Robot Mobility - State Space Modeling Exercises

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Literature

K. J. Aåström and R. M. Murray. Feedback Systems: An Introduction for Scientists and Engineers. Princeton University Press, 2008.

Exercise 1

Figure 1 illustrates the electromechanical model of a DC motor.

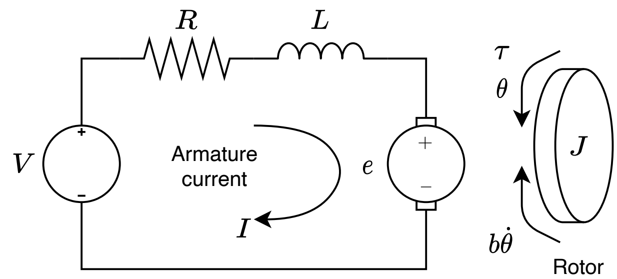

Figure 1: Sketch of a DC-motors electrical and mechanical model.

1. Derive a state space model of the DC-motor system, where voltage V is considered as input and rotational velocity $\omega = \dot{\theta}$ is considered as output.
2. Compute the eigenvalues of the system matrix. Where in the complex plane are these located and why? Use the following values: $J = 0.01$, $R = 1$, $L = 0.5$, $b = 0.1$, $K_t = 0.01$, $K_e = 0.01$.
3. Implement the model in MATLAB and simulate a step response.
4. Change the model to also output the armature current, I, and simulate a step response. What does the response indicate?
5. Simplify the model to only consider the mechanical dynamics. That is, assume $\dot{I} = 0$.
6. Simulate the simplified system and compare the step response to the original systems response. Then try to increase the inertia to $J = 0.2$. What can you conclude from this experiment?

Exercise 2

Figure 2 illustrates the classical balancing system; reversed pendulum on a cart.

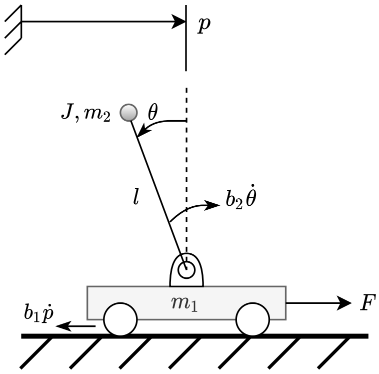

Figure 2: Inverted pendulum on a cart.

By using the Euler-Lagrange formulation the system in Figure 2 can be written in general form as

$$M(q)\ddot{q} + C(q,\dot{q}) + F(\dot{q}) + g(q) = H(q)u, \tag{1}$$

where $M(q)$ is the inertia matrix, $C(q,\dot{q})$ represents the Coriolis and centrifugal forces, $F(\dot{q})$ describes the friction forces, $g(q)$ gives the forces due to potential energy (gravity) and $H(q)$ describes how the external applied forces couple into the dynamics. For the system we choose the horizontal position and velocity of the cart, $(p, \dot{p})$, and the angle and angular velocity of the pendulum, $(\theta,\dot{\theta})$, as state variables. We let the force applied to the cart, (F), be the input. As output we choose the horizontal position of the cart, (x), and the angle of the pendulum (θ). By computing the dynamics of the system we get the following result

$$\begin{bmatrix} (m_1+m_2) & -m_2 l\cos(\theta) \\ -m_2 l\cos(\theta) & (J+m_2l^2) \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} m_2 l \sin(\theta)\dot{\theta}^2 \\ 0 \end{bmatrix} + \begin{bmatrix} b_1\dot{p} \\ b_2\dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -m_2 g l\sin(\theta) \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \tag{2}$$

where m_1 is the mass of the base, m_2 and J are the mass and moment of inertia of the pendulum, l is the distance from the base to the center of mass of the pendulum, b_1 and b_2 are coefficients of viscous friction and g is the acceleration due to gravity. We can rewrite the dynamics of the **nonlinear** system in state space form by defining the state, input, and output as follows

$$x = \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix}, \quad u = F, \quad y = \begin{bmatrix} p \\ \theta \end{bmatrix}. \tag{3}$$

Let the total mass be defined as $M_t = m_1 + m_2$ and the total inertia defined as $J_t = J + m_2 l^2$. Then the nonlinear equations can be written as

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \frac{-m_2 l s_\theta \dot{\theta}^2 + m_2 g (m_2 l^2/J_t) s_\theta c_\theta - b_1 \dot{p} - b_2 l m_2 c_\theta \dot{\theta} + u}{M_t - m_2 (m_2 l^2/J_t) c_\theta^2} \\ \frac{-m_2 l^2 s_\theta c_\theta \dot{\theta}^2 + M_t g l s_\theta - b_1 l c_\theta \dot{p} - b_2 (M_t/m_2)\dot{\theta} + l c_\theta u}{J_t (M_t/m_2) - m_2 (l c_\theta)^2} \end{bmatrix}, \tag{4}$$

where $c_\theta = \cos(\theta)$ and $s_\theta = \sin(\theta)$.

1. Linearize the system in (4) around the operating point $\bar{x} = (0, 0, 0, 0)$, $\bar{u} = 0$. You can use MATLAB's symbolic toolbox for the derivations.

2. Compute the eigenvalues of the system matrix. Where in the complex plane are these located and why? Use the following values: $m_1 = 2$, $m_2 = 1$, $J = 0.01$, $l = 1$, $b_1 = b_2 = 0.1$, $g = 9.81$.

