A crash course in linear system theory

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The purpose of this note is to introduce the ”working engineer” to fundamental concepts from linear system theory. Hence emphasis is on terminology, results and formalism not proofs. For a detailed introduction to the subject see [TSH01, ÅM08, AM06], all having editions freely available online.

1 Introduction

A large class of real world systems can be modeled as the (continuous-time) nonlinear input-output (IO) system

$$\dot{x} = f(x, u),\ y = h(x) \tag{1}$$

with[1] $x = x(t)$ the state (trajectory), $u = u(t)$ the input (trajectory), and $y = y(t)$ the output (trajectory), $\dot{x} = \frac{dx}{dt}$, $f : \mathbb{D}_x \times \mathbb{D}_u \to \mathbb{R}^n$, $h : \mathbb{D}_x \to \mathbb{R}^p$ and $\mathbb{D}_x \times \mathbb{D}_u \subseteq \mathbb{R}^n \times \mathbb{R}^m$ with $\mathbb{D}_x$ open. Unfortunately (1) can be extremely difficult to analysing in general. However, when f and h are linear, $f(x, u) = Ax + Bu,\ h(x) = Cx$ and A, B, C matrices of appropriate dimensions, very powerful methods exists for analysis and much is therefore know of such systems. This note is concerned with such linear IO systems.

It is remarked that the nonlinear IO system (1) can be approximated locally by a linear IO system: Let x' and u' be such that $0 = f(x', u')$ and $0 = h(x')$. The point (x', u') is sometimes referred to as an equilibrium point (or a steady state state-input pair) of the IO system (1). Applying a first order Taylor approximation at (x', u') yields

$$f(x, u) \approx f(x', u') + \partial_x f(x', u')(x - x') + \partial_u f(x', u')(u - u') \tag{2}$$

$$h(x) \approx h(x') + \partial_x h(x')(x - x') \tag{3}$$

with $\partial_x = \frac{\partial}{\partial x}$ etc. Hence the nonlinear IO system (1) can be approximated by the linear IO system

$$\dot{\bar{x}} = A\bar{x} + B\bar{u},\ \bar{y} = C\bar{x} \tag{4}$$

with $\bar{x} = x - x'$, $\bar{u} = u - u'$, $\bar{y} = y - h(x')$, $A = \partial_x f(x', u')$, $B = \partial_u f(x', u')$ and $C = \partial_x h(x')$. The method for obtaining this linear IO system is called linearization. Note that (4) is only an approximation of (1) locally, i.e., only for x close to x' (or equivalently $\bar{x}$ close to 0) and u close to u' (or equivalently $\bar{u}$ close to 0).

2 Input-output system

In the sequel we consider the (continuous-time) input-output (IO) system

$$\dot{x} = Ax + Bu \tag{5a}$$

$$y = Cx \tag{5b}$$

with $A \in \mathbb{R}^{n\times n}$ the state matrix, $B \in \mathbb{R}^{n\times m}$ the input matrix, and $C \in \mathbb{R}^{p\times n}$ the output matrix.

[1] here t describing (continuous) time

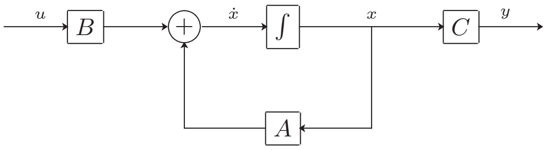


Figure 1: The IO system (5)

Other common names for the IO system (5) are linear time invariant (LTI) system, (linear) control system, or multi-input multi-output (MIMO) system. Moreover, the IO system is sometimes referred to as a single-input multi-output (SIMO) system if $m = 1$, a multi-input single-output (MISO) system if $p = 1$ or a single-input single-output (SISO) system if $m = p = 1$.

The IO systems typically represent a mathematical model of a real-world system (sometimes referred to as a plant) and the task at hand is to control (or synthesize) the output y of the IO system. That is, to choose the input u such that the output y of the IO system behave in a certain predefined way (note the special case where $y = x$ when $C = I$). The input is often referred to as a control signal or a control law. There are two types of control laws; open loop (or feedforward) control where u depend only on time, $u = u(t)$, and closed loop (or feedback) control where u depend on the output y and possibly on time, $u = u(t, y)$. The control $u = u(t, y)$ is called output feedback, and the special case $u = u(t, x)$ state feedback. It is remarked that feedback control is often preferred (sometimes in combination with feedforward) as it attenuate model uncertainties and disturbances. In the sequel we let $\mathcal{U}$ denote the set of all admissible inputs (those who make (5a) well-defined).

For given input $u \in \mathcal{U}$, note that (5a) is simply an n-dimensional system of (possibly non-autonomous) linear differential equations. Hence for given initial condition $x(0) = x_0$, the solution to (5a) is

$$x = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{6}$$

and therefore

$$y = Ce^{At}x_0 + \int_0^t \psi(t-\tau)u(\tau)d\tau \tag{7}$$

with $\psi(v) = Ce^{Av}B$ called the impulse response function. To stress that the state and output (trajectories) given by (6) and (7) depend on initial condition $x(0) = x_0$ and input u, it is sometimes convenient to write $x = x(t; x_0, u)$ and $y = y(t; x_0, u) = Cx(t; x_0, u)$ in place of $x = x(t)$ and $y = y(t)$, respectively.

As a passing remark we mention that the behavior of the IO system (5) also can be studied in the frequency domain via its transfer function $G(s) = \mathcal{L}(\psi)(s)$ obtained as the Laplace transformed $\mathcal{L}(\psi)(s) = C(sI - A)^{-1}B$ of ψ.

It is often necessary or convenient to transform the IO system (5) into another equivalent IO system. If T denote the (coordinate) transformation ($z = Tx$) from the old state x to the new state z, then the IO system in the new state become

$$\dot{z} = \bar{A}z + \bar{B}u \tag{8a}$$
$$y = \bar{C}z \tag{8b}$$

with $\bar{A} = TAT^{-1}$, $\bar{B} = TB$ and $\bar{C} = CT^{-1}$. The two IO system (5) and (8) are said to isomorphic and it is remarked that they have the same impulse response function (hence the same transfer function and IO behavior), they have identical spectra's, that is, the two set of eigenvalues $\sigma(A)$ and $\sigma(\bar{A})$ of A and $\bar{A}$ are identical, and they have the same controllability and observability property (these two concepts will be introduced below).

3 Controllability

The IO system (5) is said to be controllable at time T if for any pair (x_0, x_1) of states, is it possible to find a control law $u \in \mathcal{U}$ such that the solution $x(t) = x(t; x_0, u)$ to (5a) satisfy $x(0) = x_0$ and $x(T) = x_1$.

It turns out that the notion of controllability is independent of T and can be characterised in terms of the reachable subspace[2] $\mathbb{W}_T \subseteq \mathbb{R}^n$ define as the set of all points x_1 which can be reached from $x(0) = 0$ in

[2]The reachable space is sometimes also referred to as the Krylov subspace (mainly in the literature on numerical linear algebra).

time T by some $u \in \mathcal{U}$. From (6) we obtain

$$\mathbb{W}_T = \left\{ x \;\middle|\; x(T;0,u) = x,\ u \in \mathcal{U} \right\} = \left\{ \int_0^T e^{A(T-\tau)} Bu(\tau) d\tau \;\middle|\; u \in \mathcal{U} \right\} \tag{9}$$

which show that it is a linear subspace of $\mathbb{R}^n$. The IO system (5) is said to be reachable if $\mathbb{W}_T = \mathbb{R}^n$ (for some T).

Let $[A|B] = [B\ AB\ \cdots\ A^{n-1}B] \in \mathbb{R}^{n \times mn}$ called the reachability (or controllability) matrix, and[3]

$$X(t) = \int_0^t e^{As} BB^* e^{A^* s} ds \in \mathbb{R}^{n \times n} \tag{10}$$

called the controllability Gramian. Moreover, let $\sigma(A)$ denote the spectrum of A, that is, the set of eigenvalues of A.

Theorem 1. *It holds true that $\mathbb{W}_T = Range[A|B]$, hence $\mathbb{W} = \mathbb{W}_T$ is independent of T. Moreover, the following are equivalent*

1. $\mathbb{W} = \mathbb{R}^n$.
2. $Rank[A|B] = n$.
3. *The IO system (5) is controllable at T, for any $T > 0$.*
4. *There exists no $x \neq 0$ such that $x^* A = \lambda x^*$ and $x^* B = 0$.*
5. *All eigenvalues of A are controllable: $Rank \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$, for all $\lambda \in \sigma(A)$.*
6. *The controllability gramian is positive definite: $X(t) > 0$, for all $t > 0$.*

Hence reachability and controllability are equivalent concepts (according to conditions 1 and 3). Condition 2 is referred to as the Kalman rank condition, and the pair (A, B) is said to be controllable (or reachable) if any one of the above conditions are fulfilled. For (A, B) controllable it is remarked that for any state x_1 and time T, applying the control law

$$u(t) = B^* e^{A^*(T-t)} X(T)^{-1} x_1 \tag{11}$$

to (5a) will result in $x(T) = x_1$ whenever $x(0) = 0$. Hence the controllability Gramian $X(T)$ facilitate an analytic expression for the control law taking the initial condition $x(0) = 0$ to any given state x_1.

As a final remark we mention that $\mathbb{W}^c = \mathbb{W}$, with

$$\mathbb{W}^c = \left\{ x_0 \;\middle|\; x(T; x_0, u) = 0,\ T \geq 0,\ u \in \mathcal{U} \right\} \tag{12}$$

the set of all states x_0 which can be controlled to the zero state 0, is called the controllable subspace of the IO system (5). Hence controllability (or reachability) is equivalent to requiring that zero is attainable from any state (this will be used as the definition of controllability in the discrete-time case, see Section 8.1).

3.1 Control canonical form (SISO)

Assume that the IO system (5) is SISO[4] i.e., $m = p = 1$ and that (A, B) is controllable. For such systems it is sometimes convenient to bring them into a form called control canonical form. To do so let

$$\chi_A(s) = det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{13}$$

denote the characteristic polynomial of A. Then

$$T = [A|B]T_a^{-1}, \quad T_a = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & a_{n-1} & 1 & 0 \\ a_3 & \cdots & a_{n-1} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

[3]The symbol * denote adjoint, which in the real case degenerate to transpose.

[4]See e.g., [AM06] for the MIMO case.

transform the IO system (5) into (8) which explicit become

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{15a}$$

$$y = \begin{bmatrix} \bar{c}_0 & \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_{n-1} \end{bmatrix} z \tag{15b}$$

From (15) one may immediately construct the transfer function corresponding to the IO system:

$$G(s) = \frac{\bar{c}_{n-1}s^{n-1} + \bar{c}_{n-2}s^{n-2} + \cdots + \bar{c}_1 s + \bar{c}_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

3.2 Kalman decomposition

For uncontrollable IO systems all is not lost:

Theorem 2. *Assume that $Rank[A|B] = l < n$. Then there exists a nonsingular $P \in \mathbb{R}^{n\times n}$ such that*

$$PAP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{l\times l},\ A_{22} \in \mathbb{R}^{(n-l)\times(n-l)},\ B_1 \in \mathbb{R}^{l\times m} \tag{16}$$

with (A_{11}, B_1) controllable.

