Robot Mobility: Lecture 3
Lecture Notes

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1 Stability

The linear system Σ:

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

With the solution:

$$x = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds \tag{3}$$

First we consider if a system is stable.

$$\Sigma_1 = \dot{x} = Ax, x(0) = x_0 \to x = e^{At}x_0 \tag{4}$$

Where we use the taylor expansion to find e^{At}

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \tag{5}$$

1.1 Example

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{6}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{bmatrix} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} (\lambda_1 t)^k & 0 \\ 0 & (\lambda_2 t)^k \end{bmatrix} \tag{7}$$

Note that It is only possible to do elementwise esponentials because A is Diagonal.

$$= \sum_{k=0}^{\infty} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(\lambda_1 t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(\lambda_2 t)^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} \end{bmatrix} \tag{8}$$

And using (4)

$$x = e^{At}x_0 = \begin{bmatrix} e^{\lambda_1 t}x_{01} \\ e^{\lambda_2 t}x_{02} \end{bmatrix} \tag{9}$$

And if $\lambda_1 < 0$ and $\lambda_2 > 0$, the phase plot entails that it is not stable.

Since it is not always possible to do exponetials in (7), we set up a general ca, we set up a general case.

$$M = e^{At} \rightarrow m_{ij} = \sum_i \alpha_i t^i e^{\lambda_i t} \tag{10}$$

Where $\lambda_i \in \sigma(A)$ is the spectrum of A, namely the set of eigenvaules of A.

$\boxed{\textbf{def:}\text{ The system } \Sigma_1 \text{ is stable if for every } x_0 \; x(t; x_0) \rightarrow 0 \text{ for } t \rightarrow 0.}$

Note: If A is non-singular, then the only steady state vector is the null vector. This is the same as positive definitness:

$$\boxed{x^T B x > 0 \quad \forall x \neq 0}$$

Where B has to be symmetric.
Note that this is easy to check, becuase that B has to have positive eigenvalues.
Therefore $0 < \lambda_i \in \sigma(B)$
This entails the theorem:

$\boxed{\textbf{Theorem:}\text{ The system } \Sigma_1 \text{ is stable iff } Re(\lambda) < 0 \quad \forall \lambda \in \sigma(A)}$

Sidenote on complex numbers and imaginary axis:

$$e^{\lambda t} = e^{\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t} \tag{11}$$

Where if $\alpha < 0$ the spiral is $\rightarrow 0$, and it has a small magnitude.

Relation to Kalman Decomposition

$$\dot{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \tag{12}$$

Where

$$\dot{z_1} = A_{11} z_1 + A_{12} z_2 + B_1 u \tag{13}$$

$$\dot{z_2} = A_{22} z_2 \tag{14}$$

Note that A_{12} does not really have an effect on the system, as the z_2 is dependent on (14).

Introducing the Gain

$$\dot{x} = Ax + Bu \tag{15}$$

$$u = Kx \tag{16}$$

Where $K \in \mathbb{R}^{m \times n}$ and this entails with substitution:

$$\dot{x} = Ax + Bu = (A + BK)x \tag{17}$$

Then we can choose K s.t. $\sigma(A + BK) \subset \mathbb{C}_-$
$\boxed{\mathbb{C}_- = \{\lambda \in \mathbb{C} | Re(\lambda) < 0\}}$ Thus it is the set of all complex numbers with negative real parts.

2 Stabilizability

Control system: whenever nothing is written on summation circles, it is always +.

def: The system Σ or the pair (A, B) is stabilizable if there is a K s.t. $\sigma(A + BK) \subset \mathbb{C}_{-}$

Then: Let $\Lambda = \{\lambda_1, ..., \lambda_n\} \subset \mathbb{C}$ with Λ then $\lambda_i^* \in \Lambda$. Then (A, B) is controllable then there is a K s.t. $\sigma(A + BK) = \Lambda$

3 Reference Tracking

Given a (constant) r find u s.t. $y(t) \to r$ as $r \to \infty$
Note that $\dot{x} \to 0$ when steady state has reached. This entails that there is not dynamics at that point.
So: Given the state input pair (x', u')

$$0 = Ax' + Bu' \tag{18}$$

$$r = Cx' \tag{19}$$

Let

$$\begin{bmatrix} 0 \\ I \end{bmatrix} r = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x' \\ u' \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} x' \\ u' \end{bmatrix} r = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I' \end{bmatrix} \tag{21}$$

