Probability and Statistics January Exam 2024

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1 Question One

Assume the number of buses passing through a certain stop in 1 hour follows a Poisson distribution with average $\lambda = 5.5$. Calculate the probability that at least one bus turns up in 1 hour.

Answer: Recall that the Poisson distribution is (?). ✗

Real Answer: the Poisson distribution given $\bar{x} = \mu = \mathbb{E}(X) = \lambda$, and the $\mathbb{P}(X = k)$ is given by

$$\frac{\lambda^k e^{-\lambda}}{k!}.$$

One should consider the fact that the events are assumed to be independent. In light of this, given $\lambda = 5.5$ and $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0)$, we get,

$$1 - \frac{5.5^0 e^{-5.5}}{0!} = 0.995913... \approx 0.995,\ 99.5\%$$

2 Question Two

You roll a fair dice once

2.1 Q2 a

Calculate the probability of getting the number six.

Answer:

$$\mathbb{P}(X = 6) = \frac{|\{6\}|}{|\Omega|} = \frac{|\{6\}|}{|\{1,2,3,4,5,6\}|} = \frac{1}{6}. \checkmark$$

2.2 Q2 b

You roll a fair dice n times. Calculate the probability of getting at least one six.

Answer: To begin our investigation, let $n = 1 : \mathbb{P}(X = 6) = \frac{1}{6},\ n = 2 : \mathbb{P}(X = 2) = \frac{1}{36},\ n = 3 : \mathbb{P}(X = 6) = \frac{1}{216}$, etc. We can clearly see a climbing in the powers of six where the index linearly increases with n.

We may therefore deduce that for n-times, $\mathbb{P}(X = 6) = 6^{-n}$ ✗.

Real answer: I simply misunderstood the question. They ask at least one six, implying that there maybe more than 1 six. Therefore, we don't want to calculate $\mathbb{P}(X = 6)$, as It's actually not relevant to the question. We want to calculate $\mathbb{P}(\text{At least 1 six})$. We could cleverly set $Y = (\text{number of sixes})$ and then state $\mathbb{P}(Y \geq 1) = 1 - \mathbb{P}(Y = 0)$.

$$1 - \mathbb{P}(Y = 0) = 1 - \left(\frac{5}{6}\right)^n.$$

2.3 Q2 c

Find the values of n such that this probability is greater than or equal to 50%.

Answer: given our formula $1 - \left(\frac{5}{6}\right)^n$, we want,

$$
\begin{aligned}
1-\left(\frac{5}{6}\right)^n &\geq 0.5,\\
\left(\frac{5}{6}\right)^n &\geq 0.5,\\
\ln\left(\left(\frac{5}{6}\right)^n\right) &\geq \ln(0.5),\\
n &\geq \frac{\ln(0.5)}{\ln\left(\left(\frac{5}{6}\right)\right)},\\
n &\geq 3.8016 \approx 3.8 \approx 4. \checkmark
\end{aligned}
$$

3 Question Three

Given the random variables X and Y with pmf of,

$$
\mathbb{P}(X=k)=\begin{cases}\frac{1}{2}, & k=0,\\ \frac{1}{2}, & k=3,\\ 0, & k\notin\{0,3\}.\end{cases}
$$

$$
\mathbb{P}(Y=k)=\begin{cases}\frac{1}{3}, & k=-1,\\ \frac{2}{3}, & k=1,\\ 0, & k\notin\{-1,1\}.\end{cases}
$$

3.1 Q3 a

Provide an example of a joint probability distribution for X and Y such that X and Y are independent. You can use a table as below:

	$X=0$	$X=3$	marginal
$Y=-1$	?	?	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	?	?	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

Answer:

	$X=0$	$X=3$	marginal
$Y=-1$	$\mathbb{P}(Y=-1)\cap\mathbb{P}(X=0)$	$\mathbb{P}(Y=-1)\cap\mathbb{P}(X=3)$	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	$\mathbb{P}(Y=1)\cap\mathbb{P}(X=0)$	$\mathbb{P}(Y=1)\cap\mathbb{P}(X=3)$	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

$\downarrow$

	$X=0$	$X=3$	marginal
$Y=-1$	$1/3\times1/2$	$1/3\times1/2$	$\mathbb{P}(Y=-1)=1/3$
$Y=1$	$2/3\times1/2$	$2/3\times1/2$	$\mathbb{P}(Y=1)=2/3$
marginal	$\mathbb{P}(X=0)=1/2$	$\mathbb{P}(\mathrm{X}=3)=1/2$	

$\downarrow$

	$X = 0$	$X = 3$	marginal
$Y = -1$	$1/6$	$1/6$	$\mathbb{P}(Y = -1) = 1/3$
$Y = 1$	$1/3$	$1/3$	$\mathbb{P}(Y = 1) = 2/3$
marginal	$\mathbb{P}(X = 0) = 1/2$	$\mathbb{P}(\mathrm{X} = 3) = 1/2$	

✓

3.2 Q3 b

Provide an example of a joint probability distribution for X and Y such that X and Y are not independent.

