Probability and Statistics January Exam 2024

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1 Question One

Assume the number of buses passing through a certain stop in 1 hour follows a Poisson distribution with average $\lambda = 5.5$. Calculate the probability that at least one bus turns up in 1 hour.

Answer: Recall that the Poisson distribution is (?). ✗

Real Answer: the Poisson distribution given $\bar{x} = \mu = \mathbb{E}(X) = \lambda$, and the $\mathbb{P}(X = k)$ is given by

$$\frac{\lambda^k e^{-\lambda}}{k!}.$$

One should consider the fact that the events are assumed to be independent. In light of this, given $\lambda = 5.5$ and $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0)$, we get,

$$1 - \frac{5.5^0 e^{-5.5}}{0!} = 0.995913... \approx 0.995,\ 99.5\%$$

2 Question Two

You roll a fair dice once

2.1 Q2 a

Calculate the probability of getting the number six.

Answer:

$$\mathbb{P}(X = 6) = \frac{|\{6\}|}{|\Omega|} = \frac{|\{6\}|}{|\{1, 2, 3, 4, 5, 6\}|} = \frac{1}{6}. \checkmark$$

2.2 Q2 b

You roll a fair dice n times. Calculate the probability of getting at least one six.

Answer: To begin our investigation, let $n = 1 : \mathbb{P}(X = 6) = \frac{1}{6},\ n = 2 : \mathbb{P}(X = 2) = \frac{1}{36},\ n = 3 : \mathbb{P}(X = 6) = \frac{1}{216}$, etc. We can clearly see a climbing in the powers of six where the index linearly increases with n.

We may therefore deduce that for n-times, $\mathbb{P}(X = 6) = 6^{-n}$ ✗.

Real answer: I simply misunderstood the question. They ask at least one six, implying that there maybe more than 1 six. Therefore, we don't want to calculate $\mathbb{P}(X = 6)$, as It's actually not relevant to the question. We want to calculate $\mathbb{P}(\text{At least 1 six})$. We could cleverly set $Y = (\text{number of sixes})$ and then state $\mathbb{P}(Y \geq 1) = 1 - \mathbb{P}(Y = 0)$.

$$1 - \mathbb{P}(Y = 0) = 1 - \left(\frac{5}{6}\right)^n.$$

2.3 Q2 c

Find the values of n such that this probability is greater than or equal to 50%.

Answer: given our formula $1 - \left(\frac{5}{6}\right)^n$, we want,

$$
\begin{aligned}
1-\left(\frac{5}{6}\right)^{n} &\geq 0.5, \\
\left(\frac{5}{6}\right)^{n} &\geq 0.5, \\
\ln\left(\left(\frac{5}{6}\right)^{n}\right) &\geq \ln(0.5), \\
n &\geq \frac{\ln(0.5)}{\ln\left(\left(\frac{5}{6}\right)\right)}, \\
n &\geq 3.8016 \approx 3.8 \approx 4. \checkmark
\end{aligned}
$$

3 Question Three

Given the random variables X and Y with pmf of,

$$
\mathbb{P}(X=k)=\begin{cases}\frac{1}{2}, & k=0,\\ \frac{1}{2}, & k=3,\\ 0, & k\notin\{0,3\}.\end{cases}
$$

$$
\mathbb{P}(Y=k)=\begin{cases}\frac{1}{3}, & k=-1,\\ \frac{2}{3}, & k=1,\\ 0, & k\notin\{-1,1\}.\end{cases}
$$

3.1 Q3 a

Provide an example of a joint probability distribution for X and Y such that X and Y are independent. You can use a table as below:

	$X=0$	$X=3$	marginal
$Y=-1$	?	?	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	?	?	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

Answer:

	$X=0$	$X=3$	marginal
$Y=-1$	$\mathbb{P}(Y=-1)\cap\mathbb{P}(X=0)$	$\mathbb{P}(Y=-1)\cap\mathbb{P}(X=3)$	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	$\mathbb{P}(Y=1)\cap\mathbb{P}(X=0)$	$\mathbb{P}(Y=1)\cap\mathbb{P}(X=3)$	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

$\downarrow$

	$X=0$	$X=3$	marginal
$Y=-1$	$1/3\times1/2$	$1/3\times1/2$	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	$2/3\times1/2$	$2/3\times1/2$	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

$\downarrow$

	$X=0$	$X=3$	marginal
$Y=-1$	$1/6$	$1/6$	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	$1/3$	$1/3$	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

✓

3.2 Q3 b

Provide an example of a joint probability distribution for X and Y such that X and Y are not independent.