3. Implement the **nonlinear** model in MATLAB Simulink and simulate its behavior, with the following initial conditions $x = (0, 0.01, 0, 0)$, $u = 0$. You can get inspiration from the below links, but i urge you to implement it on your own.

- https://se.mathworks.com/help/mpc/ug/control-of-an-inverted-pendulum-on-a-cart.html
- https://se.mathworks.com/help/control/examples/control-of-an-inverted-pendulum-on-a-cart.html

Exercise 3

Figure 3 illustrates a differential drive two wheeled mobile robot seen from above.

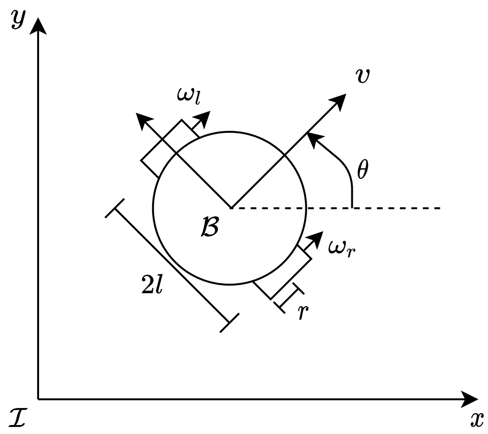

Figure 3: Two wheeled differential drive mobile robot seen from above. $\mathcal{I}$ is the inertial frame and $\mathcal{B}$ is the robots body frame.

Where

- x is the robots x-position in the inertial frame,
- y is the robots y-position in the inertial frame,
- v represents the robots linear velocity,
- θ represents the robots heading angle (rotation angle between inertial frame and body frame about the z-axis),
- ω_l represents the left wheels rotational velocity,
- ω_r represents the right wheels rotational velocity,
- l is the distance from wheel center to robot body center,
- r is the radius of the wheels.

For the above system solve the following

1. Derive an equation describing the robots linear velocity, v.
2. Derive equations describing the velocities $\dot{x}$ and $\dot{y}$ of the robot in the inertial frame.
3. Derive an equation describing the rotational velocity of the robots heading, $\dot{\theta}$.
4. Derive dynamic model describing the wheels velocity assuming they are powered by DC-motors (See Exercise 1).
5. Collect all equations in one nonlinear state space model, with the state vector given as $\xi = \begin{bmatrix} x & y & \theta & \omega_l & \omega_r \end{bmatrix}^\top$.
6. Implement the nonlinear model in MATLAB/Simulink and simulate it.
7. Linearize the model around the operating point $\bar{\xi} = (\bar{x}, \bar{y}, \bar{\theta}, \bar{\omega_l}, \bar{\omega_r})$

We will be using this model along with the pendulum model throughout the course, so don't forget them :-)

Exercise 4

Figure 4 illustrates two coupled masses (carts) connected through a lossless spring, moving on a surface with no friction.

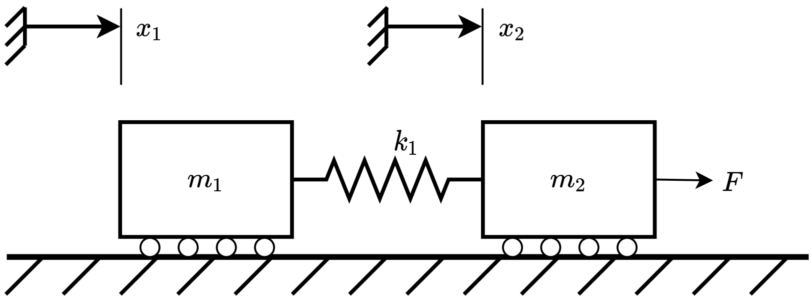

Figure 4: Two coupled carts moving on a surface with no friction.

1. Create free-body diagrams of the system in Figure 4.
2. Derive a state space model of the system. Assume that the force F is considered as input and velocity $v = \dot{x}_1$ is considered as output.
3. Compute the eigenvalues of the system matrix. Where in the complex plane are these located and why? Choose and play with different values for m_1, m_2, and k_1.
4. Implement the model in MATLAB and simulate its impulse response. Does the behavior make sense?

Exercise 5

Figure 5 illustrates two coupled masses (carts) connected through a lossless spring and a damper. Moreover, the first cart is connected through a lossless spring to an immovable wall.

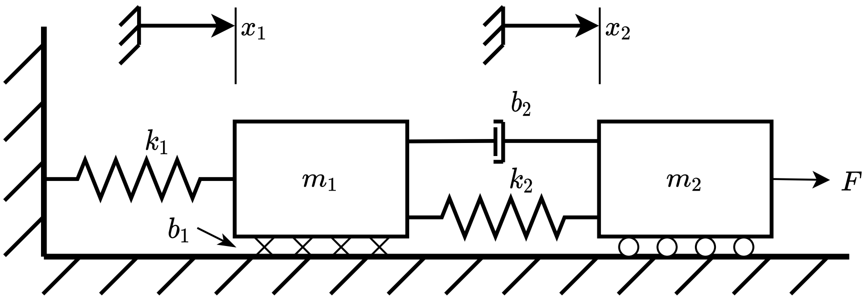

Figure 5: Two coupled carts which movements are constrained.

1. Create free-body diagrams of the system in Figure 5.
2. Derive a state space model of the system. Assume that the force F is considered as input and positions x_1 and x_2 are considered as output.
3. Compute the eigenvalues of the system matrix. Where in the complex plane are these located and why? Choose and play with different values for m_1, m_2, k_1, k_2, b_1, and b_2.
4. Implement the model in MATLAB and simulate its impulse and step response. Does the behavior make sense?