Solid angle subtended by a tetrahedron computation

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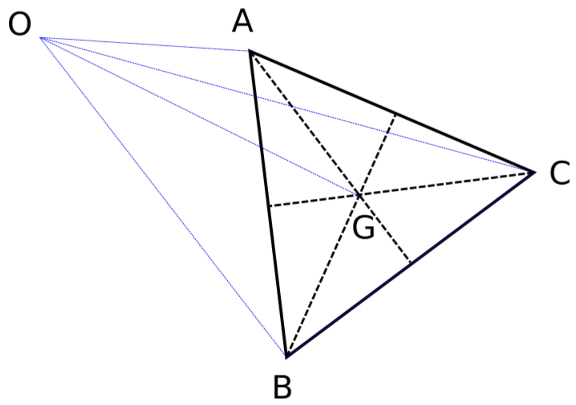

1 Notations

Let us denote, respectively:

- vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OG}$ by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{G}$,
- vectors $\overrightarrow{GA}$, $\overrightarrow{GB}$ and $\overrightarrow{GC}$ by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,
- vector norms $\|\overrightarrow{OA}\|$, $\|\overrightarrow{OB}\|$ and $\|\overrightarrow{OC}\|$ by A, B, C.

2 Computation

The solid angle Ω subtended by the triangular surface ABC is given by [1]:

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{|\mathbf{A}\ \mathbf{B}\ \mathbf{C}|}{ABC + (\mathbf{A}\cdot\mathbf{C})A + (\mathbf{A}\cdot\mathbf{C})B + (\mathbf{A}\cdot\mathbf{B})C} \tag{1}$$

where $|\mathbf{A}\ \mathbf{B}\ \mathbf{C}| = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.

2.1 Numerator

Given that $\mathbf{A} = \overrightarrow{OA} = \mathbf{G} + \mathbf{a}$ (and resp. with B and C), we get:

$$\begin{aligned}
|\mathbf{A}\ \mathbf{B}\ \mathbf{C}| &= \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\\
&= (\mathbf{G}+\mathbf{a})\cdot((\mathbf{G}+\mathbf{b})\times(\mathbf{G}+\mathbf{c}))\\
&= (\mathbf{G}+\mathbf{a})\cdot(\cancel{\mathbf{G}\times\mathbf{G}} + \mathbf{G}\times\mathbf{c} + \mathbf{b}\times\mathbf{G} + \mathbf{b}\times\mathbf{c})\\
&= \cancel{\mathbf{G}\cdot(\mathbf{G}\times\mathbf{c})} + \cancel{\mathbf{G}\cdot(\mathbf{b}\times\mathbf{G})} + \mathbf{G}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{G}\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{G}) + \cancel{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}\\
&= \mathbf{G}\cdot(\mathbf{b}\times\mathbf{c}) + \mathbf{G}\cdot(\mathbf{c}\times\mathbf{a}) + \mathbf{G}\cdot(\mathbf{a}\times\mathbf{b})\\
&= \mathbf{G}\cdot(\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b})
\end{aligned}$$

which simplifies further given that the point G is the centroid of ABC, *i.e.*

$$a + b + c = 0 \tag{2}$$

leading to

$$\begin{aligned}
|\mathbf{A}\ \mathbf{B}\ \mathbf{C}| &= \mathbf{G}\cdot(\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b})\\
&= \mathbf{G}\cdot(\mathbf{b}\times\mathbf{c} + \mathbf{c}\times(-\mathbf{b}-\mathbf{c}) + (-\mathbf{b}-\mathbf{c})\times\mathbf{b})\\
&= \mathbf{G}\cdot(\mathbf{b}\times\mathbf{c} + \mathbf{c}\times(-\mathbf{b}) + \cancel{\mathbf{c}\times(-\mathbf{c})} + \cancel{(-\mathbf{b})\times\mathbf{b}} + (-\mathbf{c})\times\mathbf{b})\\
&= \mathbf{G}\cdot(3\mathbf{b}\times\mathbf{c})\\
&= 3\mathbf{G}\cdot(\mathbf{b}\times\mathbf{c}) \qquad (3)
\end{aligned}$$

2.2 Denominator

To help with the implementation, we will try to reduce the number of terms that need to be pre-computed by removing terms containing a.