	$X=0$	$X=3$	marginal
$Y=-1$	$1/6+0.01$	$1/6+0.01$	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	$1/3+0.01$	$1/3+0.01$	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

✓

4 Question Four

The average size of a foreign exchange transaction in a certain trading desk is $\$25.1m$ with a standard deviation of $\$15m$. We assume these numbers follow a normal distribution. We are concerned about the impact of a new regulatory reporting procedure that kicks in for transactions larger than $\$50m$. Calculate the probability that a given transaction will be larger than $\$50m$.

Answer: Given that the distribution of the transactions follows a normal distribution, $\mathcal{N}(\mu, \sigma^2)$, we are concerned with the pdf of,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

We want to find, quote, "the probability that a given transaction will be larger than $\$50m$", which translates mathematically to $\mathbb{P}(X > 50m) = 1 - \mathbb{P}(X \leq 50m)$.

We can use therefore, the cdf of $\mathcal{N}(\mu, \sigma^2)$ to find $\mathbb{P}(X \leq 50m)$, which is,

$$\Phi\left(\frac{x-\mu}{\sigma}\right).$$

We therefore want to calculate $\Phi((50m - 25.1m)/(15m)) = \Phi(1.66)$. One must look this value up in the given table of values, where one obtains 0.9515.

Given that $\mathbb{P}(X \leq 50) = 0.9515$, $\mathbb{P}(X > 50m) = 1 - 0.9515 = 0.0485 \approx 0.05$. The chance of a transaction being larger than $\$50m$ is 5%. ✓✓✓

5 Question Five

This question demonstrates a special feature of the exponential distribution that makes it suitable for applications to waiting times. It is called the memoryless property. This is a characteristic of the Exponential random variable that might or might not correspond to what happens in real-life applications.

Assume that $X \sim \mathrm{Exp}(\lambda)$ models the wait time in hours to the next bus if on average there are λ busses per hour.

5.1 Q5 a

$x_1 \in \mathbb{R}_+$, what is the probability that no buses arrived before time x_1, i.e., calculate $\mathbb{P}(X > x_1) = 1 - \mathbb{P}(X \leq x_1)$.

Answer: The cdf, as needed, of the exponential distribution is $1 - e^{-\lambda x}$.

We are working with the general x_1, and with then general λ, so $1 - (1 - e^{\lambda x_1}) = 1 - 1 + e^{\lambda x_1} = e^{\lambda x_1}$. ✗

Real answer: Forgot the negative sign for lambda, $e^{-\lambda x_1}$.

5.2 Q5 b

For $0 < x_1 < x_2$, calculate the probability that the bus will arrive between the two, i.e., $\mathbb{P}(x_1 < X < x_2)$.

Answer: $\mathbb{P}(x_1 < X < x_2) = \mathbb{P}(X > x_1) + \mathbb{P}(X < x_2) = (1 - \mathbb{P}(X \leq x_1)) + \mathbb{P}(X < x_2)$.

Once again referring to the cdf, that being $1 - e^{-\lambda x}$, we get the following,

$$\big(1 - (1 - e^{-\lambda x_1})\big) + (1 - e^{-\lambda x_2}) = 1 + e^{-\lambda x_1} - e^{-\lambda x_2}. \text{ ✗}$$

Real answer: In fact we may use some basic logic to state that $\mathbb{P}(x_1 < X < x_2) = \text{cdf}(x_2) - \text{cdf}(x_1)$. Following this, we get $(1 - e^{-\lambda x_2}) - (1 - e^{-\lambda x_1}) = e^{-\lambda x_1} - e^{-\lambda x_2}$.

