LEARNING MATHEMATICS

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Abstract. To record many interesting things I learned in case I forgot. To keep myself busy ...

Contents

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1. June 2021

Something I would like to get done this month:

(1) Learn how much 2-dimensional gauge theory of finite group G tells us about representation of G. This is firstly motivated by Frobenius formula that Nam showed few months ago, interpreting number $\mathrm{Hom}(\pi_1(\sum), G)$ as sum over irreducible representations of G where $\sum$ is (closed, compact?) Riemann surface. I then found a "topological" proof using topological field theory `https://math.berkeley.edu/~qchu/TQFT.pdf` and `https://upennig.weebly.com/uploads/7/4/0/3/74037187/2d-tqft.pdf` and `https://www.math.ru.nl/~mueger/TQFT/FQ.pdf`. In topological field theory language, there is a functor Z from category of 2-cobordisms to category of vector spaces. And in our situation, it sends a manifold M to $\mathrm{Map}(\mathrm{Bun}_G(M), \mathbb{C})$ where $\mathrm{Bun}_G(M)$ is the groupoid of principal G-bundles over M, which can then be identified with $\mathrm{Hom}(\pi_1(M), G)/G$ quotient by conjugation. Roughly in the proof of Frobenius using TQFT, somehow one can cut and glue M to get the desired formula. The further question is how much $Z(M)$ tells us about representation of G, when one varies the manifold M.

More to read from: "Bartlett Categorical aspects of topological quantum field theories" (arvix); `https://arxiv.org/abs/1705.05734v1` and Kock "Frobenius algebra and 2D topological quantum field theory" `https://www.mat.uab.cat/~kock/TQFT/FS.pdf` this has book version; `https://golem.ph.utexas.edu/category/2008/06/teleman_on_topological_constru.html`.

One can also ask why we choose $\mathrm{Bun}_G(M)$ as target for our cobordism functor and expect it to tell something about representations of G. I think this is because one can interpret representations of G as bundles of some sort (see wikipedia of "induced representations").

(2) For my thesis, I am trying to understand certain self-adjoint operator in Langlands' computation of volume of fundamental domain $G(\mathbb{Z}) \setminus G(\mathbb{R})$. I don't understand this operator and all and how it is linked to Eisenstein series.

(3) Gauge theory in representation theory, geometric representation theory: `https://people.maths.ox.ac.uk/tillmann/ASPECTSbenzvi.pdf` and `https://web.ma.utexas.edu/users/benzvi/GRASP/lectures/NWTFT/nwtft.pdf`, also `https://ncatlab.org/nlab/files/BenZviGeometricI` `pdf`.

Daily learning

02/06/2021 Today I learnt roughly what is a "rigid symmetric monoidal category" and how category of n-dimensional cobordisms nCob is one, following `https://arxiv.org/abs/q-alg/9503002`. To explain roughly, "monoidal" means the category has a product $\otimes$ operation, "symmetric" means we have a map $a \otimes b \to b \otimes a$, "rigid" means every object x has a dual object x^*. The main point to take away is that one can visualise the relations/commutative diagrams in the category via cobordisms, which make it easier to remember. For example, relations between counit and unit maps is seen as straightening the curve S (see p. 5). For more examples of this, see p. 21 of `https://arxiv.org/pdf/math/0512103v1.pdf`.

Another note, on p. 4, it mentions that relations between morphisms in nCob can be understood using Morse theory, where we can stratify a bordism N (i.e. a n-manifold N) by giving a Morse function on N to pick up critical points (something relates to handle decomposition in Morse theory). *I see some familiar words like "stratification" and "Morse function" when reading about symplectic geometry, would like to learn more about this at some point*

06/06/2021 I am trying to understand first few sections of `https://ncatlab.org/nlab/files/BenZviGeometricFunct` `pdf` (with the hope of getting to know more about `https://ncatlab.org/nlab/show/geometric+infinity-function+theory`). Here is what I have so far:

Given two sets X, Y with a G-action on these two and a G-equivariant map $\phi : X \to Y$. We can pullback to give a map ϕ^* of G-equivariant complex-valued functions $Fun_G(Y)$ on Y to that $Fun_G(X)$ on X. Pushforward $\phi_* : Fun_G(X) \to Fun_G(Y)$ is a bit more tricky. Firstly, it is better to view X, Y as groupoids, then

$$\phi^* : f \in Fun_G(X) \mapsto \left(y \mapsto \sum_{x \in |\phi^{-1}(y)|} \frac{f(x)}{\#\mathrm{Aut}_{\phi^{-1}(y)}(x)} \right).$$

Here $|X|$ refers to isomorphic classes of objects in groupoid X. Note $\phi^{-1}(y)$ is also a groupoid with natural automorphisms. *How to come up with this pushforward? What condition should a good pushforward satisfy? Usually pushforward is very nontrivial to realise, unlike pullback. Is there a general rule to come up with something like this?*

The two are adjoint in following sense: One can define inner product on $Fun_G(X)$ by $(f, g) = \sum_{x \in X} \frac{f(x)\overline{g(x)}}{\#\mathrm{Aut}(x)}$ then ϕ^* and ϕ_* are adjoint with respect to this inner product. *What is the relation of this with adjointness as functors? Do we have something like Frobenius reciprocity in representation theory, where adjointness in inner product is the same as adjointness as adjoint functors due to semisimplicity of representations? What constitutes a good inner product?*

Perhaps this would help: ***https://golem.ph.utexas.edu/category/2007/03/canonical_measures_on_configur_1.html*** *or* ***https://golem.ph.utexas.edu/category/2011/09/universal_measures.html*** *or* ***https://golem.ph.utexas.edu/category/2008/07/news_on_measures_on_groupoids.html***.

06/06/2021 Regarding item 1, I managed to figure out how to describe $Z_G(\ ) : Z_G(pt) \to Z_G(S^1)$ where $N = $ is a half-phere with boundaries pt (a point) and S^1. To do this, start with the more geometric correspondence

$$\mathrm{Bun}_G(pt) \longleftarrow \mathrm{Bun}_G(N) \longrightarrow \mathrm{Bun}_G(S^1)$$

obtained by restricting G-bundles to corresponding boundaries (i.e. pull back). This gives us morphism of groupoids

$$pt \overset{p}{\leftarrow} pt \overset{q}{\to} G/G.$$

Indeed, N is just a disk so $\mathrm{Bun}_G(N) = \mathrm{Bun}_G(pt) = pt$, a point with a G action on it. We know $\pi_1(S^1) = \mathbb{Z}$ so $\mathrm{Bun}_G(S^1) = \mathrm{Hom}(\mathbb{Z}, G)/G = G/G$, groupoid with elements in G and automorphisms are conjugations by G. Then p is the identity map, q sends to the identity 1 in G (as we have group hom $\pi_1(S^1) \to 1 = \pi_1(N)$ inducing $\mathrm{Bun}_G(N) \to \mathrm{Bun}_G(S^1)$).

By definition, $Z_G(N) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Bun}_G(N), \mathbb{C})$ so this gives $Z_G(N) := q_* \circ p^* : Z_G(pt) \to Z_G(S^1)$. Note $Z_G(pt) = \mathbb{C}, Z_G(S^1) = \mathrm{Hom}(G/G, \mathbb{C}) = \mathbb{C}[G]^G$ so from pushforward described in 06/06/2021 for groupoids, we find $Z_G(N)$ sends $\lambda \in \mathbb{C}$ to $g \mapsto \lambda \delta_{g,1}/|G|$ in $\mathbb{C}[G]^G$.

A remark: Bun_G is a geometric object, but doing computation it seems to be easier to deal with $\mathrm{Hom}(\pi_1(\cdot), G)/G$.

16/06/2021 (Continued from 06/06/2021) It seems the idea of TQFT has been applied to study character varieties $\mathrm{Hom}(\pi_1(\sum), G)$ by Angel Gonzalez Prieto in `https://arxiv.org/abs/1812.11575` (his PhD thesis) or `https://arxiv.org/abs/1810.09714` (a relevant paper) , `http://www.mat.ucm.es/~joseag12/investigacion/documentos/SeminarioTesis.pdf` (PhD presentation), `http://www.mat.ucm.es/~joseag12/investigacion/documentos/TFMJAngelGonzalez.pdf` (his master thesis),... It seems there are many things unexplored here.

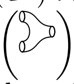
17/06/2021 (Continued from 06/06) Understand most of the computations in `https://math.berkeley.edu/~qchu/TQFT.pdf` and `https://upennig.weebly.com/uploads/7/4/0/3/74037187/2d-tqft.`

pdf. These two notes present a "topological" proof of

$$\frac{\# \operatorname{Hom}(\pi_1(\Gamma_g), G)}{|G|} = \frac{1}{|G|^{\chi(\Gamma_g)}} \sum_V (\dim V)^{\chi(\Gamma_g)}. \tag{1}$$

Here G is a finite group, Γ_g is a closed connected orientable surface of genus g, $\chi(\Gamma_g)$ is Euler characteristic of Γ_g, the sum is over all complex irreducible represention of G.

Here are the main computations:

- $Z_G($$) : Z_G(pt) \to Z_G(S^1)$. We did this on 06/06/2021. This map sends $\lambda \in \mathbb{C} = Z_G(pt)$ to $\lambda \delta_{g,1}/|G|$.
- $Z_G($$) : Z_G(S^1) \to Z_G(pt)$ sends $f \in Z_G(S^1) = \mathbb{C}[G]^G$ to $f(1)$.
- $Z_G\Big($$\Big) : Z(S^1) \otimes Z(S^1) \to Z(S^1)$ sends $f \otimes g \in \mathbb{C}[G]^G \otimes \mathbb{C}[G]^G$ to the convolution product $f * g$ ...
- $Z_G\Big($$\Big) : Z(S^1) \to Z(S^1) \otimes Z(S^1)$
- $Z_G(M_1 \sqcup_{M'} M_2) = Z(M_2) \circ Z(M_1)$ for surfaces M_1, M_2 with common boundary M'.
- As $Z_G(M) = \operatorname{Map}(\operatorname{Bun}_G(M), \mathbb{C})$ so if M is homotopic to M' then $Z_G(M) = Z_G(M')$.

From this, to compute $Z_G(\Gamma_g)$, it suffices to chop Γ_g into pieces and compute each piece, then compose everything together. On the other hand, $Z_G(\Gamma_g) = \operatorname{Hom}(\pi_1(\Gamma_g), G)/G$ so we can obtain (7).

We also have general formula when G is a Lie group (due to Witten 1991) or a quantum group (Rouchet-Szenes 2000) (I learnt this info from reading the slide https://www2.ist.ac.at/fileadmin/user_upload/group_pages/hausel/Aarhus07.pdf).

22/06/2021 (Things I would like to understand in a far way future) In this paper https://arxiv.org/pdf/1511.06271.pdf of Michael Groechenig claimed to give a generalisation to Weil's description of correspondence between groupoid of vector bundles on algebraic curve X (defined over algebraically closed field) and groupoid of the double quotient.

My impression of adeles is that it seems to be the right notion to study analysis on moduli space of bundles?

22/06/2021 (Things I would like to understand in a far way future) There seems to be some mysterious applications of p-adic integrations (and furthermore, motivic integrations) to various sorts of problems: equal Betti numbers of birational Calabi-Yau n folds https://www.math.uni-bonn.de/people/huybrech/Magni.pdf, Fundamental Lemma in Langlands program https://arxiv.org/abs/1810.06739, Topological Mirror Symmetry Conjecture by HauselThaddeus for smooth moduli spaces of Higgs bundles https://arxiv.org/abs/1707.06417v3.

22/06/2021 I attend lectures of Geordie Williamson about Spectra in representation theory and of Anna Romanov about Whittaker categories. I just want to write down what I understand (no matter how vague and imprecise it can be).

(a) In Geordie's talk: There are three ways to define cohomology $H^i(X, \mathbb{Z})$, one is via map $\triangle^n \to X$ ("maps in to X"), second is via Eilenberg-Maclane space $K(\mathbb{Z}, i)$ as $H^i(X, \mathbb{Z}) = [X, K(\mathbb{Z}, i)]$ (maps out of X), third is via constant sheaf $\underline{\mathbb{Z}}_X$, i.e. $H^i(X, \mathbb{Z}) = H^i(R\Gamma(X, \underline{\mathbb{Z}}_X))$ ("on X").
Geordie said that Grothendieck-Quillen's dream is that every generalised cohomology theory can be described as in the third way (i.e. "on X", without replying on $\triangle^n X$ or $K(\mathbb{Z}, i)$). He said Lurie has achieved this dream for K-theory. Then he goes to define spectra, which should be seen as an analogue of $\mathbb{Z}$. Then he mentioned that there is on going research trying to generalised Geometric Satake over spectra KU (instead of over

$\mathbb{Z}$) which gives the quantum group version on the RHS (instead of just representation of Langlands dual group).
What is spectra, KU rigorously? Do we have a visual(?) easier(?) explaination on why KU is an analogue of $\mathbb{Z}$? Why do we expect quantum groups on the RHS of geometric Satake, or is there some sort of deformation from $\mathbb{Z}$ to KU that explains this?
Some notes about spectra (other than Lurie ...): by Rok Gregoric `https://web.ma.utexas.edu/users/gregoric/Spectra%20Are%20Your%20Friends.pdf`, his summary of higher algebra `https://web.ma.utexas.edu/users/gregoric/Appendix.pdf`.
I found a table giving the analogy between Lurie's theory and classical theory in `https://sites.duke.edu/scshgap/files/2018/05/Pandit-Imperial.pdf`.
Actually, there is an even elementary question I don't know how to answer, as I don't know anything about K-theory: Why K-theory is considered to be an upgraded version of cohomology theory? (or something along this question...)

(b) In Anna's talk: There is a lot of technical details going on so I didn't get much out of it. But here it is: A Whittaker module is the induced representation from the upper Borel $\mathfrak{n}$ to $\mathfrak{g}$ from some character of $\mathfrak{n}$. The motivation for Whittaker model is that one has an "explicit" (?) description of representation as functions on some space. The motivation for Whittaker module (I think explained by Masoud in the talk but I may need to hear it again) is that Whittaker module of $\mathrm{GL}_n(\mathbb{F}_q)$ gives almost all the irrep(?). *What is the motivation of Whittaker story here? In what sense it is a good generalisation of irreducible representations?*
Whittaker module contains all the finite-dimensional rep of $\mathfrak{g}$ then she studies category $\mathcal{N}$, roughly similar to category $\mathcal{O}$ but with simple objects being Whittaker modules.
Travis told me that the motivation of category $\mathcal{O}$ is that it is the nicest (?) category that contains finite-dimensional rep of $\mathfrak{g}$ and the Verma module. *Why Verma module? is it because all irrep of $\mathfrak{g}$ (even infinite one) can be described from Verma module ? What sort of results do people expect when studying this category? Something like characters of irreducible modules, Kazhdan Lusztig theory?.*

23/06/2021 I learnt about some geometric constructions of representation of a group G. Let's just say G is finite for simplicity. Let X be a space with an action of G.

To obtain a representation of G, one can linearise by considering a vector space $Fun(X)$ of complex-valued functions on X (one can also choose $Fun(X, V)$ of V-valued functions on X, where V is any vector space). Then G acts on $f \in Fun(X)$ by $(g \cdot f)(x) = f(g^{-1}x)$. We will explain a geometric analogue of this construction as follows. Again, we are given a space X with an action of G. The process of "linearising", i.e. associating X with $Fun(X)$, requires the language of vector bundles.

First, we recall notion of a vector bundle of rank n. It is a surjective map $\pi : V \to X$ satisfying local triviality condition: for every $x \in X$, there exists open neighborhood U of X containing x and a homeomorphism $\pi^{-1}(U) \to U \times \mathbb{C}^n$ compatible with projections to U, such that its restriction to x induces a vector space isomorphism $V_x := \pi^{-1}(x) \cong \{x\} \times \mathbb{C}^n$. Intuitively, a vector bundle associate each point $x \in X$ a vector space of dimension n. To see vector bundles as $Fun(X, \mathbb{C}^n)$, we consider its global section $\Gamma(X, V)$, which is a vector space because the fibers are vector spaces. Hence, one can think of vector bundle on X as vector-valued functions on X. Then line bundle (i.e. vector bundle of rank 1) is $\mathbb{C}$-valued functions on X.

Now, go back to the definition of G action on $Fun(X)$ and we want to interpret this in the language of vector bundles, i.e. how much information is needed to associate an action of G on sections $\Gamma(X, V)$ of a vector bundle $\pi : V \to X$? Note that if $s \in \Gamma(X, V)$ then $s(x) \in V_x$. Now we want to say something like "$(g \cdot s)(x) = s(g^{-1} \cdot x)$" but LHS is in V_x, while the RHS is in $V_{g,x}$. Hence, what we need is a linear isomorphism $g : V_x \to V_{g\cdot x}$ for every $g \in G$, so that we can write the action as $(g \cdot s)(x) = g \cdot s(g^{-1}x)$. This motivates the notion of G-equivariant vector bundles:

A G-equivariant vector bundle over X is a vector bundle $\pi : V \to X$ with an action of G on P such that π is G-equivariant and the induced map $g : V_x \to V_{gx}$ is a linear map (a priori this is just a bijection).

Thus, we have obtained a map

$$\{G\text{-equivariant vector bundles over } X\} \longrightarrow \{G\text{-representations}\}.$$

More questions:

(a) *How to motivate notion of equivariant sheaf as generalisation of equivariant vector bundle? See Chriss Ginzburg or Achar book, wikipedia.*
(b) *What choice of space X gives you irreducible representations or all representations? For example, in Bott-Weil theory, X is a flag variety (so why flag varieties? Masoud told me that on the Lie algebra version, irreducible representations appear as quotients of Verma module, which is the induced representation from the Borel $\mathfrak{b}$; hence we expect irreducible representations to appear in bundle over $X = G/B$, which is the geometric way to describe induced representation) then considering certain line bundles over it (why line bundles but not general vector bundles? Is it because the regular representation of G on $Fun(G, \mathbb{C})$ contains all irreducible representations?) give you irreducible representations. What about Ginzburg construction of irreducible representations?*
(c) *Is there a map going backwards? Starting from a G-representation, how to get a G-equivariant vector bundle over some space X. Some ideas: Given G-module V, then $V \times G \to V$ is a principal G-bundle. Try to obtain a vector bundle out of this (for example, using associated bundle construction, or replace G with vector space $Fun(G)$, as we know action of G on V induces action of $Fun(G)$ on V).*
(d) *What representations will appear if we consider higher cohomology groups? Global sections is H^0.*

From this perspective, the construction of induced representation is quite natural. Consider a representation $\rho : H \to \mathrm{GL}(V)$ of subgroup H of G. Then we can construct a G-equivariant vector bundle as follows. Note $\pi : G \to G/H$ is a principal H-bundle, so to get a vector bundle, we use associated bundle construction: define $G \times_\rho V = G \times V/\sim$ where $(g(g')^{-1}, v) \sim (g, \rho(g')v)$ then $G \times_\rho V \to G/H$ is a vector bundle with fiber V. Note that G acts on $G \times_\rho V$ by $g(g', v) = (gg', v)$, making it into a G-equivariant vector bundle over G/H. In our analogy, this corresponds to $Fun(G/H, V)$. Taking global sections should give us back to the (analytic?) construction of induced representation.

Question:

(a) *I am still a bit confused about associated bundle construction. In particular, would the following construction be the same as associated bundle construction: Given group hom $f : G \to H$ then we have morphism $Bf : BG \to BH$ of classifying spaces, which induces a map from principal G-bundles to principal H-bundles. Now if we let $H = \mathrm{GL}(V)$ then is this the same as associated bundle construction?*
(b) *Check the details of the induced representation construction above. More is said here `https://mathoverflow.net/q/5772/89665` and here `https://math.stackexchange.com/q/1704622/58951`.*

We also have the correspondence

$$\{\text{principal G-bundles over } X\}/G \longleftrightarrow \{\text{group hom } \pi_1(X) \to G\}/G.$$

I know one direction from left to right: a principal G-bundle over X corresponds to homotopy $\phi : X \to BG$, taking fundamental group induces $\pi_1(X) \to \pi_1(BG) = G$.

Another way to describe this without using BG: consider principal G-bundle $\pi : P \to X$. Pick $x \in X, p \in P$ so $\pi(p) = x$. Consider a closed loop $\gamma : [0,1] \to X$ in X at x, i.e. $[\gamma] \in \pi_1(X, x)$. Then there exists a unique lift of γ to a curve $\tilde{\gamma}$ on P starting at p (but not necessarily closed curve). Then as $\tilde{\gamma}(1) = q$ and $\tilde{\gamma}(0) = p$ have the same fiber over x, and π is a principal G-bundle, there exists $g \in G$ so $q = p \cdot g$. We define $\pi_1(X) \to G$ by sending $[\gamma]$ to g.

Questions:

(a) *What is the reverse direction of this correspondence?*

(b) *If I choose $G = \mathrm{GL}(V)$, $X = BG$ then the RHS is representation of G. Compare with previous correspondence, can I associate principal* $\mathrm{GL}(V)$*-bundle over BG with G-equivariant vector bundles this way?*

(c) *This seems to be irrelevant, but what is the connection between principal* GL_n*-bundle and vector bundle? How to get one from the other?*

See more connection between vector bundles and representation theory at `http://www.numdam.org/item/PMIHES_1961__9__23_0.pdf`..

23/06/2021 For a Lie group G, I learnt how to describe a connection of a principal G-bundle $\pi : P \to X$ as $\mathfrak{g}$-vaulued 1-form on P.

First, we can define a vertical tangent bundle $T^vP = \{(p, v_p) : d\pi(v_p) = 0\}$. This gives us short exact sequence $0 \to T^vP \xrightarrow{f} TP \xrightarrow{g} \pi^*TX \to 0$. A connection on $P \to X$ is a choice of splitting of this short exact sequence. We have three equivantly ways to describe this:

(a) A map $s : TP \to T^vP$ s.t. $s \circ f = \mathrm{id}_{T^vP}$.

(b) A map $t : TX \to TP$ s.t. $g \circ t = \mathrm{id}_{TX}$.

(c) A direct sum decomposition $TP = T^vP \oplus H$.

If we use (a), then note that T^vP can be identified with $P \times \mathfrak{g}$, i.e. $(T^vP)_p$ can be identified with $\mathfrak{g}$ via the linear isomorphism $v_p \mapsto \frac{d}{dt}\big|_{t=0} p \cdot \exp(tv_p)$.

What is the geometric intuition of a connection for principal G-bundle?

Read more from chapter "Moduli Spaces of Flat Connections" of the book Torus Actions on Symplectic Manifolds by Audin.

2. July 2021

2.1. **07/07/2021: Kemp-Ness theorem.** From 01/07 till 07/07, we have a workshop on Heron island `https://sites.google.com/view/hiwgrt/home` about Kempf-Ness theorem, which gives a link between Geometric Invariant Theory and Symplectic Geometry.

Here are some unresolved thoughts I have during this workshop:

2.1.1. $\mathrm{GL}_n(\mathbb{C})$ *acts by conjugation on* $\mathfrak{gl}_n(\mathbb{C})$. One of the main example is letting $G = \mathrm{GL}_n(\mathbb{C})$ acting on $\mathfrak{gl}_n(\mathbb{C})$ by conjugation then stability of the action can be described as follows. Let $0 \neq V \in \mathfrak{gl}_n(\mathbb{C})$ then V can be written as $V = D + N$ where D is a diagonalisable matrix and N is nilpotent.

1. V is unstable, i.e. $0 \notin \overline{G \cdot V}$ iff $D = 0 \neq N$,
2. V is polystable, i.e. $\overline{G \cdot V}$ is closed, iff $N = 0 \neq D$.
3. V is semistable ($0 \notin \overline{G \cdot V}$) but is not polystable (note polystable implies semistable as $V \neq 0$) iff $D \neq 0 \neq N$.

