

Today Show the Hilfssatz for Nilpotent gpo.

Due to nilpotence, the commutator is proper:

$$\mathfrak{g} \ni [\mathfrak{g}, \mathfrak{g}]$$

giving a non-trivial lie algebra homomorphism $\phi: \mathfrak{g} \rightarrow \mathbb{R}$. Let $\mathfrak{g}_1 := \ker(\phi)$ then we get a subgroup $G_1 = \exp(\mathfrak{g}_1)$. choosing $x \in \mathfrak{g} \setminus \mathfrak{g}_1$ we have

$$G = G_1 \times \exp(\mathbb{R} \cdot x).$$

Proof:

$$1 \rightarrow N = \exp(\mathfrak{g}_1) \xrightarrow{\iota} G \xrightarrow{\pi} Q = \exp(RX) \rightarrow 1$$

$\curvearrowleft_{\tau = \iota}$

a split SES
 $\Rightarrow \underline{G \cong N \times Q}$

(ι is injective map, π is surjective)

$$\tau \circ \pi = \text{id}_Q$$

($\iota = \iota$ the inclusion map).

Basis for \mathfrak{g}_1 , $\beta = \{X, \underbrace{a_1, \dots, a_l}_{\text{basis for } \ker(\beta)}\}$

Define π as

$$\pi \left(\exp \left(\lambda X + \sum \alpha_i a_i \right) \right) = \exp (\lambda X)$$