5.3 Q5 c

Calculate $\mathbb{P}(X < x_2 | X > x_1)$.

Answer: $\mathbb{P}(X < x_2 | X > x_1) = (\mathbb{P}(X < x_2) \cap \mathbb{P}(X > x_1))/(\mathbb{P}(X > x_1))$. This yields, $\mathbb{P}(X < x_2) \cap (1 - \mathbb{P}(X \leq x_1))/(1 - \mathbb{P}(X \leq x_1))$.

Given that $\mathbb{P}(X > x_1) = e^{-\lambda x_1}$, and that $\mathbb{P}(X < x_2) = 1 - e^{-\lambda x_2}$, we may claim,

$$\mathbb{P}(X < x_2 | X > x_1) = \frac{(e^{-\lambda x_1}) \cap (1 - e^{-\lambda x_2})}{e^{-\lambda x_1}}. \text{ ✗}$$

Real answer: We simply need to extend out current understanding. $\mathbb{P}(X < x_2 | X > x_1) = \mathbb{P}(x_1 < X < x_2)$, so we may therefore use out previous answer to claim that,

$$\frac{e^{-\lambda x_1} - e^{-\lambda x_2}}{e^{-\lambda x_1}}.$$

5.4 Q5 d

Show that the probability above, coincides with the probability that the bus arrives before time $x_2 - x_1$, $\mathbb{P}(X < x_2 - x_1)$. This means that if I go to the bus stop, the time I will need to wait till the next bus is independent of the fact that there might have been no buses in the previous hour.

Answer: We must show that,

$$\mathbb{P}(X < x_2 - x_1) = \frac{e^{-\lambda x_1} - e^{-\lambda x_2}}{e^{-\lambda x_1}}.$$

Start by stating that $\mathbb{P}(X < x_2 - x_1) = \text{cdf}(x_2) - \text{cdf}(x_1) = (1 - e^{-\lambda x_2}) - (1 - e^{-\lambda x_1}) = e^{-\lambda x_1} - e^{-\lambda x_2}$. ✗

Real answer: It's easier, in this case, to work backwards, i.e.,

$$\frac{e^{-\lambda x_1} - e^{-\lambda x_2}}{e^{-\lambda x_1}} = e^{\lambda x_1}(e^{-\lambda x_1} e^{-\lambda x_2}) = e^0 + e\lambda(x_1 - x_2) = 1 - e^{\lambda(x_1 - x_2)} = \mathbb{P}(X < x_2 - x_1).$$

6 Question Six

We need 650 Kg of cement to build an extension. This comes in sacks of 30Kg and so we purchase 21 such sacks. However due to imperfections in the manufacturing process the weight of a cement sack is 31Kg on average, with a standard deviation of 1Kg.

6.1 Q6 a

If X_i denotes the weight of the i-th sack of cement, and X is the sum of all X_i. What is the expectation, variance and standard deviation of X.

Answer: Given that $X = \sum_{n=1}^{i} X_n$, $\mathbb{E}(X) = \mu$.

$$\mathbb{E}(X) = \frac{\sum_{n=1}^{i} X_n}{i}.$$

We may state, due to the question,

$$\mathbb{E}(X) = \frac{\sum_{n=1}^{21} X_n}{21}.$$

The variance of our random variable X would be given by,

$$\text{Var}(X) = \sqrt{(X - \mathbb{E}(X))^2}.$$

Therefore, standard deviation would be given by,

$$\sigma = (X - \mathbb{E}(X))^2.$$ ✗

Real answer: $\mathbb{E}(X) = 21 \times 31 = 651$

$\text{Var}(X) = 21 \times 1^2 = 21$ so therefore $\sigma = \sqrt{21}$.

6.2 Q6 b

Use the Central Limit Theorem to estimate the probability that X is greater or equal than 650Kg.