Here the topology of $\mathfrak{gl}_n(\mathbb{C})$ is the classical topology of $\mathbb{C}^{n^2}$. I can show the following:

1. If V is nilpotent then $0 \in \overline{G \cdot V}$. Indeed, because $G = \mathrm{GL}_n(\mathbb{C})$ acts on V by conjugation, every matrix is conjugate to an upper triangular matrix, we can assume V is upper triangular with 0's on the diagonal, $V = (a_{ij})$ where $a_{ij} = 0$ for $i \geq j$. Let $f(t)$ be a diagonal matrix whose (i, i)th entry is t^i. Then $f(t)Vf(t)^{-1} = (t^{i-j}a_{ij})_{1 \leq i,j \leq n}$. Hence, as $|t| \to \infty$ in $\mathbb{C}^\times$, $f(t)Vf(t)^{-1} \to 0$. Hence, $0 \in \overline{G \cdot V}$.
2. If V is not nilpotent then V is semistable, i.e. $0 \notin \overline{G \cdot V}$. Indeed, V has nonzero eigenvalue $\lambda \in \mathbb{C}$ and hence gVg^{-1} also has nonzero eigenvalue λ for any $g \in \mathrm{GL}_n(\mathbb{C})$. Because all norm on a finite dimensional vector space is equivalent, we can choose the operator norm $\|.\|$ on $\mathfrak{gl}_n(\mathbb{C})$, giving $\|gVg^{-1}\| \geq |\lambda| > 0$ for all $g \in \mathrm{GL}_n(\mathbb{C})$ (operator norm of a matrix is at least the spectral radius, i.e. largest eigenvalue, of that matrix). Thus $0 \notin \overline{G \cdot V}$.

I don't know how to show that if V is diagonalisable (hence we can assume V is a diagonal matrix) with at least one nonzero eigenvalue, then $G \cdot V$ is closed (and vise versa).

I can do for example when $n = 2$, $V = diag(\lambda, 0)$ with $\lambda \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} V \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{\lambda}{ad - bc} \begin{pmatrix} ad & -ab \\ cd & -bc \end{pmatrix}.$$

Then $G \cdot V = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C}) : x_{11} + x_{22} = \lambda, x_{11}x_{22} = x_{21}x_{12} \right\}$, which is closed in $\mathfrak{gl}_2(\mathbb{C})$.

What is the moment map here? Can I draw a picture for this example, showing that the moment map is some kind of critical point of some norm?

2.1.2. *Torus acting on a vector space.* Another example is letting $T = (\mathbb{C}^\times)^n$ acting on a (finite dimensional?) complex vector space V, equipped with a Hermitian inner product. Then V can be decomposed as direct sum of orthogonal vector spaces $\bigoplus_{\chi \in X^*(T)} V_\chi$ where $V_\chi = \{v \in V : t \cdot v = \chi(t)v\}$. Let $v \in V$ and $v = \sum_{i=1}^r v_i$ where $v_i \in V_{\chi_i}$, where $\chi_1, \dots, \chi_r \in X^*(T)$ are distinct characters. Note that each character can be identified with an element in $\mathbb{Z}^n$ as $X^*(T) \cong \mathbb{Z}^n$ by sending $(m_1, \dots, m_n) \in \mathbb{Z}^n$ to $(t_1, \dots, t_n) \in (\mathbb{C}^\times)^n \mapsto \prod_{i=1}^n t_i^{m_i}$. Then the stability of the action can be described as follows:

1. $v \in V$ is unstable, i.e. $0 \in \overline{T \cdot v}$, iff 0 does not lie in the convex hull of $\chi_1, \dots, \chi_r$.
2. v is polystable, i.e. $T \cdot v$ is closed, iff 0 lies in the interior of the convex hull of $\chi_1, \dots, \chi_r$.

I can prove (a). It is essentially lies in the proof of Hilbert-Mumford criterion `https://www.isibang.ac.in/~sury/hilbmumf.pdf` by B.Sury that I read in order to present to the workshop.

Let me try to decode the definition of convex hull to see why it is essentially in B. Sury's paper. One definition is that it is the set $\{\sum_{i=1}^r b_i\chi_i \in \mathbb{R}^n : 0 \le b_i \le 1, \sum b_i = 1\}$. Another way to think about this is that 0 does not lie in the convex hull iff there exists a hyperplane passing through 0, i.e. $\sum_i c_i x_i = 0$, such that χ_i's lie on the same half-plane separated by that hyperplane, i.e. $\langle \chi_i, (c_1, \dots, c_n\rangle > 0$. With this, we can choose $t_\ell = (\ell^{c_1}, \dots, \ell^{c_n})$ then $\|t_\ell \cdot v\|^2 = \sum_{i=1}^r |\ell|^{2\langle \chi_i, (c_1, \dots, c_n)\rangle} \|v_i\|^2$, which goes to 0 as $\ell \to 0$. Conversely, if $0 \in \overline{T \cdot v}$ then this is Sury's proof of Hilbert Mumford criterion for $\mathrm{GL}_n(\mathbb{C})$. *Need to write this down in case I forgot.*

How to prove (b)? Can I draw a picture of this example?

Maybe two subexamples we can draw are

(1) $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by multiplication via $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$. The orbits are $O_\lambda = \{(x, y) \in \mathbb{C}^2 : xy = \lambda\}$ for $\lambda \in \mathbb{C}^\times$, $\{(x, 0) : x \neq 0\}$ and $\{(0, y) : y \neq 0\}$. One can see that the quotient topology of this action is not Hausdorff. Ramiro mentioned that the moment map is then somesort of shortest distance to each orbit. *Need to work this out.* The non-closed orbits are the x,y-axes without the origin. If we throw out these two orbits, we expect to get a nicer topology when taking quotient. *What does this mean? Can you draw it?*

(2) $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by multiplication $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t\end{pmatrix}$. Then the orbits are $\{(0, 0)\}$ and lines through the origin but minus the origin. When we take out the bad orbit $((0, 0))$ we get $\mathbb{P}^1$.

2.1.3. *Kempf-Ness fancy version.* I don't think I have seen the relation between Kempf-Ness theorem in Kempf-Ness paper and the fancy version of this (i.e. a homeomorphism between some GIT quotient and symplectic reduction). I would like to learn this someday. For example, I don't know much about the process of throwing away bad orbits to get better quotient topology. *Maybe work out the examples above or read this paper `https://arxiv.org/pdf/math/0512411.pdf` in more details.* Geordie mentions that this Kempf-Ness paper essentially embeds in Atiyah-Bott paper and I would like to understand more what did he mean by this.

2.1.4. *Complexification, real vs complex.* This is about complexification, real form, compact form, $\mathbb{R}$ vs $\mathbb{C}$ structure, reductive etc... of Lie groups. During the workshop, we have done this for torus, i.e. what is the most natural way to get a compact torus from a complex torus and vise versa?

There are two ways to define compact torus. A non-canonical way is that it is a real Lie group diffeomorphic to $U(1)^n$ for some n, where $U(1) = S^1 = \{z \in \mathbb{C} : z\overline{z} = 1\}$. A canonical definition is that it is a connected compact abelian real Lie group.

There are also two ways to define complex torus. A noncanonical way is that it is a complex Lie group diffeomorphic to $(\mathbb{C}^\times)^n$. A canonical definition is that it is a connected reductive abelian complex Lie group (see how compact is replaced by reductive when comparing with compact torus definition, i.e. one can think of reductive complex Lie group as complexification of compact real Lie group).

From a complex torus, you can take its maximal compact real Lie subgroup, which will be unique up to conjugation (every Lie group has a unique maximal compact subgroup up to conjugation?). From a compact torus T then the exponential map $\exp : \mathfrak{t} \to T$ is surjective, i.e. $T = \exp(\mathfrak{t})$. Indeed, $\exp(t)$ is a subgroup of T (as T is abelian) and contains a neighborhood of the identity. Hence, T is a disjoint union of cosets of $\exp(\mathfrak{t})$. But as T is connected so $T = \exp(t)$. Hence, if $\Gamma = \ker(\exp)$ then T is diffeomorphic and group isomorphic to $\mathfrak{t}/\Gamma$. The complexification of T is then $T_{\mathbb{C}} = (\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C})/\Gamma$. This way of defining complexification seems to work only for torus. In general, it is more complicated. *How to define complexification in general?* Ramiro mentioned there is a universal property about this. Masoud told us another way to define complexification, which is based on the idea that categories of representations of compact Lie group and of its complexification are the same (?).

Rohin told me that real/compact form is some sort of reverse process of complexification. From a compact Lie group, you can get a complex Lie group unique in some sense, but the reverse process is not unique, and a real form refers to a "real"isation of that ...

In the first line of Kemp-Ness proof for theorem in section 4: For $\mathrm{GL}_n(\mathbb{C})$, one can give it an algebraic structure over $\mathbb{R}$ such that its real locus is $U(n)$. Indeed, write down the equation $AA^* = I$ of $U(n)$ as polynomials over $\mathbb{R}$. The claim is that over $\mathbb{C}$, this gives $\mathrm{GL}_n(\mathbb{C})$. For example, when $n = 1$, then $U(1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. And we can identify $\left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} : x^2 + y^2 = 1 \right\}$ being isomorphic to $\mathbb{C}^\times$ over $\mathbb{C}$. *Need to write this down. I have done this before but I forgot.*

2.2. **19/07/2021: Line bundles on $\mathbb{P}^1$.** Past few days I have been trying to learn about line bundles on $\mathbb{P}^1$, how to classify topological/ holomorphic/ algebraic line bundles on $\mathbb{P}^1$, how to describe their (global) sections, to compute their transition functions, how to draw some of them.

2.2.1. *Transition functions of* $\mathcal{O}(-1)$. First is the (topological) canonical line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$ over $\mathbb{C}$. Since each point in $\mathbb{P}^1$ corresponds to a line in $\mathbb{C}^2$, as a set $\mathcal{O}(-1) = \{(\ell, x) \in \mathbb{P}^1 \times \mathbb{C}^2 : x \in \ell\}$ with the obvious projection p to $\mathbb{P}^1$.

Now, I want to compute the transition functions of this line bundle. $\mathbb{P}^1$ can be thought abstractly as open cover of $U_0 = \{[x, y] \in \mathbb{P}^1 : x \neq 0\}$ and $U_1 = \{[x, y] : y \neq 0\}$. Each U_i is isomorphic to affine space $\mathbb{C}$, for example, $U_0 \cong \mathbb{C}$ via $[x, y] \mapsto y/x$.

We have an isomorphism $\pi_0 : p^{-1}(U_0) \to U_0 \times \mathbb{C}$ sending $([1 : z_\ell], (x, y))$ to $([1 : z_\ell], x)$. The inverse π_0^{-1} sends $([1 : z_\ell], c)$ to $([1 : z_\ell], c(1, z_\ell))$. Notice that here we have picked a representative $(1, z_\ell) \in \mathbb{C}^2$ of $\ell = [1 : z_\ell]$ to define π_0. The reason for this is that we want our map to be a homeomorphism. Visually, imagine $\mathcal{O}(-1)$ as collection of lines on $\mathbb{C}^2$ passing through 0. Choosing representatives for elements in U_0 as above means the representatives lie on the line $x = 1$ in $\mathbb{C}^2$, which guarantee continuity (i.e. if $|z_\ell| < 1$ and $|c| < 1$ then the image $c(1, z_\ell)$ is open in $\mathbb{C}^2$, looking like a paper fan).

Similarly, $\pi_1 : p^{-1}(U_1) \to U_1 \times \mathbb{C}$ sends $([z_\ell, 1], (x, y))$ to $([z_\ell, 1], y)$ and π_1^{-1} sends $([z_\ell, 1], c)$ to $([z_\ell, 1], c(z_\ell, 1))$.

The transition function $g_{01} : U_1 \cap U_0 \to \mathrm{GL}_1(\mathbb{C})$ is defined via

$$
\begin{aligned}
(U_1 \cap U_0) \times \mathbb{C} &\xrightarrow{\pi_0^{-1}} p^{-1}(U_1 \cap U_0) \xrightarrow{\pi_1} (U_1 \cap U_0) \times \mathbb{C} \\
(\ell = [1, z_\ell], c) &\mapsto ([1, z_\ell], c(1, z_\ell)) \mapsto ([1, z_\ell], cz_\ell)
\end{aligned}
$$

Hence, g_{01} sends $\ell = [1, z_\ell] \in U_1 \cap U_0$ to $(c \mapsto cz_\ell)$ in $\mathrm{GL}_1(\mathbb{C})$. Under identification of $U_1 \cap U_0$ with $\mathbb{C}^\times$ via indentification $U_0 \cong \mathbb{C}$, i.e. $\ell = [1, z_\ell] \mapsto z_\ell$ (we choose $U_0 \cong \mathbb{C}$ instead of $U_1 \cong \mathbb{C}$ because our map is $\pi_1 \circ \pi_0^{-1}$ so the domain $U_1 \cap U_0$ lies in U_0 originally) and $\mathrm{GL}_1(\mathbb{C})$ with $\mathbb{C}^\times$, g_{01} can be view simply as a map $\mathbb{C}^\times \to \mathbb{C}^\times$ sending z to z.

Similarly, one can check the other transition function $g_{10} : U_1 \cap U_0 \to \mathrm{GL}_1(\mathbb{C})$ defined by $\pi_0 \circ \pi_1^{-1}$ sends $z \mapsto z$, under identification of spaces $U_1 \cap U_0$ with $\mathbb{C}^\times$ via $U_1 \cong \mathbb{C}$. Let me spell this out. We have

$$
\begin{aligned}
(U_1 \cap U_0) \times \mathbb{C} &\xrightarrow{\pi_1^{-1}} p^{-1}(U_1 \cap U_0) \xrightarrow{\pi_0} (U_1 \cap U_0) \times \mathbb{C} \\
(\ell = [z_\ell, 1], c) &\mapsto ([z_\ell, 1], c(z_\ell, 1)) \mapsto ([z_\ell, 1], cz_\ell)
\end{aligned}
$$

Notice that it is true $g_{01}(\ell)g_{10}(\ell) = 1$ for all $\ell \in U_1 \cap U_0$, but upon correct identification of spaces, both these maps can be viewed as $z \mapsto z$ from $\mathbb{C}^\times$ to $\mathbb{C}^\times$.

This is the point that confuses me the most because it seems many other sources claiming that the transition map should send $z \mapsto z^{-1}$ (for example, this an this, p.30 while one reference agrees with my choice). The reason for this ambiguity seems to be because there are two ways to identify $U_1 \cap U_1$ with $\mathbb{C}^\times$, one via $U_0 \cong \mathbb{C}$ and the second via $U_1 \cong \mathbb{C}$ (see how I get g_{01} as $z \mapsto z$). To avoid ambiguity, the best way to phrase this is that g_{01} sends $[x_0, x_1] \to x_1/x_0$ and g_{10} sends $[x_0, x_1]$ to x_0/x_1 (later we will see that we can define line bundle $\mathcal{O}(k)$ with $k \in \mathbb{Z}$ with transition functions $g_{01}([x_0, x_1]) = (x_1/x_0)^k$).

One may then ask why name the bundle to be $\mathcal{O}(-1)$ instead of $\mathcal{O}(1)$? One reason, as we will later see that algebraic/holomorphic $\mathcal{O}(-1)$ has no nontrivial global section, while algebraic/holomorphic $\mathcal{O}(1)$ has global section being a $\mathbb{C}$-vector space of degree 1 homogeneous polynomials in two variables x_0, x_1. For more reasons, see `https://math.stackexchange.com/q/256482/58951`.

2.2.2. $\mathcal{O}(-1)$ *as a Mobius strip.* How do you visualise $\mathcal{O}(-1)$? I claim that you can draw this as an "infinite" Mobius strip. At least this is the picture over $\mathbb{R}$, but as a drawing, you can just pretend that any field k is just a line.

Before drawing it out, let me first explain why $\mathcal{O}(-1)$ over $\mathbb{R}$ is the infinite Mobius strip, i.e. $[0,1] \times \mathbb{R}/\sim$ where $(0,t) \sim (1,-t)$. Over $\mathbb{R}$, every line in $\mathbb{R}^2$ intersects the circle S^1 at exactly two points, so we will define a map sending $x \in [0,1]$ to $e^{\pi i x} \in S^1$ to indicate our line. Hence, $(x,t) \in [0,1] \times \mathbb{R}$ is sent to a line ℓ in $\mathbb{R}^2$ passing through $e^{\pi i x}$ and the corresponding point $te^{\pi i x}$ on that line. To get a bijection, we need identification $(0,t) \sim (1,-t)$ because both represent the same point on the x-axis.

Now, let's try to draw this out for $1 < t < 2$. We would get fig. 1 where under $(0,t) \sim (1,-t)$ we

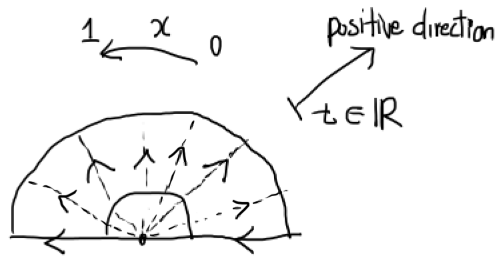


FIGURE 1. Visualise $\mathcal{O}(-1)$

need to glue the paper fan above along the arrows on the x-axis. As you let x goes from 0 to 1 to get back to the x-axis, by looking at the positive direction of each line, you can notice the twist.

Let me offer another explanation of $\mathcal{O}(-1)$ being the Mobius bundle. In the previous explanation, I take the definition of $\mathcal{O}(-1)$ as lines on $\mathbb{R}^2$, this time I want to view $\mathcal{O}(-1)$ abstractly via its transition functions. The goal would be the same, i.e. how to see the twist in $\mathcal{O}(-1)$, but I don't want to describe the homeomorphism explicitly (because it relies too much of the fact that we are in $\mathbb{R}$). If successful, I want an argument that works for any topological field.

Now, $\mathcal{O}(-1)$ is defined via its transition function $g_{01} : U_0 \cap U_1 \to \mathrm{GL}_1(\mathbb{R})$ sending $[x_0, x_1] \mapsto x_1/x_0$. This means that $\mathcal{O}(-1)$ is obtained by glueing $U_1 \times \mathbb{R}$ and $U_0 \times \mathbb{R}$ via $(U_1 \cap U_0) \times \mathbb{R} \to (U_1 \cap U_0) \times \mathbb{R}$ sending $([1,z],c) \mapsto ([1,z],zc)$. Now, I would draw $U_0 \cong \mathbb{R}$ as a circle with a point $[0,1]$ removed and $U_1 \cong \mathbb{R}$ as a circle with a point $[1,0]$ removed.

In U_0, on one side of $[1,0]$ are points $[1,z]$ with $z < 0$ and on the other side are those $[1,z]$ with $z > 0$. On each fiber of $U_0 \times \mathbb{R} \to U_0$, we choose the positive directions as pointing outwards from the circle. This choice is possible because we are working over $\mathbb{R}$. Now we look at those fibers in the circle of U_1 via the glueing $([1,z],c) \mapsto ([1,z],zc)$, keeping in mind of the positive direction of each fiber (see fig. 2).

With this, we observe that we have twisted one connected component of $(U_0 \cap U_1) \times \mathbb{R}$ by half.

Is there an easier way to see why $\mathcal{O}(-1)$ is not the trivial line bundle $\mathbb{P}^1 \times \mathbb{C}$? I would say that intuitively, because all lines/fibers of $\mathcal{O}(-1)$ "have a common point 0", while for $\mathbb{P}^1 \times \mathbb{C}$, the fibers do not intersect each other. *How to rigorously describe this phenomenon?*

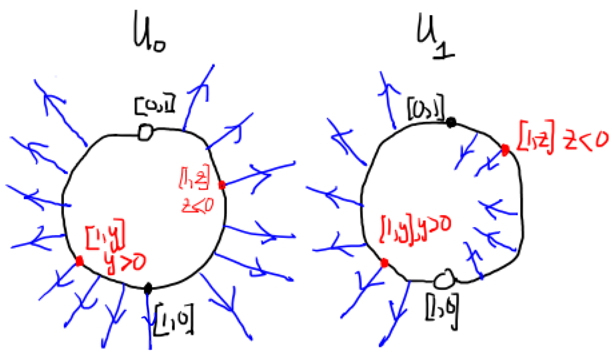


FIGURE 2. Visualise $\mathcal{O}(-1)$

2.2.3. *Dual line bundle* $\mathcal{O}(1)$. The line bundle $\mathcal{O}(1)$ is the dual line bundle to $\mathcal{O}(-1)$. What does this mean? One could guess that it means the fibers of $\mathcal{O}(1)$ are the dual spaces to the fibers of $\mathcal{O}(-1)$. But this is not a good enough description. To describe this vector bundle, we need to determine its transition functions.

In general, the dual vector bundle E^* of $E \to V$ is the vector bundle whose fibers are the dual spaces to the fibers of E. From this information, I will (heuristically) derive that the most natural vector bundle E^* that has this property is the one that has transition functions $g^*_{ij} = (g^T_{ij})^{-1}$. Because $\pi_j \circ \pi_i^{-1} : \{u\} \times \mathbb{C}^n \xrightarrow{\sim} E_u \xrightarrow{\sim} \{u\} \times \mathbb{C}^n$ is an isomorphism of vector spaces so $f_{ij} := \pi_j^{-1} \circ (g_{ij}u) \circ \pi_i \in \mathrm{GL}(E_u)$. We want the transition functions for E^*, i.e. how to cook up $\{u\} \times \mathbb{C}^n \xrightarrow{\sim} E^*_u \xrightarrow{\sim} \{u\} \times \mathbb{C}^n$? This is not possible as we don't know local trivialisations of E^*, but we can guess $f^*_{ij} : E^*_u \xrightarrow{\sim} \{u\} \times \mathbb{C}^n \xrightarrow{\sim} E^*_u$ in $\mathrm{GL}(E^*_u)$, i.e. the most natural one is $(\phi \in E^*_u) \mapsto ((v \in E_u) \mapsto \phi(f^{-1}_{ij} v))$. Notice I put f^{-1}_{ij} instead of f_{ij} because f^*_{ij} needs to satisfy the cocycle condition $f^*_{jk} f^*_{ij} = f^*_{ik}$. We will show that $f^*_{ij} = (f^{-1}_{ij})^T$ (note that for $\phi \in \mathrm{GL}_(V)$ then ϕ^T means $\phi^T \in \mathrm{GL}(V^*)$ sending $f \in V^*$ to $f \circ \phi$) and therefore implying $g^*_{ij} = (g^{-1}_{ij})^T$.

2.2.4. *Line bundles* $\mathcal{O}(k)$. We define $\mathcal{O}(k)$ to be $|k|$-times tensor product of $\mathcal{O}(1)$ if $k > 0$ or $\mathcal{O}(-1)$ if $k < 0$. We want to compute the transition functions of these bundles.

For vector bundles E_1, E_2 over V with transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_m(\mathbb{C}^m), f_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C}^n)$, the transition functions of $E_1 \otimes E_2$ are $g_{ij} \otimes f_{ij} : U_i \cap U_j \to \mathrm{GL}_{mn}(\mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^n)$.

We show $\mathcal{O}(-1) \otimes \mathcal{O}(1)$ is the trivial line bundle $\mathbb{P}^1 \otimes \mathbb{C}$. The transition function of $\mathcal{O}(-1) \otimes \mathcal{O}(1)$ is $g_{01} \otimes g^*_{01} : U_0 \cap U_1 \to \mathrm{GL}_1(\mathbb{C})$, sending $[x_0, x_1]$ to $\frac{x_1}{x_0} \otimes \frac{x_0}{x_1} = 1$ and similarly for the other transition function.