Hence, in the uncontrollable case (5a) can be decomposed ($z = Px$) into

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 + B_1 u \tag{17a}$$

$$\dot{z}_2 = A_{22}z_2 \tag{17b}$$

with (17a) describing the controllable part and (17b) describing the uncontrollable part of the IO system (5). Moreover, if let $\beta = \{e_1, \ldots, e_l\}$ denote a basis for $\mathbb{W} = Range[A|B]$ and $\{e_{l+1}, \ldots, e_n\}$ a completion of β, then $P = [e_1\ e_2\ \cdots\ e_n]^{-1}$ with e_i expressed in the old basis (so P is the (coordinate) transformation from the old basis to the new basis).

It should be clear from the above that in order to control an IO system the uncontrollable part of the system needs to be ”well behaved”. To make this precise we need the concept of stability.

4 Stabilizability

Recall that the linear system

$$\dot{x} = Ax \tag{18}$$

is said to be stable[5] if $x(t) \to 0$ as $t \to \infty$ for any initial condition $x(0) = x_0$. Stability of (18) is characterized by the spectrum $\sigma(A) \subset \mathbb{C}$ of A:

Theorem 3. *The following are equivalent*

- *The linear system* (18) *is stable.*
- *$Re(\lambda) < 0$ for all $\lambda \in \sigma(A)$, that is, all eigenvalues of A have strict negative real part.*
- *For any $X \in \mathbb{R}^{n\times n}$ with $X > 0$, there exists $P \in \mathbb{R}^{n\times n}$ with $P > 0$ such that $PA + A^*P + X = 0$.*

The expression $PA + A^*P + X = 0$ is known as the Lyapunov equation, note that one can take $X = P$. This approach to stability will be explored in section 5.

[5]It is remarked that the terminology used for linear systems is different from the one used for nonlinear system, e.g., stability as defined here would be called (locally) attractive.

For the Kalman decomposition we conclude that the uncontrollable part of the IO system needs to be stable in order to be "well behaved". That is, $Re(\lambda) < 0$ for all $\lambda \in \sigma(A_{22})$.

We can use the eigenvalue characterisation of stability to solve the following stabilization problem: Find input u such that $x(t) \to 0$ as $t \to \infty$ (implying that $y(t) \to 0$ as $t \to \infty$)

To solve the stabilization problem we introduce some terminology. The (input) system (5a), or the pair (A, B), is said to be stabilizable if there exists a (state) feedback $u = Kx$, $K \in \mathbb{R}^{m\times n}$, such that the closed loop system of (5a)

$$\dot{x} = (A + BK)x \tag{19}$$

is stable. The matrix K is sometimes referred to as a feedback gain.

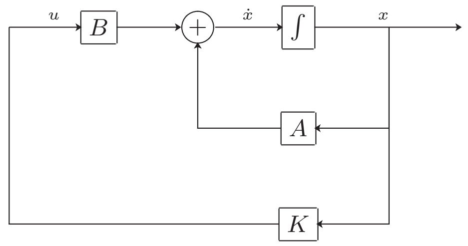


Figure 2: The closed loop system (19), resulting from the state feedback $u = Kx$

Hence, in order to stabilize the IO system on needs to find K such that $Re(\lambda) < 0$ for all $\lambda \in \sigma(A+BK)$. The pole-placement theorem below explain when this is possible:

Theorem 4. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be any set of n complex numbers with the property that if $\lambda \in \Lambda$ then $\lambda^* \in \Lambda$. Then there exists $K \in \mathbb{R}^{p\times m}$ such that $\sigma(A + BK) = \Lambda$ iff (A, B) is controllable.*

So controllability implies stabilizability, the converse is not true.

4.1 Reference tracking

In this section we specialize to the case where the number of inputs is equal to the number of outputs ($m = p$) and consider the tracking problem:[6] Find input $u \in \mathbb{R}^m$ such that the output $y \in \mathbb{R}^m$ tracks a desired constant reference (input) r: $y(t) \to r$ as $t \to \infty$. Note that for $r = 0$ this become the stabilization problem discussed above (in the case $C = I$).

To solve the tracking problem we assume that there exists a steady state state-input pair (x', u'), that is

$$0 = Ax' + Bu', \quad r = Cx' \tag{20}$$

It then follows that (x', u') can be found as

$$\begin{bmatrix} x' \\ u' \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} r = \begin{bmatrix} N_x \\ N_u \end{bmatrix} r \tag{21}$$

with $N_x \in \mathbb{R}^{n\times m}$ and $N_u \in \mathbb{R}^{m\times m}$ defined by the left-hand side. The control law solving the tracking problem is then given by

$$\begin{aligned} u &= K(x - x') + u' && \text{(22a)} \\ &= K(x - N_x r) + N_u r && \text{(22b)} \\ &= Kx + Nr && \text{(22c)} \end{aligned}$$

with the feedforward gain $N = N_u - KN_x \in \mathbb{R}^{m\times m}$ (in some literature, e.g., [FPM97] it is only N_u which is referred to as the feedforward gain, compare figure 3 and 4). The feedback gain K should be chosen as in the stabilization problem ($Re(\lambda) < 0$ for all $\lambda \in \sigma(A + BK)$) and the closed loop system takes the form

$$\dot{x} = (A + BK)x + BNr \tag{23}$$

[6] This is a special case of the servo problem

Using the coordinate transformation $z = x - x'$ it then follows that $z \to 0$ as $t \to \infty$ e.i., $x \to x'$ as $t \to \infty$.

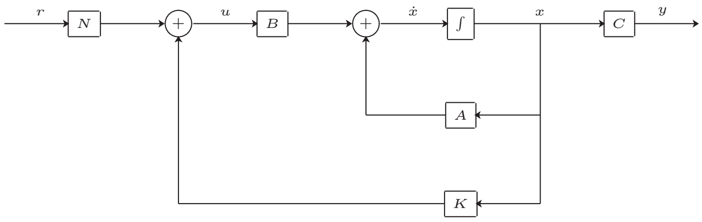


Figure 3: The closed loop system (23), resulting from the state feedback (22c). Note that this block diagram does not indicate that $N = N_u - KN_x$ depends on the feedback gain K.

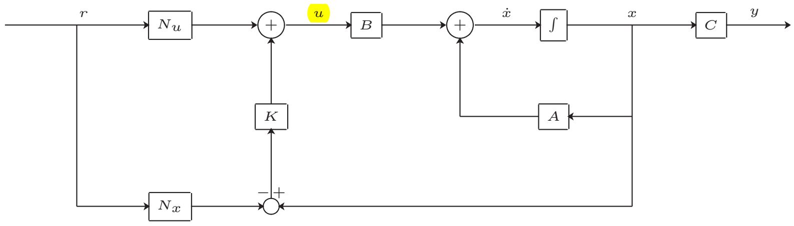


Figure 4: The closed loop system (23), resulting from the state feedback (22b).

It is remarked that one should use (22b) for implementation as changes in K only effects the $N_x r$ term (using (22a) is not convenient for implementation as one typically changes the value of the reference over time in many applications). Moreover, the above solution will typically generate steady state error ($y(t) \to r \pm e$ for some error $e \in \mathbb{R}^m$) e.g. due to incorrect system model. To mitigate the effect of steady state error we need the concept of integral action.

4.1.1 Integral action

Steady state error can be removed by introducing an extra state $x_I \in \mathbb{R}^m$ in the IO system:

$$\dot{x} = Ax + Bu \tag{24a}$$
$$\dot{x}_I = Cx - r \tag{24b}$$
$$y = Cx \tag{24c}$$

If we let $e = y - r$ denote the (tracking) error, then $x_I = \int e(s)ds$ is the integrated error.[7] Note that (24a)-(24b) can be written as

$$\dot{\begin{bmatrix} x \\ x_I \end{bmatrix}} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u - \begin{bmatrix} 0 \\ I \end{bmatrix} r = \bar{A} \begin{bmatrix} x \\ x_I \end{bmatrix} + \bar{B}u - Gr \tag{25}$$

which, for $r = 0$, is on the same form as our original IO system (5). Let (x', u') be as in (20), the control law solving the tracking problem and mitigating steady state error is then given by

$$u = \bar{K} \begin{bmatrix} x - x' \\ x_I \end{bmatrix} + u' \tag{26a}$$
$$= \begin{bmatrix} K & K_I \end{bmatrix} \begin{bmatrix} x - x' \\ x_I \end{bmatrix} + u' \tag{26b}$$
$$= \bar{K} \begin{bmatrix} x \\ x_I \end{bmatrix} + Nr \tag{26c}$$

[7]Note that the integral should be evaluated component-wise.

with the feedforward gain $N = N_u - KN_x$, and the feedback gain $\bar{K} = \begin{bmatrix} K & K_I \end{bmatrix} \in \mathbb{R}^{m\times(n+m)}$ designed to stabilize (25) with $r = 0$.[8] The closed loop system then takes the form

$$\dot{\begin{bmatrix} x \\ x_I \end{bmatrix}} = \left(\bar{A} + \bar{B}\bar{K}\right)\begin{bmatrix} x \\ x_I \end{bmatrix} + (\bar{B}N - G)r = \begin{bmatrix} A + BK & BK_I \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} BN \\ -I \end{bmatrix} r \tag{27}$$

and using the coordinate transformation $z = (x, x_I) - (x', 0)$, it follows that $z \to 0$ as $t \to \infty$ e.i., $x \to x'$ and $x_I \to 0$ as $t \to \infty$.

It is remarked that the feedforward term u' reduces the demand on the integral action, however, it may in principle be remove from the control law (26). Moreover, integral action lead to overshoot and potentially integral windup. Both can be handled by anti-windup methods which is discussed in Section 7.

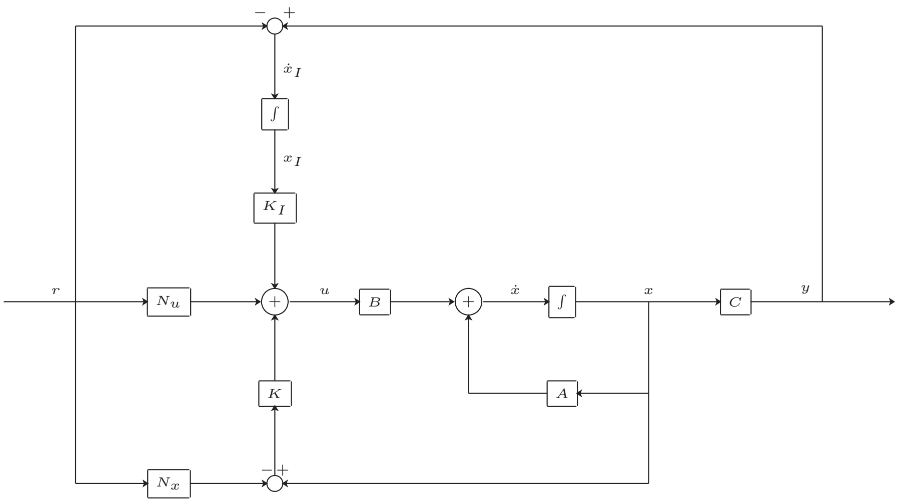


Figure 5: Reference tracking with integral action

4.2 Linear quadratic regulator

So fare we have not discussed how to choose the feedback gain. One approach is by hand-tuning i.e., choosing the gain elements based on "engineering intuition" until the system performance is acceptable. A more systematic approach is given by the linear quadratic regulator (LQR) design which we explain next.

