NSF Final

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July 23, 2024

1 Notation

1.1 Summation

The summation is a mathematical operation that adds up elements up to, and including a specified index.

$$\sum_{n=0}^{m} f(n) = N$$

In most important cases we work with $m = \infty$.

1.2 Product

The product is a mathematical operation that multiplies up elements up to, and including a specified index.

$$\prod_{n=1}^{m} f(n) = N$$

It may make sense depending on $f(n)$ to start the product at $n = 1$.

1.2.1 Factorial

The factorial is defined as,

$$m! = \prod_{n=1}^{m} n$$

2 Logic

2.1 A Statement

A statement is a sentence which may exist on its own and is either true or false.

2.2 Negation

If P is a statement, the negation of P is not P, or that of is P is false.

Take the example $x = 2$. The negation is $x \neq 2$.

2.3 Implication

Suppose P and Q are statements. 'if P, then Q' is an implication, which is written $P \implies Q$.

We have a special case where if $P \implies Q$, and P is true, then Q may not be false. One should note that the implication, unlike our spoken language, does not imply a direction causation, e.g., 'I go to sleep $\implies$ Donald Trump assassination attempt'. It should be clear that there is no causation, however mathematically this statement holds.

An important table we should therefore observe is as follows,

P	Q	$P \implies Q$
true	true	true
true	false	false
false	true	true
false	false	true

2.4 Quantifiers

There are two special mathematical objects that allow us to construct statements involving a variable in a 'general statement'.

We can write 'n is even', however is has the implicit dependency on $n \in \mathbb{Z}$, so we must state this explicitly, 'n is even $\forall\ n \in \mathbb{Z}$ '. The symbol $\forall$, literally means 'for all', and is a universal quantifier. Note that the statement we constructed is false.

2.5 Converse and Contrapositive

The converse of $P \implies Q$ is given by $Q \implies P$. The contrapositive of $P \implies Q$ is given by $\bar{Q} \implies \bar{P}$.

One should note that the contrapositive is the exact same thing as the normal statement, just 'rephrased'.

P	Q	$P \implies Q$	$\bar{P}$	$\bar{Q}$	$\bar{Q} \implies \bar{P}$
true	true	true	false	false	true
true	false	false	false	true	false
false	true	true	true	false	true
false	false	true	true	true	true

3 Proof

A proof is a logical argument that states a fact with 100% certainty.

The first part of a proof sets the context. The second part creates statements which are linked through implications. We also make assumptions throughout, like 'n is even'. The last part is the conclusion.

It maybe that to prove $P \implies Q$, we prove P, though to prove $P \iff Q$, one must prove P and Q.

A special proof technique is proof via contrapositive.

4 Analysis