You can only do reference tracking for inputs with the same number of inputs as outputs. $(n + p) \times (n + m), p = m$

$$= \begin{bmatrix} N_x \\ N_u \end{bmatrix} r \tag{22}$$

So we want to design a control input u such that y follows r. We introduce a new variable $z = x - x'$ and $v = u - u'$ and whenever $z \to 0$, $x \to x'$
Lets start by finding $\dot{z}$

$$\dot{z} = \dot{x} = Ax + Bu = Az + Ax' + Bv + Bu' = Az + Bv \tag{23}$$

Hence we chose a $v = kz$ s.t. $\sigma(A + Bk) \subset \mathbb{C}_{-}$ then:

$$z(t) = x(t) - x' \to 0 \text{that is} x(t) \to x' \text{so} y(t) = Cx(t) \to Cx' = r \tag{24}$$

Do using:
$v = kz \leftrightarrow u - u' = k(x - x') \leftrightarrow$

$u = k(x - x') + u'$
$= k(x - N_x r) + N_u r$
$= kx + Nr \quad N = Nu - KN_x$

When the model is not precise enough, there will be some **error**, namely the steady state error.

Side Note we can use the place function in matlab to place poles. Note that it assumes positivity, so add a minus before it.
For this we should use the integral action on the system. We introduce a new state:

$$X_I = \int e\, ds \quad e = y - r \tag{25}$$

$$\dot{x} = Ax + Bu \tag{26}$$

$$\dot{x_I} = e = y - r \tag{27}$$

$$y = Cx \tag{28}$$

Combining (26) - (28) entails

$$\dot{\begin{bmatrix} x \\ x_I \end{bmatrix}} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} r \tag{29}$$

And by renaming the different terms:

$$\dot{\overline{x}} = \overline{A}\overline{x} + \overline{B}u + Gr \tag{30}$$

and we do exactly the same as before:

$$z = \overline{x} - \begin{bmatrix} x' \\ 0 \end{bmatrix} \quad v = u - u' \tag{31}$$

$$\dot{z} = \overline{A}\left(z + \begin{bmatrix} x' \\ 0 \end{bmatrix}\right) + \overline{B}(v + u') + Gr = \overline{A}z + \overline{B}v \tag{32}$$

Hence

$$v = \overline{K}z \;\; u = \overline{k} \begin{bmatrix} x - x' \\ x_I \end{bmatrix} + u' \tag{33}$$

$$= \begin{bmatrix} k & k_I \end{bmatrix} \begin{bmatrix} x - x' \\ x_I \end{bmatrix} \tag{34}$$

$$= k(x - x') + u' + k_I x_I \tag{35}$$

$$= k(x - N_x r) + N_u r + k_I x_I \tag{36}$$

Note that K has to be designed at once, you cannot design K and K_I seperately.

4 Linear Quadratic Regulator

The way we choose a controller is by minimizing the control.

$$min_u \int 0^T x^* Qx + u^* Ru \, dt \tag{37}$$

s.t.

$$\dot{x} = Ax + Bu \tag{38}$$

And the solution is:

$$u = R^{-1} B^* P x \tag{39}$$

where the part before x is the gain K where

$$T < \infty : \dot{P} = -Q - PA^* - AP + PBR^{-1}B^*P, \quad P(T) = 0 \tag{40}$$

$$T = \infty : 0 = -Q - PA^* - AP + PBR^{-1}B^*P \tag{41}$$

and $Q \geq 0$ and $R > 0$ must be positive semidefinte and positive definite, respectively.
Here is the R identity and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$

$$\int_0^T x^* \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} x \ dt = \int_0^T x_1^2 + 10x_2^2 dt \tag{42}$$

The control law will try to control the second part a lot quicker than the first part.
P is called the lapynov matrix, and it is widely used in control.
When we do reference tracking:

$$(y - r)^* Q(y - r) \tag{43}$$

Then: The system $\dot{x} = Ax$ is stable iff $\exists P > 0$ s.t. $PA^* + AP < 0$

Think of it as the energy of the system. $x^T P x$ And it is used in the section on constraints.