Consider the LQR (or H_2) optimal control problem

$$\min_{u=u(t)} \int_0^T \left(y(s)^* Q y(s) + u(s)^* R u(s)\right) ds \tag{28a}$$

subject to

$$\dot{x} = Ax + Bu, \ x(0) = x_0 \tag{28b}$$

$$y = Cx \tag{28c}$$

with $Q \geq 0$ (positive semi-definite) and $R > 0$ (positive definite). If[9] (A, B) is controllable and $(A, Q^{1/2}C)$ observable[10], then the solution[11] u to (28) is given by the feedback control law

$$u = -R^{-1}B^* P x \tag{29}$$

where $P > 0$ is determined according to whether $T < \infty$ or $T = \infty$ as the solution to the

- Riccati differential equation ($T < \infty$)

$$\dot{P} = -C^* Q C - PA - A^* P + PBR^{-1}B^* P, \ P(T) = 0 \tag{30}$$

[8] Note that for this to be possible (25), with $r = 0$, needs to be stabilizable.
[9] These assumptions may be relaxed to stabilizability and detectable, see [AM06]
[10] See Chapter for this concept.
[11] derived from e.g., Pontryagin's maximum principle

- algebraic Riccati equation ($T = \infty$)

$$0 = C^*QC + PA + A^*P - PBR^{-1}B^*P \tag{31}$$

It is remarked that the control law (29) solve the stabilization problem. Hence, we may used the above in synthesizing a stabilizing control law by using the matrices Q and R as tuning parameters. Moreover, it can be shown that $P_T \to P$ as $T \to \infty$ with P_T the solution to (30) and P the solution to (31).

In the case $C = I$, a common tuning method, known as Bryson's rule, consist of punishing the ith state x_i and the ith control u_i by choosing Q and R as diagonal matrices with the ith diagonal entry Q_{ii} and R_{ii} equal to

$$Q_{ii} = \frac{1}{\text{maximum value of } x_i^2} \quad \text{and} \quad R_{ii} = \frac{1}{\text{maximum value of } u_i^2}$$

Using LQR in the context of reference tracking with integral action is straight forward. On simply use the IO system consisting of (25) with $r = 0$ and an output matrix equal to the identity (in place of (28b)-(28c)) to produce the feedback gain $\bar{K} = -R^{-1}\bar{B}P$ and control law (26).

5 Constraints

A real world system is always subject to constraints, for the IO system (5) this can be expressed as

1. **Bounded state:** $|x_i(t)| \leq \bar{x}_i$ for given $\bar{x}_i$
2. **Bounded input:** $|u_i(t)| \leq \bar{u}_i$ for given $\bar{u}_i$
3. **Bounded output:** $|y_i(t)| \leq \bar{y}_i$ for given $\bar{y}_i$

In may cases it is possible to design the feedback gain such that 1-3 is satisfied. The synthesis of such feedback gains is based on the concept of linear matrix inequalities (LMIs). We will not describe the theory behind LMIs in this note, but focus only on the application.

Before addressing the constraints 1-3 we explain how LMIs can be used for stabilization. Recall the Lyapunov equation $PA+A^*P+X = 0$ with $P, X > 0$ from Theorem 3. This may be written equivalently as the two LMIs $P > 0$ and $PA + A^*P < 0$ in the (matrix) variable P. Hence stability of $\dot{x} = Ax$ is equivalent to find $P > 0$ such that $PA + A^*P < 0$. Solving the two LMIs, that is finding P such that P is positive definite and $PA + A^*P$ is negative definite, can be done by various software tools e.g., YALMIP.

Moreover, from Theorem 3 we concluded that the control law $u = Kx$ stabilize the IO system (5) if $Re(\lambda) < 0$ for all $\lambda \in \sigma(A+BK)$. The feedback gain K can be obtained via the solution to the following two LMIs[12] in the (matrix) variables Q and Y

$$Q > 0 \quad \text{and} \quad AQ + QA^* + BY + Y^*B^* < 0, \tag{32}$$

with $Y = KQ$ (hence $K = YQ^{-1}$). One advantage of find K by means of (32) is that beside making the state converge to zero we also get

$$x(0) \in \Omega_\gamma = \{x \mid x^T Px \leq \gamma,\ P = Q^{-1}\} \ \Rightarrow x(t) \in \Omega_\gamma \ \forall t \geq 0$$

Hence the state is constrained to be in the ellipsoid Ω_γ (or equivalently, Ω_γ is positive invariant). Moreover, by modifying (32) slightly to

$$Q > 0 \quad \text{and} \quad AQ + QA^* + BY + Y^*B^* + \alpha Q < 0, \tag{33}$$

for given (decay rate) $\alpha \geq 0$, we also get the following bound on the convergence of the state:

$$||x(t)|| \leq Ce^{-\alpha t}||x(0)||$$

for some constant C. Making the system respect bounds on states, inputs, outputs and initial conditions can also be synthesised via LMIs:

[12] These are obtained by rewriting $P(A + BK) + (A + BK)^*P < 0$ using the change of variables $Q = P^{-1}$.

- **Bounded state:** $|x_i(t)| \le \bar{x}_i$ for given $\bar{x}_i$; if $x(0) \in \Omega_\gamma$ and the following LMI in Q is solvable

$$\begin{bmatrix} Q/\gamma & Qe_i \\ e_i^* Q & \bar{x}_i^2 \end{bmatrix} \ge 0$$

  with e_i the i'th unit vector.

- **Bounded input:** $|u_i(t)| \le \bar{u}_i$ for given $\bar{u}_i$; if the following LMIs in X, Y, Q are solvable

$$\begin{bmatrix} X & Y \\ Y^* & Q \end{bmatrix} \ge 0 \qquad X_{ii} \le \bar{u}_i^2$$

- **Bounded output:** $|y_i(t)| \le \bar{y}_i$ for given $\bar{y}_i$; if the following LMI in Y and Q is solvable

$$\begin{bmatrix} Q & (AQ+BY)^* C_i^* \\ C_i(AQ+BY) & \bar{y}_i^2 \end{bmatrix} \ge 0$$

  with C_i denoting the ith row of matrix C.

- **Initial condition:** $x(0) \in \Omega_\gamma$; if the following LMI in Q is solvable

$$\begin{bmatrix} \gamma & x(0)^* \\ x(0) & Q \end{bmatrix} \ge 0$$

In Section 7 we discuss a control synthesis related to constrained input called anti-windup.

6 Observability

So fare we have (implicit) assumed that we can used all states (all the components of x) in our control design. That is, we have assumed that we can measure x such that we can apply the control input $u = Kx$ to our IO system. For real world system this is typically not the case e.g., one can measure the position but not the velocity. The states which can not be measured can in many cases be estimated via the outputs and inputs, through what is known as a state observer. To describe the observer design we first need to discuss the concept of observability.

Two distinct states x_0 and x_0' are distinguishable on $[0,T]$ if there exists input u such that $y(t;x_0,u) \ne y(t;x_0',u)$ for some $t \in [0,T]$. They are indistinguishable if they are not distinguishable, that is, if for any input u we have $y(t;x_0,u) = y(t;x_0',u)$ for all $t \in [0,T]$. So we can not see the difference between two indistinguishable states based on the corresponding outputs.

The IO system (5) is said to be observable (on $[0,T]$) if any pair of distinct states are distinguishable (on $[0,T]$).

Let $\mathbb{V}_T \subseteq \mathbb{R}^n$ denote the set of all indistinguishable states on $[0,T]$ (that is, $x, x' \in \mathbb{V}_T$ implies that x and x' are indistinguishable on $[0,T]$),

$$(A|C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{pn \times n}$$

called the observability matrix, and

$$Y(t) = \int_0^t e^{A^* s} C^* C e^{As} ds \in \mathbb{R}^{n \times n} \tag{34}$$

called the observability Gramian. Note that $(A|C)^* = [A^*|C^*]$. It turns out that the notion of observability is independent of T and can be characterised in terms of the unobservable subspace[13] $Null(A|C) = Range[A^*|C^*]^\perp \subseteq \mathbb{R}^n$.

[13] here $\perp$ indicate orthogonal complement

Theorem 5. *It holds true that $\mathbb{V}_T = Null(A|C)$, hence $\mathbb{V} = \mathbb{V}_T$ is independent of T. Moreover, the following are equivalent*

1. $\mathbb{V} = \{0\}$
2. $Rank(A|C) = Rank[A^*|C^*] = n$
3. *The IO system* (5) *is observable on* $[0, T]$ *for any* $T > 0$.
4. *There exists no* $x \neq 0$ *such that* $Ax = \lambda x$ *and* $Cx = 0$
5. *All eigenvalues of* A *are observable:* $Rank \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \ \forall \lambda \in \sigma(A)$
6. *The observability gramian is positive definite,* $Y(t) > 0$*, for any* $t > 0$.

Condition 4 is sometimes referred to as the Popov-Belevitch-Hautus test, and the pair (A, C) is said to be observable if any one of the above conditions are fulfilled (note that specifying the interval $[0, T]$ is no longer necessary, and that observability is independent of control input).

There are two other (equivalent) characterisation of observability, the first one being in terms of reconstructibility: A state x_0 is unreconstructible if $y(t; x_0, 0) = 0$ for all $t \leq 0$, that is, if x_0 is indistinguishable, on $(-\infty, 0]$, from the zero state 0. The set $\mathbb{V}^-$ of all unreconstructible states is referred to as the unreconstructible subspace, and we have $\mathbb{V}^- = \mathbb{V}$. Moreover, if we let $\mathbb{V}^+$ denote the set of of all states x_0 with $y(t; x_0, 0) = 0$ for all $t \geq 0$, that is, all states which are indistinguishable, on $[0, \infty)$, from the zero state 0, then $\mathbb{V}^+ = \mathbb{V}$. This last characterisation of the unobservable subspace $\mathbb{V} = Null(A|C)$ will be used to define observability in the discrete-time case (see section 8.4).

From Theorem 1 and Theorem 5 we see that there is a relation between the concepts of controllability and observability. Indeed, (A, C) is observable iff (A^*, C^*) is reachable/controllable. This relation is called duality and can be used to transform results on observability to results on reachability/controllability and vice versa.