	$X = 0$	$X = 3$	marginal
$Y = -1$	$1/6 + 0.01$	$1/6 + 0.01$	$\mathbb{P}(Y = -1) = 1/3$
$Y = 1$	$1/3 + 0.01$	$1/3 + 0.01$	$\mathbb{P}(Y = 1) = 2/3$
marginal	$\mathbb{P}(X = 0) = 1/2$	$\mathbb{P}(\mathrm{X} = 3) = 1/2$	

✓

4 Question Four

The average size of a foreign exchange transaction in a certain trading desk is \$$25.1m$ with a standard deviation of \$$15m$. We assume these numbers follow a normal distribution. We are concerned about the impact of a new regulatory reporting procedure that kicks in for transactions larger than \$$50m$. Calculate the probability that a given transaction will be larger than \$$50m$.

Answer: Given that the distribution of the transactions follows a normal distribution, $\mathcal{N}(\mu, \sigma^2)$, we are concerned with the pdf of,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

We want to find, quote, "the probability that a given transaction will be larger than \$$50m$", which translates mathematically to $\mathbb{P}(X > 50m) = 1 - \mathbb{P}(X \leq 50m)$.

We can use therefore, the cdf of $\mathcal{N}(\mu, \sigma^2)$ to find $\mathbb{P}(X \leq 50m)$, which is,

$$\Phi\left(\frac{x-\mu}{\sigma}\right).$$

We therefore want to calculate $\Phi((50m - 25.1m)/(15m)) = \Phi(1.66)$. One must look this value up in the given table of values, where one obtains 0.9515.

Given that $\mathbb{P}(X \leq 50) = 0.9515$, $\mathbb{P}(X > 50m) = 1 - 0.9515 = 0.0485 \approx 0.05$. The chance of a transaction being larger than \$$50m$ is 5%. ✓✓✓

5 Question Five

This question demonstrates a special feature of the exponential distribution that makes it suitable for applications to waiting times. It is called the memoryless property. This is a characteristic of the Exponential random variable that might or might not correspond to what happens in real-life applications.

Assume that $X \sim \mathrm{Exp}(\lambda)$ models the wait time in hours to the next bus if on average there are λ busses per hour.

5.1 Q5 a

$x_1 \in \mathbb{R}_+$, what is the probability that no buses arrived before time x_1, i.e., calculate $\mathbb{P}(X > x_1) = 1 - \mathbb{P}(X \leq x_1)$.

Answer: The cdf, as needed, of the exponential distribution is $1 - e^{-\lambda x}$.

We are working with the general x_1, and with then general λ, so $1-(1-e^{\lambda x_1}) = 1 - 1 + e^{\lambda x_1} = e^{\lambda x_1}$. ✗

Real answer: Forgot the negative sign for lambda, $e^{-\lambda x_1}$.

5.2 Q5 b

For $0 < x_1 < x_2$, calculate the probability that the bus will arrive between the two, i.e., $\mathbb{P}(x_1 < X < x_2)$.

Answer: $\mathbb{P}(x_1 < X < x_2) = \mathbb{P}(X > x_1) + \mathbb{P}(X < x_2) = (1 - \mathbb{P}(X \leq x_1)) + \mathbb{P}(X < x_2)$.

Once again referring to the cdf, that being $1 - e^{-\lambda x}$, we get the following,

$$\big(1-(1-e^{-\lambda x_1})\big) + (1 - e^{-\lambda x_2}) = 1 + e^{-\lambda x_1} - e^{-\lambda x_2}. \text{ ✗}$$

Real answer: In fact we may use some basic logic to state that $\mathbb{P}(x_1 < X < x_2) = \mathrm{cdf}(x_2) - \mathrm{cdf}(x_1)$. Following this, we get $(1 - e^{-\lambda x_2}) - (1 - e^{-\lambda x_1}) = e^{-\lambda x_1} - e^{-\lambda x_2}$.

5.3 Q5 c

Calculate $\mathbb{P}(X < x_2 | X > x_1)$.

Answer: $\mathbb{P}(X < x_2 | X > x_1) = (\mathbb{P}(X < x_2) \cap \mathbb{P}(X > x_1))/(\mathbb{P}(X < x_2) \cup \mathbb{P}(X > x_1))$. This yields,

6 Question Six

7 Question Seven