The transition function of $\mathcal{O}(2)$ is $g_{01} \otimes g_{01}$, sending $[x_0, x_1]$ to $(x_1/x_0)^2$. Now, if you try to draw out $\mathcal{O}(2)$ as I did for $\mathcal{O}(-1)$, you will notice that topologically, it is just the trivial line bundle $\mathbb{P}^1 \times \mathbb{R}$. Similarly, $\mathcal{O}(1)$ over $\mathbb{R}$ is also just a Mobius bundle. In fact, over $\mathbb{P}^1$ (or S^1), there are just two topological line bundles up to isomorphism!

Can I interpret $\mathcal{O}(k)$ where k means number of twists? Just like $\mathcal{O}(-1)$ is the Mobius strip over $\mathbb{R}$ (and I think $\mathcal{O}(1)$ over $\mathbb{R}$ is also a Mobius strip)? See `https://math.stackexchange.com/q/220203/58951` or `https://math.stackexchange.com/q/2219539/58951` for example. For example, can you draw $\mathcal{O}(2)$ or $\mathcal{O}(-2)$?.

2.2.5. *Only two topological line bundles over $\mathbb{P}^1(\mathbb{R})$.* Read `https://ayoucis.wordpress.com/2014/12/12/line-bundles-on-the-circle/`.

2.2.6. *What about topological line bundles over other fields?*

2.2.7. $\mathcal{O}(-1)$ *as an algebraic line bundle over* $\mathbb{C}$. So far, we have describe $\mathcal{O}(-1)$ as a topological line bundle, i.e. a continuous map $p : \mathcal{O}(-1) \to \mathbb{P}^1$ with local trivialisations $\pi_0 : p^{-1}(U_0) \xrightarrow{\sim} U_0 \times \mathbb{C}$ and $\pi_1 : p^{-1}(U_1) \xrightarrow{\sim} U_1 \times \mathbb{C}$ that commutes with projections to U_0, U_1, respectively. Furthermore, these local trivialisations must induce isomorphisms of vector spaces $p^{-1}(\ell) \cong \{\ell\} \times \mathbb{C}$ of each fiber of p.

How do we describe $\mathcal{O}(-1)$ as an algebraic/holomorphic/smooth/etc line bundle? `https://math.stackexchange.com/q/3481443/58951`.

2.2.8. $\mathbb{P}^1$ *as a scheme.* Before, we describe $\mathbb{P}^1$ abstractly as a topological space over $\mathbb{C}$, in fact the construction works over any field k, by glueing two k's, namely $\mathbb{A}_0 = k$ and $\mathbb{A}_1 = k$. This is glued via $\mathbb{A}_0 \setminus \{0\} = k^\times \to k^\times = \mathbb{A}_1 \setminus \{0\}$ sending $z \mapsto z^{-1}$. In particular, U is open in $\mathbb{P}^1$ if $U \cap \mathbb{A}_0$ and $U \cap \mathbb{A}_1$ are both open.

Now we wish to equip $\mathbb{P}^1$ with an extra structure, i.e. $\mathbb{P}^1$ as a scheme. I will explain how to construct $\mathbb{P}^1$ over field k as a sheaf of rings by glueing two affine pieces $\mathbb{A}_0 = \operatorname{Spec} k[t_0]$ and $\mathbb{A}_1 = \operatorname{Spec} k[t_1]$. By gluing, I mean we have two do two things: glue two topological spaces $\mathbb{A}_0$ and $\mathbb{A}_1$ and then glue their structure sheaves $\mathcal{O}_{\mathbb{A}_1}$ and $\mathcal{O}_{\mathbb{A}_0}$.

Compute global section of $\mathcal{O}_{\mathbb{P}^1}$.

2.2.9. *Classifying algebraic vector bundles on* $\mathbb{P}^1$ *via double cosets.* Algebraic vector bundles on $\mathbb{P}^1$ of degree n are classified via the transition functions $g_{ij} : \operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} \mathcal{O}(\mathrm{GL}_n)$ which corresponds to a point in $\mathrm{GL}_n(k[t, t^{-1}])$ ...

2.2.10. $\mathcal{O}(2)$ *as tangent bundle of* $\mathbb{P}^1$.

2.2.11. *Global sections of algebraic bundles.*

2.3. **23/07/2021: Categorial measure theory.** Masoud mentioned that I only defined pushforward measures but not pullback measures in my thesis. When I googled, I found this explanation `https://mathoverflow.net/q/122704/89665` saying there is no pullback of general maps. One needs extra conditions, such as being able to integrate on fibers. There is also a way to develop measure theory categorically with pushforward and pullback available this way, from `https://mathoverflow.net/a/20820/89665`. But the language is quite forgein to me. *Need to read more at some point.*

2.4. **23/07/2021: Sheaf of solutions of ODE is a local system.** I want to claim that local existence and uniqueness of a differential equation is the same as saying that its sheaf of solutions is a constant sheaf.

We have the following result for local existence and uniqueness of ODEs (cited from `https://www.math.utah.edu/~milicic/Eprints/de.pdf`, theorem 1.2): Let Ω be a simply connected region in $\mathbb{C}$, $z_0 \in \Omega$ and $A : \Omega \to \mathrm{GL}_n(\mathbb{C})$ a holomorphic map. For any $Y_0 \in \mathbb{C}^n$, there exists a unique holomorphic function $Y : \Omega \to \mathbb{C}^n$ such that $\frac{dY}{dz} = AY$ in Ω, and $Y(z_0) = Y_0$.

First, we can construct a sheaf $\mathcal{F}$ of solutions over Ω for this equation by letting $\mathcal{F}(U)$ to be set of all $Y : U \to \mathbb{C}$ satisfying the ODE. The claim is that $\mathcal{F}$ is a constant sheaf $\mathbb{C}^n$ (note that because Ω is simply connected so any continuous $\Omega \to \mathbb{C}^n$ with $\mathbb{C}^n$ having the discrete topology must be a constant function, meaning $\underline{\mathbb{C}^n}_\Omega(\Omega) = \mathbb{C}^n$). Indeed, $\mathcal{F}(\Omega) \to \mathbb{C}^n$ sending $Y \mapsto Y(z_0)$.

Now, if Ω is not simply connected then $\mathcal{F}$ is a $\mathbb{C}$-local system, i.e. Ω is union of its connected components and $\mathcal{F}$ restricted to each component is a constant sheaf, as shown below.

2.5. **23/07/2021: Shrawan Kumar's SMRI talk.** I attend Shrawan Kumar'S talk "Root components for tensor product of affine Kac-Moody Lie algebra modules" for the Sydney Mathematical Research Institute. Here are something news I learnt:

(1) Some history about tensor product decomposition problem for finite/affine/Kac-Moody Lie algebras.
(2) There is a problem that I think I can work out the proof: For integral dominant weight λ, μ, let $V(\lambda), V(\mu)$ be the corresponding highest weight reps. By complete irreducibility, we can decompose $V(\lambda) \otimes V(\mu)$ as direct sum of $V(\nu)$. Let $n_{\lambda,\mu}^{\nu}$ be the multiplicity of $V(\nu)$ in $n_{\lambda,\mu}^{\nu}$. The problem is that:

    Show, if $n_{\lambda,\mu}^{\nu} \neq 0$ then $n_{m\lambda,m\mu}^{m\nu} \neq 0$ for all positive integer m.

    Kumar claimed that there are two proofs of this, one use standard representation theoretic method and the second via Borel-Weil theorem, and I would like to know how to prove this for the above two methods.
(3) Do some google on "ample line bundle" as this terminology is mentioned in the talk. Roughly, a line bundle L on a proper [1] scheme X over field k is ample if $L^{\otimes n}$ has enough global sections to give a closed immersion (i.e. closed embedding) $X \to \mathbb{P}^N$ where $N = \dim H^0(X, L^{\otimes n}) - 1$. From what I know, this definition is important in order to classify algebraic varieties, i.e. describe a variety with certain properties as subvariety of certain projective space defined by equations of certain degrees. See `https://www.math.ucla.edu/~totaro/papers/public_html/algebraic.pdf` for example.

    Now, let me try to explain more about the part of "having enough global sections gives a morphism $X \to \mathbb{P}^N$". I read this from wikipedia `https://en.wikipedia.org/wiki/Ample_line_bundle`. Choose global sections $a_0, \dots, a_{N-1} \in H^0(X, L)$ then we can defined $f : X \to \mathbb{P}^N$ sending $x \mapsto [a_0(x), \dots, a_{N-1}(x)]$. Note that this is well-defined if over any $x \in X$, at least one of $a_i(x)$ is non-zero, i.e. intersection of zero sets of all global sections is empty. This is what is called "basedpoint-free" line bundle. "Semi-ample" line bundle L is when $L^{\otimes n}$ is basedpoint-free for some n. "Very-ample L is when $X \to \mathbb{P}^N$ is a closed immersion. "Ample" L is when $L^{\otimes r}$ is very-ample. So it's a bunch of definitions. *Workout the examples in wikipedia: $\mathcal{O}(d)$ on $\mathbb{P}^1$ is based-point free iff $d \geq 0$, and very ample iff $d \geq 1$.*

    What is the role of being "ample" in representation theory, Kumar mentioned that $L(\lambda)$ as ample line bundle corresponding to a dominant weight λ, what does this mean?

2.6. **27/07/2021: Discriminant and different of field extension.** Let A be a Dedekind domain with field of fraction K, L/K is a finite separable extension and B is integral closure of A in L. Recall a prime $\mathfrak{q}|\mathfrak{p}$ of L is unramified if $e_{\mathfrak{q}} = 1$ and $B/\mathfrak{q}$ is separable extension of $A/\mathfrak{p}$. A prime $\mathfrak{p}$ of K is unramified if every prime $\mathfrak{q}|\mathfrak{p}$ lying above it are unramified.

In this notes, we will define *different* $\mathcal{D}_{B/A}$ and *discriminant* $D_{B/A}$ and show that these encodes information about ramification of L/K. In particular, we would like to explain the following:

The different is B-ideal that is divisible by the ramified primes $\mathfrak{q}$ of L, and the discriminant is A-ideal that is divisible by the ramified primes $\mathfrak{p}$ of K. The valuation $v_{\mathfrak{q}}(\mathcal{D}_{B/A})$ will give us information about the ramification index $e_{\mathfrak{q}}$ and its exact value when $\mathfrak{q}$ is tamely ramified.

This is created to summarised Lecture 12 in `https://math.mit.edu/classes/18.785/2019fa/lectures`.

2.6.1. *The different.* First, we need to define these two objects. We have trace pairing $L \times L \to K$ defined by $(x, y) \mapsto \mathrm{Trace}_{L/K}(xy)$. When L/K is separable, this is a perfect pairing (i.e. it induces K-module isomorphism L with $L^{\vee} = \mathrm{Hom}_K(L, K)$). B is a A-lattice in L (i.e. finitely generated

[1] finite dimensional of global section of line bundles on X?

A-module that spans L as K-vector space) and we have a corresponding *dual lattice* for B, defined as

$$B^* := \{x \in L : \mathrm{Trace}_{L/K}(xb) \in A \ \forall b \in B\}.$$

It is an A-lattice in L isomorphic to dual A-module $M^\vee := \mathrm{Hom}_A(M, A)$. One can show $B^* \in \mathcal{I}_B$ ($\mathcal{I}_B$ is the *ideal class group* of B, i.e. group of invertible fractional ideals of B, i.e. finitely generated B-submodules lying of L; here fractional ideal I being invertible means $IJ = B$ for some fractional ideal J). We define the *different* $\mathcal{D}_{L/K}$ of L/K (or the different $\mathcal{D}_{B/A}$ of B/A) to be the inverse of B^* in $\mathcal{I}_B$. Explicitly, we have

$$\mathcal{D}_{L/K} := \mathcal{D}_{B/A} := (B^*)^{-1} = \{x \in L : xB^* \subset B\}.$$

Note $B \subset B^*$ as $\mathrm{Trace}_{L/K}(ab) \in A$ for any $a, b \in B$, we find that the different is an ideal of B, not just fractional ideal.

Different respects localisation and completion:

- Let S be multiplicative subset of A. Then $S^{-1}\mathcal{D}_{B/A} = \mathcal{D}_{S^{-1}B/S^{-1}A}$. To prove this, it suffices to show inverses and duals commutes with localisation.
- Let $\mathfrak{q}|\mathfrak{p}$ be a prime of B. Then $\mathcal{D}_{\hat{B}_\mathfrak{q}/\hat{A}_\mathfrak{p}} = \mathcal{D}_{B/A}\hat{B}_\mathfrak{q}$ as $\hat{B}_\mathfrak{q}$-ideals. Here $\hat{B}_\mathfrak{q}$ and $\hat{A}_\mathfrak{p}$ are completions of B and A at $\mathfrak{q}, \mathfrak{p}$, respectively.

2.6.2. *The discriminant.* Let $n := [L : K]$. For B an A-lattice in L, we can define the *discriminant of L/K (or of B/A)* to be the A-module $D_{L/K}$ (or $D_{B/A}$) generated by

$$\mathrm{disc}(x_1, \ldots, x_n) := \det[\mathrm{Trace}_{B/A}(x_i x_j)]_{ij} \in A$$

where $x_1, \ldots, x_n \in B$. This is infact a fractional ideal of A. When B is free A-lattice in L (such as when $A = \mathbb{Z}$) then $D_{B/A}$ is a principal fractional ideals, generated by $\mathrm{disc}(e_1, \ldots, e_n)$ where $e_1, \ldots, e_n$ is K-basis for L in B.

Depending on the situations, we have few ways to compute the discriminant:

- Let Ω/K be field extension for which there are discint $\sigma_1, \ldots, \sigma_n \in \mathrm{Hom}_K(L, \Omega)$ then for any $e_1, \ldots, e_n \in L$, we have
$$\mathrm{disc}(e_1, \ldots, e_n) = \det[\sigma_i(e_j)]_{ij}^2,$$
- For polynomial $f(x) = \prod_i (x - \alpha_i)$ of degree n, then the discriminant of extension $A[x]/(f)$, where α is the image of x in $A[x]/(f)$, is generated by
$$\mathrm{disc}(1, \alpha, \ldots, \alpha^{n-1}) = \prod_{i<j}(\alpha_i - \alpha_j)^2.$$
People also define this to be the *discriminant of f*.

Discriminant also respects localisation and completion:

- For S multiplicative subset of A then $S^{-1}D_{B/A} = D_{S^{-1}B/S^{-1}A}$.
- For prime $\mathfrak{q}|\mathfrak{p}$ of B then *Need to learn more ... Check Serre after this.*

2.7. **28/07/2021: Representations of $\mathfrak{sl}_2$.** Recall on 23/07/2021, I found the following question on Kumar's talk that he mentioned as an "easy observation" (see `https://youtu.be/gph8XNkpdBM?t=368`):

Problem 1. *Show that if $V(\nu)$ appears in the direct sum decomposition of $V(\lambda) \otimes V(\mu)$ into irreducible representations, then $V(m\nu)$ appears in the direct sum decomposition of $V(m\lambda) \otimes V(m\mu)$ for any positive integer m.*

As a first step to answer a question, I just want to review the construction of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ via Verma modules.

The Lie algebra $\mathfrak{sl}_2$ has basis $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ with relations $[h,e] = 2e, [h,f] = -2f, [e,f] = h$.

The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{GL}(\mathfrak{g})$ taking $(x \in \mathfrak{g}) \mapsto (\mathrm{ad}\ x : y \mapsto [x,y])$ gives us the root system Φ for $\mathfrak{g}$ and the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. For $\mathfrak{sl}_2$, we find $\mathfrak{sl}_2 = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f$ where $\alpha \in \Phi = \{\pm\alpha\} \subset \mathfrak{h}^*$ is defined by $\alpha(h) = 2$ (because $[h,e] = 2e$). We choose α to be a simple root, so $\Phi^+ = \{\alpha\}$.

Next, I want to determine the coroots, (co)weights. In order to do this, I first need an identification between $\mathfrak{h}$ and $\mathfrak{h}^*$. We have the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ sending $\kappa(a,b) = \mathrm{Trace}(\mathrm{ad}\ a \circ \mathrm{ad}\ b)$ which is nondegenerate when restricting to $\mathfrak{h} \times \mathfrak{h}$. This gives us an inner product on $\mathfrak{h}$. In our case of $\mathfrak{sl}_2$, we find $\kappa(h,h) = 8$.

Because of nondegeneracy of κ, we have an isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$ sending $\lambda \mapsto t_\lambda \in \mathfrak{h}$ defined by $\lambda(h) = \kappa(t_\lambda, h)$. Hence, we can turn κ into an inner product $(\cdot,\cdot)$ on $\mathfrak{h}^*$ by $(\alpha,\beta) := \kappa(t_\alpha, t_\beta)$. In the literatures, there are two ways to define coroots. One is Bourbaki style, where the coroots lying inside $\mathfrak{h}$ while in Humphreys' style, the coroots lying in $\mathfrak{h}^*$. In Bourbaki's style, the coroot h_λ of $\lambda \in \Phi$ (in Bourbaki, this notation is $\lambda^\vee$, but we will save this for Humphreys style) is the unique element in $\mathfrak{h}$ such that $\lambda(h_\lambda) = 2$ (in particular, $h_\lambda = 2t_\lambda/(t_\lambda, t_\lambda)$). In Humphreys style, coroot of $\lambda \in \Phi$ is $\lambda^\vee := \frac{2\lambda}{(\lambda,\lambda)}$. One can show that $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} = \alpha(h_\beta) \in \mathbb{Z}$ for all $\beta \in \Phi$. In our case of $\mathfrak{sl}_2$, with simple root $\alpha \in \Phi$, we find $t_\alpha = \frac{h}{4}$ as $\alpha(h) = 2$. Hence $h_\alpha = h$ and $\alpha^\vee = \alpha$.

From now on, we will stick to Humphreys style. We can then define the (integral) weight lattice by

$$\Lambda := \{\alpha \in \mathfrak{h}^* | \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}\ \forall \beta \in \Phi\}.$$

In the case of $\mathfrak{sl}_2$, we find $\Lambda = \frac{1}{2}\mathbb{Z}\alpha$. Let $\omega = \alpha/2$ then ω is the fundamental weight of α, i.e. $\langle \omega, \alpha^\vee \rangle = 1$. Upon a choice of positive roots Φ^+, we can define the dominant (integral) weight lattice $\Lambda^+ := \{\alpha \in \mathfrak{h}^* | \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}_{>0}\ \forall \beta \in \Phi^+\}$. In the case of $\mathfrak{sl}_2$, we find $\Lambda^+ = \mathbb{Z}_{>0}\omega$.

Next, we will construct the Verma module $M(\lambda)$ for every $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \mathfrak{n}$ to be the Borel subalgebra corresponding to the Cartan subalgebra $\mathfrak{h}$. In $\mathfrak{sl}_2$, $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e, \mathfrak{n} = \mathbb{C}e, \mathfrak{n}^- = \mathbb{C}f$. We define $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$ is a 1-dimensional representation of $\mathfrak{b}$ with a trivial action by $\mathfrak{n}$ and $\mathfrak{h}$ acts on $\mathbb{C}_\lambda$ by λ. Concrely, let $v^+ := 1 \otimes 1 \in M(\lambda)$ then $M(\lambda)$ has $\mathbb{C}$-basis $U(\mathfrak{n}^-)v^+$, with $\mathfrak{n} \cdot v^+ = 0$ and $h \cdot v^+ = \lambda(h)v^+$. For the case $\mathfrak{sl}_2$, $M(\lambda) = \mathrm{span}_{\mathbb{C}}\{f^i v^+\}$, $e \cdot v^+ = 0$ and $h \cdot v^+ = \lambda(h)v^+$. Letting $v_i := \frac{f^i v^+}{i!}$ for $i = 0, 1, \ldots$ then we can show that (here we have abused of notation to write λ for $\lambda(h)$):

$$\begin{aligned} h \cdot v_i &= (\lambda - 2i)v_i, \\ e \cdot v_i &= (\lambda - i + 1)v_{i-1}, \\ f \cdot v_i &= (i+1)v_{i+1}. \end{aligned}$$

$M(\lambda)$ always has a maximal proper submodule $L(\lambda)$ with quotient being a simple module $V(\lambda)$. $V(\lambda)$ is finite dimensional iff $\lambda \in \Lambda^+$. To see this for the case of $\mathfrak{sl}_2$, we observe:

(1) If $\lambda \notin \mathbb{Z}_{\geq 1}$ then $\lambda - i + 1 \neq 0$ for all $i = 1, \ldots$, implying $M(\lambda)$ is irreducible. Indeed, starting with any $v \in M(\lambda)$, one can keep applying e to get v_0.
(2) If $\lambda \in \mathbb{Z}_{\geq 1}$ then $e \cdot v_{\lambda+1} = 0$. This means $U(\mathfrak{g}) \cdot v_{\lambda+1} = \mathrm{span}_{\mathbb{C}}(v_{\lambda+1}, v_{\lambda+2}, \ldots)$ is a maximal submodule of $M(\lambda)$ that is isomorphic to $M(-\lambda-2) \cong V(-\lambda-2)$. Its quotient $V(\lambda)$ has dimension $\lambda + 1$ and is irreducible. In $V(\lambda)$, we have $f^{\langle \lambda, \alpha^\vee \rangle + 1} v_0 = f^{\lambda+1} v_0 = 0$.

Now, coming back to Kumar's problem on 23/07/2021:

Problem 2. *Show that if $V(\nu)$ appears in the direct sum decomposition of $V(\lambda) \otimes V(\mu)$ into irreducible representations, then $V(m\nu)$ appears in the direct sum decomposition of $V(m\lambda) \otimes V(m\mu)$ for any positive integer m.*

I want to do an example for $\mathfrak{sl}_2$, based on the description of $V(\lambda)$ above.

Example 3. For $\mathfrak{sl}_2$, I will show that $V(1) \otimes V(2) = V(3) \oplus V(1)$. From the problem, I then should have $V(2)$ in $V(2) \otimes V(4)$.

Denote $V(m) := \text{span}_{\mathbb{C}}\{v_{im} : i = 0, 1 \ldots\}$ where the v_{im}'s is defined above. Then $v_{01} \otimes v_{02}$ is a highest weight vector in $V(1) \otimes V(2)$ of weight 3, implying $V(3)$ appears in $V(1) \otimes V(2)$. Because $V(1) \otimes V(2)$ has dimension $2 \times 3 = 6$, $V(3)$ has dimension 4 so it can only be $V(1) \otimes V(2) = V(3) \oplus V(1)$. In fact, we can describe $V(1)$ explicitly as a submodule of $V(1) \otimes V(2)$. Note that $v_{01} \otimes v_{12}$ and $v_{11} \otimes v_{02}$ have weight 1. When applying e to both of these, we find a highest weight vector $v_{01} \otimes v_{12} - 2v_{11} \otimes v_{02}$ of weight 1.

We can play the same game to see how $V(2)$ appears in $V(2) \otimes V(4)$. In particular, $v_{i2} \otimes v_{j4}$ for $i + j = 2, 0 \le i \le 2, 0 \le j \le 4$ are weight vectors of weight 2 in $V(2) \otimes V(4)$. By applying e to these vectors:

(1) $e(v_{02} \otimes v_{24}) = (4 - 2 + 1)v_{02} \otimes v_{14}$,
(2) $e(v_{12} \otimes v_{14}) = (2 - 1 + 1)v_{02} \otimes v_{14} + (4 - 1 + 1)v_{12} \otimes v_{04}$,
(3) $e(v_{22} \otimes v_{04}) = (2 - 2 + 1)v_{12} \otimes v_{04}$.

