Analysis Work Sheet Final

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Contents

1 Sequences

1.1 Q1

Find the formula for the n-th term of the sequence $1, -4, 9, -16, 25$.

Answer: Given $1, -4, 9, -16, 25$, we can clearly see that it's a sequence of squares with an alternating negative sign,

$$a_n = (-1)^{n-1}(n^2), \text{ for } n \geq 1.$$

1.2 Q2

Find a formula for the nth term of the sequence in terms of n, where the sequence is $1, 0, 1, 0, 1, \ldots$.

Answer: The sequence is a periodic one which oscillates from 1 to 0. Let us create a table of values to assist in this,

1	2	3	4	5	6
1	0	1	0	1	0
odd	even	odd	even	odd	even

We can see that when n is even, we get a 0, and else is 1. This implies we will have a formula making use of $n/2$.

Let $a_n = n/2 : a_n = 1/2, 1, 3/2, 2, 5/2, \ldots$. It seems that we get closer to our desired result if we let $a_n = (n+1)/2 : a_n = 1, 3/2, 2, 5/2, 3, \ldots$.

I have just realized that the most suitable function would be $\sin(x)$,

$$a_n = \sin\left(\frac{n\pi}{2}\right)^2.$$

1.3 Q3

Determine if the sequence $\{a_n\}$ converges or diverges. Find the limit if the sequence converges. The sequence is $a_n = 4 + (0.3)^n$

Answer:

$$\lim_{n\to\infty} 4 + 0.3^n = 4$$

1.4 Q4

Determine if the sequence $\{a_n\}$ converges or diverges. Find the limit if the sequence converges. The sequence is $a_n = \left(\frac{n+6}{7n}\right)\left(1 - \frac{6}{n}\right)$.

Answer:

$$\begin{aligned}
\text{Given } a_n &= \left(\frac{n+6}{7n}\right)\left(1 - \frac{6}{n}\right), \\
\lim_{n\to\infty}\left(\frac{n+6}{7n}\right)\left(1 - \frac{6}{n}\right) &= \lim_{n\to\infty}\left(\frac{n+6}{7n}\right)\lim_{n\to\infty}\left(1 - \frac{6}{n}\right), \\
&= \lim_{n\to\infty}\left(\frac{n+6}{7n}\right)(1) = \lim_{n\to\infty}\left(\frac{\frac{d}{dn}(n+6)}{\frac{d}{dn}(7n)}\right), \\
&= \lim_{n\to\infty}\left(\frac{1}{7}\right) = \frac{1}{7}.
\end{aligned}$$

1.5 Q5

Determine if the sequence $\{a_n\}$ converges or diverges. Find the limit if the sequence converges. The sequence is $a_n = \sqrt[n]{4^n n}$.

Answer:

$$
\begin{aligned}
&\lim_{n\to\infty} \sqrt[n]{4^n n} = L, \\
&\ln(L) = \ln\left(\lim_{n\to\infty} (4^n n)^{1/n}\right) = \lim_{n\to\infty}\left(\ln (4^n n)^{\frac{1}{n}}\right), \\
&\ln(L) = \lim_{n\to\infty} \frac{1}{n} \ln (4^n) + \ln(n) = \lim_{n\to\infty} \frac{1}{n}(n\ln(4) + \ln(n)), \\
&\ln(L) = \frac{\lim_{n\to\infty}(n\ln(4)+\ln(n))}{n} = \lim_{n\to\infty} \ln(4) + \lim_{n\to\infty} \frac{\ln(n)}{n}, \\
&\ln(L) = \ln(4) + \lim_{n\to\infty} \frac{\frac{d}{dn}(\ln(n))}{\frac{d}{dn}(n)} = \ln(4) + \lim_{n\to\infty} \frac{1}{n}, \\
&\therefore \ln(L) = \ln(4) \implies L = 4.
\end{aligned}
$$

1.6 Q6

Use the definition of convergence to prove the given limit.

