Robot Mobility: Lecture 2

Lecture Notes

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September 18, 2023

1 Recap

A system of first order ODE's

$$\dot{z} = f(z, v) \tag{1}$$
$$\overline{y} = h(x)$$

1.1 Linearization

where $\dot{z}$ is a hidden state. Today we will only look at (1)
First we look at the equilibrium points $f(\overline{z}, \overline{v})$ and we end up with a system $\dot{x} = Ax + Bu$.

$\boxed{A \text{ is the jacobian of } f \text{ with respect to } z \text{ of } (\ \overline{z}, \overline{v})}$

Note: There always n states, m inputs and p outputs.

Remember the left side by: MOI * acceleration (Newtons law for rotating bodies)

$$ml^2\ddot{\theta} = mglsin\theta - \beta\dot{\theta} + mlcos\theta\ddot{x} \tag{2}$$

First we isolate $\ddot{\theta}$ (divide through with ml^2 (note that the constants are simplified to a, b, c

$$\ddot{\theta} = asin\theta - b\dot{\theta} + ccos\theta u \tag{3}$$

$\Downarrow$

We write up differentials up untill the one below the highest order of derivatives. in this case it is up to velocity, since (3) uses acceleration.

$$z_1 = \theta, z_2 = \dot{z}$$

$$\dot{z_1} = z_2, z_2 = asinz_1 - bz_2 + ccosz_1$$

The above two equations is the definition of $\dot{z}f(z, v)$

1.2 Equlibrium point

Now we want to find the equilibrium point. $0 = f(\overline{z}, \overline{v})$ Emmidiately we can see that the velocities are 0, this means that $z_2 = \dot{z_1} = 0$. Thus

$$0 = asin\overline{z_1} + ccos\overline{z_1}\overline{v}$$

since we want the pendulumm to be upright, where $\theta = 0$.

$$z_1 = 0 \rightarrow \overline{v} = 0$$

This entails that we want to find the Jacobian at 0
$\boxed{\text{The first row of the jacobian is the gradient of } \dot{z_1} = z_2}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{4}$$

$$x = z - \overline{z} = z$$

$$u = v - \overline{v} = v$$

2 Controlability

$$\Sigma : \dot{x} = Ax + Bu, x(0) = x_0, y = Cx \tag{5}$$

where $x(0) = x_0$ is the initial condition
We can use
TODO intdef expands at in.

$$x = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{6}$$

We have the following definition:

def:
The system Σ, or the pair (A, B), is controllable (at time T) if for every (x_0, x_1) there exist a $u \in U$
the notation is: $\boxed{x = x(t) = x(t, \overline{x}, u)}$ where $\overline{x}$ is the state which i have to be in!

$$x_0 = x(0; x_0, u) \tag{7}$$

$$x_1 = x(T; x_0, u) \tag{8}$$

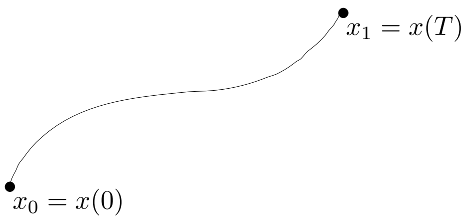

Figure 1: reachability

3 Reachability

Reachable subspace:

$$W_T = \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau | u \in U \subseteq \mathbb{R}^n \tag{9}$$

def: Σ is reachable if $W_T = \mathbb{R}^n$
This entails that $W_T = Range[A|B]$
Where the reachability matrix: $[A|B] = [BABAb^2 \cdots A^{n-1}B] \in \mathbb{R}^{n \times nm}$

$\boxed{\Sigma \text{ is reachable} \leftrightarrow \Sigma \text{ is controlable}}$

Pendulum

$$[A|B] = \begin{bmatrix} 0 & c \\ c & -bc \end{bmatrix} \tag{10}$$

hence is controlable since it has full rank. It looses rank if $c = 0$ so the condition is: (if $c \neq 0$)
Example: $a = 2, b = c = 1$. Now based on (4)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

We compute the eigenvalues:

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = 1 \rightarrow Av_1 = \lambda_1 v_1$$

$$v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \lambda_2 = -2 \rightarrow Av_2 = \lambda_2 v_2$$

This system is unstable by nature.