Hence, $2v_{02} \otimes v_{24} - 3v_{12} \otimes v_{14} + 12v_{22} \otimes v_{04}$ is a highest weight vector of weight 2.

However, I don't know how to see $V(2)$ lying inside $V(2) \otimes V(4)$ from the fact that $V(1)$ lies inside $V(1) \otimes V(2)$.

Maybe it will help if I go a bit more general, suppose $V(c)$ appears in $V(a) \otimes V(b)$ for $\mathfrak{sl}_2$. This means there exists a highest weight vector of weight c in $V(a) \otimes V(b)$. Such vector has the form $\sum_{i+j=(a+b-c)/2} c_{ij} v_{ia} \otimes v_{jb}$. For this to be a highest weight vector, we need

$$\begin{aligned} 0 &= e \cdot \left(\sum_{i+j=(a+b-c)/2} c_{ij} v_{ia} \otimes v_{jb} \right), \\ &= \sum_{i+j=\ell} c_{ij} \left((a - i + 1) v_{i-1,a} \otimes v_{j,b} + (b - j + 1) v_{i,a} \otimes v_{j-1,b} \right), \ \ell = (a + b - c)/2 \\ &= \sum_{i+j=\ell-1} \left(c_{i+1,j}(a - i) + c_{i,j+1}(b - j) \right) v_{i,a} \otimes v_{j,b}. \end{aligned}$$

To show $V(mc)$ appears in $V(ma) \otimes V(mb)$ for some positive integer m, I want to find a weight vector $\sum_{i+j=m\ell} d_{ij} v_{i,ma} \otimes v_{j,mb}$ of weight mc in $V(ma) \otimes V(mb)$, where $d_{ij} \in \mathbb{C}, \ell = m(a+b-c)/2$, such that it is of highest weight, i.e. for all $i + j = m\ell - 1$ then $d_{i+1,j}(ma - i) + d_{i,j+1}(mb - j) = 0$.

Now this is where I got stuck for the $\mathfrak{sl}_2$ case ...

2.8. **More unresolved questions.**

2.8.1. *Functional equation of Riemann zeta function. What is the proof of the functional equation of the Riemann zeta function using Poisson summation formula.* I read the proof from Terence Tao's blog `https://terrytao.wordpress.com/2008/07/27/tates-proof-of-the-functional-equation/`, `https://math.bu.edu/people/jsweinst/Teaching/MA843/TatesThesis.pdf` or `https://people.reed.edu/~jerry/361/lectures/mats.html` (the continuation and functional equation sections).

Need to explain Mellin transform is Fourier transform on $(\mathbb{R}_{>0}, \cdot)$ *by transfering the usual Fourier transform via* $\exp : \mathbb{R} \to \mathbb{R}_{>0}$. To learn more, read `https://people.reed.edu/~jerry/`

361/lectures/mats.html (the Mellin transform section), https://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf, https://mathoverflow.net/q/79868/89665.

Relate this with Tate's thesis, read Fourier analysis on Number fields, https://math.stackexchange.com/q/25090/58951.

2.8.2. *Grothendieck's proof of classification of line bundles on $\mathbb{P}^1$.* See Marielle Ong's https://drive.google.com/file/d/1yfe9lTjF48a0UiZJqEqb8PvRY5yUC2Os/view or Sabin Cautis notes Vector bundles on Riemann surfaces.

See more things about $\mathbb{P}^1$ at https://math.berkeley.edu/~qchu/Notes/256B.pdf or https://math.stanford.edu/~vakil/725/class21.pdf.

2.8.3. *Fourier transform.* I want to first explain that Poisson transform is some sort of change of basis formula, i.e. given $f(x)$, we want to write it with respect to some basis $e^{2\pi ix}$ (this choice of basis is invariant under translation in some sense) and this is what the Fourier transform indicates .... *Where can I read more something along this line?* See Jacob Lurie https://www.youtube.com/watch?v=w3f8KEcv4RE&t=2497s.

2.8.4. *Algebraic groups.* I just want to verify the following facts: SO_n is connected semisimple but is not simply connected.

Some more things to learn

(1) Learn Langlands' proof of Tamagawa: Fourier inversion https://people.reed.edu/~jerry/311/mats.html, Mellin transform https://people.reed.edu/~jerry/361/lectures/mats.html https://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf, Riemann-Zeta functions, spectral theory https://www.math.nagoya-u.ac.jp/~richard/teaching/s2019/Operators.pdf, https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/specthm.pdf, https://en.wikipedia.org/wiki/Spectral_theorem.
(2) Geometry:
    (a) Learn about classification of vector bundles (Hatcher http://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html), characteristic classes (e.g. Tu's book, https://web.ma.utexas.edu/users/a.debray/lecture_notes/u17_characteristic_classes.pdf), connections, equivariant cohomology (Tu's book, Geordie's note, see folder) -¿ Read Atiyah, Bott paper.

3.1. **04/08/2021: Tamagawa number for GL_1 over $\mathbb{Q}$.** I will define Tamagawa measure for GL_1 over $\mathbb{Q}$ and then compute its Tamagawa number.

The ideles $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^\times$ is the restricted product of $\mathbb{Q}_v^\times$'s with respect to its compact open $\mathbb{Z}_v^\times$. We have a norm map

$$|\cdot| : \mathbb{A}^\times \to \mathbb{R}_{>0}$$

defined by sending $a = (a_v) \in \mathbb{A}^\times$ to $\prod_v |a_v|_v$. The product is well-defined because $a_v \in \mathbb{Z}_v^\times$ for almost all places v of $\mathbb{Q}$. Its kernel is denoted $\mathbb{A}^1$, which is closed in $\mathbb{A}^\times$. In fact, we have a homeomorphism

$$\begin{aligned} \phi : \mathbb{A}^\times &\to \mathbb{A}^1 \times \mathbb{R}_{>0}, \\ a = (a_v) &\mapsto \left((a_\infty/|a|, a_2, a_3, \dots), |a|\right), \\ ra &\mapsfrom (a, r). \end{aligned}$$

Proof that ϕ is a homeomorphism. The map $(a, r) \mapsto ra$ from $\mathbb{A}^1 \times \mathbb{R} > 0 \to \mathbb{A}^\times$ is continuous because $\mathbb{A}^\times$ is a topological ring with multiplication map being continuous, and that the two maps $\mathbb{A}^1 \hookrightarrow \mathbb{A}^\times$ and $\mathbb{R}_{>0} \hookrightarrow \mathbb{A}^\times$ are continuous.

We just need to show its inverse ϕ is also continuous. $\mathbb{A}^1$ has basis of open sets

$$U_S := \left\{ a = (a_v) \in U_\infty \times \prod_{p \in S \setminus \{\infty\}} b_p(1 + p^{k_p}\mathbb{Z}_p) \times \prod_{v \notin S} \mathbb{Z}_v^\times : \prod_{v \in S} |a_v|_v = 1 \right\}.$$

where S is a finite set of places of $\mathbb{Q}$ containing the infinite place. Let U be an open subset in $\mathbb{R}_{>0}$. Then the preimage $\phi^{-1}(U_S \times U)$ is

$$U_\infty U \times \prod_{p \in S \setminus \{\infty\}} b_p(1 + p^{k_p}\mathbb{Z}_p) \times \prod_{v \notin S} \mathbb{Z}_v^\times$$

which is open. $\square$

To define the Tamagawa measure $\mu_{\mathrm{GL}_1, \mathbb{Q}}$ on $\mathbb{A}^\times$, we choose a left-invariant differential form $\omega = x^{-1}dx$ on $\mathrm{GL}_1(\mathbb{Q})$. It is left-invariant because left-multiplication by $a \in \mathbb{Q}$ gives $L_a(x^{-1}dx) = (ax)^{-1}d(ax) = x^{-1}dx$. Over each completion of $\mathbb{Q}$, this induces a left-invariant Haar measure $\mu_{\mathrm{GL}_1(\mathbb{Q}_v), \omega} = d|\omega|_v$ on $\mathbb{Q}_v^\times$ by integrating over ω. It is left-invariant because we have the change of variables formula, even over $\mathbb{Q}_p$, i.e.

$$\int_{\mathbb{Q}_v^\times} f(x)|x|_v^{-1} d|x|_v = \int_{\mathbb{Q}_v^\times} f(x) d|\omega|_v = \int_{\mathbb{Q}_v^\times} f(ax) d|L_a \omega|_v = \int_{\mathbb{Q}_v^\times} f(ax)|x|_v^{-1} d|x|_v,$$

where $f(x)$ is a complex-valued continuous function with compact support on $\mathbb{Q}_v^\times$, $d|x|_v$ is the Haar measure on $\mathbb{Q}_v$. Here $\mathbb{Q}_\infty^\times$ means $\mathbb{R}^\times$. For example, we can compute

$$\mu_{\mathbb{Q}_p^\times, \omega}(\mathbb{Z}_p^\times) = \int_{\mathbb{Z}_p^\times} |x|_p^{-1} d|x|_p = \int_{\mathbb{Z}_p^\times} d|x|_p = \sum_{i=1}^{p-1} \int_{i + p\mathbb{Z}_p} d|x|_p = (p-1) \int_{p\mathbb{Z}_p} d|x|_p = \frac{p-1}{p}.$$

Let the Tamagawa measure on $\mathbb{A}^\times$ over $\mathbb{Q}$ to be essentially the product measure

$$\mu_{\mathbb{R}^\times, \omega} \times \prod_p \left(1 - \frac{1}{p}\right)^{-1} \mu_{\mathbb{Q}_p^\times, \omega}$$

In other words, there is a unique Haar measure on $\mathbb{A}^\times$ such that over the open set $\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{Z}_v^\times$ where S is a finite set of places of $\mathbb{Q}$ containing the infinite place, the measure is by taking the product of measures of each local part.

Because of the homeomorphism $\mathbb{A}^\times \cong \mathbb{A}^1 \times \mathbb{R}_{>0}$, if we give $\mathbb{R}_{>0}$ the natural Haar measure obtained by integrating $x^{-1}dx$, this determines a Haar measure da_1 on $\mathbb{A}^1$ satisfying

$$\int_{\mathbb{A}^\times} f(x)\mu_{\mathrm{GL}_1,\mathbb{Q}}(x) = \int_{\mathbb{R}_{>0}} \int_{\mathbb{A}^1} f(a_1 t)t^{-1} da_1 dt.$$

As $\mathbb{Q}^\times$ is discrete in $\mathbb{A}^1$, this induces a measure $\mu'_{\mathrm{GL}_1,\mathbb{Q}}$ on $\mathbb{Q}^\times \setminus \mathbb{A}^1$. We define the Tamagawa number for GL_1 over $\mathbb{Q}$ to be

$$\tau(\mathrm{GL}_{1,\mathbb{Q}}) := \int_{\mathbb{Q}^\times \setminus \mathbb{A}^1} \mu'_{\mathrm{GL}_1,\mathbb{Q}}.$$

To compute $\tau(\mathrm{GL}_{1,\mathbb{Q}})$, we will determine a fundamental domain for $\mathbb{Q}^\times \setminus \mathbb{A}^1$, based on the following proposition:

Proposition 4. *We have*

(a) $\mathbb{Q}^\times$ *is dense in* $\mathrm{GL}_1(\mathbb{A}^\infty)$*, implying* $\mathbb{Q}^\times \setminus (\mathbb{A}^\times)^\infty / \widehat{\mathbb{Z}}^\times = \{1\}$.

(b) We have a homeomorphism $\mathbb{Q}^\times \setminus \mathbb{A}^\times \cong (\{\pm 1\} \setminus \mathbb{R}^\times) \times \widehat{\mathbb{Z}}^\times$.

Proof. (a) We need to show every basis of open sets of $\mathrm{GL}_1(\mathbb{A}^\times)$ contains an element in $\mathbb{Q}^\times$. A basis of open sets of $\mathrm{GL}_1(\mathbb{A}^\times)$ consists of $\prod_{p\in S\setminus\{\infty\}} a_p(1+p^{k_p}\mathbb{Z}_p) \times \prod_{p\notin S\cup\{\infty\}} \mathbb{Z}_p^\times$, where S is a finite set of places of $\mathbb{R}$ containing the infinite place, $k_p \in \mathbb{Z}_{\geq 1}$ (as then $1+p^{k_p}\mathbb{Z}_p$ is open neighborhood of 1 in $\mathbb{Q}_p^\times$), $a_p \in \mathbb{Q}^\times$ (for any $a_p \in \mathbb{Q}_p^\times$, you can always find $a_p' \in \mathbb{Q}^\times$ so $a_p - a_p' \in p^{k_p}\mathbb{Z}_p$). By Chinese Riemander Theorem, there exists $q \in \mathbb{Q}^\times$ such that $q \equiv a_p \pmod{p^{k_p}}$ for all $p \in S \setminus \{\infty\}$ and q only has primes $p \in S \setminus \{\infty\}$ in its prime factorisation. This implies q lies in the desired open set.

To see $(\mathbb{A})^\infty = \mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times$, as $\widehat{\mathbb{Z}}^\times$ is compact open in $(\mathbb{A})^\times$, any $a \in (\mathbb{A})^\infty$ then the open set $a\widehat{\mathbb{Z}}^\times$ must contains an element in $\mathbb{Q}^\times$, as desired.

(b) We define the map

$$\begin{aligned} \phi : (\{\pm 1\} \setminus \mathbb{R}^\times) \times \widehat{\mathbb{Z}}^\times &\to \mathbb{Q}^\times \setminus \mathbb{A}^\times, \\ (r + \{\pm 1\}, z) &\mapsto rz. \end{aligned}$$

Note that $\{\pm 1\} = \widehat{\mathbb{Z}}^\times \cap \mathbb{Q}^\times$, one can then show that ϕ is indeed a homeomorphism. Be careful that $\widehat{\mathbb{Z}}^\times$ and $\mathbb{Q}^\times$ are embedded differently into $\mathbb{A}^\times$. ☐

From this proposition, as $\mathbb{A}^1 \cong \mathbb{A}^\times/\mathbb{R}_{>0}$ so $\mathbb{Q}^\times \setminus \mathbb{A}^1 \cong \mathbb{Q}^\times \setminus \mathbb{A}/\mathbb{R}_{>0} \cong \widehat{\mathbb{Z}}^\times$. Thus, we have

$$\begin{aligned} \tau(\mathrm{GL}_{1,\mathbb{Q}}) &= \int_{\widehat{\mathbb{Z}}^\times} \mu_{\mathrm{GL}_1,\mathbb{Q}}, \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{-1} \int_{\mathbb{Z}_p^\times} \mu_{\mathbb{Q}_p^\times,\omega}, \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{-1} \frac{p-1}{p}, \\ &= 1. \end{aligned}$$

Now, something I would like to learn next:

(1) How to write the above description but for GL_n.
(2) What happen over other number fields/function fields?

3.2. **07/08/2021: Global left-invariant top form of SL_2.** GL_2 has a global left-invariant top form $\det(x_{ij})^{-2} dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22}$. I want to use this and the map $\mathrm{SL}_2 \times \mathbb{G}_m \to \mathrm{GL}_2$ to find a global left-invariant top form of SL_2.

3.2.1. *Sheaf of differentials.* We want to minic the following construction in differential geometry to algebraic geometry language: Given a smooth map $\phi : X \to Y$ of smooth manifolds, this induces map $\phi^* : T^*Y \to T^*X$ of cotangent bundles.

I read this from Neron models book, chapter 2, p. 33.

First, we will describe the sheaf of differentials (i.e. the cotagent bundles in differential geometry) $\Omega_{X/k}$ for an affine scheme $X = \operatorname{Spec} A$. To do this, I will need to introduce the *module of differentials* $\Omega_{A/k}$. It is an A-module equipped with a k-derivation $d : A \to \Omega_{A/k}$ (i.e. a k-linear map such that $d(fg) = fd(g) + gd(f)$ for $g, f \in A$) such that it is universal among those A-modules M with A-derivation $d_M : A \to M$. The *sheaf of differentials* $\Omega_{X/k}$ is then the sheaf of $\mathcal{O}_X$-modules corresponding to the module of differential $\Omega_{A/k}$. In particular, over open set $D(f)$ of X where $f \in A$, its sections are $\Gamma(D(f), \Omega_{X/k}) = (\Omega_{A/k})_f$.

We start with $\phi : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ a morphism of affine schemes over k. This gives us a ring map $B \to A$ and a morphism $\mathcal{O}_Y \to \phi_*\mathcal{O}_X$ of sheaves of rings over Y [2].

I claim that we have a morphism

$$\Phi : \phi^*\Omega_{Y/k} \to \Omega_{X/k}$$

of sheaves of $\mathcal{O}_X$-modules that resembles $T^*Y \to T^*X$ in differential geometry.

I will describe $\phi_*\Omega_{Y/k}$ first. It is a sheaf of $\mathcal{O}_X$-modules obtained by pulling back $\Omega_{Y/k}$ along $\phi : X \to Y$. Its sections over open U of X forms a $\mathcal{O}_X(U)$-module $\Omega_{Y/k}(\phi^{-1}(U)) \otimes_{\mathcal{O}_Y(\phi^{-1}(U))} \mathcal{O}_X(U)$. In other words, it is a sheaf of $\mathcal{O}_X$-modules corresponding to the A-module $\Omega_{B/k} \otimes_B A$.

Thus, to describe Φ, we just need to know a map $\Omega_{B/k} \otimes_B A \to \Omega_{A/k}$ of A-modules.

By composing the ring map $B \to A$ with d_A, we get a k-derivation map corresponding to the B-module $\Omega_{A/k}$, hence by universal property of $\Omega_{B/k}$, this induces a map of B-modules

$$\begin{aligned} \Omega_{B/k} &\to \Omega_{A/k} \\ f d_B(g) &\mapsto \phi(f) d_A(\phi(g)),\ f, g \in B. \end{aligned}$$

We then also have a morphism of A-modules

$$\Omega_{B/k} \otimes_B A \to \Omega_{A/k}.$$

3.2.2. *Global left-invariant top form of* GL_2. I want to determine a left-invariant global differential form of top degree for GL_2.

Firstly, the module of differential $\Omega_{\mathcal{O}(\mathrm{GL}_2)/k}$ is the $\mathcal{O}(\mathrm{GL}_2)$-module generated by dx_{ij}, dt for $1 \le i, j \le 2$, modulo the relation

$$0 = d(t(x_{11}x_{22} - x_{21}x_{12}) - 1) = td(x_{11}x_{22} - x_{21}x_{12}) + (x_{11}x_{22} - x_{12}x_{21})dt.$$

As in $\mathcal{O}(\mathrm{GL}_2)$, $(x_{11}x_{22} - x_{12}x_{21})t = 1$ so we can write

$$dt = -t^2 d(x_{11}x_{22} - x_{21}x_{12}). \tag{2}$$

Consider left-multiplication by $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}_2(k)$ via

$$\begin{aligned} L_a : \mathcal{O}(\mathrm{GL}_2) &\to \mathcal{O}(\mathrm{GL}_2), \\ x_{ij} &\mapsto a_{i1}x_{1j} + a_{i2}x_{2j}, \\ t &\mapsto (a_{11}a_{22} - a_{12}a_{21})^{-1}t. \end{aligned}$$

[2] At some point, I was confused on whether the arrow should be $\mathcal{O}_Y \to \phi_*\mathcal{O}_X$ or the other way around. To convince myself on this, just view $\mathcal{O}_Y$ as functions on Y and $\mathcal{O}_X$ as functions on X, and we know a function on Y induces a function on X by precomposing with ϕ.

This induces a morphism of $\mathcal{O}(\mathrm{GL}_2)$-modules

$$L_a : \bigwedge^2 \Omega_{\mathcal{O}(\mathrm{GL}_2)/k} \to \bigwedge_{i,j=1}^2 \Omega_{\mathcal{O}(\mathrm{GL}_2)/k},$$

$$f \bigwedge_{i,j=1}^2 dx_{ij} \mapsto L_a(f) \bigwedge_{i,j=1}^2 (a_{i1}dx_{1j} + a_{i2}dx_{2j}).$$

Note that we only need to specify what $\bigwedge_{ij} dx_{ij}$ is sent to because dt is determined from (2). I want to find $f \in \mathcal{O}(\mathrm{GL}_2)$ so $L_a(f \bigwedge_{i,j} dx_{ij}) = f \bigwedge_{ij} dx_{ij}$, or

$$L_a(f)\det(a_{ij})^2 dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} = f dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22}.$$

Thus, we want $f \in \mathcal{O}(\mathrm{GL}_2)$ such that $L_a(f)\det(a_{ij})^2 = f$ for all $a \in \mathrm{GL}_2(k)$. Note that $f = t^2$ satisfies this.

3.2.3. *Global left-invariant top form of* SL_2. Now, I want to focus on an example $\phi : \mathrm{SL}_2 \times \mathbb{G}_m \to \mathrm{GL}_2$, defined by

$$\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, t\right) \mapsto \begin{pmatrix} tx_{11} & tx_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

From this map and the corresponding left-invariant top form of GL_2, I want to obtain a left-invariant global top differential form of SL_2.

We first have an isomorphism $\phi' : \mathcal{O}(\mathrm{GL}_2) \to \mathcal{O}(\mathrm{SL}_2) \otimes \mathcal{O}(\mathbb{G}_m)$ of k-algebras given by

$$\begin{aligned} \phi' : k[x_{ij}, t]/(t(x_{11}x_{22} - x_{21}x_{22}) - 1) &\to k[x_{ij}]/(x_{11}x_{22} - x_{21}x_{12} - 1) \otimes_k k[t, t^{-1}], \\ x_{11} &\mapsto tx_{11}, \\ x_{12} &\mapsto tx_{12}, \\ x_{21} &\mapsto x_{21}, \\ x_{22} &\mapsto x_{22}, \\ t &\mapsto t^{-1}. \end{aligned}$$

The module of differentials $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is the $\mathcal{O}(\mathrm{SL}_2)$-module generated by dx_{ij} for $1 \le i, j \le 2$, modulo the relation

$$0 = d(x_{11}x_{22} - x_{12}x_{21} - 1) = x_{11}dx_{22} + x_{22}dx_{11} - x_{12}dx_{21} - x_{21}dx_{12}.$$

And $\Omega_{\mathcal{O}(\mathbb{G}_m)/k}$ is the $\mathcal{O}(\mathrm{GL}_1)$-module generated by dt with the k-derivation $dt^{-1} := -t^{-2}dt$.