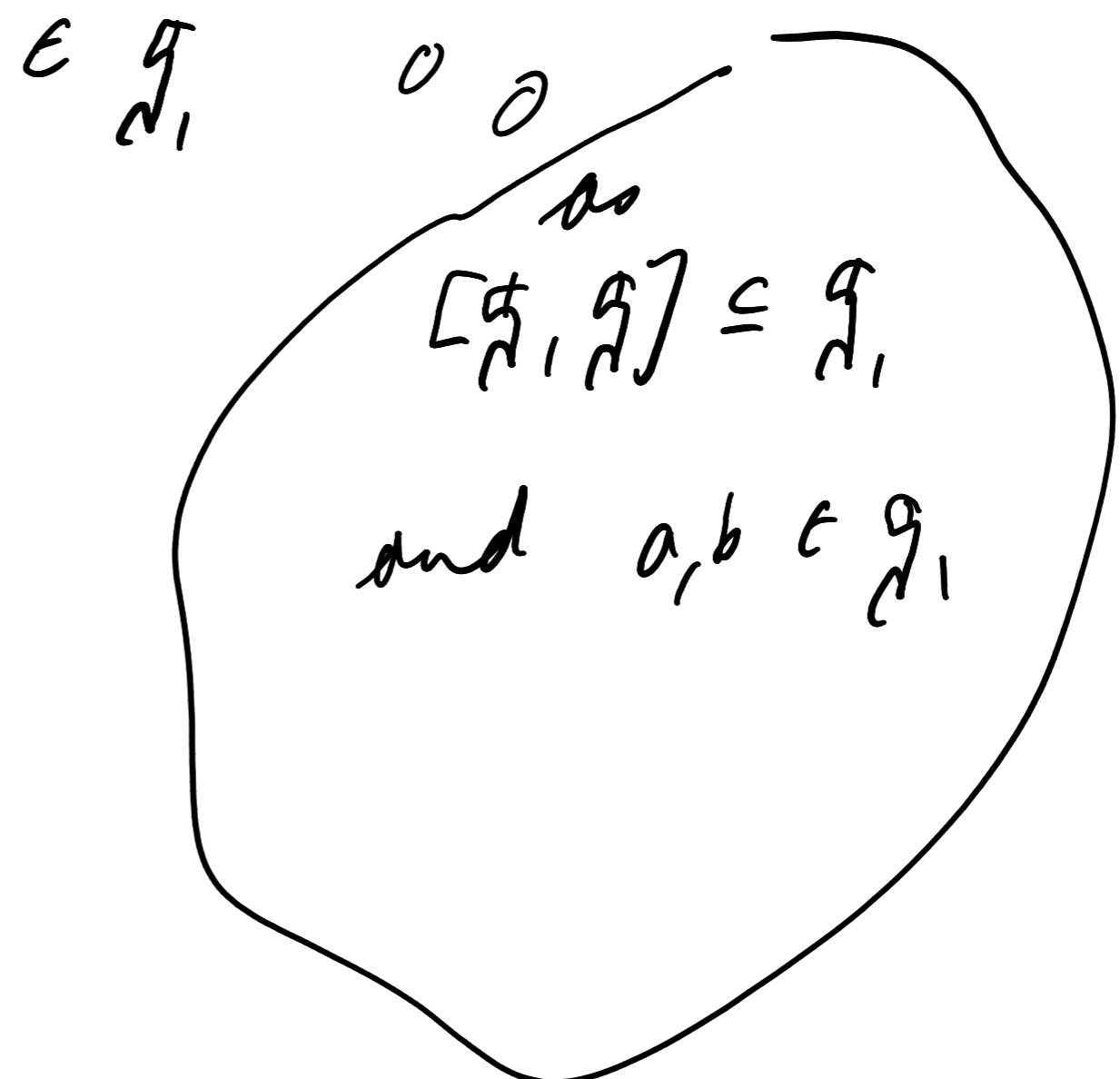
It is a grp hom. as

$$\exp(\lambda X + a) \exp(\sigma X + b)$$

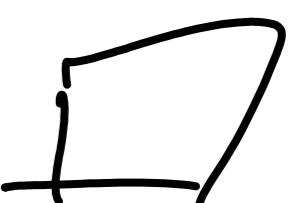
$$= \exp \left((\lambda + \sigma)X + \underbrace{(a+b) + \text{comm}}_{} \right)$$

$$= \exp((\lambda + \sigma)X)$$

$$= \exp(\lambda X) \exp(\sigma X)$$



Note: $\ker(\pi) = \exp(\mathfrak{g}_1) = N$



Idea: Proceed by induction. Take a rep π of G , restrict it to G_1 , and then decompose it into irreducibles by using the theory of disintegration.

- Restriction remains irreducible.
- Splits up into a single orbit of an irred. σ of G_1 under the action of IR .

Obtain map $\text{pr}: \mathfrak{g}^* \rightarrow \mathfrak{g}_1^*$

$\text{IR} \curvearrowright G_1 \rightsquigarrow \text{IR} \curvearrowright$ space of irreps of $G_1 \ni \sigma$
(semidirect product)

$\sigma \xrightarrow[\text{Kirillov}]{} \text{orbit } \overline{\sigma} \subset \mathfrak{g}_1^*$

$\cup (\exp(R \cdot x) \cap \text{pr}^{-1}(\bar{0}) \subset \mathfrak{g}^*)$
wadj.

= single orbit of $G \backslash \mathfrak{g}^*$

\rightsquigarrow π
Kirillov orbit
there

en

Heisenberg gr.

To do § 7.2 for

$$[\mathfrak{g}, \mathfrak{g}] = \{z, y-z\}$$

§ 7.3 constructing the rep corresponding to an orbit ($0 \mapsto \pi_0$)

Fix $\lambda \in \mathbb{C}$, note action $G \curvearrowright \mathcal{A}$:

$$g\lambda(gx) = \lambda(x) \quad (\#)$$

since

$$\langle x, \text{ad}_g^* \lambda \rangle := \langle \text{ad}_{g^{-1}} x, \lambda \rangle \iff g\lambda(x) = \lambda(g^{-1}x)$$

