Solid angle subtended by a tetrahedron computation

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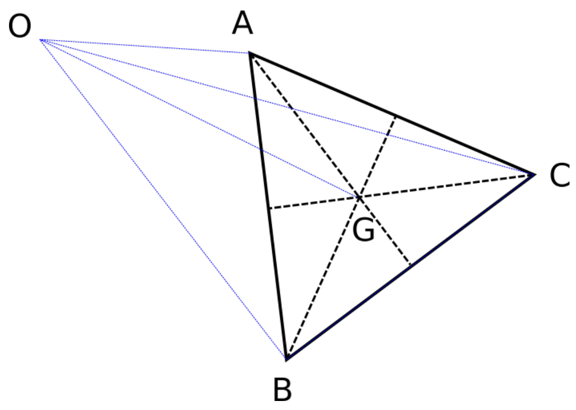

1 Notations

Let

- A be the vector $\vec{OA}$
- B be the vector $\vec{OB}$
- C be the vector $\vec{OC}$
- a be the vector $\vec{GA}$
- b be the vector $\vec{GB}$
- c be the vector $\vec{GC}$
- $\mathbf{A}$ be the magnitude of the vector $\vec{OA}$
- $\mathbf{B}$ be the magnitude of the vector $\vec{OB}$
- $\mathbf{C}$ be the magnitude of the vector $\vec{OC}$

2 Computation

The solid angle Ω subtended by the triangular surface ABC is:

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{|A\ B\ C|}{\mathbf{ABC} + (A\cdot B)\,\mathbf{C} + (A\cdot C)\,\mathbf{B} + (B\cdot C)\,\mathbf{A}}$$

where $|A\ B\ C| = A\cdot(B\times C)$

2.1 Numerator

Given that $A = \vec{OA} = G + a$ (and resp. with B and C), we get

$$\begin{aligned}
|A\ B\ C| &= A\cdot(B\times C)\\
&= A\cdot((G+b)\times(G+c))\\
&= (G+a)\cdot(\cancel{G\times G} + G\times c + b\times G + b\times c)\\
&= \cancel{G\cdot(G\times c)} + \cancel{G\cdot(b\times G)}\\
&\quad + G\cdot(b\times c) + a\cdot(G\times c) + a\cdot(b\times G) + \cancel{a\cdot(b\times c)}\\
&= G\cdot(b\times c) + G\cdot(c\times a) + G\cdot(a\times b)\\
&= G\cdot(b\times c + c\times a + a\times b)
\end{aligned}$$

since G is the centroid of ABC,

$$a + b + c = 0 \tag{1}$$

. We obtain

$$\begin{aligned}
|\underline{\mathrm{A}}\ \underline{\mathrm{B}}\ \underline{\mathrm{C}}| &= \underline{\mathrm{G}} \cdot (\underline{\mathrm{b}} \times \underline{\mathrm{c}} + \underline{\mathrm{c}} \times \underline{\mathrm{a}} + \underline{\mathrm{a}} \times \underline{\mathrm{b}}) \\
&= \underline{\mathrm{G}} \cdot (\underline{\mathrm{b}} \times \underline{\mathrm{c}} + \underline{\mathrm{c}} \times (-\underline{\mathrm{b}} - \underline{\mathrm{c}}) + (-\underline{\mathrm{b}} - \underline{\mathrm{c}}) \times \underline{\mathrm{b}}) \\
&= \underline{\mathrm{G}} \cdot (\underline{\mathrm{b}} \times \underline{\mathrm{c}} + \underline{\mathrm{c}} \times (-\underline{\mathrm{b}}) + \cancel{\underline{\mathrm{c}} \times (-\underline{\mathrm{c}})} + \cancel{(-\underline{\mathrm{b}}) \times \underline{\mathrm{b}}} + (-\underline{\mathrm{c}}) \times \underline{\mathrm{b}}) \\
&= \underline{\mathrm{G}} \cdot (3\,\underline{\mathrm{b}} \times \underline{\mathrm{c}}) \\
&= 3\,\underline{\mathrm{G}} \cdot (\underline{\mathrm{b}} \times \underline{\mathrm{c}})
\end{aligned} \tag{2}$$

2.2 norms

We can write:

$$\begin{aligned}
A^2 &= \underline{\mathrm{A}} \cdot \underline{\mathrm{A}} \\
&= (\underline{\mathrm{G}} + \underline{\mathrm{a}}) \cdot (\underline{\mathrm{G}} + \underline{\mathrm{a}}) \\
&= G^2 + 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{a}} + a^2
\end{aligned}$$

Hence:

$$\begin{aligned}
(ABC)^2 &= (G^2 + 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{a}} + a^2)(G^2 + 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + b^2)(G^2 + 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + c^2) \\
&= (G^2 + 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{a}} + a^2)(G^4 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + G^2c^2 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + 4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}) + 2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + b^2G^2 + 2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + b^2c^2)
\end{aligned}$$