First let's see the term in A

$$\begin{aligned}(\mathbf{B}\cdot\mathbf{C})A &= (\mathbf{G}+\mathbf{b})\cdot(\mathbf{G}+\mathbf{c})A \\ &= (G^2+\mathbf{G}\cdot\mathbf{b}+\mathbf{G}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{b})A\end{aligned} \tag{4}$$

for which there is no need for simplification. For the term in B, we can use eq. (2):

$$\begin{aligned}(\mathbf{A}\cdot\mathbf{C})B &= (\mathbf{G}+\mathbf{a})\cdot(\mathbf{G}+\mathbf{c})B \\ &= (G^2+\mathbf{G}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{G}+\mathbf{a}\cdot\mathbf{c})B \\ &= (G^2+\mathbf{G}\cdot\mathbf{c}+(-\mathbf{b}-\mathbf{c})\cdot\mathbf{G}+(-\mathbf{b}-\mathbf{c})\cdot\mathbf{c})B \\ &= (G^2-\mathbf{G}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{c})B\end{aligned}$$

where the last term can be expanded, recalling that $\mathbf{c}=\mathbf{C}-\mathbf{G}$

$$\begin{aligned}\mathbf{c}\cdot\mathbf{c} &= (\mathbf{C}-\mathbf{G})\cdot(\mathbf{C}-\mathbf{G}) \\ &= C^2-2\mathbf{G}\cdot\mathbf{C}+G^2 \\ &= C^2-2\mathbf{G}\cdot(\mathbf{G}+\mathbf{c})+G^2 \\ &= C^2-2\mathbf{G}\cdot\mathbf{c}-G^2\end{aligned}$$

finally yielding

$$(\mathbf{A}\cdot\mathbf{C})B = (2G^2-\mathbf{G}\cdot\mathbf{b}+2\mathbf{G}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}-C^2)B. \tag{5}$$

Similarly, term in C can be expressed as:

$$(\mathbf{A}\cdot\mathbf{B})C = (2G^2+2\mathbf{G}\cdot\mathbf{b}-\mathbf{G}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}-B^2)C. \tag{6}$$

Finally, combining eqs. (4), (5) and (6), the denominator reads:

$$ABC+(G^2+\mathbf{G}\cdot\mathbf{b}+\mathbf{G}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{b})A+(2G^2-\mathbf{G}\cdot\mathbf{b}+2\mathbf{G}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}-C^2)B+(2G^2+2\mathbf{G}\cdot\mathbf{b}-\mathbf{G}\cdot\mathbf{c}-\mathbf{b}\cdot\mathbf{c}-B^2)C \tag{7}$$

$$ABC+(A+2B+2C)G^2+(A-B+2C)\mathbf{G}\cdot\mathbf{b}+(A+2B-C)\mathbf{G}\cdot\mathbf{c}+(A-B-C)\mathbf{b}\cdot\mathbf{c}-(B+C)BC \tag{8}$$

3 Final formula

The initial formula eq. (1)

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{|\mathbf{A}\ \mathbf{B}\ \mathbf{C}|}{ABC+(\mathbf{A}\cdot\mathbf{C})A+(\mathbf{A}\cdot\mathbf{C})B+(\mathbf{A}\cdot\mathbf{B})C}$$

can finally be rewritten by combining eqs. (3) and (8) as

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3\mathbf{G}\cdot(\mathbf{b}\times\mathbf{c})}{ABC+(A+2B+2C)G^2+(A-B+2C)\mathbf{G}\cdot\mathbf{b}+(A+2B-C)\mathbf{G}\cdot\mathbf{c}+(A-B-C)\mathbf{b}\cdot\mathbf{c}-(B+C)BC}$$

References

[1] Adriaan Van Oosterom and Jan Strackee. The solid angle of a plane triangle. *IEEE transactions on Biomedical Engineering*, (2):125–126, 1983.