Let $g = e^{tY}$ then $(\#)$:

$$\text{ad}_{e^{tY}}^* \lambda (\text{ad}_{e^{tY}} X) = \lambda(X)$$

taking the derivative at $t=0$:

$$(\text{d}\text{ad}^*)_1(Y) \lambda (\text{ad}_1 X) + \text{ad}_1^* \lambda ((\text{d}\text{ad})_1(Y) X) = 0$$

$$\Rightarrow \text{ad}_Y^* \lambda(X) + \lambda(\text{ad}_Y X) = 0$$

$$\Leftrightarrow Y\lambda(X) + \lambda([Y, X]) = 0.$$

$$(\text{d}\text{ad}_1(Y)) = : \text{ad}_Y :$$

$$(\text{d}\text{ad}^*)_1(Y) = : \text{ad}_Y^* :$$

We see from this that

$$Y\lambda(x) = -\lambda([Y, x])$$

so if $y, x \in \mathfrak{g}_\lambda^0$,

$$\mathfrak{g}_\lambda := \text{Lie}(\{g \in G \mid \text{ad}_g^* \lambda = \lambda\})$$

so $\text{ad}_x^* \lambda = 0$, $g = e^{tx}$.

[cf. Zalk 3]

Lie (stab_G(λ))

$$\lambda([x, y]) = 0$$

Meaning that $\lambda([\mathfrak{g}_\lambda, \mathfrak{g}_\lambda]) = 0$ so we have a
Lie alg. homomorphism

$$\lambda : \mathfrak{g}_\lambda \rightarrow \mathbb{R}$$

exp. (both simply connected)

[cf. Wikipedia - Lie alg. correch.]

$$e^{i\lambda} : G_\lambda \rightarrow S^1$$

This is a character of $\text{stab}_G(\lambda) = \mathfrak{g}_\lambda$.

Note: $x, y \mapsto \lambda([x, y])$

$\rho \downarrow$

$$\omega: \mathfrak{g}/\mathfrak{g}_\lambda \times \mathfrak{g}/\mathfrak{g}_\lambda \rightarrow \mathbb{R}$$

which is alternating and non-degenerate.

So under the identification $\mathfrak{g}/\mathfrak{g}_\lambda \cong T_\lambda O$ we get the canonical symplectic structure ω .

Induce the rep

Need to first extend the character as much as possible.

Want to find sub-algebra $\mathfrak{g} \supset \mathfrak{g}_\lambda$ s.t. $\lambda: \mathfrak{g} \rightarrow \mathbb{R}$ is a Lie alg. hom., i.e. $\lambda([g, g]) = 0$. So it has isotropic image in $\mathfrak{g}/\mathfrak{g}_\lambda$, i.e.

$\omega|_{P(g)} = 0$ (where $(\mathfrak{g}/\mathfrak{g}_\lambda, \omega)$ is a symplectic structure).

$$P: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}_\lambda$$

$t_1, t_2 \in T_\lambda O; x, y \in \mathfrak{g}$

s.t. $t_1 = x \cdot \lambda, t_2 = y \cdot \lambda$

then

$$\omega(t_1, t_2) := \lambda([x, y]).$$

$$\begin{array}{c|c} \mathbb{R} & \\ \hline 0 & -I \\ \hline I & 0 \end{array}$$

Def A polarization is a Lie subalg. $\mathfrak{g}_\lambda \subset \mathfrak{g} \subset \mathfrak{g}$ s.t. $\mathfrak{g}/\mathfrak{g}_\lambda$ is maximal isotropic (Lagrangian) for the form

$$X, Y \mapsto \pi[X, Y].$$

Fact (Kirillov): Polarizations exist.

Let \mathfrak{g} be a polarization, $Q := \exp(\mathfrak{g})$ and let

$$\chi: Q \rightarrow S^1$$

$$\chi(e^y) := e^{i\chi(y)}, \quad y \in \mathfrak{g}.$$

$$Q \xrightarrow{\chi} S^1$$

Then $\text{Ind}_{Q^\lambda}^G \chi$ is irreducible and gives π_0 . To verify this we compute the character.

Remaining Q's:

- What happens if you extend to the induced rep at G_7 for H_3 instead of at Q ?