6.1 Observable canonical form (SISO)

Assume that the IO system (5) is SISO[14] i.e., $m = p = 1$ and that (A, C) is observable. In this case one can obtain a dual version of the control canonical form called the observable canonical form. Indeed, let T_a be as in (14), then $T = \left(([A^*|C^*]T_a^{-1})^*\right)^{-1}$ transform the IO system (5) into (8) which explicit become

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & \vdots & \vdots & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_{n-1} \end{bmatrix} u \tag{35a}$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} z \tag{35b}$$

with a_i's coming from the characteristic polynomial (13) of A. The corresponding transfer function is then given by

$$G(s) = \frac{\bar{b}_{n-1}s^{n-1} + \bar{b}_{n-2}s^{n-2} + \cdots + \bar{b}_1 s + \bar{b}_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

For $k = 0, 1, \ldots$ let $\mu_k = \frac{d^k y}{dt^k}\big|_0$ with $\mu_0 = y(0)$, hence μ_k is the k'th time-derivative of the output y evaluated at $t = 0$. It is then remarked that we may use (35) to determine the initial state $z(0) = (z_{10}, z_{20}, \ldots, z_{n0})$ from the initial data $\mu_k, \ k = 0, 1, \ldots, n-1$ as follows: From (35b) we have $y = z_n$, and taking time derivative of this relation and using (35a) with $u = 0$ we get $\dot{y} = z_{n-1} - a_{n-1}z_n, \ \ddot{y} = z_{n-2} - a_{n-2}z_n, \ldots$. Evaluating these relations at $t = 0$ gives

$$\mu_0 = z_{n0}, \ \mu_1 = z_{(n-1)0} - a_{n-1}z_{n0}, \ \mu_2 = z_{(n-2)0} - a_{n-2}z_{n0}, \ldots, \mu_{n-1} = z_{10} - a_1 z_{n0}$$

[14] See e.g., [AM06] for the MIMO case.

which determines $z(0)$ from $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$. This procedure illustrate the following equivalent definition of observability: The IO system (5) is said to be observable if for any $T > 0$, the initial state $x(0)$ can be determined from the time history of the input $u = u(t)$ and the output $y = y(t)$ in the interval $[0, T]$.

6.2 Kalman decomposition

Using duality we also have a decomposition for unobservable IO systems:

Theorem 6. *Assume that $Rank[A^*|C^*] = l < n$. Then there exists a nonsingular $P \in \mathbb{R}^{n\times n}$ such that*

$$PAP^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \qquad CP^{-1} = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad A_{11} \in \mathbb{R}^{l\times l},\ A_{22} \in \mathbb{R}^{(n-l)\times(n-l)},\ C_1 \in \mathbb{R}^{p\times l} \tag{36}$$

with (A_{11}, C_1) observable.

Hence, in the unobservable case (5) can be decomposed ($z = Px$, $PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$) into

$$\dot{z}_1 = A_{11}z_1 + B_1u \tag{37a}$$
$$\dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2u \tag{37b}$$
$$y = C_1z_1 \tag{37c}$$

with $\dot{z}_2 = A_{22}z_2$ from (37b) describing the unobservable subsystem. Moreover, if let $\beta = \{e_{n-l+1}, \ldots, e_n\}$ denote a basis for $Null(A|C) = Range[A^*|C^*]^\perp$ and $\{e_1, \ldots, e_{n-l}\}$ a completion of β, then $P = [e_1\ e_2\ \cdots\ e_n]^{-1}$ with e_i expressed in the old basis (so P is the (coordinate) transformation from old basis to the new basis).

6.3 Observer

We can now begin the (state) observer design. Let P denote a plant which can be modeled as an IO system having input u and output y. An observer, sometimes called a Luenberger observer, is an IO system having inputs u and y, and outputs an estimate, denoted $\hat{x}$, of the state x of P. This is depicted in Figure 6(a).

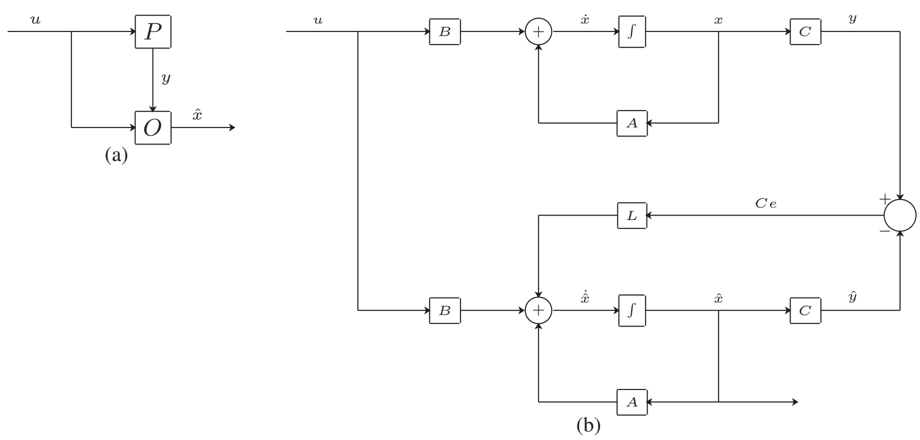


Figure 6: Observer 6(a) with plant P and observer O. Detailed block diagram 6(b) of 6(a), with the upper system being a model of the plant P and the lower system being the observer O containing a copy of P together with the observer gain L

If the plant P can be described by (5), then the equations governing the observer is

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{38a}$$
$$\hat{y} = C\hat{x} \tag{38b}$$

with the (observer) gain $L \in \mathbb{R}^{n\times p}$ chosen such that the (estimation) error $e(t) = x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. This can be accomplished by observing that

$$\dot{e} = (A - LC)e \tag{39}$$

so L needs to be chosen such that (39) is stable: $Re(\lambda) < 0$ for all $\lambda \in \sigma(A - LC)$. If such an observer gain L exists the IO system (5), or the pair (A, C), is called detectable. Note that observability implies detectability since by duality (A^*, C^*) is controllable and therefore stabilizable.

6.3.1 Stabilization by dynamic measurement feedback

With the concept of an observer now available we can revisit the stabilization procedure from section 4. The procedure described below is sometimes referred to as observer based control or stabilization by dynamic measurement feedback.

In place of the control law $u = Kx$, which is possible to implement only when we can measure all states, we can now use the observer to generate the input $u = K\hat{x}$ to the plant P yielding

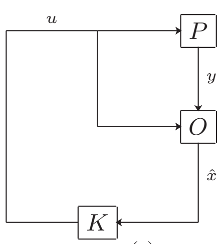
$$\dot{x} = Ax + BK\hat{x} \tag{40}$$

To find a stabilizing feedback gain K we combine (40) with (39) to obtain

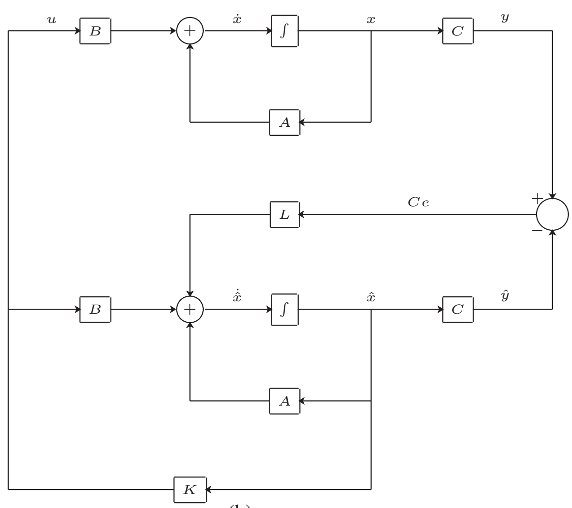
$$\dot{\begin{bmatrix} x \\ e \end{bmatrix}} = \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{41}$$

which show that in order to stabilize the plant P using only the outputs we need to chose the gains K and L such that (19) and (39) are stable (in this case the IO system (5) is said to be stabilizable). That is, such that all eigenvalues of $A + BK$ and $A - LC$ have negative real part. This is known as the separation principle which can be stated formally as:

Theorem 7. *The IO system* (5) *is stabilizable iff* (A, B) *is stabilizable and* (A, C) *is detectable.*



(a)



(b)

Figure 7: Stabilization by dynamic measurement feedback

It is remarked that as a rule of thumb one should always choose the observer gain L such that the eigenvalues of the closed loop system $A - LC$ are 5 to 10 times that of the eigenvalues of the closed loop system $A + BK$.

Finally, to use an observer in a reference tracking situation one only needs to replace x by $\hat{x}$ in the feedback formula (22) if no integral action is required, or in (26) if integral action is required. This is illustrated in the block diagram in figure 8 in the case of integral action. It should be noted that one should always include integral action when using an observer since otherwise this may lead to instability due to model mismatch.

Similar to (27), the closed loop system corresponding to figure 8 then takes the form

$$\begin{bmatrix} \dot{x} \\ x_I \\ e \end{bmatrix} = \begin{bmatrix} A+BK & BK_I & -BK \\ C & 0 & 0 \\ 0 & 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ x_I \\ e \end{bmatrix} + \begin{bmatrix} BN \\ -I \\ 0 \end{bmatrix} r \tag{42}$$

with $BNr = BN_u r - BKN_x r = Bu' - BKx'$ and (x', u') the steady state state-input pair. Note that the (extended) state matrix is stable, since it is upper block diagonal and that the blocks are stable by design of $\bar{K} = \begin{bmatrix} K & K_I \end{bmatrix}$ and L. That is, we have a separation principle also in the case of an observer combined with reference tracking and integral action (find L and $\bar{K}$ such that (39) and (27), with $r = 0$, is stable).

Moreover, using the coordinate transformation $z = (x, x_I, e) - (x', 0, 0)$, it follows that $z \to 0$ as $t \to \infty$ e.i., $x \to x'$, $x_I \to 0$ and $e \to 0$ as $t \to \infty$.

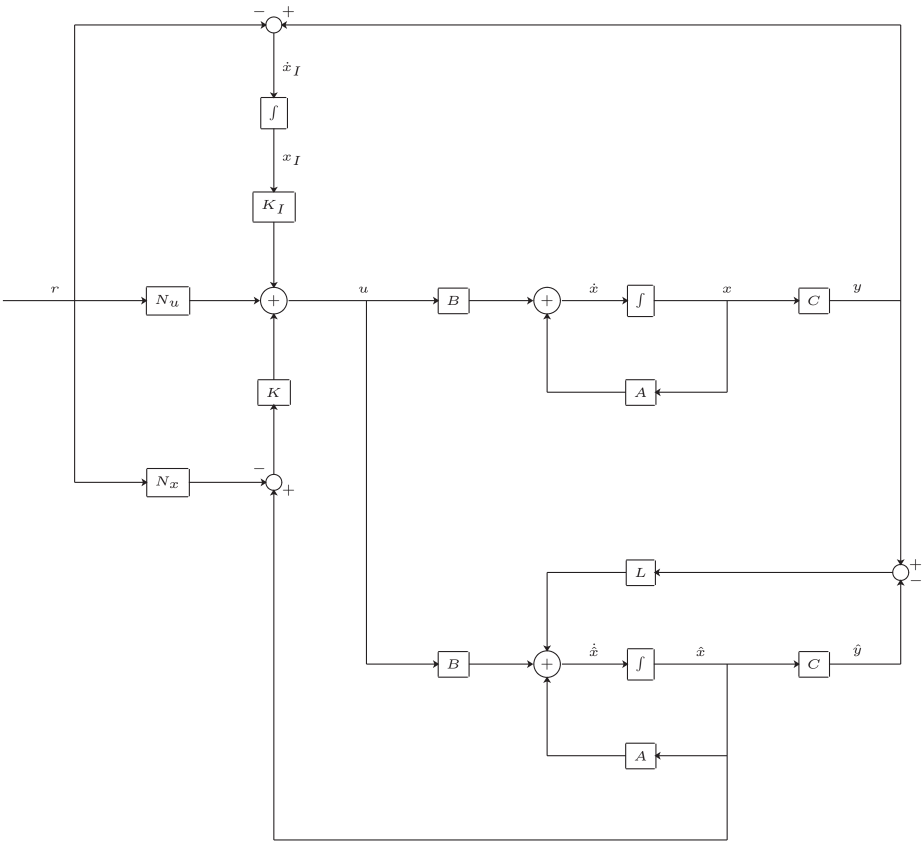


Figure 8: Reference tracking with observer and integral action

The estimator-controller system described by figure 8 is on the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{43a}$$
$$\dot{x}_I = y - r \tag{43b}$$
$$u = K\hat{x} + K_I x_I + Nr \tag{43c}$$

If we apply (43c) in (43a) and collect terms we get

$$\dot{\bar{x}} = F\bar{x} + G_r r - G_y y \tag{44a}$$
$$u = \bar{K}\bar{x} + Nr \tag{44b}$$

with

$$\bar{x} = \begin{bmatrix} \hat{x} \\ x_I \end{bmatrix}, \quad F = \begin{bmatrix} A - LC + BK & BK_I \\ 0 & 0 \end{bmatrix}, \quad G_r = \begin{bmatrix} BN \\ -I \end{bmatrix}, \quad G_y = \begin{bmatrix} -L \\ -I \end{bmatrix} \tag{44c}$$

The block diagram corresponding to (44) is depicted Figure 9. This form will be convenient in Section 7 when we discuss anti-windup.