Given that: $\underline{\mathrm{a}} = -(\underline{\mathrm{b}} + \underline{\mathrm{c}}) \Rightarrow a^2 = b^2 + 2\,\underline{\mathrm{b}} \cdot \underline{\mathrm{c}} + c^2$
We can write:

$$\begin{aligned}
(ABC)^2 &= (G^2 + 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{a}} + a^2)(G^4 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + G^2c^2 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + 4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}) + 2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + b^2G^2 + 2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + b^2c^2) \\
&= (G^2 - 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} - 2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + b^2 + 2\,\underline{\mathrm{b}} \cdot \underline{\mathrm{c}} + c^2) \\
&\quad (G^4 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + G^2c^2 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + 4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}) + 2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + b^2G^2 + 2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + b^2c^2) \\
&= G^2\left(G^4 + \cancel{2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}} + G^2c^2 + \cancel{2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}}} + \cancel{4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}})} + \cancel{2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}}} + b^2G^2 + \cancel{2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}} + b^2c^2\right) \\
&\quad -2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}}\left(\cancel{G^4} + \cancel{2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}} + \cancel{G^2c^2} + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + 4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}) + 2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + \cancel{b^2G^2} + \cancel{2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}} + \cancel{b^2c^2}\right) \\
&\quad -2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}\left(\cancel{G^4} + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + \cancel{G^2c^2} + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + 4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}) + \cancel{2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}}} + \cancel{b^2G^2} + 2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + \cancel{b^2c^2}\right) \\
&\quad +2\,\underline{\mathrm{b}} \cdot \underline{\mathrm{c}}\left(G^4 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + G^2c^2 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + 4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}) + 2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + b^2G^2 + 2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + b^2c^2\right) \\
&\quad +b^2\left(G^4 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + G^2c^2 + \cancel{2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}}} + \cancel{4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}})} + \cancel{2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}}} + b^2G^2 + 2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}} + b^2c^2\right) \\
&\quad +c^2\left(G^4 + \cancel{2G^2\,\underline{\mathrm{G}}} \cdot \underline{\mathrm{c}} + G^2c^2 + 2G^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + \cancel{4(\underline{\mathrm{G}} \cdot \underline{\mathrm{b}})(\underline{\mathrm{G}} \cdot \underline{\mathrm{c}})} + 2c^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{b}} + b^2G^2 + \cancel{2b^2\,\underline{\mathrm{G}} \cdot \underline{\mathrm{c}}} + b^2c^2\right)
\end{aligned}$$

2.3 Denominator

First let's see the term in $\mathbf{C}$

$$\begin{aligned}
(A \cdot B)\,\mathbf{C} &= (G + a) \cdot (G + b)\,\mathbf{C} \\
&= (\mathbf{G}^2 + G \cdot b + a \cdot G + a \cdot b)\mathbf{C}
\end{aligned}$$

Using (1)

$$\begin{aligned}
(A \cdot B)\,\mathbf{C} &= (\mathbf{G}^2 + G \cdot b + (-b - c) \cdot G + (-b - c) \cdot b)c \\
&= (\mathbf{G}^2 - G \cdot c - c \cdot b)\mathbf{C}
\end{aligned} \tag{3}$$

Now, the term in $\mathbf{B}$

$$\begin{aligned}
(A \cdot C)\,\mathbf{B} &= (G + a) \cdot (G + c)\,\mathbf{B} \\
&= (\mathbf{G}^2 + G \cdot c + a \cdot G + a \cdot c)\mathbf{B}
\end{aligned}$$

Using (1)

$$\begin{aligned}
(A \cdot C)\,\mathbf{B} &= (\mathbf{G}^2 + G \cdot c + (-b - c) \cdot G + (-b - c) \cdot c)b \\
&= (\mathbf{G}^2 - G \cdot c - c \cdot b)\mathbf{B}
\end{aligned} \tag{4}$$

For the third term there is no simplification

$$(B \cdot C)\,\mathbf{A} = (G+b)\cdot(G+c)\,\mathbf{A} \tag{5}$$
$$= (\mathbf{G}^2 + G\cdot b + G\cdot c + c\cdot b)\mathbf{A} \tag{6}$$

For the computation of the norms, we can use:

$$\begin{aligned}\mathbf{A}^2 &= ||OG||^2 + \mathbf{a}^2 + 2G\cdot a\\ &= (G+a)\cdot(G+a)\\ &= (G-b-c)\cdot(G-b-c)\end{aligned}$$

and respectively with B and C, we obtain

$$(\mathbf{ABC}) = ||G-b-c||\,||G+b||\,||G+c|| \tag{7}$$

and

$$(A\cdot B)\,\mathbf{C} + (A\cdot C)\,\mathbf{B} + (B\cdot C)\,\mathbf{A} = \mathbf{G}^2(\mathbf{A}+\mathbf{B}+\mathbf{C}) + c\cdot b(\mathbf{A}-\mathbf{B}-\mathbf{C}) + G\cdot b(\mathbf{A}-\mathbf{B}) + G\cdot c(\mathbf{A}-\mathbf{C}) \tag{8}$$

3 Final formula

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G\cdot(b\times c)}{\mathbf{ABC} + (A\cdot B)\,\mathbf{C} + (A\cdot C)\,\mathbf{B} + (B\cdot C)\,\mathbf{A}}$$

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G\cdot(b\times c)}{\mathbf{ABC} + (\mathbf{G}^2 - G\cdot c - c\cdot b)\mathbf{C} + (\mathbf{G}^2 - G\cdot c - c\cdot b)\mathbf{B} + (G+b)\cdot(G+c)\,\mathbf{A}}$$