Number, Sets and Functions 2023 Exam

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Contents

1 Question One

Let X = {1,3,4,6,9} and let Y = {2,3,5,8,9}.
Write down each of the following sets. *No justification is needed*,

(a) $X \cup Y$,
answer: $\{1, 2, 3, 4, 5, 6, 8, 9\}$. ✓

(b) $X \triangle Y$,
answer: $\{1, 2, 4, 5, 6, 8\}$. ✓

(c) $\{X \in x : x + 2 \notin X\}$,
answer: $\{3, 6, 9\}$. ✓

(d) $\{y + 2 : y \in Y \wedge y - 2 \in X\}$,
answer: $\{5, 7, 10\}$. ✓

Write down the supremum of each of the following sets

(e) $\{x^2 : -2 \leq x \leq 1\}$,
answer: 1. 7

$$\text{Real answer is } 4, \text{ Simply square } -2.$$

(f) $\left\{\dfrac{n}{n+1} : n \in \mathbb{N}\right\}$,
answer: $\dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}}$,

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$ ✓

(g) $\{\sin(x) : x \in \mathbb{Q}\}$,
answer: 1. ✓

2 Question Two

(a) Define precisely what it means for a function $f : A \to B$ to be injective.
answer: Injective means that both A and B are the same, i.e., the domain and codomain are the same. 7

Real answer: $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$.

(b) Define precisely what it means for a function $f : A \to B$ to be surjective.
answer: Surjective means ? 7

Real answer: $\forall\ b \in B\ \exists\ a \in A : f(a) = b$.

Are the following injective or not?
(c) $f : \mathbb{Z} \to \mathbb{Z}, f(n) = 20n + 22$.
answer: given our definition of injectivity, $\forall\ m, n \in \mathbb{Z}, f(m) = 20m + 22 = 20n + 22 = f(n) \implies m = n$.
$f : \mathbb{Z} \to \mathbb{Z}, f(n) = 20n + 22$ is injective. ✓

(d) $f : \mathbb{Z} \to \mathbb{Z}, f(n) = n(n + 1)$.
answer: Given that $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$,
Let $n, m \in \mathbb{Z} : f(n) = f(m) \equiv n(n + 1) = m(m + 1) \equiv m = n \implies f$ is injective. 7

Real answer: counter proof by counter example, $f(-1) = 0 = f(0) = 0(0 + 1) = 0$.

(e) $f : \mathcal{P}(\mathbb{Q}) \to \mathcal{P}(\mathbb{Q}), f(a) = a \cup \{1, 2, 3\}$
answer: First lets understand the function. We are working within the power series of the rationals.
This means that we are working with sets and not numbers.
$f(A) = A \cup \{1, 2, 3\}$ takes the intersection of two sets within $\mathbb{Q}$.
Recalling that for f to be injective, $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$,
Let $n = \{1\}, m = \{2\} : f(n) = \{1, 2, 3\}, f(m) = \{1, 2, 3\} \not\Longrightarrow n = m$ as $\{1\} \neq \{2\}$.
This suggests that f is not injective. ✓

(f) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}, f(m, n) = (m^2 + n^2, m^2 - n^2)$.
answer: Given $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$,
$f(1, 1) = (2, 0), f(-1, -1) = (2, 0)$ but $(1, 1) \neq (-1, -1)$.
By such logic, f is not injective. 7

Real answer: Notice the domain is $\mathbb{N} \times \mathbb{N}$ which means that $(-1, -1) \notin \mathbb{N} \times \mathbb{N}$.
This is the mistake made. If we notice any input is a positive integer, the proof follows,
$n, m, p, q \in \mathbb{N} \times \mathbb{N} : f(n, m) = f(p, q)$.
$\implies m^2 + n^2 = p^2 + q^2$ and $m^2 - n^2 = p^2 - q^2$.
$m^2 = p^2 - q^2 + n^2 \implies p^2 - q^2 + n^2 + n^2 = p^2 + q^2 \implies n^2 = q^2$ or $n = q$.
Given $n = q, m^2 = p^2 \implies m = q$. Index positions match, and we are only concerned with positive inputs.
Given $m = p, n = q \impliedby f(n, m) = f(p, q),\ f$ is injective.

3 Question Three

(a) Suppose P, Q, R are statements. Complete the following truth table for the statement $P \implies Q \wedge Q \implies \bar{R}$.
answer:

P	Q	R	$(P \implies Q) \wedge (Q \implies \bar{R})$
t	t	t	f
t	t	f	$\underline{\text{f}}$
t	f	t	$\underline{\text{f}}$
t	f	f	f
f	t	t	f
f	t	f	$\underline{\text{t}}$
f	f	t	$\underline{\text{t}}$
f	f	f	t

✗

Real answer:

P	Q	R	$(P \implies Q) \wedge (Q \implies \bar{R})$
t	t	t	f
t	t	f	$\underline{\text{t}}$
t	f	t	f
t	f	f	f
f	t	t	f
f	t	f	t
f	f	t	t
f	f	f	t

trivial mistake only.

