Mobility: Lecture 1

Lecture Notes

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1 Introduction

4 hours on next 6 mondays. Shahab will tell us on ordinary system theory after John.
We are going to try to make a turtlebot move in the end of the course. We need to use the theory. The exercises and the turtlebot will be taken care off by Mirhan.

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2 Input-Output systems

The standard differential equation.

$$\Sigma : \dot{x} = f(x, u)y = h(x)$$

where $f : D_x \times D_u \rightarrow \mathbb{R}^n, D_x, \subset \mathbb{R}^n, D_u \subset \mathbb{R}^m$

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3 First Exercise: Cart Pendulumn

3.1 Lagrange (d'Alembert)

$L = T - U$
Where T is the kinetic energy, and U is the potential energy.
The coordinates: You would need 2 coordinates to describe this cart system.

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{1}$$

$$r_c = \begin{bmatrix} x \\ 0 \end{bmatrix} \tag{2}$$

$$v_c = \dot{r_c} = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} \tag{3}$$

Where $v_c^2 = v_c \cdot v_c$ is the inner product. It gives a scalar. $= v_c^T v_c = \dot{x}^2$

$$r_p = \begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} x - l \cdot sin(\theta) \\ l \cdot cos(\theta) \end{bmatrix}$$

$$v_p = \begin{bmatrix} \dot{x} - l \cdot cos(\theta)\dot{\theta} \\ -l \cdot sin(\theta)\dot{\theta} \end{bmatrix} \begin{aligned} v_p^2 = v_p \cdot v_p = v_p^T v_p &= (\dot{x} - l \cdot cos(\theta\dot{\theta})^2 + (l \cdot sin(\theta\dot{\theta})) \\ &= \dot{x}^2 + l^2 cos^2(\theta \cdot \dot{\theta}^2) - 2\dot{x}lcos(\theta)\dot{\theta} + l^2 sin^2(\theta\dot{\theta}^2 \\ &= \dot{x}^2 + l^2\dot{\theta}^2 - 2\dot{x}lcos(\theta\dot{\theta}) \end{aligned} \tag{4}$$

$\theta + 2$
$\mathcal{L}\{sin(t)\} = \frac{1}{s+1}$

3.1.1 Kinetic energy: T

The kinetic energy of the cart, T_c, and the pendulum, T_p

Cart

$$T_c = \frac{1}{2}m_c v_c^2 = \frac{1}{2}m_c\dot{x}^2 \tag{5}$$

Pendulum

$$T_p = \frac{1}{2}m_p v_p^2 = \frac{1}{2}m_p(\dot{x}^2 + l^2\dot{\theta}^2 - 2\dot{x}lcos(\theta\dot{\theta}) \tag{6}$$

$$T_c = \frac{1}{2}m_c v_c^2 \tag{7}$$

The Total kinetic energy $T = T_c + T_p$:

$$T = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\dot{\theta}^2 - m_p l^2\dot{\theta}^2 - m_p\dot{x}lcos\theta\dot{\theta} \tag{8}$$

and $m = m_c + m_p$

3.1.2 The Potential Energy

Cart
The potential energy of the cart is 0, since it is constrained in the y direction. The potential energy is given relative to some point, and since we just assume the origin in at the same height as the cart, the potential energy $u_c = 0$

$$u_c = m_c g h_c = m_c g 0 = 0$$

Pendulum

$$u_p = m_p g h_p = m_p g l cos\theta$$

Total potential energy

$$u = u_c + u_p = u_p = m_p g l cos\theta \tag{9}$$

From this we now have the equations of motion (EOM), which we can use to derive the lagrange equation:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\dot{q}} - \frac{\partial\mathcal{L}}{q} \tag{10}$$

Where $\frac{\partial\mathcal{L}}{q}$ is the conservative forces.

Conservative forces are forces that do not change the energy that is going in or out of the system.

Non-conservative forces change the energy of the system, e.g. friction. $\boxed{f_r = -\nabla u(r)}$

Note: That the lagrange equation has 4 independent variables, since $q = \begin{bmatrix} x \\ \theta \end{bmatrix}$, thus $\mathcal{L}(x, \dot{x}, \theta, \dot{\theta})$.
If we start to derive the lagrange equation w.r.t. q, then we get:

$$x : \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\dot{x} : \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} - m_p l cos\theta\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x} - m_p l sin\theta\dot{\theta}^2 - m_p l cos\theta\ddot{\theta}$$

In general we have $\boxed{\mathcal{L} = T - U}$

$$\mathcal{L} = \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}m_p\dot{\theta}^2 - m_p l^2\dot{\theta}^2 - m_p\dot{x}lcos\theta\dot{\theta} - m_p glcos\theta \tag{11}$$

Where the last term is U
This results in the following equations of motion:

$$m\ddot{x} + m_p l sin\theta\dot{\theta}^2 - m_p l cos\theta\ddot{\theta} = -\alpha_x\dot{x} + u \tag{12}$$

Where the right term is any non conservative force; in this case it is viscous friction, with the friction coefficient α_x