\mathsf{y}_1 + \frac{1}{3}\mathsf{y}_2. \end{aligned}$$

Problem 3.6

Suppose we observe a random N-dimensional vector $\mathbf{y}$, whose components are independent, identically-distributed Gaussian random variables, each with mean x_1 and non-zero variance x_2.

(a) Suppose x_1 is unknown but x_2 is known. Does an efficient estimator exist? Find the maximum likelihood estimate of x_1 based on observation of $\mathbf{y}$. Evaluate the bias and the mean-square error of this estimator.

(b) Now suppose x_1 is known but x_2 is unknown. Does an efficient estimator exist? Find the maximum likelihood estimate of x_2 based on observation of $\mathbf{y}$. Evaluate the bias and the mean-square error of this estimator.

(c) Finally, suppose both x_1 and x_2 are unknown.

  (i) Does an efficient estimator exist? Find $\hat{\mathbf{x}}_{\mathrm{ML}}(\mathbf{y}) = [\hat{x}_{1,\mathrm{ML}}(\mathbf{y}), \hat{x}_{2,\mathrm{ML}}(\mathbf{y})]^T$, the maximum likelihood estimate of

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

  based on observation of $\mathbf{y}$.

  (ii) Evaluate $\mathbb{E}\left[\hat{\mathbf{x}}_{\mathrm{ML}}(\mathbf{y})\right]$ and $\mathbb{E}\left[(\hat{x}_{1,\mathrm{ML}}(\mathbf{y}) - x_1)^2\right]$. Compare with your results from parts (a) and (b).

  (iii) Evaluate $\mathbb{E}\left[(\hat{x}_{2,\mathrm{ML}}(\mathbf{y}) - x_2)^2\right]$ assuming $N = 2$. Compare with your result from part (b).

Problem 3.7 (practice)

Let w and u be two jointly Gaussian random variables, such that:

$$\begin{aligned} \mathbb{E}[w] &= 0, \qquad \mathbb{E}[u] = 0 \\ \mathbb{E}[w^2] &= \sigma^2, \qquad \mathbb{E}[u^2] = \sigma^2 \\ \mathbb{E}\left[wu\right] &= \theta\sigma^2 \end{aligned}$$

θ is a deterministic, unknown parameter satisfying $|\theta| < 1$.
σ^2 is a deterministic, known parameter.
For parts (a) and (b), $y = u + w + \mu$ where μ is a deterministic parameter.

(a) Assume μ is **known**, and we want to estimate θ from the measurement $y = y$.

  (i) Calculate the Cramér-Rao bound for estimating θ with an unbiased estimator.

(ii) Does an efficient estimator exist? If it does, find it.

(iii) Find the maximum likelihood estimator for θ.

(b) Now assume μ is **unknown**, and we want to estimate (μ, θ) from the measurement $\mathsf{y} = y$.

(i) Calculate the Cramér-Rao bound for estimating (μ, θ) with an unbiased estimator.

(ii) Does an efficient estimator exist? If it does, find it.

Now we will estimate θ directly from the measurement $(\mathsf{u}, \mathsf{w}) = (u, w)$.

(c) Does an efficient estimator exist? If it does, find it.
Hint: Finding an explicit formula for the Fisher information for this part can be difficult. You might be able to answer the question without doing it.

(d) We are interested in the estimator

$$\hat{\theta} = \frac{uw}{\sigma^2}$$

(i) Find the bias for the proposed estimator.

(ii) Find the mean square error (MSE) for the proposed estimator.

(iii) Compare the MSE performance of this estimator with that in part (a). Should we expect that one of them always outperforms the other? Explain.
Hint: $\mathbb{E}\left[\mathsf{u}^2 \mathsf{w}^2\right] = \sigma^4 \left(1 + 2\theta^2\right)$