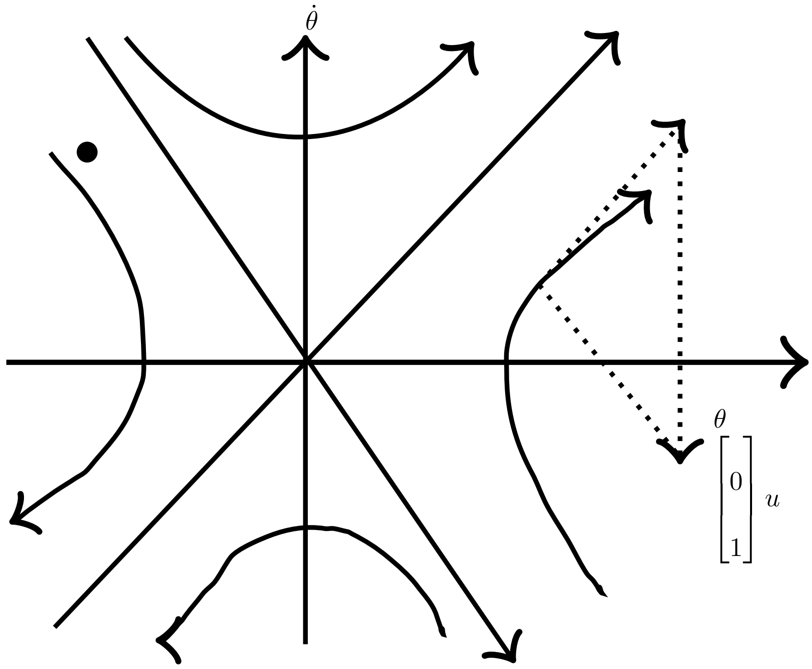

Figure 2: Eigenspace

If you shut of the control $\dot{x} = Ax$, and give it some initial condition. Then Figure 2 tells me how the system will behave. Only a few initial conditions, will bring the system to 2, and these are along the eigenvectors.
Note that the straight arrows are the subspace spanned by the eigenvectors v_1 and v_2

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4 Kalman decomposition

What happens if we loose controlability:
i.e. if $Rank[A|B] = l < n$ then there exists a matrix $P \in \mathbb{R}^{n \times n}(z = Px)$ s.t.

$$\begin{aligned} \dot{z} = P\dot{x} = PAx + PBu = PAP^{-1}z + PBu \\ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} z + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u \end{aligned}$$

with (A_{11}, B_1) controlable. Moreover $P = [e_1 \cdots e_l e_{l+1} \cdots e_n]^{-1}$
with $Span\{e_1, \cdots, e_l\} = Range[A|B]$ and $Span\{e_1, \cdots e_l\} = \mathbb{R}^n$
If we constrain the environment of reachability, then we can still control it. It is still reachable in the l-dimensional subspace.
TODO: Add split and gathered to equation environments

$$\begin{gathered} \dot{z_1} = A_{11}z_1 + A_{12}z_2 + B_1u \quad A_{11} \in \mathbb{R}^{l \times l} \\ \dot{z_2} = A_{22}z_2 \end{gathered} \tag{11}$$

Note that $\dot{z_2}$ will either approach 0 or ∞

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow B = \begin{bmatrix} -1 \\ 2 \end{bmatrix} v_2$

$$[A|B] = [BAB] = [v_2 \lambda_2 v_2] = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \quad Rank[A|B] = 1 < 2$$

Lets do an orthogonal basis to v_2:

$$P = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}^{-1} \tag{12}$$

Now we compute PAP^{-1}
(12) implies that:

$$\begin{gathered} \dot{z} = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ \dot{z_1} = -2z_1 - z_2 + u \\ \dot{z_2} = z_2 \rightarrow z_2 = e^t z_{20} \end{gathered} \tag{13}$$

where z_{20} is an initial condition of z_2. The value of $\dot{z_2} \rightarrow \infty$
In order to reach controllability, we need the points to be on the subspaces spanned by the eigenvectors. These are the controlable subspaces.
next we do v_1

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 \rightarrow \dot{z} = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

note that, the -2, results in the t in (13) is negative, thus it is controlable.