Answer: We define the CLT as $Y \sim \mathcal{N}(\mu, \frac{\sigma}{\sqrt{n}})$, and so we must remember the following two formulae for $\mathcal{N}$,

$$\text{pdf}_{\mathcal{N}}(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

and,

$$\text{cdf}_{\mathcal{N}}(X) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Given the following we may calculate our statistic using the CLT. First lets convert our information into a form which the CLT accepts.

We want $\mathbb{P}(X \geq 650)$, given $\mathbb{E}(X) = 651$, $\sigma = \sqrt{21}$, $n = 21$. This yields $Y \sim \mathcal{N}\left(651, \frac{\sqrt{21}}{\sqrt{21}}\right) = Y \sim \mathcal{N}(651, 1)$. We therefore want $1 - \mathbb{P}(X < 650) = 1 - \text{cdf}_{\mathcal{N}}(650)$. This produces,

$$1 - \text{cdf}_{\mathcal{N}}(650) = \Phi\left(\frac{650 - 651}{\sqrt{21}}\right).$$

Using our table of values we get $1 - \text{cdf}_{\mathcal{N}}(650) = 1 - 0.5871 = 0.4129$.

Real answer: Everything is indeed correct until the end. We get a negative for $\Phi\left(\frac{650-651}{\sqrt{21}}\right)$, we simply ignore or add in the -1. So in this case, $\Phi(-0.218218) \approx 58\%$.

7 Question Seven

You flip a coin 25 times and count the number of heads, X.

7.1 Q7 a

Write down the `R` instruction to calculate the probability that X is greater than or equal to 10 and less than 15.

Answer: I don't fucking know?

Real answer: $\mathbb{P}(10 \leq X < 15) = \mathbb{P}(X < 15) - \mathbb{P}(X < 9)$.

`R: pbinom(14, 25, 1/2)-pbinom(8, 25, 1/2).`

7.2 Q7 b

Calculate the mean, variance and standard deviation of X.

Answer: Given that X counts heads, we may model this case with the Bernoulli distribution, given by,

$$f(k,p) = \begin{cases} p & \text{if } k = 1, \\ q = 1 - p & \text{if } k = 0. \end{cases}.$$

Due to there only being two outcomes, which are all independent, we may state that,

$$\mathbb{E}(X) = n\mathbb{P}(Y = \text{ heads}) = \frac{25}{2}.$$

$$\text{Var}(X) = n\mathbb{P}(Y = \text{ heads})(1 - \mathbb{P}(Y = \text{ heads})) = (25)(0.5)(0.5) = 6.25.$$

$$\sigma = \sqrt{6.25}.$$

7.3 Q7 c

We now use Central Limit Theorem to approximate X by a normal random variable Y . Write down the R instruction to calculate the probability that Y is greater than or equal to 10 and less than 15. Remember that in this case you must use the continuity adjustment.

Answer: Recalling that $Y \sim \mathcal{N}(\mu, \frac{\sigma}{\sqrt{n}})$, which makes use of,

$$\text{pdf}_{\mathcal{N}}(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

and,

$$\text{cdf}_{\mathcal{N}}(X) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

We want to know $\mathbb{P}(10 \leq Y < 15) = \mathbb{P}(Y < 15) - \mathbb{P}(Y < 9)$, which is given by,

$$\Phi\left(\frac{15 - 12.5}{\sqrt{6.25}}\right) - \Phi\left(\frac{9 - 12.5}{\sqrt{6.25}}\right),$$

$$= \Phi(0.4) - \Phi(-0.56) = \Phi(0.4) - (1 - \Phi(0.56)) = 0.6554 - (1 - 0.7123) = 0.3677.$$

This implies that $\mathbb{P}(10 \leq Y < 15) \approx 37\%$. ✗

Real answer: I forgot to account for continuity correction, where one simply minuses a half from the test value, i.e., $\mathbb{P}(10 \leq X < 15) = \mathbb{P}(9.5 \leq Y < 14.5)$. From there the same process carries out.