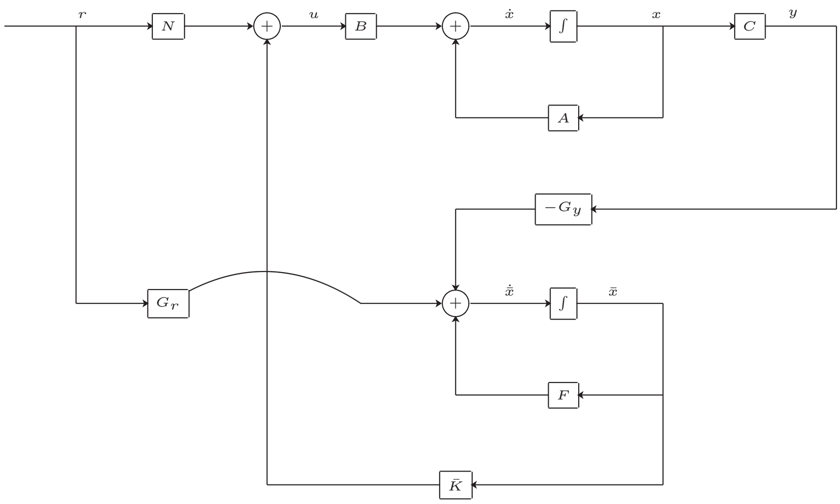


Figure 9: Reference tracking with observer and integral action (equivalent to figure 8)

6.3.2 Reference tracking using zero assignment estimator

In this section we briefly explain another method for synthesizing reference tracking (see figure 10). As previously we assume that the number of inputs is equal to the number of outputs ($m = p$) and the reference $r \in \mathbb{R}^m$ is a constant vector. The estimator-controller system can be written as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + BK\hat{x} + L(y - \hat{y}) + Mr && \text{(45a)} \\ &= (A + BK - LC)\hat{x} + Ly + Mr && \text{(45b)} \\ u &= K\hat{x} + Nr && \text{(45c)} \end{aligned}$$

with $N \in \mathbb{R}^{m\times m}$ and $M \in \mathbb{R}^{n\times m}$ to be chosen. There are several strategies for selecting N and M e.g., autonomous estimator, tracking error estimator, and zero assignment estimator. As a matter of fact, what was presented above, Figure 8 without integral action, is precisely the autonomous estimator. To obtain Figure 10 from Figure 8 (without integral action), we simply set $N = N_u - KN_x$ and $M = BN$ (compare (45) and (44)). Below we outline the zero assignment estimator (see [FPEN94] for a full account of these methods).

For a given IO system (5) choose feedback gain K such that $A + BK$ has stable eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, and observer gain L such that $A - LC$ has stable eigenvalues $\{\mu_1, \dots, \mu_n\}$, hence (41) is stable (and $\max\{\lambda_i\} < \min\{\mu_j\}$). Now let $A_z = A + BK - LC$ and choose $\hat{M} \in \mathbb{R}^{n\times m}$ such that $A_z - \hat{M}K$ has eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$.[15] The feedforward gains N and M can now be determined by

$$\begin{aligned} N &= -\left(\begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} A & BK \\ LC & A + BK - LC \end{bmatrix}^{-1} \begin{bmatrix} B \\ \hat{M} \end{bmatrix}\right)^{-1} && \text{(46a)} \\ M &= \hat{M}N && \text{(46b)} \end{aligned}$$

In figure 10 the block diagram corresponding to the zero assignment method is depicted. Note that if integral action is needed then one should add an integral block similar to e.g., the one in Figure 8.

[15] This procedure assign zeros at $\{\lambda_1, \cdots, \lambda_n\}$ for the transfer function from r to y which cancel the poles $\{\lambda_1, \cdots, \lambda_n\}$ assigned by K

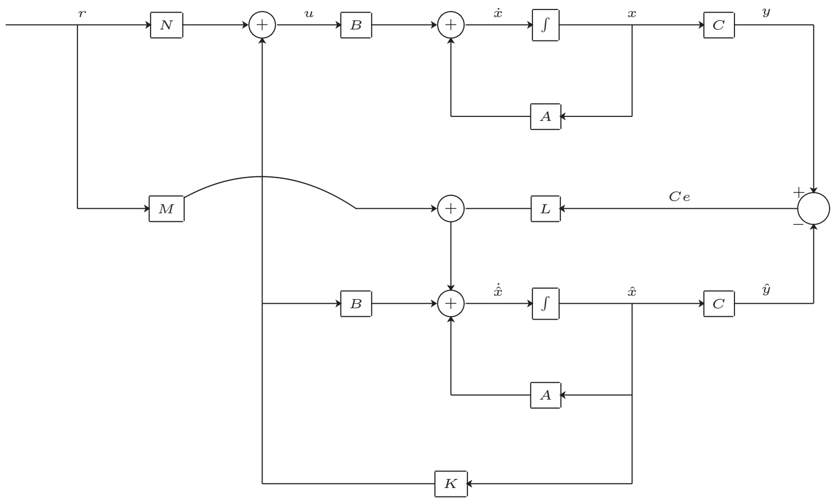


Figure 10: Reference tracking using zero assignment estimator

7 Anti-windup

As mentioned in Section 5 it is always the case that a real world system is subject to constraints. In this section we address the problem of input saturation, that is, the case where the possible values of the input signal $u(t)$ lie in some interval, say, $I = [-k, k]$ for some constant $k > 0$. If a control law containing integral action is designed without taking input saturation into account this may result in accumulation of (tracking) error during saturation. This phenomena is know as windup and can lead to degradation in performance and even instability, hence an anti-windup scheme is required.

To analyze the windup phenomena we introduce the saturation function

$$sat : \mathbb{R}^m \to I^m; \quad u \mapsto sat(u) = (sat_1(u), \dots, sat_m(u))$$

with the coordinate function sat_i given by

$$sat_i(u) = \begin{cases} m & u_i \geq k \\ u & u_i \in I \\ -m & u_i \leq -k \end{cases} \tag{47}$$

The saturation function is sometimes referred to as a saturation nonlinearity, a terminology which will become clear below. Under the assumption of saturated input the IO system (5) then become

$$\dot{x} = Ax + Bsat(u) \tag{48a}$$
$$y = Cx \tag{48b}$$

which is also depitced in Figure 11 below

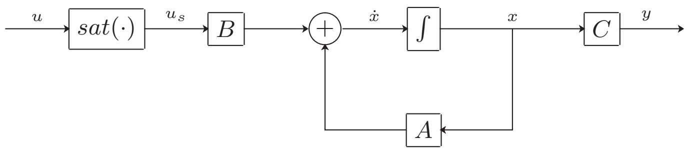


Figure 11: The saturated IO system (48)

This seemingly small change of replacing u with $sat(u)$ has in general a profound impact on the control synthesis. Most importantly, the introduction of the saturation nonlinearity (47) turn the linear IO system (5) into the nonlinear IO system (48). Such systems was briefly mentioned in Section 1, for a detailed discussion on nonlinear IO system and the corresponding control synthesis see e.g., [Kha14].

In the sequel we present the anti-windup synthesis discussed in [Run90]. For a detailed account of modern anti-windup synthesis for state-space systems with control algorithms based on LMI's see [ZT11] or [SP05].

Let us assume that we are to synthesis a reference tracking for the IO system (48). The synthesis consist of two steps: In the first step we ignored the presence of input saturation. We are thus in the situation depicted in Figure 9 (or equivalently Figure 8) and can proceed to chosen the feedback gain $\bar{K} = \begin{bmatrix} K & K_I \end{bmatrix}$ and observer gain L such that the closed loop system (42) is stable. In the second step (the anti-windup synthesis) we introduce the input saturation sat and gain $\bar{M} \in \mathbb{R}^{(n+p)\times m}$ as shown in Figure 12, and note that if $u \in I^m$ then $u = sat(u)$ so $v = u_s - u = sat(u) - u = 0$ implying that $\bar{M}$ can be ignored and therefore that Figure 9 and Figure 12 are identical. Hence in the case $u \in I^m$ we obtain reference tracking due to the choice of $\bar{K}$ and L. If the input saturates, $u \notin I^m$, then $u_s = \pm k$ is constant and the estimator-controller system is

$$\dot{\bar{x}} = F\bar{x} + G_r r - G_y y + \bar{M}(u_s - u) \tag{49a}$$
$$= (F - \bar{M}\bar{K})\bar{x} + (G_r - \bar{M}N)r - G_y y + \bar{M}u_s \tag{49b}$$
$$u = \bar{K}\bar{x} + Nr \tag{49c}$$
$$u_s = sat(u) \tag{49d}$$

If we ignore all terms in (49b) except the first we see that if the pair $(F, \bar{K})$ is observable then $F - \bar{M}\bar{K}$ can be assigned any eignvalues by means of $\bar{M}$. In fact it turns out that in this cases the saturation effect can be mitigated if a Nyquist like criteria is fulfilled:

Theorem 8 ([Run90]). *Consider the transfer function $G = \frac{G_c G_p - W}{1+W}$ with*

$$G_p = C(sI - A)^{-1}B, \;\; G_c = \bar{K}(sI - F)^{-1}G_y + D_y, \;\; W = \bar{K}(sI - F)^{-1}\bar{M}$$

If the linear system $G(s)$ has all poles in the open left half plane and has nonlinear feedback from a saturation the closed loop is absolutely stable provided that a straight line through the origin can be given a nonzero slope such that $G(i\omega) + 1$ is strictly to the right of the line.

