TTLM Contest Problems

To The Limit Maths

QUICK REMINDER. . .

- This is a 15-question, 2-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
- No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and the internet are not permitted** (other than for submitting answers).
- **Do not** discuss problems or solutions either orally or digitally until the February 6th, 12:00 AM (after the contest window is over).
- Good luck, and have fun!

For questions, join our discord and ping Head Tutors, or email us at tothelimitmathinc@gmail.com.

Problem 1

What is the area of the smallest $n \times m$ grid (with n and m both positive integers) such that the number of paths from the bottom left to the top right is divisible by 2021?

Problem 2

A set $\{x_1, x_2, \ldots, x_n\}$ is called "godly" if it has the property that

$$\prod_{i=1}^{n} x_i = x_j^2$$

for a j such that $1 \leq j \leq n$. How many such "godly" sets are there with the largest element as 2022, where all elements are greater than 1?

Problem 3

A rectangular prism has one side of length $\left(3+\sqrt{7}\right)$. Given that the volume and perimeter are integers and surface area is 40 and x is the length of the space diagonal (the line segment connecting two vertices that do not share a common face), find x^2.

Problem 4

Let $r(n)$ represent the number of divisors of n. Evaluate

$$\sum_{k=1}^{100} r(k)$$

Problem 5

Given that $r_1, r_2, \ldots, r_8$ are the roots of

$$x^8 - 2\sqrt{x^9} + 2x$$

and $r_9, r_{10}, \ldots, r_{16}$ are the roots of

$$x^8 + 2\sqrt{x^9} + 2x$$

find

$$|r_1^{14} + r_2^{14} + \cdots + r_{16}^{14}|.$$

Problem 6

Bob goes through an obstacle course of 5 total obstacles, where he must successfully complete all 5 obstacles in a row to pass. The probability he successfully completes each obstacle is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{5}$, respectively, and each attempt takes 1 second. He is allowed to choose the order of the obstacles. Assuming he chooses the optimal order, what is the expected time (in seconds) it takes for him to pass the obstacle course?

Problem 7

Let p be a prime and b be a positive integer such that $130 \geq p > b$. Find the number of pairs of integers (p, b) such that

$$b(b-1)(b-2)...21_p = 123...(b-1)b_{p^2}$$

where the number on the left is written in base p and the number on the right is written in base p^2 ($b, b-1, ..., 1$ are all digits).

Problem 8

Let $\theta_n = \arctan(\frac{2}{a_{n-1}})$ and $a_n \cos\theta_n = a_{n-1}$. If both a_0 and θ_1 are positive integers (θ_1 is written in degrees) and the minimum value of $a_{50} = x$, then find x^2.

Problem 9

Suppose that

$$A = \sum_{x=0}^{15} \sum_{y=0}^{15-x} \left(30 - \frac{7}{5}x - y\right).$$

Find the remainder when A is divided by 1000.

Problem 10

A primitive Pythagorean triple is called *happy* if

- all the sides are integers,
- it has 1 even leg and 1 odd leg,
- and the even leg is more than twice the length of the odd leg.

How many *happy* Pythagorean triples are there such that the even leg is less than 200?

Problem 11

Let $P(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}$, with n distinct complex roots. If the distinct roots are labeled $a_1, a_2, a_3, ..., a_n$, find

$$(1 + a_1)^{2n} + (1 + a_2)^{2n} + ... + (1 + a_n)^{2n}$$

Problem 12

Let ω_1 be a circle with center A and radius 10. Let ω_2 be a circle with center D and radius 20, which passes through point A. Let line AD intersect ω_1 at B and C with B in between A and D. Let E and F be the intersections of ω_1 and ω_2. Let line ED intersect ω_2 at H such that $H \neq E$ and let line CD intersect ω_2 at I such that $I \neq A$. If the area of pentagon $CFHIE$ can be written as $\frac{m\sqrt{n}}{p}$ where m and p are coprime and n is square-free, then find $m + n + p$.

Problem 13

Many AIME problems are structured such that the answer to a question is a/b with a and b relatively prime and $a, b > 0$. Some problems ask for $a + b$ as the desired value, while others ask for $a \cdot b$ as the desired value. $A(n)$ is defined as the number of possible solutions to a/b given $a + b = n$. $B(n)$ is defined as the number of possible solutions to a/b given $a \cdot b = n$ ($n/1$ counts as a solution). Find the largest $k < 1000$ such that $A(k) = B(k)$.

Problem 14

We are given that

$$x = \sum_{k=0}^{30} \arcsin\left(\frac{-2k-3}{\sqrt{(k+1)^4+1}\sqrt{(k+2)^4+1}}\right).$$

If $\sin\left(\frac{\pi}{4} + x\right)$ can be written as $\frac{a}{\sqrt{b}}$ where b is not a perfect square, find the remainder when $a + b$ is divided by 1000.

Problem 15

Bob walks a path starting from $(0, 0, 0)$, where each move, he walks from (x, y, z) to $(x+1, y, z)$, $(x, y+1, z)$, or $(x, y, z+1)$. Let X be the number of ways there are to get to $(5, 5, 5)$ given that each move, he cannot move in the same direction as the previous move. Find the remainder when X is divided by 1000.