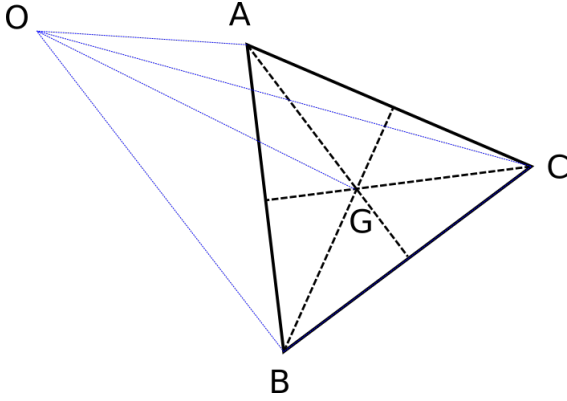


Solid angle subtended by a tetrahedron computation

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1 Notations

Let

- A be the vector \vec{OA}
- B be the vector \vec{OB}
- C be the vector \vec{OC}
- a be the vector \vec{GA}
- b be the vector \vec{GB}
- c be the vector \vec{GC}
- \mathbf{A} be the magnitude of the vector \vec{OA}
- \mathbf{B} be the magnitude of the vector \vec{OB}
- \mathbf{C} be the magnitude of the vector \vec{OC}

2 Computation

The solid angle Ω subtended by the triangular surface ABC is:

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{|A \ B \ C|}{\mathbf{A}\mathbf{B}\mathbf{C} + (A \cdot B)\mathbf{C} + (A \cdot C)\mathbf{B} + (B \cdot C)\mathbf{A}}$$

where $|A \ B \ C| = A \cdot (B \times C)$

2.1 Numerator

Given that $A = \vec{OA} = G + a$ (and resp. with B and C), we get

$$\begin{aligned} |A \ B \ C| &= A \cdot (B \times C) \\ &= A \cdot ((G + b) \times (G + c)) \\ &= (G + a) \cdot (\cancel{G \times G} + G \times c + b \times G + b \times c) \\ &= \cancel{G \cdot (G \times c)} + \cancel{G \cdot (b \times G)} \\ &\quad + G \cdot (b \times c) + a \cdot (G \times c) + a \cdot (b \times G) + \cancel{a \cdot (b \times c)} \\ &= G \cdot (b \times c) + G \cdot (c \times a) + G \cdot (a \times b) \\ &= G \cdot (b \times c + c \times a + a \times b) \end{aligned}$$

since G is the centroid of ABC ,

$$a + b + c = 0 \quad (1)$$

. We obtain

$$\begin{aligned} |A \ B \ C| &= G \cdot (b \times c + c \times a + a \times b) \\ &= G \cdot (b \times c + c \times (-b - c) + (-b - c) \times b) \\ &= G \cdot (b \times c + c \times (-b) + \cancel{c \times (-c)} + \cancel{(-b) \times b} + (-c) \times b) \\ &= G \cdot (3b \times c) \\ &= 3G \cdot (b \times c) \end{aligned} \quad (2)$$

2.2 Denominator

First let's see the term in \mathbf{C}

$$\begin{aligned} (A \cdot B) \mathbf{C} &= (G + a) \cdot (G + b) \mathbf{C} \\ &= (\mathbf{G}^2 + G \cdot b + a \cdot G + a \cdot b) \mathbf{C} \end{aligned}$$

Using (1)

$$\begin{aligned} (A \cdot B) \mathbf{C} &= (\mathbf{G}^2 + G \cdot b + (-b - c) \cdot G + (-b - c) \cdot b) \mathbf{C} \\ &= (\mathbf{G}^2 - G \cdot c - c \cdot b - \mathbf{b}^2) \mathbf{C} \end{aligned} \quad (3)$$

Now, the term in \mathbf{B}

$$\begin{aligned} (A \cdot C) \mathbf{B} &= (G + a) \cdot (G + c) \mathbf{B} \\ &= (\mathbf{G}^2 + G \cdot c + a \cdot G + a \cdot c) \mathbf{B} \end{aligned}$$

Using (1)

$$\begin{aligned} (A \cdot C) \mathbf{B} &= (\mathbf{G}^2 + G \cdot c + (-b - c) \cdot G + (-b - c) \cdot c) \mathbf{B} \\ &= (\mathbf{G}^2 - G \cdot b - c \cdot b - \mathbf{c}^2) \mathbf{B} \end{aligned} \quad (4)$$

For the third term there is no simplification

$$(B \cdot C) \mathbf{A} = (G + b) \cdot (G + c) \mathbf{A} \quad (5)$$

$$= (\mathbf{G}^2 + G \cdot b + G \cdot c + c \cdot b) \mathbf{A} \quad (6)$$

For the computation of the norms, we can use:

$$\begin{aligned} \mathbf{A}^2 &= \|OG\|^2 + \mathbf{a}^2 + 2G \cdot a \\ &= (G + a) \cdot (G + a) \\ &= (G - b - c) \cdot (G - b - c) \end{aligned}$$

and respectively with \mathbf{B} and \mathbf{C} , we obtain

$$(\mathbf{ABC}) = \|G - b - c\|^2 \|G + b\|^2 \|G + c\|^2 \quad (7)$$

and

$$\begin{aligned} (A \cdot B) \mathbf{C} + (A \cdot C) \mathbf{B} + (B \cdot C) \mathbf{A} \\ = \mathbf{G}^2(\mathbf{A} + \mathbf{B} + \mathbf{C}) + c \cdot b(\mathbf{A} - \mathbf{B} - \mathbf{C}) + G \cdot b(\mathbf{A} - \mathbf{B}) + G \cdot c(\mathbf{A} - \mathbf{C}) - \|b\|^2 \mathbf{C} - \|c\|^2 \mathbf{B} \end{aligned} \quad (8)$$

3 Final formula

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G \cdot (b \times c)}{\mathbf{A}\mathbf{B}\mathbf{C} + (A \cdot B)\mathbf{C} + (A \cdot C)\mathbf{B} + (B \cdot C)\mathbf{A}}$$

we get

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G \cdot (b \times c)}{\mathbf{A}\mathbf{B}\mathbf{C} + (\mathbf{G}^2 - G \cdot c - c \cdot b)\mathbf{C} + (\mathbf{G}^2 - G \cdot c - c \cdot b)\mathbf{B} + (G + b) \cdot (G + c)\mathbf{A}}$$

or

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G \cdot (b \times c)}{\mathbf{A}\mathbf{B}\mathbf{C} + \mathbf{G}^2(\mathbf{A} + \mathbf{B} + \mathbf{C}) + c \cdot b(\mathbf{A} - \mathbf{B} - \mathbf{C}) + G \cdot b(\mathbf{A} - \mathbf{B}) + G \cdot c(\mathbf{A} - \mathbf{C})}$$