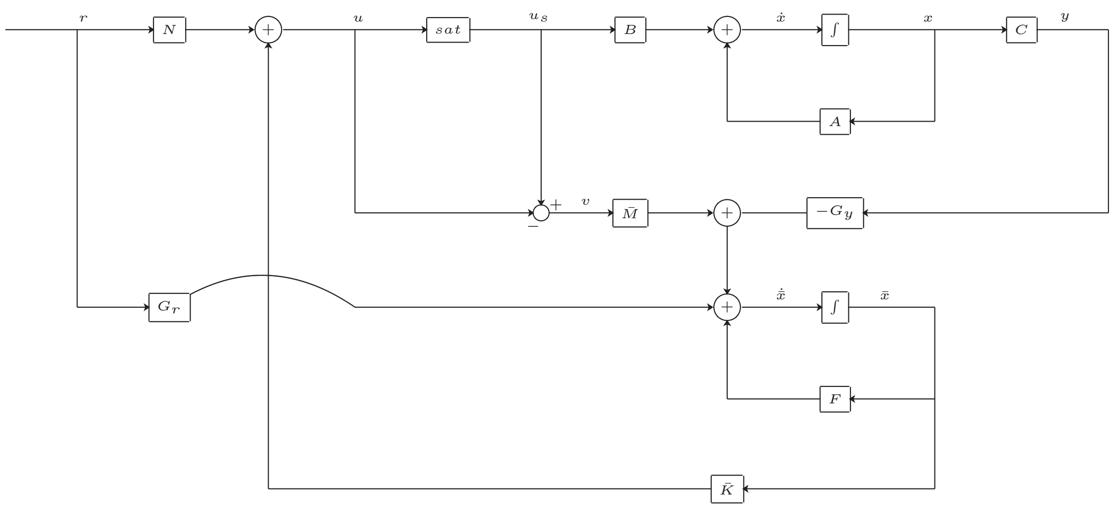


Figure 12: Reference tracking with observer, integral action and anti-windup.

8 Discrete time systems

So fare we have been discussing continuous-time IO systems, in the rest of this note we will be concerned with IO systems described in discrete time (see e.g., [FPM97, ÅW90, AM06] for a detailed discussion of discrete-time control systems).

A discrete-time (linear) IO system can be modeled as

$$x(k+1) = Ax(k) + Bu(k) \tag{50a}$$
$$y(k) = Cx(k) \tag{50b}$$

with $k \in \mathbb{N}$ describing (discrete) time. Terminology identical to the continuous-time case described by (5) is used in (50), e.g., $A \in \mathbb{R}^{n\times n}$ is called the state matrix. Note that the input u is now a sequence of points in $\mathbb{R}^m$ (typically $u(0), u(1), u(2), \dots$) as a pose to the continuous-time case where it was a function of (continuous) time t. In the discrete-time case we let $\mathcal{U}$ denote the set of all admissible inputs (that is, the set of all sequences). Many of the continuous-time results from above translate almost directly to the discrete-time case, but as we will see later there are some important differences (for a detailed account see [Ant04]).

For given initial condition $x(0) = x_0$, the solution to (50a) is

$$x(k+1) = x(k+1; x_0, u) = A^{k+1}x_0 + \sum_{j=0}^{k} A^{k-j}Bu(j) \tag{51}$$

and therefore

$$y(k) = y(k; x_0, u) = CA^k x_0 + \sum_{j=0}^{k-1} h(k-j)u(j) \tag{52}$$

with $h(k) = 0,\ k \leq 0$ and $h(k) = CA^{k-1}B,\ k > 0$ called the impulse response function. As a passing remark we mention that the behaviour of the IO system (50) also can be studied in the z-domain via its transfer function $H(z) = \mathcal{Z}(\{h(k)\})(z)$ obtained as the z-transformed (or discrete-time Laplace transformed) $\mathcal{Z}(\{h(k)\})(z) = C(zI-A)^{-1}B$ of the sequence $\{h(k)\}$.

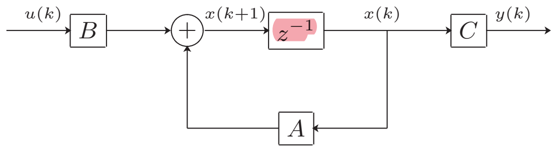


Figure 13: The IO system (50), with z^{-1} representing the backwards shift operator: $z^{-1}\mathcal{Z}(\{x(k+1)\}) = \mathcal{Z}(\{x(k)\})$

Real world systems is often described using first principles (Newton's laws) which then leads to a continuous-time model representing the system. However, as all digital devices operate in discrete-time it is often desirable to implement a discretised version of the continuous-time model. There are several ways to obtain the discrete-time system (50) as a discretised version of the continuous-time IO system (5). In the sequel we explain two way, exact and approximative discretisation. To do so we change notation slightly by indexing the continuous-time system matrices by c i.e., A_c, B_c, C_c, and the discrete-time system matrices by d i.e., A_d, B_d, C_d.

Exact discretisation: This discretisation scheme departures from the exact solution (6) and use a zero-order-hold assumption to obtain the discrete-time IO system (50) from the continuous-time system (5). In somewhat more details, let h denote the sampling time[16] and set $x(k) = x(kh)$ for $k \in \mathbb{N}$. Using (6) and assuming that the control input $u(t)$ is constant in the interval $[kh, (k+1)h)$, that is, $u(kh) = u(t)$ for $t \in [kh, (k+1)h)$, one can show that

$$x(k+1) = e^{A_c h}x(k) + \int_0^h e^{A_c\tau}d\tau B_c u(k) \tag{53}$$

Hence the system matrices corresponding to the discrete-time IO system is: $A_d = e^{A_c h}$, $B_d = \int_0^h e^{A_c\tau}d\tau B_c$ or $B_d = A_c^{-1}(e^{A_c h} - I)B_c$ if A_c is invertable, and $C_d = C_c$. When using exact discretisation it is noted that A_c and A_d have the same eigenvectors, and the eigenvalues λ_c of A_c are related to the eigenvalues λ_d of A_d by $\lambda_d = e^{h\lambda_c}$.

[16] where $h < \frac{1}{2b}$ with b the bandwidth the (SISO) system, that is, b is the first frequency where the gain drops below 70.79 percent (-3 dB) of its DC value.

Another way of computing the discretised state-space matrices which avoid the integral in (53) is to use the following formula[17]

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} h\right) \tag{54}$$

with the zero matrix in the lower left corners being $m \times n$.

Approximative discretisation (Euler's method): Due to the matrix exponents and the integral in (53), exact discretisation is often numerical infeasible. In these cases one may obtain an approximative discretisation by using that $\dot{x}(t) \approx \frac{x(t+h)-x(t)}{h}$ with $t = kh$ in (5). This immediately yields: $A_d = (I + A_c h)$, $B_d = hB_c$, and $C_d = C_c$. It is remarked that the same expressions could be obtained by using the approximation $e^{Ah} \approx I + Ah$ directly in (53). Finally, when using approximative discretisation it is noted that A_c and A_d have the same eigenvectors, and the eigenvalues λ_c of A_c are related to the eigenvalues λ_d of A_d by $\lambda_d = 1 + h\lambda_c$.

8.1 Reachability and controllability

Identical to the continuous-time case we define the reachable space

$$\mathbb{W}_T = \left\{ \sum_{j=0}^{T} A^{T-j} B u(j) \;\middle|\; u \in \mathcal{U} \right\} \tag{55}$$

as the set of all points x_1 which can be reached from $x(0) = 0$ in time $T \in \mathbb{N}$ by some $u \in \mathcal{U}$.

Theorem 9. *It holds true that $\mathbb{W}_T = Range[A|B]$, hence $\mathbb{W} = \mathbb{W}_T$ is independent of T. Moreover, the following are equivalent*

1. $\mathbb{W} = \mathbb{R}^n$
2. $Rank[A|B] = n$
3. *Any $x \in \mathbb{W}$ can be reached in at most n time steps.*
4. *There exists no $x \neq 0$ such that $x^*A = \lambda x^*$ and $x^*B = 0$*
5. *All eigenvalues of A are controllable: $Rank\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n, \; \forall \lambda \in \sigma(A)$*
6. *The controllability Gramian*

$$X(k) = [A|B][A|B]^* = \sum_{j=0}^{k-1} A^j B B^* (A^*)^j$$

is positive definite, $X(k) > 0$, for any $k > 0$.

The pair (A, B), or the IO system (50), is said to be reachable if any one of the above conditions are fulfilled. Moreover, we remark that if $x_0, x_1 \in \mathbb{W}$ then $x_1 = x(k; x_0, u)$ for some $u \in \mathcal{U}$ and $k \leq n$, that is, any two states in the reachable space are reachable. It is also noted that item 5 is similar to the continuous time case (item 5 of theorem 1). When using Euler's method with sampling time h we get $\begin{bmatrix} A_d - \lambda_d I & B_d \end{bmatrix} = h \begin{bmatrix} A_c - \lambda_c I & B_c \end{bmatrix}$ with (A_d, B_d) the discrete-time system matrices obtained from the continuous-time system matrices (A_c, B_c). Hence reachability is not effected by (approximative) discretisation.

We now turn to the concept of controllability: The pair (A, B), or the IO system (50), is said to be controllable if $\mathbb{W}^c = \mathbb{R}^n$, with

$$\mathbb{W}^c = \left\{ x_0 \;\middle|\; x(T; x_0, u) = 0, \; T \in \mathbb{N}, \; u \in \mathcal{U} \right\}$$

the controllable subspace, defined similar to (12). Recall that in the continuous-time case reachability ($\mathbb{W} = \mathbb{R}^n$) is equivalent to controllability ($\mathbb{W}^c = \mathbb{R}^n$). In the discrete-time case this is no longer the case since

$$\mathbb{W}^c = \mathbb{W} + Null A^n \tag{56}$$

[17]follows by considering the solution to $\dot{x} = Ax + Bz, \; \dot{z} = 0$

Hence, in the discrete-time case reachability is equivalent to controllability iff A is non-singular. Finally, in the non reachable case one may use a Kalman decomposition identically to the continuous-time case (cf. section 3.2).

8.2 Stabilizability

As in the continuous-time case, stability of

$$x(k+1) = Ax(k) \tag{57}$$

can be characterized by the spectrum $\sigma(A) \subset \mathbb{C}$ of A.

Theorem 10. *The following are equivalent*

- *The linear system* (57) *is stable, that is,* $x(k) \to 0$ *as* $k \to \infty$ *for any initial condition* $x(0) = x_0$
- $|\lambda| < 1$ *for all* $\lambda \in \sigma(A)$*, that is, all eigenvalues of* A *are inside the unit circle in* $\mathbb{C}$
- *For any* $X \in \mathbb{R}^{n\times n}$ *with* $X > 0$*, there exists* $P \in \mathbb{R}^{n\times n}$ *with* $P > 0$ *such that* $A^*PA - P + X = 0$*.*

The equation $A^*PA - P + X = 0$ is called the discrete Lyapunov equation. Moreover, it follows that in order to stabilize (50a) one can chose the control law $u = Kx$ such that $\sigma(A+BK) \subset \mathbb{C}$ is inside the unit circle (in this case (50a), or the pair (A, B) is said to be stabilizable). The existence of such a gain K is guaranteed if (A, B) is reachable.