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Answer: Let $\epsilon \in \mathbb{R}_+ :\ \exists\ N : n > N \implies |a_n - L| < \epsilon$ for L. Given the max of $|\sin n| = 1$, $L = 0 \because |a_n| < \epsilon\ \forall\ n > \left\lceil \frac{1}{\epsilon} \right\rceil$.

1.7 Q7

Find a formula for the nth partial sum of the series and use it to find the series' sum if the series converges.

The sum takes form,

$$\frac{17}{1\times 2} + \frac{17}{2\times 3} + \frac{17}{3\times 4} + \cdots + \frac{17}{n(n+1)} + \ldots$$

Answer:

$$
\begin{aligned}
&\frac{17}{1\times 2} + \frac{17}{2\times 3} + \frac{17}{3\times 4} + \cdots + \frac{17}{n(n+1)}, \\
=&17\left(\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)}\right), \\
=&\left(\frac{17}{(n+1)!}\right) = S_n.
\end{aligned}
$$

✗

Real answer: I made a very silly mistake and assumed the denominators were multiplied, though they obviously are not.

$$=17\left(\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)}\right),$$

If we produce a table of values we may spot a pattern,

n	1	2	3	4	5	6
S_n	17/2	34/3	51/4	68/5	85/6	102/7
Simplified	17(1/2)	17(2/3)	17(3/4)	17(4/5)	17(5/6)	17(6/7)

We can now very easily see that,

$$S_n = \frac{17n}{n+1}.$$

Now to find the sum, we take the limit,

$$\lim_{n\to\infty} \frac{17n}{n+1} = \lim_{n\to\infty} \frac{17}{1} = 17.$$

1.8 Q8

Determine if the geometric series converges or diverges. If a series converges, find its sum.

$$\frac{1}{3} + \frac{1}{3}^2 + \frac{1}{3}^3 + \frac{1}{3}^4 + \dots$$

Answer:

$$\frac{1}{3} + \frac{1}{3}^2 + \frac{1}{3}^3 + \frac{1}{3}^4 + \dots,$$

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{243} + \dots$$

Lets try to find the n-th term, $a_n = \frac{1}{3^n}$.

Now we take the limit,

$$\lim_{n\to\infty} \frac{1}{3^n} = 0.$$

The series therefore does converge to zero. ✗

Real answer: We need either a formula for the n-th partial sum or we can use the fact that,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \text{ for } |r| < 1.$$

In light of this, we can recall that a geometric series follows the pattern $a + ar + ar^2 + \dots$. We, in this case, let $a = 1/3$, and $r = 1/3$.

$|r| < 1$ holds, so,

$$S_\infty = \frac{1/3}{1 - 1/3} = 0.5.$$

1.9 Q9

Use the nth-term test for divergence to show that the series is divergent, or state that the test is inconclusive.

$$\sum_{n=1}^{\infty} \cos\left(\frac{18}{n}\right).$$

Answer: Recall that the n-th term test asks,

$$\text{if } \lim_{n\to\infty} a_n \neq 0 \text{ or } \lim_{n\to\infty} a_n \text{ is undefined, } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

In light of this,

$$\lim_{n\to\infty} \cos\left(\frac{18}{n}\right) = 1 \implies \text{ Sum diverges.}$$

1.10 Q10

Determine whether the series converges or diverges. If it converges, find its sum.

$$\sum_{n=0}^{\infty} e^{\frac{-5n}{2}}.$$

Answer: We can see that the series converges because the limit tends to 0.

$$\sum_{n=0}^{\infty} e^{\frac{-5n}{2}} = \sum_{n=0}^{\infty} \frac{1}{e^{\frac{5n}{2}}}.$$

This series is indeed a geometric one, as shown, with $a = 1$, $r = 1/e^{\frac{5}{2}}$. This therefore means that,

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{e^{\frac{5}{2}}}} = \frac{e^{\frac{5}{2}}}{e^{\frac{5}{2}}-1}.$$

1.11 Q11