(b) Suppose $x, y, z \in \mathbb{R}$. Write down the contrapositive of the following,
$x^2 > y^2 \implies \exists\, w \in \mathbb{R} : x < w \lor w < z$.
answer: $\forall\, w \in \mathbb{R} : w \leq x \land w \geq z \implies x^2 \leq y^2$ ✓

There is a better way to state this, $z \leq w \leq x \forall\, w \in \mathbb{R} \implies x^2 \leq y^2$

(c) Define the sequence $a_1, a_2, a_2, \ldots$ of integers by
$a_1 = 0, a_n = 4a_{n-1} + 12$ for $n \geq 2$.
Prove by induction that $a_n = 4^n - 4 \forall\, n \in \mathbb{N}$.
answer: Let us recall what proof by induction is. First we state our base case. This is $P(1)$.
Let us also remember that $P(n) : a_n = 4^n - 4 \forall\, n \in \mathbb{N}$.
Our inductive hypothesis is: Given $P(1)$ holds, show that for $n \geq 2,$ if $P(n-1)$ holds, then $P(n)$ also holds.
We therefore need to prove $P(n-1)$.

$P(n-1) : a_{n-1} = 4^{n-1} - 4 \forall\, n \in \mathbb{N}$. Recall $a_n = 4(a_{n-1} + 12)$.
$\implies a_n = 4(4^{n-1} - 4) + 12 = 4^n + 4$.
By induction, we have shown that $a_n = 4^n + 4 \forall\, n \in \mathbb{N} : n \geq 2$. ✓

(d) Explain why the following "proof" is not true
Suppose x is a real number satisfying $(x-2)^3 + 3(x-2)^2 + 2x = 4$.
Then $x = 0 \lor 1$.

answer: The proof (which is omitted) states that one should divide though by a defined variable y,
Which is not allowed. This is not allowed because when trying to find solutions,
we have eliminated some by the division operation. This means that there are potentially more solutions.
In this case, the proof misses the solution $x = 2$. ✓

4 Question Four

(a) Suppose $d, n \in \mathbb{N}$. What does it mean for $d \mid n$?
answer: $d \mid n \iff k \in \mathbb{N} : n = dk.$ ✓

(b) Suppose p, q are prime and that $p \neq q$.how many divisors does $p^3 \times q^3$ have?
answer: q^3 has divisors $1, q, q^2, q^3$ and p^3 has divisors $1, p, p^2, p^3$.
We may deduce that the number of divisors is the number of possible combinations or the divisors of p and q.
This is $4^2 = 16$ divisors. ✓

(c) Use Euclid's algorithm to find $\gcd(198, 82)$.
answer: Given $a = qb + r \implies 198 = 2(82) + 34,$

$82 = 2(32) + 18,$
$32 = 1(18) + 14,$
$18 = 1(14) + 4,$
$14 = 3(4) + 2,$
$4 = 2(2) + 0 \implies \gcd(198, 82) = 2.$ ✓

(d) Suppose $a, b \in \mathbb{N}$ and that $a \mid b$. Prove that $a^2 \mid b^2$.
answer: $a \mid b \implies b = ak\ a \in \mathbb{N}$. $a^2 \mid b^2 \implies b^2 = a^2 k,$

$$k = \frac{b^2}{a^2} \implies b = a\frac{b^2}{a^2} \implies \frac{1}{b} = \frac{1}{a} \implies a = b$$ 7

Real answer: Given $b = ak$ from $a \mid b$, we get $b^2 = a^2k^2, k^2 \in \mathbb{N}$ $\square$
The error arises from the fact that I used $b = ak$ and $b^2 = a^2k,$ but the k's are different.

(e) Define the relation $\mathcal{R}$ on $\mathbb{N} : a\mathcal{R}b \iff a \mid 2b$. Is $\mathcal{R}$ transitive?
answer: A transitive relation is given by $a\mathcal{R}b$ and $b\mathcal{R}c \implies a\mathcal{R}c$.
Let $2b = ak_1 : a\mathcal{R}b$ and let $2c = bk_2 : b\mathcal{R}c$.
$a\mathcal{R}c \iff a \mid 2c$ which yields $a \mid bk_2$.
Its already given that $bk_2 \in \mathbb{N} \implies \mathcal{R}$ is transitive. 7

Unsure as to why I stopped the proof/ disproof, bk_2 depends on $2b = ak_1$ where ak_1 is a multiple of 2, which may not be true. Therefore, $\mathcal{R}$ is not transitive.

5 Question Five

(a)
answer: