Number, Sets and Functions 2023 Exam

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1 Question One

Let $X = \{1, 3, 4, 6, 9\}$ and let $Y = \{2, 3, 5, 8, 9\}$.
Write down each of the following sets. *No justification is needed*,

(a) $X \cup Y$,
answer: $\{1, 2, 3, 4, 5, 6, 8, 9\}$. ✓

(b) $X \triangle Y$,
answer: $\{1, 2, 4, 5, 6, 8\}$. ✓

(c) $\{X \in x : x + 2 \notin X\}$,
answer: $\{3, 6, 9\}$. ✓

(d) $\{y + 2 : y \in Y \wedge y - 2 \in X\}$,
answer: $\{5, 7, 10\}$. ✓

Write down the supremum of each of the following sets

(e) $\{x^2 : -2 \leq x \leq 1\}$,
answer: 1. ✗

Real answer is 4, Simply square -2.

(f) $\left\{\dfrac{n}{n+1} : n \in \mathbb{N}\right\}$,
answer: $\dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}}$,

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$ ✓

(g) $\{\sin(x) : x \in \mathbb{Q}\}$,
answer: 1. ✓

2 Question Two

(a) Define precisely what it means for a function $f : A \to B$ to be injective.
answer: Injective means that both A and B are the same, i.e., the domain and codomain are the same. ✗

Real answer: $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$.

(b) Define precisely what it means for a function $f : A \to B$ to be surjective.
answer: Surjective means ? ✗

Real answer: $\forall\ b \in B\ \exists\ a \in A : f(a) = b$.

Are the following injective or not?
(c) $f : \mathbb{Z} \to \mathbb{Z}, f(n) = 20n + 22$.
answer: given our definition of injectivity, $\forall\ m, n \in \mathbb{Z}, f(m) = 20m + 22 = 20n + 22 = f(n) \implies m = n$.
$f : \mathbb{Z} \to \mathbb{Z}, f(n) = 20n + 22$ is injective. ✓

(d) $f : \mathbb{Z} \to \mathbb{Z}, f(n) = n(n + 1)$.
answer: Given that $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$,
Let $n, m \in \mathbb{Z} : f(n) = f(m) \equiv n(n + 1) = m(m + 1) \equiv m = n \implies f$ is injective. ✗

Real answer: counter proof by counter example, $f(-1) = 0 = f(0) = 0(0 + 1) = 0$.

(e) $f : \mathcal{P}(\mathbb{Q}) \to \mathcal{P}(\mathbb{Q}), f(a) = a \cup \{1, 2, 3\}$
answer: First lets understand the function. We are working within the power series of the rationals.
This means that we are working with sets and not numbers.
$f(A) = A \cup \{1, 2, 3\}$ takes the intersection of two sets within $\mathbb{Q}$.
Recalling that for f to be injective, $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$,
Let $n = \{1\}, m = \{2\} : f(n) = \{1, 2, 3\}, f(m) = \{1, 2, 3\} \not\Longrightarrow n = m$ as $\{1\} \neq \{2\}$.
This suggests that f is not injective. ✓

(f) $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}, f(m, n) = (m^2 + n^2, m^2 - n^2)$.
answer: Given $\forall\ a, b \in A$ if $f(a) = f(b) \implies a = b$,
$f(1, 1) = (2, 0), f(-1, -1) = (2, 0)$ but $(1, 1) \neq (-1, -1)$.
By such logic, f is not injective. ✗

Real answer: Notice the domain is $\mathbb{N} \times \mathbb{N}$ which means that $(-1, -1) \notin \mathbb{N} \times \mathbb{N}$.
This is the mistake made. If we notice any input is a positive integer, the proof follows,
$n, m, p, q \in \mathbb{N} \times \mathbb{N} : f(n, m) = f(p, q)$.
$\implies m^2 + n^2 = p^2 + q^2$ and $m^2 - n^2 = p^2 - q^2$.
$m^2 = p^2 - q^2 + n^2 \implies p^2 - q^2 + n^2 + n^2 = p^2 + q^2 \implies n^2 = q^2$ or $n = q$.
Given $n = q, m^2 = p^2 \implies m = q$. Index positions match, and we are only concerned with positive inputs.
Given $m = p, n = q \impliedby f(n, m) = f(p, q)$, f is injective.

3 Question Three