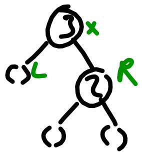
Op3) Binary trees BT

- $() \in BT$
- $L, R \in BT$, some value x, then $(L, x, R) \in BT$

Define $f : BT \to \mathbb{N}$

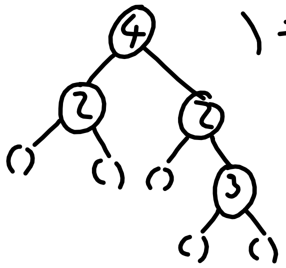
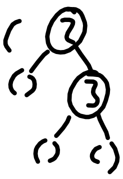
- $f(()) = 0$
- $f((L,x,R)) = \begin{cases} 1, & \text{if } L = R = () \\ f(L) + f(R), & \text{otherwise} \end{cases}$ (*)

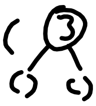
The same as $L = ()$ and $R = ()$

a) $f((\underbrace{()}_{L}, \underbrace{3}_{x}, \underbrace{((),2,())}_{R})) \equiv f($  $)$

$= f(()) + f((\underbrace{()}_{L}, \underbrace{2}_{x}, \underbrace{()}_{R}))$

$= 0 + 1 = 1$

$f($  $) = f($  $) + f($  $)$

$= 1 + [f(()) + f($  $)]$

$= 1 + [0 + 1]$

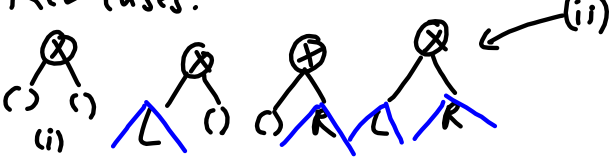
$= 2$

b) Show that f computes the number of nodes in a tree which have two empty children.

Proof by induction:

Base case: $f(()) = 0$ ✓

other cases:



We can assume that f has this property for L, R in those cases.

(i) $f($  $) = 1$ ✓

(ii) $f($  $) \overset{(*)}{=} f(L) + f(R)$, for $L \neq ()$, $R \neq ()$, because we handled that in the separate case (i).

Ex.: Given the following function

$g: \mathbb{N} \to \mathbb{N}$

$g(0) = 0$

$g(1) = 1$

(*) $g(n+2) = g(n)$, $n \geq 0$ $\equiv$ $g(n) = g(n-2)$, $n \geq 2$

Show that for all $n \in \mathbb{N}$,

$$g(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

Proof by induction:

Base cases: $g(0) = 0$, 0 even ✓
$g(1) = 1$, 1 odd ✓

Induction: Assume that our property holds up to n, show that

$$g(n+1) = \begin{cases} 0, & \text{if } n+1 \text{ is even} \\ 1, & \text{if } n+1 \text{ is odd} \end{cases}$$

Proof:

$$g(n+1) \underset{(*)}{=} g(n-1) \underset{\text{by IH}}{=} \begin{cases} 0, & \text{if } n-1 \text{ even} \\ 1, & \text{if } n-1 \text{ odd} \end{cases}$$

Case distinction:

i) $n-1$ even $\boxed{\leadsto}$ $n+1$ is also even $\leadsto$ $g(n+1) = g(n-1) = 0$ ✓

ii) $n-1$ odd $\boxed{\leadsto}$ $n+1$ is also odd $\leadsto$ - - - - - $= 1$ ✓

by high-school math

Now use this property to show that g is idempotent, i.e. $g(g(n)) = g(n)$ for all $n \in \mathbb{N}$.