For reference tracking we find $N_x \in \mathbb{R}^{n\times m}$ and $N_u \in \mathbb{R}^{m\times m}$ by

$$\begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{58}$$

and, as in the continuous-time case, let $u = K(x - N_x r) + N_u r$ (with K such that $\sigma(A+BK) \subset \mathbb{C}$ is inside the unit circle) which is the control law solving the tracking problem ($y(k) \to r$ as $k \to \infty$). Integral action is introduced via the (integral) state x_I fulfilling the difference equation

$$x_I(k+1) = x_I(k) + e(k) = x_I(k) + Cx(k) - r$$

which result in the augmented system

$$\begin{bmatrix} x(k+1) \\ x_I(k+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} \begin{bmatrix} x(k) \\ x_I(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) - \begin{bmatrix} 0 \\ I \end{bmatrix} r \tag{59}$$

The control law solving the tracking problem and mitigating steady state error is then given by

$$u = \bar{K} \begin{bmatrix} x - x' \\ x_I \end{bmatrix} + u' = \begin{bmatrix} K & K_I \end{bmatrix} \begin{bmatrix} x - x' \\ x_I \end{bmatrix} + u' \tag{60a}$$

with $\bar{K}$ stabilizing (59) with $r = 0$, similar to the continuous-time case.

To mitigate input saturation, with say $u(k) \in [u_l, u_u]$, an anti-windup scheme can be devices by means of a simple if/else statement as follows: IF $u(k) \in [u_l, u_u]$ THEN $x_I(k+1) = x_I(k) + e(k)$ ELSE $x_I(k+1) = x_I(k)$. Versions of this is often seen in industrial applications.

8.3 Linear quadratic regulator

In the discrete-time case the LQ optimal control problem takes the form

$$\min_{u=\{u(k)\}} \sum_{k=0}^{N-1} \left(y(k)^*Qy(k) + u(k)^*Ru(k)\right) + y(N)^*Gy(N) \tag{61a}$$

subject to

$$x(k+1) = Ax(k) + Bu(k),\ y(k) = Cx(k),\ x(0) = x_0 \tag{61b}$$

with $Q, G \geq 0$ (positive semi-definite) and $R > 0$ (positive definite). If (A, B) is reachable and $(A, Q^{1/2}C)$ observable, then the solution (that is, the optimal input sequence $u(0), u(1), \dots, u(N-1)$) is given by the feedback control law

$$u(k) = -L(k)x(k) \tag{62}$$

with

$$L(k) = (R + B^*S(k+1)B)^{-1}B^*S(k+1)A \tag{63}$$

and

$$\begin{aligned} S(k) &= C^*QC + L^*(k)RL(k) + (A - BL(k))^*S(k+1)(A - BL(k)) && (64) \\ &= C^*QC + A^*S(k+1)A - A^*S(k+1)B(R + B^*S(k+1)B)^{-1}B^*S(k+1)A && (65) \\ &= C^*QC + A^*S(k+1)(A - BL(k)) && (66) \\ S(N) &= C^*GC && (67) \end{aligned}$$

Hence to calculate the optimal input sequence one needs to start with (67) and recursively calculate $L(N-1), L(N-2), \dots, L(0)$. Note that (64) and (66) are best suited for this as they do not contain any matrix inversions. See also remark 1 on page 23.

If the case $N = \infty$ is of interest then the last term in (61a) and (67) should be disregarded, and one should set $S(k) = S(k+1) = S > 0$ which leads to an algebraic riccati equation in S obtained from (65). See also footnote 9 on page 25.

Moreover, one may also use the LQ approach over an indefinite time period by applying the (constant) feedback $u(k) = -L(0)x(k)$. This is sometimes referred to as receding horizon since at each time step one is applying a control law which is based on N samples ahead in time. Note that this can be viewed as an approximation to the $N = \infty$ case.

8.4 Observability

As in the continuous-time case let

$$\mathbb{V}^+ = \{x \mid y(k; x, 0) = 0,\ k \geq 0\}$$

be the set of of all states which are indistinguishable, on $[0, \infty)$, from the zero state 0.

Theorem 11. *It holds true that $\mathbb{V}^+ = Null(A|C)$. Moreover, the following are equivalent*

1. $\mathbb{V}^+ = \{0\}$
2. $Rank(A|C) = Rank[A^*|C^*] = n$
3. *There exists no $x \neq 0$ such that $Ax = \lambda x$ and $Cx = 0$*
4. *All eigenvalues of A are observable: $Rank \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n,\ \forall \lambda \in \sigma(A)$*
5. *The observability gramian*

$$Y(k) = [A^*|C^*][A^*|C^*]^* = \sum_{j=0}^{k-1} (A^*)^j C^* C A^j$$

is positive definite, $Y(k) > 0$, for any $k > 0$.

The system (50), or the pair (A, C), is said to be observable if any one of the above conditions are fulfilled. It is remarked that the relation between the unobservable subspace $\mathbb{V}^+ = Null(A|C)$ and the unreconstructible subspace $\mathbb{V}^- = \{x \mid y(k; x, 0) = 0,\ k \leq 0\}$ is not as in the continuous-time case since now

$$\mathbb{V}^+ = \mathbb{V}^- \cap Range A^n$$

It is also noted that item 4 is similar to the continuous time case (item 5 of theorem 5). When using Euler's method with sampling time h we get $A_d - \lambda_d I = h(A_c - \lambda_c I)$ with A_d the discrete-time system matrice obtained from the continuous-time system matrice A_c. Hence observability is not effected by (approximative) discretisation.

8.5 Observer

Similar to the continuous-time case let P denote a plant described by (50). In the discrete-time case there are (at least) two way of designing an observer: using old measurements (prediction observer), or using the most recent measurements (current observer). In the sequel we discuss both design methods.

The equations governing the prediction observer is

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \tag{68a}$$
$$\hat{y}(k) = C\hat{x}(k) \tag{68b}$$

with the (observer) gain $L \in \mathbb{R}^{n\times p}$ chosen such that the error $e(k) = x(k) - \hat{x}(k) \to 0$ as $k \to \infty$. This can be accomplished by observing that

$$e(k+1) = (A - LC)e(k) \tag{69}$$

so L needs to be chosen such that (69) is stable: $|\lambda| < 1$ for all $\lambda \in \sigma(A - LC)$. Note that the time evolution of the estimate $\hat{x}$ is

$$\text{time} = 0 \ : \ \hat{x}(0) \tag{70a}$$
$$\text{time} = 1 \ : \ \hat{x}(1) = A\hat{x}(0) + Bu(0) + L(y(0) - \hat{y}(0)) \tag{70b}$$
$$\text{time} = 2 \ : \ \hat{x}(2) = A\hat{x}(1) + Bu(1) + L(y(1) - \hat{y}(1)) \tag{70c}$$
$$\vdots$$

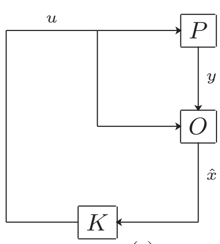
The prediction observer can now be used to generate the input $u = K\hat{x}$ to the plant P which then evolves according to

$$x(k+1) = Ax(k) + BK\hat{x}(k) \tag{71}$$

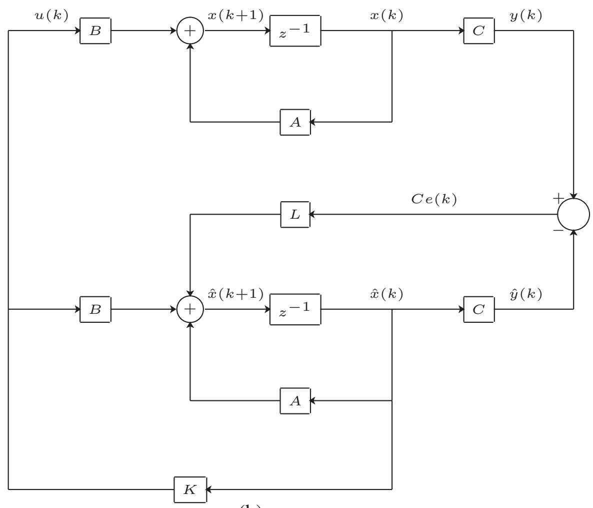
Combining this with (69) we obtain

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} \tag{72}$$

which, exactly as in the continuous-time case, show that in order to stabilize the plant P using only the outputs we need to chose the gains K and L such that $A+BK$ and $A-LC$ both have all their eigenvalues inside the unit circle in $\mathbb{C}$. This is doable when (A, B) is reachable and (A, C) is observable.



(a)



(b)

Figure 14: Stabilization by dynamic measurement feedback using prediction observer

Including a prediction observer in the reference tracking synthesis is similar to the continuous-time case described in section 6.3.

When using the prediction observer the estimate $\hat{x}(k+1)$ depend on the old measurement $y(k)$. The current observer use the most recent (or current) measurement $y(k+1)$ to estimate $\hat{x}(k+1)$ by introducing an auxiliary variable $\bar{x}$ in the observer design:

$$\hat{x}(k+1) = \bar{x}(k+1) + L(y(k+1) - \bar{y}(k+1)) \tag{73a}$$
$$\bar{x}(k+1) = A\hat{x}(k) + Bu(k) \tag{73b}$$
$$\bar{y}(k) = C\bar{x}(k) \tag{73c}$$

In this case the evolution of the error $e(k) = x(k) - \bar{x}(k)$ is

$$e(k+1) = (A - ALC)e(k) \tag{74}$$

so if $\bar{L} = AL$ is chosen such that $|\lambda| < 1$ for any $\lambda \in \sigma(A - \bar{L}C)$, and A is nonsingular then (74) will be stable with $L = A^{-1}\bar{L}$. In this case we have $e(k) = x(k) - \bar{x}(k) \to 0$ as $k \to \infty$ implying that $\bar{x}(k) \to x(k)$ as $k \to \infty$. Combining this with (73a) we conclude that $\hat{x}(k) \to x(k)$ as $k \to \infty$ as desired.

For the current observer the time evolution of the estimate $\hat{x}$ is

$$\text{time} = 0 \ : \ \bar{x}(1) = A\hat{x}(0) + Bu(0) \tag{75a}$$
$$\bar{y}(1) = C\bar{x}(1) \tag{75b}$$
$$\text{time} = 1 \ : \ \hat{x}(1) = \bar{x}(1) + L(y(1) - \bar{y}(1)) \tag{75c}$$
$$\bar{x}(2) = A\hat{x}(1) + Bu(1) \tag{75d}$$
$$\bar{y}(2) = C\bar{x}(2) \tag{75e}$$
$$\text{time} = 2 \ : \ \hat{x}(2) = \bar{x}(2) + L(y(2) - \bar{y}(2)) \tag{75f}$$
$$\bar{x}(3) = A\hat{x}(2) + Bu(2) \tag{75g}$$
$$\bar{y}(3) = C\bar{x}(3) \tag{75h}$$
$$\vdots$$

As above we use the current observer to generate the input $u = K\hat{x}$ to the plant P which then yields

$$\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A + BK & -BK - LC \\ 0 & A - ALC \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix} \tag{76}$$

showing that stabilizing the plant P amounts to choosing the gains K and L such that $A + BK$ and $A - ALC$ both have all their eigenvalues inside the unit circle in $\mathbb{C}$. This is doable when A is nonsingular (A, B) is reachable and (A, C) is observable.

For the current observer it can be an advantage to introduce the following notation: Let $\hat{x}(k+i|k)$ denote (the future) value of $\hat{x}$ at time $k+i$, assumed at (the current) time k. With this notation (73) can be written as

$$\hat{x}(k|k) = \bar{x}(k|k-1) + L(y(k) - \bar{y}(k|k-1)) \tag{77a}$$
$$\bar{x}(k+1|k) = A\hat{x}(k|k) + Bu(k) \tag{77b}$$
$$\bar{y}(k|k-1) = C\bar{x}(k|k-1) \tag{77c}$$

which agree with (75). It should be remarked that this notation is widely used whenever receding horizon is applied, in particular in the literature on Model Predictive Control (MPC).

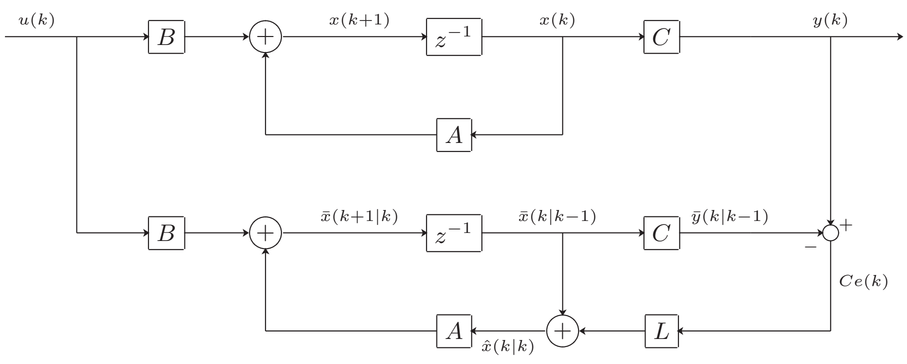


Figure 15: Current observer

9 The Kalman filter

In this section we describe, what is safe to say, the most widely used observer; the Kalman filter. It is very easy to implement (see the algorithm given by (82)-(83)) and has ”build-in” noise mitigation.

Consider the IO system (50) with noise:

$$x(k+1) = Ax(k) + Bu(k) + Gw(k) \tag{78a}$$
$$y(k) = Cx(k) + v(k) \tag{78b}$$
$$w(k) \sim N(0, Q(k)), \;\; v(k) \sim N(0, R(k)) \tag{78c}$$

where $N(\mu, \Sigma)$ denote multivariate normal (or Gaussian) distribution with mean vector μ and covariance matrix Σ, $G \in \mathbb{R}^{n\times m}$ called the process noise (gain) matrix, $w(k) \in \mathbb{R}^m$ the (Gaussian) process noise, and $v(k) \in \mathbb{R}^p$ the (Gausian) measurement noise. The (discrete-time) stochastic processes $w(k)$ and $v(k)$ are assumed to be white[18] and independent. Moreover, the covariance matrices $Q(k) \in \mathbb{R}^{m\times m}$ and $R(k) \in \mathbb{R}^{p\times p}$ are assumed know.

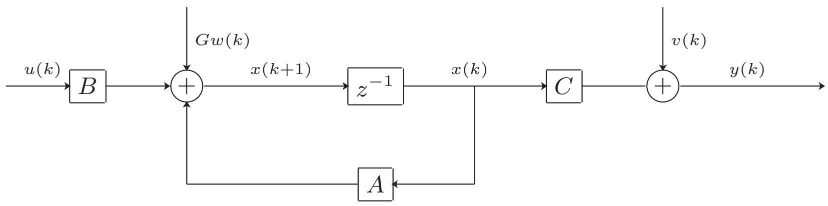


Figure 16: The discrete-time IO system (78)

Remark 1. *Before proceeding to the Kalman filter we remark that if the dynamics* (61b) *in the LQR problem is replaced by* (78) *and* (79b)*, and one takes the expectation of the cost* (61a)*, then the solution to this stochastic optimal control problem (the linear quadratic gaussian, LQG) is the same as the (deterministic) solution* (62)-(66)*, see e.g., [Åst70].*

Given the measurements and initial value

$$Y_0^k = y(0), y(1), y(2)\ldots, y(k) \tag{79a}$$
$$x(0) \sim N(\hat{x}, P) \tag{79b}$$

with $x(0)$ independent of $w(k)$ and $v(k)$. The estimation problem is to find the estimate $\hat{x}(k)$ that minimizes the mean square error (MSE)

$$E\left((x(k) - \hat{x}(k))^*(x(k) - \hat{x}(k))\right) \tag{80}$$

It is well known that the general solution is the conditional mean value

$$\hat{x}(k) = E\left(x(k)\middle|Y_0^k\right) \tag{81}$$

The Kalman filter provides the solution to the estimation problem and can be formulated algorithmically as (82)-(83) below. To formulate the algorithm we use the notation introduced in section 8.5 and let $\hat{x}(k{+}1|k)$, the a priori estimate, denote the estimate of $x(k+1)$ based on the observations up to time k, that is, based on Y_0^k, and likewise for the a posteriori estimate $\hat{x}(k|k)$.

The Kalman algorithm, with initial conditions $\hat{x}(0|-1)$ and $P(0|-1)$, consist of the following recursive two-step procedure:

The measurement update (or update step) at time k is performed as follows:

$$K(k) = P(k|k-1)C^*(CP(k|k-1)C^* + R(k))^{-1} \tag{82a}$$
$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)(y(k) - C\hat{x}(k|k-1)) \tag{82b}$$
$$P(k|k) = (I - K(k)C)P(k|k-1)(I - K(k)C)^* + K(k)R(k)K(k)^* \tag{82c}$$
$$= (I - K(k)C)P(k|k-1) \tag{82d}$$

[18] That is, have zero mean and independent samples (hence $w(k)$ and $v(k)$ are i.i.d processes if the covariance matrices $Q(k)$ and $R(k)$ are independent of time k)

The time update (or prediction step) from k to $k+1$ is performed as follows:

$$\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k) \tag{83a}$$
$$P(k+1|k) = AP(k|k)A^* + GQ(k)G^* \tag{83b}$$

The term $\hat{e}(k) = y(k) - C\hat{x}(k|k-1)$ is called the innovation variable and the expression (82c) for the (covariance) update $P(k|k) \in \mathbb{R}^{n\times n}$ is sometimes referred to as the Joseph form. The expression (82d) is computationally cheaper compared to the Joseph form but might cause numerical instability when arithmetic precision is low. Finally, $K(k) \in \mathbb{R}^{n\times p}$ from (82a) is typically referred to as the Kalman gain. It is informative to note that the Kalman filter is precisely the current observer from section 8.5 together with a method of choosing the observer gain L (namely as the Kalman gain).

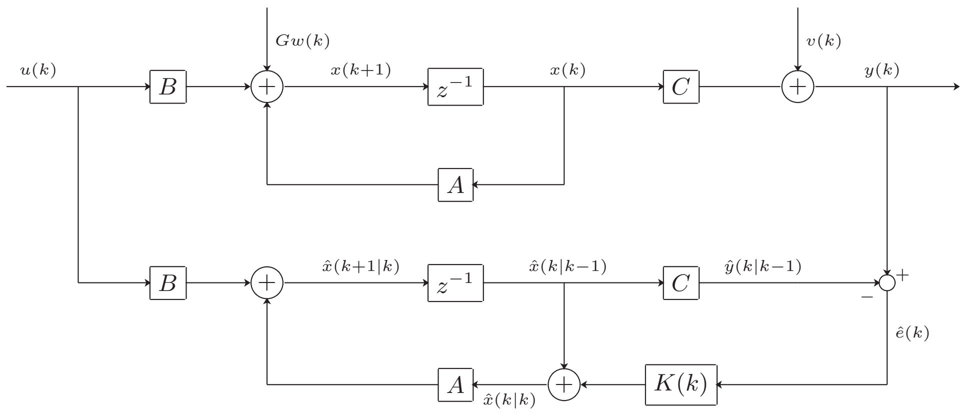


Figure 17: Kalman filter

It is, of course, important to mention that the terms in algorithm (82)-(83) have probabilistic interpretations:[19]

$$\hat{x}(k|k-1) = E\left(x(k)\middle|Y_0^{k-1}\right) \tag{84a}$$
$$\hat{x}(k|k) = E\left(x(k)\middle|Y_0^{k}\right) \tag{84b}$$
$$P(k+1|k) = Cov\left(x(k+1)\middle|Y_0^{k}\right) \tag{84c}$$
$$= E\left((x(k+1) - \hat{x}(k+1|k))(x(k) - \hat{x}(k+1|k))^*\middle|Y_0^{k}\right), \tag{84d}$$
$$P(k|k) = Cov\left(x(k)\middle|Y_0^{k}\right) \tag{84e}$$
$$= E\left((x(k) - \hat{x}(k|k))(x(k) - \hat{x}(k|k))^*\middle|Y_0^{k}\right) \tag{84f}$$

It should be noted that the covariance matrices $Cov\left(x(k+1)\middle|Y_0^k\right)$ and $Cov\left(x(k)\middle|Y_0^k\right)$ are in fact independent of the observed measurements Y_0^k (as see by (82)-(83)). Moreover, if the covariance matrices $Q(k)$ and $R(k)$ are independent of time k, $(A, GQ^{1/2})$ is reachable, and (A, C) is observable then the covariance $P(k+1|k)$ and Kalman gain $K(k)$ will converge to steady state values P_s and K_s, respectively.[20] Indeed, under the assumptions above the steady state value of the Kalman gain is given by

$$K_s = P_s C^*(CP_sC^* + R)^{-1} \tag{85}$$

with $P_s > 0$ the unique solution to the riccati equation

$$P = A(P - PC^*(CPC^* + R)^{-1}CP)A^* + GQG^* \tag{86}$$

Hence these steady state values can be found offline and one typically only use the steady state value of the Kalman gain in implementation (together with (77)). Note that (86) is just (83b) combined with (82d) and using steady state values. Moreover, (86) is the dual to the (infinite horizon) LQR riccati equation, that is,

[19]For random vectors x, y and z, the conditional covariance matrix and conditional variance are denoted $Cov(x,y|z) = E\left((x - E(x|z))(y - E(y|z))^*\middle|z\right)$ and $Cov(x|z) = Cov(x,x|z)$, respectively.
[20]This is true since none of the system matrices are time dependent.

in (65) with $S(k) = S(k+1) = P$ set A to A^*, B to C^*, and C to G^* to obtain (86). Also, AK_s is dual to the (infinite horizon) adjoint LQR gain L^*, with L given by (63). It should be remarked that sometimes AK_s is referred to as the Kalman gain and that it correspond to the gain $\bar{L}$ from the current observer, see below (74).

Finally note that under assumption (79b) we obtain

$$x(0) \sim N(\hat{x}(0|-1), P(0|-1)) \tag{87}$$

from (84a) and (84c).

We will end the discussion of the Kalman filter with comments on ”common practices” design. The initial values $\hat{x}(0|-1)$ and $P(0|-1)$ can be chosen based on a guess according to (87) e.g., $\hat{x}(0|-1) = 0$ and $P(0|-1) = GQ(0)G^*$ to copy $Gw(0)$. Alternative, one can simply set $\hat{x}(0|-1) = 0$ and $P(0|-1) = I$. The gain matrix G is typically modeled/assumed to be the identity matrix. The covariance matrices $Q(k)$ and $R(k)$ are often assumed time independent and diagonal, thus only the variances Q_{ii} and R_{ii} of $w_i(k)$ and $v_i(k)$ influence the Kalman filter. In this case, one can often estimate the covariance matrices R based on some time series of measurements, and Q is viewed as a tuning parameter with the rule that Q_{ii} should be chosen as large as possible without the state estimate $\hat{x}$ becoming to large.

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