The module of differentials $\Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k}$ is the $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)$-module generated by dx_{ij} for $1 \le i, j \le 2$ and dt modulo the relation $d(x_{11}x_{22} - x_{21}x_{12}) = 0$, with the differential $dt^{-1} := -t^{-2}dt$. In particular, one can show that

$$\Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k} \cong \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \otimes_{\mathcal{O}(\mathrm{SL}_2)} \mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m) \oplus \Omega_{\mathcal{O}(\mathbb{G}_m)/k} \otimes_{\mathcal{O}(\mathbb{G}_m)} \mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m),$$

which we will write $\Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k} \cong \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \oplus \Omega_{\mathcal{O}(\mathbb{G}_m)/k}$ for convenience. This induces an isomorphism of $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)$-modules

$$\bigwedge^4 \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k} \cong \Omega_{\mathcal{O}(\mathbb{G}_m)/k} \wedge \bigwedge^3 \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}.$$

From ϕ', we have a morphism of $\mathcal{O}(\mathrm{GL}_2)$-module

$$\Omega_{\mathcal{O}(\mathrm{GL}_2)/k} \to \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k},$$

inducing a morphism of $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)$-module

$$\Omega_{\mathcal{O}(\mathrm{GL}_2)/k} \otimes_{\mathcal{O}(\mathrm{GL}_2)/k} (\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)) \to \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k}.$$

This is the map obtained by taking global sections of

$$\Phi : \phi^*\Omega_{\mathrm{GL}_2/k} \to \Omega_{\mathrm{SL}_2 \times \mathbb{G}_m/k}.$$

With this, we have a morphism of top forms $\bigwedge^4 \phi^*\Omega_{\mathrm{GL}_2/k} \to \bigwedge^4 \Omega_{\mathrm{SL}_2 \times \mathbb{G}_m/k}$, whose global sections sends

$$\begin{aligned}\bigwedge_{i,j=1}^{2} dx_{ij} \mapsto d(tx_{11}) \wedge d(tx_{12}) \wedge dx_{21} \wedge dx_{22}, \\ &= (tdx_{11} + x_{11}dt) \wedge (tdx_{12} + x_{12}dt) \wedge dx_{21} \wedge dx_{22}, \\ &= t^2 dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} \\ &\quad + tx_{11}dt \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} \\ &\quad - tx_{12}dt \wedge dx_{11} \wedge dx_{21} \wedge dx_{22}.\end{aligned}$$

Note that $x_{11}x_{22}-x_{21}x_{12} = 1$ in the $\mathcal{O}(\mathrm{SL}_2)\otimes_k\mathcal{O}(\mathbb{G}_m)$-module $\bigwedge^4 \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k}$ and that $d(x_{11}x_{22}-x_{21}x_{12}) = 0$ so

$$dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} = (x_{11}x_{22} - x_{21}x_{12})dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} = 0.$$

In the end, we find

$$t^2 \bigwedge_{i,j=1}^{2} dx_{ij} \mapsto t^{-1}dt \wedge (x_{11}dx_{12} \wedge dx_{21} \wedge dx_{22} - x_{12}dx_{11} \wedge dx_{21} \wedge dx_{22}).$$

From previous section, $t^2 \bigwedge_{i,j} dx_{ij}$ is a global left-invariant top form of GL_2. We also know $t^{-1}dt$ is the global left-invariant top form of $\mathbb{G}_m$. This should imply that

$$\omega = x_{11}dx_{12} \wedge dx_{21} \wedge dx_{22} - x_{12}dx_{11} \wedge dx_{21} \wedge dx_{22}$$

is a global left-invariant top form of SL_2. .... Is this correct? If yes, what is the quickest way to check this?

3.3. **11/08/2021: $V(m\nu)$ in $V(m\lambda) \otimes V(m\mu)$.** I want to continue to solve the problem I had on 28/07/2021. I learnt this proof from Travis.

Lemma 5. *There is a unique copy $V(m\lambda)$ in $V(\lambda)^{\otimes m}$ that contains $v^{\otimes m}$ for every $v \in V(\lambda)$.*

Proof. We denote v_λ to be the highest weight vector of weight λ in $V(\lambda)$. Then $v_\lambda^{\otimes m}$ is a highest weight vector of weight $m\lambda$ in $V(\lambda)^{\otimes m}$ so $V(\lambda)^{\otimes m}$ contains a copy of $V(m\lambda)$.

There is only one copy of $V(m\lambda)$ in $V(\lambda)^{\otimes m}$ because $v_\lambda^{\otimes m}$ is the only vector of highest weight λ in $V(\lambda)^{\otimes m}$ (up to linear independence). Indeed, suppose $\sum_{i=1}^n c_i v_{i1} \otimes v_{i2} \otimes \cdots \otimes v_{im}$ is a highest weight vector of weight $m\lambda$ where $0 \neq c_i \in \mathbb{C}$ and v_{ik} is a weight vector of weight λ_{ik}, then from $h \cdot \sum_{i=1}^n c_i v_{i1} \otimes v_{i2} \otimes \cdots \otimes v_{im} = (m\lambda)(h) \sum_{i=1}^n c_i v_{i1} \otimes v_{i2} \otimes \cdots \otimes v_{im}$, we find $m\lambda = \sum_{k=1}^m \lambda_{ik}$ for every $1 \leq i \leq n$. Because $\lambda_{ik} \leq \lambda$ for every i, k so we find $\lambda_{ik} = \lambda$ for every i, k, as desired.

$V(m\lambda)$ in $V(\lambda)^{\otimes m}$ contains $v^{\otimes m}$ for every $v \in V(\lambda)$ because $U(\mathfrak{g}) \cdot v^{\otimes m}$ contains $v_\lambda^{\otimes m}$. Indeed, let $v = \sum_{i=1}^{\ell} a_i v_i$ where v_i is a weight vector of weight $\mu_i \leq \lambda$ ($\mu_i \neq \mu_j$ for $i \neq j$) and $a_i \in \mathbb{C} \setminus \{0\}$. Let $\lambda - \mu_j = \sum_{i=1}^k c_{ij}\alpha_i$ where α_i's are the simple roots, $c_{ij} \in \mathbb{Z}_{\geq 1}$.

We can choose a μ_1 among μ_j's such that $c_{11} = \max_j\{c_{1j}\}$; c_{21} is maximal among those c_{2j}'s of μ_j's satisfying $c_{1j} = c_{11}$; c_{31} is maximal among those c_{3j}'s of μ_j's satisfying $c_{1j} = c_{11}, c_{21} = c_{2j}$; ...

With this, by letting $e_i \in \mathfrak{n}$ that corresponds to the simple root α_i, we find $e_1^{c_{11}} \cdots e_k^{c_{k1}} v_j = 0$ for $j \neq 1$ and $0 \neq e_1^{c_{11}} \cdots e_k^{c_{k1}} v_1 \in \mathbb{C}v_\lambda$. Thus, when $g = e_1^{c_{11}} \cdots e_k^{c_{k1}}$ then $gv \in \mathbb{C}v_\lambda$, while $g^2 v = 0$. This follows $g^m v^{\otimes m} = m!(gv)^{\otimes m}$, as desired. $\square$

Lemma 6. *For $\mathfrak{g}$-modules V, W then $V \otimes W \cong W \otimes V$ by sending $v \otimes w \mapsto w \otimes v$.*

Proof. It suffices to check that the map $f : V \otimes W \to W \otimes V$ is a $\mathfrak{g}$-module homomorphism. Indeed, $f(g(v\otimes w)) = f((gv)\otimes w + v\otimes(gw)) = w\otimes(gv)+(gw)\otimes v = g(w\otimes v) = g\cdot f(v\otimes w)$. This argument is the same as saying that $U(\mathfrak{g})$ is cocommutative, i.e. comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, defined by $\Delta(g) = g \otimes 1 + 1 \otimes g$ for $g \in \mathfrak{g}$, commutes with $\tau : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, where τ is the permutation map. ☐

Back to the problem, we want to show that $V(m\lambda) \otimes V(m\mu)$ contains a copy of $V(m\nu)$. There exists a highest weight vector of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. By the construction of tensor product, we can write it as $v \otimes w$ where $v \in V(\lambda), w \in V(\mu)$.

We also know that $(V(\lambda) \otimes V(\mu))^{\otimes m} \cong V(\lambda)^{\otimes m} \otimes V(\mu)^{\otimes m}$ sends $(v \otimes w)^{\otimes m}$ to $v^{\otimes m} \otimes w^{\otimes m}$. We have $(v \otimes w)^{\otimes m}$ is of highest weight $m\nu$. From the previous lemma, we know $v^{\otimes m} \in V(m\lambda)$ and $w^{\otimes m} \in V(m\mu)$ so $U(\mathfrak{g}) \cdot v^{\otimes m} \otimes w^{\otimes m}$ is a submodule of $V(m\lambda) \otimes V(m\mu)$ in $V(\lambda)^{\otimes m} \otimes V(\mu)^{\otimes m}$. With this, we conclude $V(m\lambda) \otimes V(m\mu)$ contains a copy of $V(m\nu)$.

3.4. **12/08/2021: Tate vector spaces.** I read something interesting called Tate vector spaces. It refers to an infinite dimensional vector space V equipped with a set of lattices in V such that V is isomorphic to the inverse limit of V/L where L runs through lattices in V.

This notion generalises other spaces such as the Grassmannians, adeles, $k(\!(t)\!)$.

There are dimension theory (i.e. assign each lattice in V a number, called dimension) and determinant theory (i.e. assign each each lattice in V a line) for Tate vector space V generalise certain constructions in the above mentioned examples. For the determinant theory, one can use it to construct central extensions of $\mathrm{GL}(V)$ and somehow people want to do this ...

Need to read more at `http://page.mi.fu-berlin.de/groemich/chicago.pdf` *by Michael Groechenig and* `https://people.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Nov3-10(CentExt).pdf` *by Dustin Clausen.*

3.5. **13/08/2021: Inner product and Hom.** Madeline pointed out to the category theory reading group that in Etingof's book on representation theory (p. 189 of `http://www-math.mit.edu/~etingof/reprbook.pdf`), there is a myterious dictionary between a category and a vector space V equipped with a nondegenerate inner product. In particular, an inner product (x, y) in V is analogous to $\mathrm{Hom}(X, Y)$ in a category. I wonder if we can make this analogy more formal. I suspect it is some sort of (de)categorification, although I am not too sure about this.

Let me try to describe this analogy in the case our category is $\mathrm{Rep}(G)$, i.e. the category of finite-dimensional complex representations of a finite group G. Then the vector space is vector space V of class functions, i.e. complex-valued functions on G that is invariant under conjugation action of G. This the space where all characters of representations live. One can equipp V with a Hermitian inner product. For example, if χ_V, χ_W are characters of representations V, W of G, then $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}(V, W)$. In some sense, $\mathrm{Rep}(G)$ is the categorification of V, where orthonormal basis of V corresponds to irreducible representations.

This reminds me of a question I had on 06/06/2021 about the relation between adjoint operators and adjoint functors.

Nasos told us that John Baez have done something related to this dictionary in this paper of his: Higher-Dimensional Algebra II: 2-Hilbert Spaces `https://arxiv.org/pdf/q-alg/9609018.pdf`.

3.6. **14/08/2021: Principal G-bundles.** I want to digest the definition of principal G-bundles, as it seems to me that there are many ways to phrase this notion and different sources define principal bundles differently, depending on how nice the space of consideration is.

Update 30/09/2021: Brian Conrad wrote some notes with many examples on taking quotient by a group action `https://math.stanford.edu/~conrad/diffgeomPage/handouts/qtmanifold.pdf`.

In this discussion, every topological space is assumed to be Hausdorff.

We start off with one definition of principal G-bundles, taken from Cohen's lecture notes on The Topology of Fiber Bundles or Stephen A. Mitchell's notes `https://sites.math.washington.edu/~mitchell/Notes/prin.pdf`.

Definition 7. *For a topological group G, a principal G-bundle P over X is a (surjective) continuous map $\pi : P \to X$ satisfying the following conditions:*

(1) G acts on the right on P;
(2) Local trivialisation: There is an open cover $\{U\}$ of X such that for each such U, we have a homeomorphism $\pi_U : \pi^{-1}(U) \to U \times G$ that is G-equivariant satisfying

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim", "\pi_U"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

Here g acts on $U \times G$ by $(u, g)g' = (u, gg')$.

Firstly, π_U being G-equivariant tells us that G acts on fiber P_x. Indeed, for a local trivialisation (U, π_U) at x, if $p \in P_x$ so $\pi_U(p) = (x, g)$, we find $\pi_U(pg') = \pi_U(p)g' = (x, gg')$. Hence, $pg' \in P_x$. To say G acts on fibers is the same as saying $\pi : P \to X$ is G-equivariant with trivial G-action on X.

Furthermore, π_U being bijective implies G acts simply transitively on P_x. Indeed, choose a local trivialisation (U, π_U) of x, for any $y, z \in P_x$, let $\pi_U(y) = (x, g_y)$ and $\pi_U(z) = (x, g_z)$ then there exists a unique $g \in G$ so $g_y g = g_z$, implying $\pi_U(z) = \pi_U(y)g = \pi_U(yg)$. Hence, there exists unique $g \in G$ so $z = yg$.

On the other hand, π_U being a G-equivariant bijection implies that G acts freely (i.e. trivial stabiliser) on P. Indeed, if we have $p \in P$ and $g \in G$ so $pg = p$ then by choosing a local trivialisation (U, π_U) of $\pi(p)$, we find $(\pi(p), g') = \pi_U(p) = \pi_U(pg) = \pi_U(p)g = (\pi(p), g'g)$. Hence, $g' = g'g$ so $g = 1$.

Because G acts simply transitively on P_x, we know that for any $y \in P_x$, the map $G \to P_x$ defined by $g \mapsto yg$ is a continuous bijection [3]. However, we cannot say anything more than this. In particular, this does not imply that G is homeomorphic to P_x, i.e. P_x is a G-torsor, which is what we want when we define principal G-bundles [4]. Hence, we would like to add the following condition to the definition

Definition 8. *We add the following condition to our definition of principal G-bundle $\pi : P \to X$.*

(1) For every $y \in P_x$, the morphism $G \to P_x$ defined by $g \mapsto yg$ is a homeomorphism.

In the case where our spaces are smooth manifolds and G is a Lie group, this condition automatically holds, see p.5 `https://www.mathi.uni-heidelberg.de/~lee/MenelaosSS16.pdf`. The reason roughly is that the map $G \to P_x$ is of constant rank and bijective, hence a diffeomorphism.

[3] One can show $f_y : G \to P_x$ is continuous when assuming P_x (or P) is Hausdorff. Indeed, since G acts continuously on P_x via $\phi : P_x \times G \to P_x$, for any closed subset U of P_x, we know $\phi^{-1}(U) \cap (\{y\} \times G)$ is closed, implying $\phi_y : G \to P_x$, defined by $\phi_y(g) = yg$, is continuous

[4] For example, consider $G = \mathbb{R}$ with discrete topology acts on $P = \mathbb{R}$ with the usual topology, then G is not homeomorphic to $P_x = \mathbb{R}$.

An equivalent way to phrase this condition is that the map $\phi : P \times G \to P \times P$, defined by $(x, g) \mapsto (x, xg)$, is a homeomorphism. Indeed, for closed subset U in G then $\phi(\{x\} \times U)$ is closed. Hence, $\phi_x : g \mapsto xg$ is a closed map. We know ϕ_x is a continuous bijection so this implies ϕ_x is a homeomorphism.

An equivalent way to say $\pi : P \to X$ has local trivialisations is to say it has local sections.

If $\pi : P \to X$ admits local sections, i.e. for every x there is a continuous section $s : U \to P$ on some open neighborhood U of x, then this will imply local trivialisation condition. Indeed, we can define $\pi_U^{-1}(u, g) = s(u) \cdot g$. The map is bijective since G acts simply transitively on fibers. Indeed, if $s(u')g = s(u)g'$ then $u = u'$ and for any $p \in \pi^{-1}(U)$, there exists a unique $g \in G$ so $s(\pi(p)) \cdot g = p$. The map is continuous as it is the composition of $U \times G \xrightarrow{s \times \mathrm{id}_G} P \times G \to P$. *I cannot seem to show that this map is a homeomorphism, i.e. it is open/closed with the current assumption on X, P?* It seems for this to be true, s needs to be homeomorphic onto its image, then π_U^{-1} is an open map. For this to work, I can assume extra condition that X is locally compact Hausdorff, which means we can assume U is compact (or restrict U to a compact neighborhood). Then we have $s : U \to s(U)$ is a bijective continuous map from compact U to Hausdorff $s(U)$, implying s is a homeomorphism onto its image. Thus, π_U is a local trivialisation.

Conversely, if we are given a local trivialisation (U, π_U) of a principal G-bundle $\pi : P \to X$, we can define a local section $s : U \to P$ by $s(u) := \pi_U^{-1}(u, 1)$. To check s is a continuous map, given a closed set $V \subset \pi^{-1}(U)$ of $p \in \pi^{-1}(U)$, $V \cap P_{\pi(p)}$ is also a closed set of p, we have $s^{-1}(V \cap P_{\pi(p)})$ is either $\{\pi(p)\}$ or $\varnothing$, hence is closed.

We know if $\pi : P \to X$ is a principal G-bundle then G acts freely on P. In the other direction, if we are given a space P with a free G-action on the right of P, it is not enough to say $\pi : P \to P/G$ is a principal G-bundle. For example, let $\mathbb{R}$ acts on the torus $\mathbb{R}^2/\mathbb{Z}^2$ by translating $(1/2, a)$ where a is an irrational number. Even though the action is free, the orbits are dense and the quotient space is not even Hausdorff. One needs to add extra condition, for example, π having local sections. Some references would call P a (right) *free* G-space if $\pi : P \to P/G$ is a principal G-bundle.

Given a principal G-bundle $\pi : P \to X$ then by the universal property of quotient spaces, this induces a continuous bijective map $\phi : P/G \to X$, defined by $[p] \in P/G \mapsto \pi(p)$. In our setup, I don't think we can show ϕ is a homeomorphism. However, if we assume our spaces are smooth manifolds, G is a Lie group, then ϕ is a diffeomorphism (see for example `https://www.mathi.uni-heidelberg.de/~lee/MenelaosSS16.pdf`).

Some more references `https://web.ma.utexas.edu/users/dafr/M392C-2017/Notes/lecture13.pdf`, `https://ncatlab.org/nlab/show/principal+bundle`

Principal G-bundles in topological setting is discussed at Tammo tom Dieck's book Algebraic Topology

3.7. **14/08/2021: Gufang Zhao's first lecture: Bundles as double quotient space.**

Just want to take some notes on Gufang Zhao lectures. The summary of his lecture series is as follows: One can construct certain bundles on Riemann surfaces (bundles of conformal blocks on the configuration space of points of the Riemann surface), equipped it with a flat connection, called Knizhnik-Zamolodchikov (KZ) connection. The solution of of the KZ equations can be obtained by counting algebraic curves.

His first lecture is about constructing such bundles.

3.7.1. *Representations from sections of equivariant vector bundles.*

Given a space X with a free (right) G-action (i.e. so that $\pi : X \to X/G$ is a principal G-bundle) and a representation $\rho : G \to \mathrm{GL}(V)$ of G, we can construct a vector bundle $X \times_G V \to X/G$ by letting $X \times_G V := (X \times V)/\sim$

where the equivalence relation is $(xg, v) \sim (x, \rho(g^{-1})v)$ [5]. It has a natural projection $p : X \times_G V \to X/G$, making it a vector bundle over X/G with fiber isomorphic to V. We can describe its global sections $H^0(X/G, X \times_G V)$ as the space of G-equivariant functions $s : X \to V$, i.e. $s(xg) = g \cdot s(x)$ for all $g \in G$. Indeed, given such s, we can define $\overline{s} : X/G \to X \times_G V$ by $\overline{s}(\overline{x}) = (x, s(x))$ where $\pi(x) = \overline{x}$. This is well-defined because $(xg^{-1}, s(xg^{-1})) \sim (x, g \cdot s(xg^{-1})) \sim (x, s(x))$.

If we also have an action of a group H on X that commutes with the G-action, we can turn $X \times_G V \to X/G$ into a H-equivariant vector bundle by letting H act on $X \times_G V$ by $h(x, v) := (hx, v)$ and on X/G by $h(xG) = hxG$. This is well-defined because the action of H on X commutes with action of G on X. This induces a representation of H on the space of global sections $H^0(X/G, X \times_G V)$. Indeed, let $\overline{s} : X/G \to X \times_G V$ be in $H^0(X/G, X \times_G V)$ then $(h \cdot \overline{s})(x) := h \cdot s(h^{-1}x)$. As $\overline{s}$ corresponds to a G-equivariant $s : X \to V$, we have $(h \cdot s)(x) = s(h^{-1}x)$.

Example 9. Consider $X = \mathbb{C}^2 \setminus \{0\}$ and $G = \mathbb{C}^\times$ acts on X by $t(x, y) = (tx, ty)$, then X/G is $\mathbb{P}^1(\mathbb{C})$.

For every $d \in \mathbb{Z}$, we can construct a representation $\mathbb{C}(d)$ of $G = \mathbb{C}^\times$ on $V = \mathbb{C}$ by $t \cdot x = t^d x$.

Thus, this gives us a vector bundle $\mathcal{O}(d) := \mathbb{C}^2 \setminus \{0\} \times_{\mathbb{C}^\times} \mathbb{C}(d)$ over $\mathbb{P}^1$. Its global sections can be described as

$$H^0(\mathbb{P}^1, \mathcal{O}(d)) = \{f : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C} | f(tx) = t^d f(x)\ \forall t \in \mathbb{C}^\times\}.$$

When we only care about holomorphic/algebraic sections then

$$H^0(\mathbb{P}^1, \mathcal{O}(d)) = \{f \in \mathbb{C}[x, y] : f(tx, ty) = t^d f(x, y)\ \forall t \in \mathbb{C}^\times\}.$$

This is the space of homogeneous polynomials of degree d over two variables x, y.

Let $H = \mathrm{SL}_2(\mathbb{C})$ acts on $X = \mathbb{C}^\times \setminus \{0\}$ by left-mutiplication. Then $\mathrm{SL}_2(\mathbb{C})$ acts on $H^0(\mathbb{P}^1, \mathcal{O}(d))$ by $(a \cdot f)(x, y) = f(a^{-1}(x, y))$.

3.8. **15/08/2021: Tamagawa number for SL_n over $\mathbb{Q}$.** I would like to sketch the computation that the Tamagawa number of SL_n over $\mathbb{Q}$ is 1, i.e. $\mu_{\mathrm{SL}_n, \mathbb{Q}}(\mathrm{SL}_n(\mathbb{Q}) \setminus \mathrm{SL}_n(\mathbb{A})) = 1$. I learnt this from Garrett's notes `https://www-users.cse.umn.edu/~garrett/m/v/volumes.pdf` and Andre Weil's book Adeles and Algebraic Groups (p. 47, §3.4). From these sources, I know that Siegel came up with this proof for $\mathrm{SL}_n(\mathbb{Z}) \setminus \mathrm{SL}_n(\mathbb{R})$, which was then adapted by Weil to prove it for $\mathrm{SL}_n(\mathbb{Q}) \setminus \mathrm{SL}_n(\mathbb{A})$.

At the moment, to focus on describing the main idea of the proof, I will avoid analytic issues such as convergence of integrals, measure-preserving homeomorphisms and normalisations of measures. This is also because I haven't managed to figure out all these technical details.

For $G_n = \mathrm{SL}_n$, we are able to find a copy of SL_{n-1} in SL_n via the action of SL_n on k^n. This then allows us to use induction on n. More precisely, let $G_n(k)$ acts on k^n by left-multiplication. Let $G'_n(k)$ be the stabiliser of $e_1 = (1, 0, 0, \dots, 0)^t \in k^n$ in $G(k)$. We then have the following identification of spaces

(1) $G'_n(k) \setminus G_n(k)$ is continuously bijective to $G_n(k)e_1$ for $k = \mathbb{Q}_v$ or $\mathbb{A}$. When k is a division algebra, $G_n(k)e_1 = k^n \setminus \{0\}$ as one can write out explicitly $M \in G_n(k)$ such that $Me_1 = y$ for any $y \in k^n \setminus \{0\}$. Note that $\mathbb{A}$ is not a division algebra, but if we only care about integrating over this quotient space, then it is good enough to know that the set $\mathbb{A}^n - G_n(\mathbb{A})e_1$ has measure 0.
(2) G'_n is the semidirect product of G_{n-1} with $\mathbb{G}_a^{n-1}$. In particular, elements in $G'_n(k)$ can be described as
$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$
where $u \in k^{n-1}$, $x \in G_{n-1}(k)$.

[5]Some references define this equivalence relation as $(xg, v) \sim (x, \rho(g), v)$. I don't think it matters which one we choose, but we need to modify this discussion accordingly

We start with the following formula that describe integration over $G'_n(\mathbb{Q}) \setminus G_n(\mathbb{A})$ in two ways

$$\int_{x\in G_n(\mathbb{Q})\setminus G_n(\mathbb{A})} \int_{y\in G'_n(\mathbb{Q})\setminus G_n(\mathbb{Q})} f(xy)dxdy = \int_{z\in G'_n(\mathbb{A})\setminus G_n(\mathbb{A})} \int_{t\in G'_n(\mathbb{Q})\setminus G'_n(\mathbb{A})} f(zt)dzdt. \tag{3}$$

As $G'_n = \mathbb{G}_a^{n-1} \ltimes G_{n-1}$, by inductive hypothesis on G_n, we find the Tamagawa number of G'_n (i.e. the volume of $G'_n(\mathbb{Q}) \setminus G'_n(\mathbb{A})$) is

$$\tau(G'_n) = \tau(\mathbb{G}_a^{n-1})\tau(G_{n-1}) = 1.$$

Therefore, if we choose f to be a function on $G'_n(\mathbb{A}) \setminus G_n(\mathbb{A})$, i.e. trivial on $G'_n(\mathbb{A})$, and from the identification of $G'_n(k) \setminus G_n(k)$, we can rewrite (4) as

$$\int_{x\in G_n(\mathbb{Q})\setminus G_n(\mathbb{A})} \sum_{y\in\mathbb{Q}^n\setminus\{0\}} f(yx)dxdy = \int_{\mathbb{A}^n} f(z)dz. \tag{4}$$

The Fourier transform of $f : \mathbb{A}^n \to \mathbb{C}$ is

$$\widehat{f}(y) = \int_{\mathbb{A}^n} f(x)\chi_{\mathbb{A}}(y^t \cdot x)dx,$$

where $\chi_{\mathbb{A}}$ is the standard unitary character on $\mathbb{A}$. Applying (4) for $\widehat{f}$, we find

$$\int_{x\in G_n(\mathbb{Q})\setminus G_n(\mathbb{A})} \sum_{y\in\mathbb{Q}^n\setminus\{0\}} \widehat{f}(yx)dx = \int_{\mathbb{A}^n} f(z)dz.$$

On the other hand, noting that for $x \in G_n(\mathbb{A})$, as $\det x = 1$, we find $f(yx) = \widehat{f}(y(x^t)^{-1})$ for $y \in \mathbb{A}^n$. Thus, combining the above two equations, we find

$$\int_{\mathbb{A}^n} (f(x) - \widehat{f}(x))dx = \int_{G_n(\mathbb{Q})\setminus G_n(\mathbb{A})} (\widehat{f}(0) - f(0))dx.$$

By Fourier inversion formula, the left-hand side is $\widehat{f}(0) - f(0)$. One can choose f such that $f(0) \neq \widehat{f}(0)$, giving $G_n(\mathbb{Q}) \setminus G_n(\mathbb{A})$ volume 1, as desired.

Remark 10. What I found surprising about this proof is the following:

(1) It works (almost) verbatim if we work with $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ instead. An induction argument can be used to show the answer is $\zeta(1)\zeta(2)\cdots\zeta(n)$.
(2) Comparing this proof with the one given by describing a fundamental domain for $\mathrm{SL}_n(\mathbb{Q}) \setminus \mathrm{SL}_n(\mathbb{A})$, for the later one, we need to compute the volume of $\mathrm{SL}_n(\mathbb{Z}_p)$, but this is nowhere to be seen for this proof. In some sense, the appearance of $\mathbb{Q}_p$ is completely suppressed.
(3) It seems to me that because this is an inductive argument, one does not have to make an explicit choice of a left-invariant top form for G. This proof also works for any global field I believe.
(4) The proof works mainly because for $G_n = \mathrm{SL}_n$, one can find a nice subgroup $G'_n = \mathbb{G}_a^{n-1} \ltimes G_{n-1}$. I wonder if this phenomenon holds for other groups. For example, it is mentioned in the references that the inductive strategy of this proof works for Sp_{2n}, (any more?)

Update 22/09/2021: The group G_n is called the mirabolic subgroup of SL_n (mirabolic = miraculous parabolic). It is important in establishing "multiplicity-one" results in automorphic representation theory, (see `https://mathoverflow.net/q/196006/89665`).

3.9. **22/08/2021: Tamagawa number of** Sp_{2n}**.** Let $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ be a $2n$-by-$2n$ matrix. The symplectic group Sp_{2n} is defined as

$$\mathrm{Sp}_{2n}(k) = \{M \in M_{2n\times 2n}(k) : M^t J_n M = J_n\}.$$

In this section, we show that the Tamagawa number of the symplectic group Sp_{2n} is 1. We prove this by induction on n. For $n = 1$ then $\mathrm{Sp}_2 = \mathrm{SL}_2$ so $\tau(\mathrm{Sp}_2) = \tau(\mathrm{SL}_2) = 1$.

Let $G_n = \mathrm{Sp}_{2n}$ acts on k^{2n} by left-multiplication. Let $G_n'(k)$ be the stabiliser of $e = (1, 0, \dots, 0)^t \in k^{2n}$. Elements in $G_n'(k)$ can be written as

$$\begin{pmatrix} 1 & * & * & * \\ 0 & a & * & b \\ 0 & 0 & 1 & 0 \\ 0 & c & * & d \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{n-1}(k)$ and other entries are suitably chosen.

Determine elements in G_n'. To determine other entries, let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D \in M_{n\times n}$, then such matrix lies in Sp_{2n} iff $A^tC - C^tA = B^tD - D^tB = 0$ and $A^tD - C^tB = I_n$. For X to stabilises e means $A = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ and $C = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$. Hence, we find $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x & x_2 \\ 0 & a & y_1 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & y_2 & d \end{pmatrix}$ where $(x_1, x_2) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^t \begin{pmatrix} 0 & I_{n-1} \\ -I_{n-1} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n-2}$. □

The set g of elements in $G_n'(k)$ so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_{(2n-2)\times(2n-2)}$ is a normal subgroup of $G_n'(k)$. The set of elements in g so $(x_1, x_2) = 0$ is is a normal subgroup of g. Thus, we find that G_n' is isomorphic to the semidirect product $(\mathbb{G}_a \ltimes \mathbb{G}_a^{2n-2}) \ltimes G_{n-1}$. Thus, by inductive hypothesis, the Tamagawa number of G_n' is $\tau(G_n') = \tau(G_{n-1})\tau(\mathbb{G}_a)^{2n-1} = 1$.

We can identify $G_n'(k) \setminus G_n(k)$ with $G_n'(k)e$. When k is a division algebra, $G_n'(k)e = k^{2n} \setminus \{0\}$. When $k = \mathbb{A}$ then $\mathbb{A}^{2n} \setminus G_n'(\mathbb{A})e$ has measure 0. Proceeding exactly as in SL_n case, we obtain $\tau(\mathrm{Sp}_{2n}) = 1$.

3.10. **22/08/2021: Tamagawa measure and restriction of scalars.** I will discuss on how to define a measure on the adelic points of a variety. After this, we will focus on the case for linear algebraic groups and define the Tamagawa measure on the adelic points of such groups.

The references we follow are: Weil's Adeles and Algebraic Groups, first chapter of Gaitsgory and Lurie's book Weils Conjecture for Function Fields I.

3.10.1. *Tamagawa measure on smooth schemes.* Let X be a separated [6] smooth (affine) scheme of finite type over k. A *volume form* on X is a nowhere-vanishing algebraic differential form of top degree on X. Suppose X has a volume form ω (in other words, the canonical line bundle on X has a nonzero global section).

Over a complete valued field k, $X(k)$ has a canonical structure of a k-analytic manifold. When k is a local field equipped with a Haar measure, from the volume form of X, one can define a measure on $X(k)$.

[6] I don't know that well of this separatedness condition for schemes but it will guarantee that over complete valued field k, $X(k)$ is Hausdorff

Let k be now a global field. Let k_v to be the completion of k with respect to a place v of k. Let $\mathcal{O}_{k_v}$ to be the ring of integers of k_v. Let μ_{k_v} to be standard Haar measure on k_v (i.e. when k_v is a nonarchimedean local field, we normalise μ_{k_v} so $\mu_{k_v}(\mathcal{O}_v) = 1$; when $k_v = \mathbb{R}$ then it is the Lebesgue measure; when $k_v = \mathbb{C}$ then μ_{k_v} is twice the Lebesgue measure).

From a volume form ω on X, we can define a measure ω_v on $X(k_v)$, where v is a place of k and k_v is the completion of k with respect to v. Denote $\mathcal{O}_{k_v}$ to be the ring of integers of k_v.

Let S be a nonempty finite set of places of k that contains every archimedean places of k and let $\mathcal{O}_S := \{x \in k : x \in \mathcal{O}_{k_v}\ \forall v \notin S\}$. Suppose there exists a smooth scheme $\overline{X}$ over $\mathcal{O}_S$ such that $\overline{X}_k = X$ (here $\overline{X}_k$ is an affine scheme over k such that its coordinate ring is $\mathcal{O}(\overline{X}_k) = \mathcal{O}(\overline{X}) \otimes_{\mathcal{O}_S} k$) [7], and suppose that $\overline{X}$ has a volume form $\overline{\omega}$. Then this induces a volume form $\overline{\omega}|_X$ for X. As $\mathcal{O}_S \hookrightarrow \mathcal{O}_v$ or Spec $\mathcal{O}_v \to$ Spec $\mathcal{O}_S$ [8], we can make sense of $\mu_{\omega,v}(\overline{X}(\mathcal{O}_v))$ where $\mu_{\omega,v}$ is the induced measure from ω on $X(k_v) \supset \overline{X}(\mathcal{O}_v)$. We can also regard

$$\prod_{v \notin S} \overline{X}(\mathcal{O}_v) \times \prod_{v \in S} X(k_v)$$

as an open subgroup of $X(\mathbb{A}_k)$.

If the product

$$\prod_{v \notin S} \mu_{\omega,v} \overline{X}(\mathcal{O}_v)$$

converges absolutely to a nonzero real number, we say X *admits a Tamagawa measure.*

If X admits a Tamagawa measure, a Tamagawa measure μ_ω on $X(\mathbb{A}_k)$ with respect to ω is such that over open sets $\prod_{x \notin S} \overline{X}(\mathcal{O}_v) \times \prod_{v \in S} X(k_v)$ of $X(\mathbb{A}_k)$, μ_ω is the product measure

$$\tau(\mathbb{G}_a)^{-\dim X} \prod_v \mu_{\omega,v}.$$

Here $\tau(\mathbb{G}_a)$ is the Tamagawa number for $\mathbb{A}$ (i.e. we define the measure $\mu_{\mathbb{A}}$ on $\mathbb{A}$ to be the one such that over open sets $\prod_{v \notin S} \mathcal{O}_v \times \prod_{v \in S} k_v$, it is the product measure. We then let $\tau(\mathbb{G}_a) = \int_{\mathbb{Q} \setminus \mathbb{A}} \mu_{\mathbb{A}}$, which is well-defined because $\mathbb{Q} \setminus \mathbb{A}$ is compact).

This definition of μ_ω does not depend on the choice of a set S of places of k or the choice of an integral model $\overline{X}$ for X (because two choice of integral models become isomorphic after enlarging S).

The reason why we add a factor $\tau(\mathbb{G}_a)^{-\dim X}$ is because of the following result [9]

Theorem 11. *Let K/k be a finite and separable extension. Let V be a separated smooth scheme of finite type over K. Let W to be the Weil's restriction of scalars of V with respect to K/k, i.e. W is a scheme over k such that $W(R) := V(R \otimes_k K)$ where R is a k-algebra. Under the restriction of scalars, there is a canonical isomorphism $W(\mathbb{A}_k) \cong V(\mathbb{A}_K)$. We then have*

(1) W admits a Tamagawa measure iff V admits a Tamagawa measure.
(2) If the previous condition holds, the canonical isomorphism $W(\mathbb{A}_k) \cong V(\mathbb{A}_K)$ is a measure preserving map.

What to do next:

[7] One says $\overline{X}$ is a *model* for X over $\mathcal{O}_S$. So the point of a model is as follows: for a linear algebraic group G over $\mathbb{Q}$. We would like to describe $G(\mathbb{Z})$ without having to embedd G to GL_n and write $G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$, which is less functorial

[8] For example, when $k = \mathbb{Q}$, $S = \{\infty, 2, 3\}$ then $\mathcal{O}_S = \mathbb{Z}[1/2, 1/3]$ and for $v = 5 \notin S$, as $1/2$ and $1/3$ are invertible in $\mathbb{Z}_5$, we have a map $\mathbb{Z}[1/2, 1/3] \to \mathbb{Z}_5$.

[9] Another way for people to define Tamagawa measure is to choose Haar measures on local fields k_v such that $\tau(\mathbb{G}_a) = 1$. A natural choice of Haar measures on k_v would be the one that is self-dual with respect to its Pontryagin dual $\widehat{k_v}$.

(1) Do some more examples over number fields and function fields.
(2) Examples with restriction of scalars.

3.10.2. *Tamagawa measure for semisimple algebraic groups.* Let G be a connected semisimple linear algebraic group over a global field k. The above discussion of Tamagawa measure applies for ω being a left-invariant volume form for G. In particular, G does admit a Tamagawa measure (in this case, Tamagawa measure for G is a Haar measure) and furthermore, such measure does not depend on the choice of ω.

3.11. **27/08/2021: Affine Grassmannian.** Today I attend a WiSe talk about affine Grassmannian by Alex Weeks `https://sites.google.com/view/mathwise` and I just want to write down something I learnt from this talk.

We start by defining the ring of formal power series $\mathcal{O} = \mathbb{C}[[z]] = \{a_0 + a_1 z + a_2 z^2 + \dots | a_i \in \mathbb{C}\}$ and its fraction field, the field of formal Laurent series $K = \mathbb{C}((z)) = \{a_n z^n + a_{n+1} z^{n+1} + \dots | n \in \mathbb{Z}, a_i \in \mathbb{C}\}$.

An $\mathcal{O}$-lattice $L \subset K^n$ is a finitely-generated $\mathcal{O}$-module [10] such that $K \otimes_{\mathcal{O}} L \cong K^n$ As a set, the affine Grassmannian of GL_n is defined to be

$$\mathrm{Gr}_{\mathrm{GL}_n} = \{L \subset K^n : L \text{ is an } \mathcal{O}-\text{lattice}\}.$$

We will later see that $\mathrm{Gr}_{\mathrm{GL}_n}$ has more structure.

For example, let $n = 1$, then $\mathrm{Gr}_{\mathrm{GL}_1} \cong \mathbb{Z}$, where $n \in \mathbb{Z} \mapsto z^n \mathcal{O}$. The set $\{z^n \mathcal{O} : n \in \mathbb{Z}\}$ are all the $\mathcal{O}$-lattices in K because if L is an $\mathcal{O}$-lattice, there exists an element of lowest degree $a_n z^n + a_{n+1} + \dots$ in L, where $a_n \neq 0$. This follows $z^n \in L$ and hence $L = z^n \mathcal{O}$.

We can contruct $\mathcal{O}$-lattice L_λ, where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, in K^n, by $L_\lambda = z^{\lambda_1} \mathcal{O} e_1 \oplus \dots \oplus z^{\lambda_n} \mathcal{O} e_n$. Here $e_i = (0, \dots, 1, 0 \dots, 0)$.

The ind-scheme structure of $\mathrm{Gr}_{\mathrm{GL}_n}$ comes from the following observation: For any $L_1, L_2 \in \mathrm{Gr}_{\mathrm{GL}_n}$, we can find $a, b \in \mathbb{Z}_{\geq 0}$ such that $z^a L_1 \subset L_2 \subset z^{-b} L_1$. To see this, we can choose $\mathcal{O}$-basis $\{v_1, \dots, v_n\}$ for L_2 and $\mathcal{O}$-basis $\{w_1, \dots, w_n\}$ for L_1. We then can write $w_i = \sum a_{ij} w_j$ with $a_{ij} \in K$. We can choose sufficiently large $n \in \mathbb{Z}$ such that $z^n w_i \in L_2$ for every $1 \leq i \leq n$. This implies $z^n L_1 \subset L_2$. For the case $n = 1$, the following observation just says that for any integer n, there exists $a, b \in \mathbb{Z}$ so $a \leq n \leq b$ (which trivially holds ...).

From this observation, given $z^a \mathcal{O}^n \subset L \subset z^{-b} \mathcal{O}^n$, we find $L / z^a \mathcal{O}^n \subset z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$. Furthermore, $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$ is a finite dimensional complex vector space of dimension $(a+b)n$. Indeed, it has basis $z^{-b} e_i, z^{-b+1} e_i, \dots, z^{a-1} e_i$ over all i. If we write $d = \dim_{\mathbb{C}} (L / z^a \mathcal{O}^n)$, we find $L / z^a \mathcal{O}^n \in \mathrm{Gr}(d, z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n)$, our usual Grassmannian of d-dimensional subspaces in $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$.

Now, we wonder if a d-dimensional subspace $U \in \mathrm{Gr}(d, z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n)$ can be used to build an $\mathcal{O}$-lattice in K^n?

To see this, we first can define an operator T on $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$ by multiplying by z. This operator T is nilpotent with Jordan type $(a+b, \dots, a+b)$. Indeed, $T^{a+b}(z^j e_i) = 0$ where $1 \leq i \leq n, -b \leq j \leq a$ and $T(z^j e_i) = z^{j+1} e_i$ for $-b \leq j \leq a-2$.

If L is an $\mathcal{O}$-lattice where $z^a \mathcal{O}^n \subset L \subset z^{-b} \mathcal{O}^n$ then $L / z^a \mathcal{O}^n$ is a T-invariant subspace of $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$. Back to our question: The converse also holds, i.e. $U \in \mathrm{Gr}(d, z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n)$ defines an $\mathcal{O}$-lattice in K^n if it is T-invariant.

Indeed, let L to be the kernel of

$$L := \ker \left(z^b \mathcal{O}^n \to z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n \to \left(z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n \right) / U \right)$$

[10] Zhu's note on affine Grassmannian put finitely generated *projective* $\mathcal{O}$-module, which is a free module as any finitely generated projective module over a PID (e.g. $\mathcal{O}$) is free. This then means we can choose an $\mathcal{O}$-basis $e_1, \dots, e_n$ for L, i.e. $L = \mathcal{O} e_1 \oplus \dots \oplus \mathcal{O} e_n$.

We claim that L is a $\mathcal{O}$-lattice in K^n. Firstly, L is an $\mathcal{O}$-module because U is T-invariant. It is finitely generated because it is an $\mathcal{O}$-submodule of finitely generated module $z^b\mathcal{O}^n$ ($\mathcal{O} = \mathbb{C}[[z]]$ is Noetherian as $\mathbb{C}$ is, and any submodule of finitely genenerated module over a Noetherian ring is also finitely generated). L contains $z^a\mathcal{O}^n$ so $L \otimes_{\mathcal{O}} K = K^n$, as desired.

In summary, we have a bijection from d-dimensional $\mathcal{O}$-lattices $z^a\mathcal{O}^n \subset L \subset z^{-b}\mathcal{O}^n$ to d-dimensional subspaces of $z^{-b}\mathcal{O}^n/z^a\mathcal{O}^n$.

Thus, the task of finding an $\mathcal{O}$-lattice L (of dimension d) between $z^a\mathcal{O}^n$ and $z^{-b}\mathcal{O}^n$ boils down to find a d-dimensional subspace of $z^{-b}\mathcal{O}^n/z^a\mathcal{O}^n$.

Example 12. When $n = 2, a = 2, b = 0$ then $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ by identifying basis ze_1, e_1, ze_2, e_2 of $z^2\mathcal{O}^2/\mathcal{O}^2$ with (e_1, e_2, e_3, e_4) of $\mathbb{C}^4$. To find $\mathcal{O}$-lattices $z^2\mathcal{O}^2 \subset L \subset \mathcal{O}^2$ of dimension 2, we want to find T-invariant 2-dimensional subspaces of $\mathbb{C}^4$. Note that $\ker T$ is 2-dimensional, so it is one candidate. If T-invariant subspace U is not $\ker T$, then there exists $v \in U$ so $T(v) \neq 0$. Then $U = \operatorname{span}_{\mathbb{C}}(v, Tv)$ because T has only eigenvalue 0 with eigenvector Tv, but $v \notin \ker T$ so v and Tv are linearly independent.

These are all the 2-dimensional subspaces of $\mathbb{C}^4$ that are T-invariant.

Our problem of finding 2-dimensional T-invariant subspaces is related to the nilpotent cone. Let Nil_2 to be the set of all 2×2 nilpotent matrices over $\mathbb{C}$. Equivalently, we can write

$$\mathrm{Nil}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) : a^2 + bc = 0 \right\}.$$

It is called a "cone" because under change of variables $b = x - y, c = x + y$ then the equation becomes $a^2 + x^2 = y^2$, which is a cone over the real points.

We can define a map

$$\begin{aligned} \mathrm{Nil}_2 &\to \mathrm{Gr}_{\mathrm{GL}_2} \\ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} &\mapsto L := \mathcal{O}(z + a, c) \oplus \mathcal{O}(b, -a + z). \end{aligned}$$

We notice $z^2\mathcal{O}^2 \subset L \subset \mathcal{O}^2$. L is a free $\mathcal{O}$-module because $a^2 + bc = 0$. To see $\dim_{\mathbb{C}}(L/z^2\mathcal{O}^2) = 2$, by identifying $z^2\mathcal{O}^2/\mathcal{O}^2 \cong \mathbb{C}^4$ via $(ze_1, e_1, ze_2, e_2) \mapsto (e_1, e_2, e_3, e_4)$, we can write

$$L/z^2\mathcal{O}^2 = \mathbb{C}(1, a, 0, c) \oplus \mathbb{C}(0, b, 1, -a).$$

This vector space is T-invariant because $a^2 + bc = 0$.

3.12. **Some unanswered questions.**

(1) Masoud asked what happen if we do not choose the standard character in identifying k with $\widehat{k}$ where k is a local field, what would be the dual Haar measure on k and $\widehat{k}$ in the Fourier inversion formula.
(2) What happen to Haar measure on local field under field extension. For example $\mathbb{C}/\mathbb{R}$?
(3) Learn Borel-Weil-Bott construction of irreducible representations. See book Complex geometry and Representation theory. See notes by Joel Kammitzer (already print out).
(4) Why the non-split maximal torus of $\mathrm{GL}_2(\mathbb{F}_q)$ is $\left\{\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}\right\}$ where $x, y, \varepsilon \in \mathbb{F}_q$ and ϵ is non-square.
(5) Learn about moduli (very short note by Ivan Mirkovic algebraic geometry course).
(6) Why adeles can be used to compute cohomology? See 12/08/2021 and `https://webusers.imj-prg.fr/~matthew.morrow/Morrow,%20M.,%20Introduction%20to%20HLF.pdf`.

4. September 2021

4.1. **01/09/2021: Algebraic function fields.** I want to take some notes on ways to describe algebraic function fields. I learned this from Salvador's book Topics in the Theory of Alebraic Function Fields.

A global function field k is a field of characteristic p that is a finitely generated extension of transcendental degree 1 over $\mathbb{F}_p$.

Some digestion on the definition of transcendance: Let L/K be a field extension. A *transcendental basis* of L over K is a maximal subset of L algebraically independent over K (i.e. for any n elements $s_1, \ldots, s_n$ in the basis, there does not exist $f \in K[x_1, \ldots, x_n]$ so that $f(s_1, \ldots, s_n) = 0$). One can show that S is a transcendental basis of L over K iff $L/K(S)$ is an algebraic extension [11], and the cardinality of S is called the *transcendental degree of* L *over* K. For example, the field $L = K(X)[Y]/(X^2 - Y - 1)$ has transcendental basis $\{x\}$ or $\{y\}$, where $x = X \mod (X^2 - Y - 1)$, as there does not exist $f \in K[x]$ so $f(x) = 0$ in L. Furthermore, $L/K(x)$ is algebraic as $y^2 = x + 1$.

Back to our global function field $k/\mathbb{F}_p$. If $T \in k$ is transcendental over $\mathbb{F}_p$ then $k/\mathbb{F}_p(T)$ is a finite (algebraic) extension [12]. If we choose a different transcendental basis T' for $k/\mathbb{F}_p$ then $[k : \mathbb{F}_p(T)]$ does not necessarily equal to $[k : \mathbb{F}_p(T')]$. This is one of the principal differences with number fields, i.e. there is no natural choice of a base field. For example, take $L = \mathbb{Q}(x)[z]/(x^2 + z^4 - 1)$ then $[L : \mathbb{Q}(x)] = 4, [L : \mathbb{Q}(z)] = 2, [L : \mathbb{Q}(x^2)] = 8$.

4.2. **03/09/2021: Tamagawa number of orthogonal groups.** In this section, we will show that the Tamagawa number of orthogonal group is 2. We follow Igusa's chapter 4 of Lectures on Forms of Higher Degrees and Hida's notes on Siegel-Weil Formulas.

4.2.1. *Orthogonal groups.* Let V be a vector space over $\mathbb{Q}$ of dimension $n \geq 3$. A map $q : V \to \mathbb{Q}$ is called a *quadratic form on V over $\mathbb{Q}$* if it satisfies the following conditions:

(1) The function $b : V \times V \to \mathbb{Q}$, defined by
$$b(x, y) := q(x + y) - q(x) - q(y),$$
is a symmetric bilinear form.

(2) For $\lambda \in \mathbb{Q}$ and $x \in V$, we have $q(\lambda x) = \lambda^2 q(x)$.

Throughout this section, we will always assume that q is nondegenerate, i.e. b is nondegenerate.

A morphism between two quadratic forms q, q' is a linear map $f : V \to V$ such that $q' \circ f = q$. The automorphism group of a quadratic form q over $\mathbb{Q}$ is denoted as $\mathrm{O}_q(\mathbb{Q})$, called the *orthogonal group* of (V, q).

For any $\mathbb{Q}$-algebra R, q induces a quadratic form $q_R : V \otimes_{\mathbb{Q}} R \to R$ over R by extension of scalars. Its automorphism group is denoted by $\mathrm{O}_q(R)$. Thus, we have defined an algebraic group O_q over $\mathbb{Q}$ corresponding to the quadratic form $q : V \to \mathbb{Q}$.

Furthermore, O_q is an affine algebraic group. Indeed, we fix a choice of basis $\{e_1, \ldots, e_n\}$ for V, a quadratic form q on V then corresponds to a symmetric matrix B_q defined by $(B_q)_{ij} := \frac{1}{2}(q(e_i + e_j) - q(e_i) - q(e_j))$. One can show that two quadratic forms q, q'' on V are isomorphic over $\mathbb{Q}$ if $B_{q'} = T^t B_q T$ for some $T \in \mathrm{GL}_n(\mathbb{Q})$. Thus, for any $\mathbb{Q}$-algebra R, we can describe $\mathrm{O}_q(R)$ as
$$\mathrm{O}_q(R) = \{x \in \mathrm{GL}_n(R) : B_q = x^t B_q x\}.$$

[11] If $L/K(S)$ is an algebraic extension then for any $x' \notin S$, there exists $f \in K(S)[x]$ so $f(x') = 0$, i.e. $\sum_i f_i(s_1, \ldots, s_n) x'^i = 0$ for some $s_1, \ldots, s_n \in S$ and $f_i \in K[x_1, \ldots, x_n]$. This means $\{x'\} \cup S$ is algebraically dependent, i.e. if S is an algebraically independent set of L/K, then S is a transcendental basis of L

[12] finitely generated algebraic extension is finite

4.2.2. *Tamagawa measure of* O_q. As an algebraic group, $\mathrm{O}_{q,\mathbb{Q}}$ has an algebraic left-invariant differential form ω of top degree over $\mathbb{Q}$. Hence, for each place v of $\mathbb{Q}$, ω defines a left-invariant Haar measure $\mu_{\mathrm{O}_q(\mathbb{Q}_v),\omega}$ on $\mathrm{O}_q(\mathbb{Q}_v)$. As we know that *Need to state this for general semisimple group*

$$\prod_{p<\infty} \mu_{\mathrm{O}_q(\mathbb{Q}_p),\omega}(\mathrm{O}_q(\mathbb{Z}_p)) < \infty$$

so this defines a Tamagawa measure $\mu_{\mathrm{O}_q,\mathbb{Q}}$ on $\mathrm{O}_q(\mathbb{A})$.

4.2.3. *Tamagawa number of* O_q. In this section, we will prove the following result

Theorem 13. $\tau_{\mathbb{Q}}(\mathrm{O}_q) = 2$.

To prove this, we induct on $n = \dim V$. We will take for granted that $\tau_{\mathbb{Q}}(\mathrm{O}_q) = 2$ for $n = 3, 4$. The proof for these cases can be found at Weil's book, p. 65, theorem 3.7.1.

4.2.4. *Stabilisers of the action of* O_q *on* $\mathbb{G}_a^n$. For any $\mathbb{Q}$-algebra R, we denote

$$\mathrm{O}_{v,q}(R) = \{g \in \mathrm{O}_q(R) : vg = v\}$$

to be the stabiliser of $v \in V_R := V \otimes_{\mathbb{Q}} R$. In particular, for $v \in V_{\mathbb{Q}}$, $\mathrm{O}_{v,q}$ is an affine algebraic group over $\mathbb{Q}$. In this section, we will describe the structure of $\mathrm{O}_{v,q}(R)$ in the two cases where $q(v) = 0$ and $q(v) \neq 0$. For convenience, we will restrict the discussion for $\mathrm{O}_{v,q}(\mathbb{Q})$, as the case $\mathrm{O}_{v,q}(R)$ for any $\mathbb{Q}$-algebra R is completely similar.

If $q(v) \neq 0$ then we will show O_v is an orthogonal group of dimension $n-1$. Let $W_v = (\mathbb{Q}v)^{\perp} := \{v' \in V : q(v, v') = 0\}$ then W_v is a $\mathbb{Q}$-vector space of dimension $n-1$. Indeed, we know $q(v) \neq 0$ so $q(v, v) \neq 0$. Hence, for any basis $v, v_1, \dots, v_{n-1}$ of V, one can choose $c_i \in \mathbb{Q}$ so that $q(v, v_i + c_i v) = 0$. This means $W_v = \mathrm{span}_{\mathbb{Q}}\{v_1 + c_1 v, \dots, v_{n-1} + c_{n-1} v\}$.

We note that if $g \in \mathrm{O}_v$ then g preserves W_v, as $0 = q(v, v') = q(gv, gv') = q(v, gv')$ for $v' \in W_v$. This means $\mathrm{O}_v \subset \mathrm{O}_{q|_{W_v}}$. Conversely, given $g \in \mathrm{O}_{q|_{W_v}}$ then we can extend g to action on V by letting $gv = v$, as $v \notin W_v$. Thus, $\mathrm{O}_v = \mathrm{O}_{q|_{W_v}}$, i.e. O_v is the orthogonal group corresponding to the quadratic form $q|_{W_v}$.

If $q(v) = 0$ but $v \neq 0$, q restricts to $\mathbb{Q}v$ is trivial. Since q is nondegenerate, there exists $v' \in V$ independent from v such that $q(v, v') = 1$. Then $q(v' - xv) = \frac{1}{2} q(v' - xv, v' - xv) = q(v') - x$ for $x \in \mathbb{Q}$. Thus, by taking $x = q(v')$ and replace v' by $v' - xv$, we may assume that $q(v') = 0$. This follows $W = (\mathbb{Q}v \oplus \mathbb{Q}v')^{\perp}$ has dimension $n-2$, as for any $w \in W$, we can find $c, c' \in \mathbb{Q}$ so $q(w + cv + c'v', v) = q(w + cv + c'v', v') = 0$. Thus, under a choice of basis $v, v_1, \dots, v_{n-2}, v'$ of V where $v_1, \dots, v_{n-2}$ is a basis of W, q has the matrix form

$$B_q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where S is a $(n-2) \times (n-2)$ symmetric matrix corresponding to $q|_W$. From this, one can show that $\mathrm{O}_{v,q}$ is the semidirect product $\mathbb{G}_a^{n-2} \ltimes \mathrm{O}_{q|_W}$, i.e. element in $\mathrm{O}_{v,q}(\mathbb{Q})$ has the form

$$\begin{pmatrix} 1 & w & -\frac{1}{2} w^t S w \\ 0 & 1 & -S^t w \\ 0 & 0 & 1 \end{pmatrix}$$

where $w \in \mathbb{G}_a^{n-2}(\mathbb{Q})$.

4.2.5. *An evaluation of integral.* We will assume the following results

(1) Hasse-Minkowski theorem: For every $n \geq 1$, if $q_{\mathbb{A}}^{-1}(x) \neq \emptyset$ then $q_{\mathbb{Q}}^{-1}(x) \neq \emptyset$ for every $x \in \mathbb{Q}$.
(2) Witt's theorem: For $n \geq 3$, if we have $0 \neq v_i \in q_{\mathbb{Q}}^{-1}(i)$ then $v_i \cdot \mathrm{O}_q(\mathbb{Q}) = q_{\mathbb{Q}}^{-1}(i) - \{0\}$ and $v_i \cdot \mathrm{O}_q(\mathbb{A}) = q_{\mathbb{A}}^{-1}(i) - \{0\}$ for every $i \in \mathbb{Q}$.

Next, similarly to the case of SL_n and Sp_{2n}, we want to rewrite the following integral

$$\int_{\mathrm{O}_q(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{A})} \left(\sum_{x \in \mathbb{Q}^n} f(xg) \right) d\mu_{\mathrm{O}_q}(g).$$

where $f : V_{\mathbb{A}} \to \mathbb{C}$ is a Schwatz-Bruhat function, so that we can see the appearance of $\tau_{\mathbb{Q}}(\mathrm{O}_q)$.

From Witt's theorem, we have

$$V_{\mathbb{Q}} - \{0\} = \bigsqcup_{i \in \mathbb{Q}} \left(q_{\mathbb{Q}}^{-1}(i) - \{0\} \right) = \bigsqcup_{i \in \mathbb{Q}} v_i \cdot \mathrm{O}_q(\mathbb{Q}).$$

Furthermore, $\mathrm{O}_{v_i}(\mathbb{Q}) \setminus \mathrm{O}_q(\mathbb{Q}) \cong v_i \cdot \mathrm{O}_q(\mathbb{Q})$ via $\gamma \mapsto v_i\gamma$, we find

$$\begin{aligned}
\int_{\mathrm{O}_q(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{A})} \left(\sum_{x \in V_{\mathbb{Q}} \backslash \{0\}} f(xg) \right) d\mu_{\mathrm{O}_q}(g) &= \int_{\mathrm{O}_q(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{A})} \sum_{i \in \mathbb{Q}} \sum_{v \in q_{\mathbb{Q}}^{-1}(i) - \{0\}} f(vg) d\mu_{\mathrm{O}_q}(g), \\
&= \sum_{i \in \mathbb{Q}} \sum_{\gamma \in \mathrm{O}_{v_i}(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{Q})} \int_{\mathrm{O}_q(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{A})} f(v_i\gamma g) d\mu_{\mathrm{O}_q}(g), \\
&= \sum_{i \in \mathbb{Q}} \int_{\mathrm{O}_{v_i}(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{A})} f(v_i g) d\mu_{\mathrm{O}_q}(g), \\
&= \sum_{i \in \mathbb{Q}} \tau_{\mathbb{Q}}(\mathrm{O}_{v_i}) \int_{\mathrm{O}_{v_i}(\mathbb{A})\backslash \mathrm{O}_q(\mathbb{A})} f(v_i g) d\mu_{\mathrm{O}_q}(g), \\
&= \sum_{i \in \mathbb{Q}} \tau_{\mathbb{Q}}(\mathrm{O}_{v_i}) \int_{q_{\mathbb{A}}^{-1}(i) \backslash \{0\}} f(v) dv, \\
&= \sum_{i \in \mathbb{Q}} \tau_{\mathbb{Q}}(\mathrm{O}_{v_i}) \int_{q_{\mathbb{A}}^{-1}(i)} f(v) dv
\end{aligned}$$

By induction on n and from the previous descriptions of O_{v_i}, we find $\tau_{\mathbb{Q}}(\mathrm{O}_{v_i}) = 2$ for all $0 \neq v_i$, implying

$$\int_{\mathrm{O}_q(\mathbb{Q})\backslash \mathrm{O}_q(\mathbb{A})} \left(\sum_{x \in \mathbb{Q}^n} f(xg) \right) d\mu_{\mathrm{O}_q}(g) = f(0)\tau_{\mathbb{Q}}(\mathrm{O}_q) + 2 \sum_{i \in \mathbb{Q}} \int_{q_{\mathbb{A}}^{-1}(i)} f(v) dv. \tag{5}$$

4.2.6. *A second evaluation of integral.* We show that

$$\sum_{i \in \mathbb{Q}} \int_{q_{\mathbb{A}}^{-1}(i)} f(v) dv = \sum_{i \in \mathbb{Q}} \int_{V_{\mathbb{A}}} f(v) \exp_{\mathbb{A}}(q(v)x) dv.$$

For any $x, y \in \mathbb{A}^\times$ and $0 \neq v \in q_{\mathbb{A}}^{-1}(x)$, there exists $g \in \mathrm{GL}_n(\mathbb{A})$ such that $q(vg) = y$. Hence, $q_{\mathbb{A}}^{-1}(x) \times \mathbb{A}^\times \xrightarrow{\sim} V_{\mathbb{A}} - q_{\mathbb{A}}^{-1}(0)$ by sending $(v, y) \mapsto vg$. This follows that from a volume form $\omega = dx_1 \wedge \cdots \wedge dx_n$ on $V_{\mathbb{A}}$, one can split $\omega = \omega_x \wedge dx$ where ω_x is the volume form on $q_{\mathbb{A}}^{-1}(x)$.

For $x = 0$, if there exists $0 \neq v \in V_{\mathbb{A}}$ so $q(v) = 0$, then there also exists a volume form on $q_{\mathbb{A}}^{-1}(0)$. Indeed, ... *Need to learn this*

With this, we can define $F_f(x) = \int_{q_{\mathbb{A}}^{-1}(x)} f d|\omega_x|$, we find that

$$\widehat{F_f}(x) = \int_{\mathbb{A}} F_f(y)\exp(yx)dx = \int_{V_{\mathbb{A}}} f(v)\exp_{\mathbb{A}}(q(v)x)dv.$$

By the Poisson summation formula, we find

$$\sum_{i\in\mathbb{Q}} F_f(i) = \sum_{i\in\mathbb{Q}} \widehat{F_f}(i).$$

Thus, (5) can be rewritten as

$$\int_{\mathrm{O}_q(\mathbb{Q})\backslash\mathrm{O}_q(\mathbb{A})} \left(\sum_{x\in\mathbb{Q}^n} f(xg)\right) d\mu_{\mathrm{O}_q}(g) = f(0)\tau_{\mathbb{Q}}(\mathrm{O}_q) + 2\sum_{i\in\mathbb{Q}} \int_{V_{\mathbb{A}}} f(v)\exp_{\mathbb{A}}(q(v)i)dv. \tag{6}$$

4.2.7. *A Poisson summation formula.* Letting $\phi_g(x) := f(xg)$, we have

$$\begin{aligned}
\widehat{\phi_g}(x) &:= \int_{V_{\mathbb{A}}} f(yg)\exp_{\mathbb{A}}(q(x,y))dy,\\
&= \int_{V_{\mathbb{A}}} f(yg)\exp_{\mathbb{A}}(q(xg,yg))dy \quad (y\mapsto yg^{-1}),\\
&= \int_{V_{\mathbb{A}}} f(y)\exp_{\mathbb{A}}(q(xg,y))dy,\\
&= \widehat{\phi}(xg)
\end{aligned}$$

By the Poisson summation formula, we find

$$\sum_{x\in V_{\mathbb{Q}}} f(xg) = \sum_{x\in V_{\mathbb{Q}}} \widehat{f}(xg).$$

This says that the left hand side of (6) is invariant under $f\mapsto \widehat{f}$.

On the other hand, we know that

$$\int_{V_{\mathbb{A}}} \widehat{f}(v)\exp_{\mathbb{A}}(q(v)i)dv = \int_{V_{\mathbb{A}}} \widehat{f}(v)\exp_{\mathbb{A}}(-i^{-1}q(v))dv.$$

This follows

$$\begin{aligned}
\sum_{i\in\mathbb{Q}} \int_{V_{\mathbb{A}}} f(v)\exp_{\mathbb{A}}(iq(v))dv - \sum_{i\in\mathbb{Q}} \int_{V_{\mathbb{A}}} \widehat{f}(v)\exp_{\mathbb{A}}(iq(v))dv &= \int_{V_{\mathbb{A}}} (f(v)-\widehat{f}(v))dv,\\
&= \widehat{f}(0) - f(0).
\end{aligned}$$

Hence, it follows from (6) that

$$(\tau_{\mathbb{Q}}(\mathrm{O}_q) - 2)(f(0)-\widehat{f}(0)) = 0.$$

We can choose $f\in\mathcal{S}(V_{\mathbb{A}})$ such that $f(0)\neq\widehat{f}(0)$, implying $\tau_{\mathbb{Q}}(\mathrm{O}_q) = 2$.

4.3. **05/09/2021: Holomorphic maps between Riemann surfaces.** I want to give a summary of properties of holomorphic maps between two Riemann surfaces, learned from Forster's Lectures on Riemann Surfaces.

Our Riemann surfaces are always assumed to be connected and Hausdorff.

4.3.1. *Global behaviour.* There are a few results on the global behaviour of a holomorphic function, obtained by looking at its domain of definition:

(1) Riemann's Removable Singularities Theorem: Let U be an open subset of a Riemann surface X and $a \in U$. Suppose f is a holomorphic function on $U \setminus \{a\}$ that is bounded on some neighborhood of a. Then f can be extended uniquely to a holomorphic function $\tilde{f}$ on U.
(2) Identity theorem: Let f_1, f_2 be two holomorphic maps which coincide on a set A having a limit point $a \in X$. Then f_1 and f_2 are identically equal.

These properties can be obtained via similar results for holomorphic functions on $\mathbb{C}$.

What are the higher dimensional results for the above?

4.3.2. *Local behaviour: Ramification.* Let $f : X \to Y$ be a non-constant holomorphic map. We would like to study the behaviour of f locally at a point in X. A point $a \in X$ is called a *ramification point* of f if f is not locally injective at a. We call $b \in Y$ a *branch point* of f if $f^{-1}(b)$ has a ramification point. If f has no ramification point then f is called *unramified/unbranched* [13].

There is another way to describe ramification for non-constant holomorphic maps by noting that for any $a \in X$, there exists $k \in \mathbb{Z}_{\geq 1}$ such that under certain choices of charts near a and $f(a)$, f is given by $z \mapsto z^k$ (Forster §2.1). Then f is ramified at $a \in X$ iff $k \geq 2$, and such k is called the *ramification index* of $a \in X$. Indeed, observe that the map $p_k : \mathbb{C} \to \mathbb{C}$, defined by $z \mapsto z^k$ for $k \in \mathbb{Z}_{\geq 2}$, has only one ramification point 0, as for any $\varepsilon > 0$, if $a \in B_{\varepsilon > 0}(0)$ then $ae^{2\pi in/k} \in B_\varepsilon(0)$ where $n \in \mathbb{Z}, 1 \leq n \leq k$.

An example of an unramified map is the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$. This map is unramified because for any $a \in \mathbb{C}$, one can choose $\varepsilon > 0$ so $B_\varepsilon(a)$ does not contains any two points that differ by an integer multiple of $2\pi i$. Here is another example of unramified morphism: Let Γ be a lattice in $\mathbb{C}$, i.e. $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$, then the projection $p : \mathbb{C} \to \mathbb{C}/\Gamma$ is a local homeomorphism under the quotient topology on $\mathbb{C}/\Gamma$. One can equipp $\mathbb{C}/\Gamma$ with a complex structure inherited from $\mathbb{C}$ via p, making it into a Riemann surface and p is then unramified (as p is a local homeomorphism).

The local behaviour $z \mapsto z^k$ of f allows one to deduce many local properties of f. For example, we find that f is an open map. Some more properties from this local behaviour when you add extra conditions of f, X and Y:

(1) If f is injective and holomorphic, then $f : X \to f(X)$ is biholomorphic.
(2) If X is compact then f is surjective and Y compact. Indeed, as $f(X)$ is both open (since f is open) and closed (since X is compact) and Y is connected, we find $f(X) = Y$ is compact.
    In particular, every holomorphic function on a compact Riemann surface is constant.

Study some more specific examples of holomorphic functions: Why meromorphic functions on $\mathbb{P}^1$ *is rational?; What do functions* $f : \mathbb{C}/\Gamma \to \mathbb{P}^1$ *on the torus look like?; What is Weierstrass functions?;What are some example of ramified morphism?*

4.3.3. *Local behaviour: Covering maps.* Let $f : X \to Y$ be a non-constant holomorphic map. In contrast to the previous section, we would like to study the behaviour of f locally at a point in Y. Firstly, f is a discrete map, i.e. the fiber $f^{-1}(y)$ for any $y \in Y$ is discrete in X, or else f would be equal to y by the Identity theorem.

[13] Some references such as Griffths and Harris or Forster would not distinguish between branch points and ramification points, i.e. both are elements in X that are not locally injective

Locally at a ramification point $x \in X$, f can be represented by $z \mapsto z^k$ for some $k \in \mathbb{Z}_{>1}$, and as that 0 is the only ramification point of $p_k : \mathbb{C} \to \mathbb{C}, p_k(z) = z^k$, the set A of all ramification points of f is discrete and closed in X.

If we require that f is *proper* [14] i.e. the preimage of every compact set is compact, then the set $B := f(A)$ of branch points of f is also closed and discrete in Y [15]. Let $Y' := Y \setminus B$ and $X' := X \setminus f^{-1}(B)$ then $f : X' \to Y'$ is an unramified proper holomorphic map.

A proper, non-constant unramified holomorphic map $f : X \to Y$ is a *n-sheeted covering map*, i.e.

(1) There exists a positive integer n such that for any $y \in Y$, $f^{-1}(y)$ is a discrete set consisting of n points.
(2) For any $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_n\}$, there exists an open neighborhood U of y so that $f^{-1}(U)$ is a disjoint union of open neighborhoods U_i's of x_i's, and $f|_{U_i}$ is a homeomorphism for every $1 \leq i \leq n$.

We give a brief explanation on why a proper, non-constant, unramified holomorphic map $f : X \to Y$ has these properties:

(1) Because f is proper and discrete, we find $f^{-1}(y)$ is finite for every $y \in Y$. In particular, this does not use the unramified condition.
(2) Because f is unramifed, f is a local homeomorphism. Combining with the fact that $f^{-1}(y)$ is finite and discrete, we find that f is a (finite) covering map.
(3) Our Riemann surface Y is connected (hence path connected because Y is locally path connected) and f is a finite covering map so from the theory of covering maps (where one has curve lifting properties), $f^{-1}(y)$ and $f^{-1}(y')$ have the same cardinality for any $y, y' \in Y$. (See Forster §4 for more elaborations).

Thus, for a non-constant proper surjective holomorphic map $f : X \to Y$, by removing all the branch points, we find that $f : X' \to Y'$ is a n-sheeted covering map. One may wonder if $f^{-1}(y)$ also has cardinality n for a branch point $y \in Y$. If one takes into account of ramification index, the answer is yes. For $x \in X$, we let $\nu(f, x)$ to be the ramification index of $x \in X$ with respect to f.

Theorem 14. *Let $f : X \to Y$ be a proper, non-constant holomorphic maps between connected Riemann surfaces, then there exists a positive integer n, called the number of sheets of f, such that*

$$n = \sum_{x \in f^{-1}(y)} \nu(f, x)$$

for any $y \in Y$.

Proof. Note that if $x \in X$ is unramified then $\nu(f, x) = 1$. Hence, we let n be the number of sheets for the unramified map $f : X' \to Y'$ that is obtained from $f : X \to Y$.

Let $y \in Y$ be a branched point, $f^{-1}(y) = \{x_1, \dots, x_r\}$ and $k_j := \nu(f, x_j)$. From the local behaviour of non-constant holomorphic map, for any $1 \leq j \leq r$, there exists disjoint neighborhoods U_j of x_j and V_j of y such that for any $c \in V_j \setminus \{y\}$, the set $f^{-1}(c) \cap U_j$ contains exactly k_j points.

[14] See `https://en.wikipedia.org/wiki/Proper_map` for the motivation of properness

[15] $f(A)$ is closed because any proper mapping between locally compact spaces $f : X \to Y$ is closed. Indeed, let C be a closed subset of X, we want to show $f(C)$ is closed in Y or $Y \setminus f(C)$ is open. Let $y \in Y \setminus f(C)$ and because Y is locally compact, y has an open neighborhood V with compact closure $\overline{V}$. Because f is proper, $f^{-1}(\overline{V})$ is compact, hence $E = f^{-1}(\overline{V}) \cap C$ is also compact (it is closed in the compact set $f^{-1}(\overline{V})$). It follows that $f(E)$ is compact, hence closed in Y (as Y is Hausdorff). Let $U = V \setminus f(E)$ then U is a neighborhood of y that is disjoint from $f(C)$, as desired.
On the other hand, as A is discrete, every subset of A is closed, hence every subset of $f(A)$ is closed, meaning $f(A)$ is discrete.

Because f is proper, hence closed, we can find a neighborhood $V \subset V_1 \cap \cdots \cap V_r$ of y such that $f^{-1}(V) \subset U_1 \cup \cdots \cup U_r$. Indeed, let $X \setminus \cup_j U_j$ is closed in X, hence $f(X \setminus \cup_j U_j)$ is closed and does not contain y, implying $V := Y \setminus f(X \setminus \cup_j U_j)$ is the desired open neighorborhood of y.

This means for every unbranch point $c \in V$, $f^{-1}(c)$ consists of $k_1 + \cdots + k_r$ points, which should be n. We are done. □

To do: Learn some examples and calculations:

(1) https://math.stackexchange.com/q/403923/58951
(2) https://math.stackexchange.com/q/702538/58951
(3) https://math.stackexchange.com/q/119699/58951
(4) https://math.stackexchange.com/q/144748/58951
(5) https://math.stackexchange.com/q/740414/58951
(6) https://en.wikipedia.org/wiki/Analytic_continuation
(7) https://en.wikipedia.org/wiki/Covering_space
(8) https://people.math.wisc.edu/~robbin/951dir/algebraicCurves.pdf
(9) http://www.tjsullivan.org.uk/pdf/MA475_Riemann_Surfaces.pdf
(10) https://people.brandeis.edu/~igusa/Math101bS07/Math101b_notesB2.pdf
(11) https://users.math.yale.edu/~td276/lecture3.pdf

4.4. **07/09/2021: Adeles over function fields.** *Need to define this from perspective of $\mathbb{P}^1$ over $\mathbb{F}_p$.*

4.4.1. *Completions of $\mathbb{F}_p(T)$.* The rational function field $\mathbb{F}_p(T)$ over $\mathbb{F}_p$ has the following equivalence classes of nontrivial absolute values and completions

(1) Fix a monic irreducible polynomial $Q \in \mathbb{F}_p[T]$. Any $f \in \mathbb{F}_p(T)$ is uniquely written as $f = Q^r g$ for some $r \in \mathbb{Z}$ and $g \in \mathbb{F}_p(T)$ such that Q does not divide the numerator or denominator of g. We can define an absolute value $|\cdot|_Q$ on $\mathbb{F}_p(T)$ by $|f|_Q = p^{-r\deg(Q)}$ and $|0|_Q = 0$.

    The completion of $\mathbb{F}_p(T)$ with respect to $|\cdot|_Q$ is the field of formal Laurent series $\mathbb{F}_q(\!(Q)\!)$ in the variable Q with coefficients in $\mathbb{F}_q$, where $q = p^{\deg Q}$:
$$\mathbb{F}_q(\!(Q)\!) = \left\{ \sum_{i=N}^{\infty} a_i Q^i : N \in \mathbb{Z}, a_i \in \mathbb{F}_q \right\}.$$
    The absolute value on $\mathbb{F}_q(\!(Q)\!)$ is defined by
$$\left| \sum_{i=N}^{\infty} a_i Q^i \right|_Q = q^{-N} \ (a_N \neq 0).$$

(2) For $f = P_1/P_2 \in \mathbb{F}_p(T)$ where $P_1, P_2 \in \mathbb{F}_p(T)$ then we define
$$|f|_\infty = p^{\deg(P_1)-\deg(P_2)}, \ |0|_\infty = 0.$$
    The completion of $\mathbb{F}_p(T)$ with respect to this absolute value $|\cdot|_\infty$ is $\mathbb{F}_p(\!(1/T)\!)$, the field of formal Laurent series in the variable $1/T$. This field has an absolute value defined by
$$\left| \sum_{i=N}^{\infty} a_i \left(\frac{1}{T}\right)^i \right|_\infty = p^{-N} \ (a_N \neq 0).$$

For any $f \in \mathbb{F}_p(T)^\times$, one can check that $\prod_v |f|_v = 1$ where the product runs over all places v of $\mathbb{F}_p(T)$. In fact, this is simply a restatement of the fact that if $f = \prod_Q Q^{e_Q} \in \mathbb{F}_p(T)^\times$, then $\deg f = \sum_Q e_Q \deg Q$, where the sum/product is over all monic irreducible polynomials in $\mathbb{F}_p[T]$.

4.4.2. *Adeles over $\mathbb{F}_p(T)$.* For a place v of $k = \mathbb{F}_p(T)$, denote $\mathcal{O}_v := \{x \in k_v : |x|_v \leq 1\}$ to be the valuation ring of k_v. In particular, we find
$$\mathcal{O}_Q = \left\{ \sum_{i=0}^{\infty} a_i Q^i : a_i \in \mathbb{F}_q \right\},$$
$$\mathcal{O}_\infty = \left\{ \sum_{i=0}^{\infty} a_i \left(\frac{1}{T}\right)^i : a_i \in \mathbb{F}_p \right\},$$
where $Q \in \mathbb{F}_p[T]$ is a monic irreducible polynomial. We define the adeles $\mathbb{A}_{\mathbb{F}_p(T)}$ over $\mathbb{F}_p(T)$ to be the restricted product of $\mathbb{F}_p(T)_v$'s with respect to the subgroups $\mathcal{O}_v$. We can embed $\mathbb{F}_p(T) \hookrightarrow \mathbb{A}_{\mathbb{F}_p(T)}$ diagonally.

Proposition 15. *Under the action of $\mathbb{F}_p(T)$, $\mathbb{F}_p(T) \setminus \mathbb{A}_{\mathbb{F}_p(T)}$ has a fundamental domain*
$$(T^{-1}\mathcal{O}_\infty) \times \prod_Q \mathcal{O}_Q.$$

Proof. We will first show that $\mathbb{F}_p(T)$ is dense in $\mathbb{A}^\infty_{\mathbb{F}_p(T)}$, i.e. for any non-empty finite set S of monic irreducible polynomials in $\mathbb{F}_p[T]$, then any open set

$$\prod_{Q\in S}(a_Q + b_Q\mathcal{O}_Q) \times \prod_{Q\notin S\cup\{\infty\}} \mathcal{O}_Q$$

of $\mathbb{A}^\infty_{\mathbb{F}_p(T)}$, where $a_Q, b_Q \in \mathbb{F}_q(\!(Q)\!)$, contains an element of $\mathbb{F}_p(T)$. Without loss of generality, we can assume that $b_Q = Q^{c_Q}$ for $c_v \in \mathbb{Z}$, and $a_v \in \mathbb{F}_p(T)$. By the Chinese Riemander Theorem, there exists $f \in \mathbb{F}_p(T)$ such that $f \equiv a_Q \pmod{Q^{c_Q}}$ for all $v \in S$ and f does not have monic irreducible polynomials $Q \notin S$ as factors. This implies f lies in the desired open set.

Because $\mathbb{F}_p(T)$ is dense in $\mathbb{A}^\infty_{\mathbb{F}_p(T)}$, and $\prod_Q \mathcal{O}_Q$ is an open set of $\mathbb{A}^\infty_{\mathbb{F}_p(T)}$, we find

$$\mathbb{F}_p(T) + \prod_Q \mathcal{O}_Q = \mathbb{A}^\infty_{\mathbb{F}_p(T)}.$$

Next, we will show that

$$\mathbb{F}_p(T) + (T^{-1}\mathcal{O}_\infty) \times \prod_Q \mathcal{O}_Q = \mathbb{A}_{\mathbb{F}_p(T)}.$$

Indeed, we consider $(x_\infty, x) \in \mathbb{A}_{\mathbb{F}_p(T)}$, where $x_\infty \in \mathbb{F}_p(\!(1/T)\!)$ and $x \in \mathbb{A}^\infty_{\mathbb{F}_p(T)}$. We can write $x = x_1 + x_2$ where $x_1 \in \mathbb{F}_p(T)$ and $x_2 \in \prod_Q \mathcal{O}_Q$. Furthermore, we can choose $x_3 \in \mathbb{F}_p(T)$ such that $x_3 - x_\infty \in T^{-1}\mathcal{O}_\infty$ and $x_3 - x_1 \in \mathbb{F}_p[T]$, i.e. there exists a positive integer N and $x_3 = \sum_{i=-N}^N a_i T^i$ with $a_i \in \mathbb{F}_p$ so that $\sum_{i=-N}^{-1} a_i T^i - x_1 \in \mathbb{F}_p[T]$ and $\sum_{i=0}^N a_i T^i - x_\infty \in T^{-1}\mathcal{O}_\infty$. Thus, we find $(x_\infty, x) = x_3 + (x_\infty - x_3, x_1 - x_3 + x_2)$ lies in $\mathbb{F}_p(T) + (T^{-1}\mathcal{O}_\infty) \times \prod_Q \mathcal{O}_Q$, as desired.

Finally, note that if $x, x' \in (T^{-1}\mathcal{O}_\infty) \times \prod_Q \mathcal{O}_Q$ and $x - x' \in \mathbb{F}_p(T)$ then as $\mathbb{F}_p(T) \cap (T^{-1}\mathcal{O}_\infty) \times \prod_Q \mathcal{O}_Q = \{0\}$, we find $x = x'$. □

4.4.3. *Unitary characters.* For a place v of $\mathbb{F}_p(T)$, we can define the standard unitary characters on $\mathbb{F}_p(T)_v$ and on $\mathbb{A}_{\mathbb{F}_p(T)}$ as follows:

(1) If $v = \infty$ then $\psi_\infty : \mathbb{F}_p(\!(1/T)\!) \to \mathbb{C}^\times$ is defined by

$$\psi_\infty\left(\sum_{i=N}^\infty a_i (1/T)^i\right) = e^{-2\pi i a_1/p}.$$

In this definition, a_1 is any lift of $a_1 \in \mathbb{F}_p$ to $\mathbb{Z}$. Note that $\psi_\infty(T^{-2}\mathcal{O}_\infty) = 1$.

(2) If v corresponds to a monic irreducible polynomial $Q \in \mathbb{F}_q[T]$ then $\psi_Q : \mathbb{F}_q(\!(Q)\!) \to \mathbb{C}^\times$ is defined by

$$\psi_Q\left(\sum_{i=N}^\infty a_i Q^i\right) = e^{2\pi i (a_{-1} \bmod T)/p}.$$

In this definition, we have identified $a_{-1} \in \mathbb{F}_q$ with an element in $\mathbb{F}_p[T]/(Q(T))$, then $a_{-1} \bmod T \in \mathbb{F}_p$. Note that $\psi_Q(\mathcal{O}_Q) = 1$.

(3) For $x = (x_v) \in \mathbb{A}_{\mathbb{F}_p(T)}$, we can define an adelic unitary character $\psi : \mathbb{A}_{\mathbb{F}_p(T)} \to \mathbb{C}^\times$ by

$$\psi(f) = \prod_v \psi_v(x_v),$$

where ψ_v's are the local characters on $\mathbb{F}_p(T)_v$ as defined before. It is clear that ψ is nontrivial.

Proposition 16. *For $f \in \mathbb{F}_p(T)$, we have $\psi(f) = 1$.*

Proof. Observe that $\psi_\infty(f)$ only carries the information about the coefficient of $1/T$ in the expansion of f with respect to the variable $1/T$. To see how this relates to ψ_Q, we first write f as

$$f(T) = \sum_Q \frac{b_Q(T)}{Q(T)^{e_Q}} = \sum_Q \sum_{i=1}^{e_Q} \frac{b_{i,Q}(T)}{Q(T)^i}$$

where the finite sum is over all monic irreducible polynomials such that the order of Q in the factorisation of f into irreducible polynomials is $-e_Q < 0$; $\deg b_{i,Q}(T) < \deg Q(T)$.

Note that $\psi_Q(f) = e^{2\pi i\cdot[T^{\deg Q-1}]b_{1,Q}/p}$, where $[T^{\deg Q-1}]b_{1,Q} \in \mathbb{F}_p$ denotes the coefficient of $T^{\deg Q-1}$ in $b_{1,Q}$. *This does not look right*

We have $Q(T) = T^{\deg Q}F_Q(1/T)$ for some $F_Q \in \mathbb{F}_p[T]$, hence

$$\frac{b_{i,Q}(T)}{Q(T)^i} = \frac{b_{i,Q}(T)T^{-i\deg Q}}{F_Q(1/T)^i}$$

As $F_Q(1/T)^{-i} \in \mathcal{O}_\infty$ (furthermore, as Q is monic, $F_Q(1/T)^{-i}$ has constant coefficient 1 in the expansion with respect to the variable $1/T$) and as $\deg b_{i,Q}(T) < \deg Q(T)$, we find that the coefficient of $1/T$ in the expansion of $b_{i,Q}(T)Q(T)^{-i}$ with respect to $1/T$ is 0 if $i \geq 2$ and if $i = 1$, it is the coefficient of $T^{\deg(Q)-1}$ in $b_{1,Q}(T)$. $\square$

4.4.4. *Fourier transform.* From the standard unitary characters ψ_v on $\mathbb{F}_p(T)_v$, we can define the Fourier transforms on $\mathbb{F}_p(T)_v$ as follows:

Proposition 17. *Let μ_∞ be a Haar measure on on $\mathbb{F}_p((1/T))$ so that $\mu_\infty(\mathcal{O}_\infty) = p$ and for each monic irreducible polynomial $Q \in \mathbb{F}_p[T]$, let μ_Q be a Haar measure on $\mathbb{F}_q((Q))$ such that $\mu_Q(\mathcal{O}_Q) = 1$. For each place v of $\mathbb{F}_p(T)$, we can define the Fourier transform of a Bruhat-Schwartz function $f : \mathbb{F}_p(T)_v \to \mathbb{C}$ by*

$$\widehat{f}(x) = \int_{\mathbb{F}_p(T)_v} f(y)\psi_v(xy)d\mu_v(y).$$

Then $\widehat{f}$ is Bruhat-Schwartz and that the Fourier inversion formula holds

$$\hat{\hat{f}}(x) = f(-x) \quad \forall x \in \mathbb{F}_p(T)_v.$$

Proof. A Bruhat-Schwartz function $f : \mathbb{F}_p(T)_v \to \mathbb{C}$ is a locally constant compactly supported function. Every such function on $\mathbb{F}_p(T)_v$ is a finite linear combination of characteristic functions on compact open sets $a + Q^n\mathcal{O}_Q$ with $a \in \mathbb{F}_p(T)_Q$ and $n \in \mathbb{Z}$ (when $v = \infty$ then the open sets are $a + (1/T)^n\mathcal{O}_\infty$). Hence, it suffices to prove this proposition for the case $f = 1_{a+Q^n\mathcal{O}_Q}$ when $v = Q$ or $f = 1_{a+(1/T)^n\mathcal{O}_\infty}$ when $v = \infty$.

We check this for the case $v = Q$. The other case $v = \infty$ is completely similar. Firstly, we will show that

$$\int_{Q^n\mathcal{O}_Q} \psi_Q(xy)d\mu_Q(x) = \begin{cases} q^{-n} & y \in Q^{-n}\mathcal{O}_Q, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, if $y \in Q^{-n}\mathcal{O}_Q$ then $xy \in \mathcal{O}_Q$ for all $x \in Q^n\mathcal{O}_Q$. As $\psi_Q(\mathcal{O}_Q) = 1$, the integral becomes $\mu_Q(Q^n\mathcal{O}_Q) = q^{-n}$. If $y \notin Q^{-n}\mathcal{O}_Q$ then there exists $x_0 \in \mathbb{F}_q((Q))$ such that $\psi_Q(x_0y)$ is nontrivial. By invariance of the Haar measure μ_Q, we find

$$\int_{Q^n\mathcal{O}_Q} \psi_Q(xy)d\mu_Q(x) = \int_{Q^n\mathcal{O}_Q} \psi_Q((x+x_0)y)d\mu_Q(x) = \psi_Q(x_0y)\int_{Q^n\mathcal{O}_Q} \psi_Q(xy)d\mu_Q(x).$$

As $\psi_Q(x_0y)$ is nontrivial, we find the integral for $y \notin Q^{-n}\mathcal{O}_Q$ is 0.

From this calculation, we find

$$\begin{aligned}\widehat{1}_{a+Q^n\mathcal{O}_Q}(x) &= \int_{\mathbb{F}_q((Q))} 1_{a+Q^n\mathcal{O}_Q}(y)\psi_Q(xy)d\mu_Q(y),\\ &= \int_{a+Q^n\mathcal{O}_Q} \psi_Q(xy)d\mu_Q(y),\\ &= \int_{Q^n\mathcal{O}_Q} \psi_Q(x(y+a))d\mu_Q(y),\\ &= \psi_Q(xa)q^{-n}\cdot 1_{Q^{-n}\mathcal{O}_Q}(x)\end{aligned}$$

Hence, we have

$$\begin{aligned}\hat{\hat{1}}_{a+Q^n\mathcal{O}_Q}(x) &= \int_{\mathbb{F}_q((Q))} \hat{1}_{a+Q^n\mathcal{O}_Q}(y)\psi_Q(xy)d\mu_Q(y),\\ &= q^{-n}\int_{Q^{-n}\mathcal{O}_Q} \psi_Q(x(y+a))d\mu_Q(y),\\ &= 1_{Q^n\mathcal{O}_Q}(y+a) = 1_{a+Q^n\mathcal{O}_Q}(-y).\end{aligned}$$

We are done. ☐

This proprosition allows us to define a Haar measure on $\mathbb{A}_{\mathbb{F}_p(T)}$ and define the Fourier transform on the space of Bruhat-Schwartz functions on $\mathbb{A}_{\mathbb{F}_p(T)}$. We also have the Fourier inversion formula, which gives us the Poisson summation formula.

Proposition 18 (Poisson summation formula)**.** *Let $f \in \mathcal{S}(\mathbb{A}_{\mathbb{F}_p(T)})$ be a Bruhat-Schwartz function on $\mathbb{A}_{\mathbb{F}_p(T)}$. Then*

$$\sum_{x\in\mathbb{F}_p(T)} f(x) = \sum_{x\in\mathbb{F}_p(T)} \widehat{f}(x).$$

Similar to the number field case, the Poisson summation formula implies that the volume of $\mathbb{F}_p(T)\setminus\mathbb{A}_{\mathbb{F}_p(T)}$ (with respect to the Haar measure on $\mathbb{A}_{\mathbb{F}_p(T)}$) is 1.

4.4.5. *Riemann-Roch theorem.* We will prove the Riemann-Roch theorem for $\mathbb{P}^1$ over $\mathbb{F}_p$ using the Poisson summation formula.

The *divisor group* of $\mathbb{P}^1$ is the free abelian group generated by the the places v of $\mathbb{F}_p(T)$. A divisor D can be written $\sum_v n_v v$ with $d_v \in \mathbb{Z}$ and $d_v = 0$ for all but finitely many v. We define the *degree* of a place v to be

$$\deg(v) := \begin{cases} 1, & v=\infty,\\ \deg(Q), & v = Q \text{ is a monic irreducible polynomial.}\end{cases}$$

The degree of a divisor $D = \sum_v n_v v$ is defined by

$$\deg(D) := \sum_v n_v \deg(v).$$

Any nonzero $\omega \in \Omega_{\mathbb{P}^1/\mathbb{F}_p}$ corresponds to a divisor $K = \sum_v \kappa_v v$, where κ_v is the "order of vanishing of ω at v" *Need to find the precise definition of κ_v.* For example, $\omega \in \Omega_{\mathbb{P}^1/\mathbb{F}_p}$ defined by $-du \in \Gamma(U_0, \Omega_{\mathbb{P}^1/\mathbb{F}_p})$ and $w^{-2}dw \in \Gamma(U_1, \Omega_{\mathbb{P}^1/\mathbb{F}_p})$ gives rise to a divisor $K_\omega = -2\cdot\infty$.

For a divisor $D = \sum_v n_v v$, we can define $\mathcal{O}_{\mathbb{A}}(D) := \prod_v \mathfrak{p}_v^{-n_v} \subset \mathbb{A}$. Let $L(D) := K \cap \mathcal{O}_{\mathbb{A}}(D)$ then $L(D)$ is $\mathbb{F}_q$-vector space of functions on $\mathbb{P}^1$ having at worst a pole of order n_v at v for each v. Indeed, for $f, g \in L(D)$ and $\alpha \in \mathbb{F}_p$ then

$$v_v(f+g) \geq \min\{v_v(f), v_v(g)\} = -n_v, v_v(\alpha f) = v_v(f).$$

Furthermore, $L(D)$ is finite-dimensional since for $f \in L(D)$ with factorisation $f = \prod_Q Q^{e_Q}$ into irreducible monic polynomials, then

$$e_Q \geq -n_Q, \deg(f) = \sum_Q e_Q \deg(Q) \leq n_\infty.$$

We can then let $\ell(D) = \dim_{\mathbb{F}_p} L(D)$. For example, $\ell(K_\omega) = 0$ is the *genus* of $\mathbb{P}^1$.

Theorem 19 (Riemann-Roch)**.** *Let $K = -2 \cdot \infty$ to be the divisor corresponding to the differential $\omega \in \Omega_{\mathbb{P}^1/\mathbb{F}_p}$, defined by $-du \in \Gamma(U_0, \Omega_{\mathbb{P}^1/\mathbb{F}_p})$ and $w^{-2}dw \in \Gamma(U_1, \Omega_{\mathbb{P}^1/\mathbb{F}_p})$. Then for any divisor D on $\mathbb{P}^1$,*

$$\ell(D) - \ell(K - D) = \deg(D) + 1.$$

Proof. For a divisor $D = \sum_v n_v v$, we find that

$$\widehat{1}_{\mathfrak{p}_v^{-n_v}} = p^{n_v \deg v - \kappa_v/2} 1_{\mathfrak{p}_v^{n_v - \kappa_v}},$$

which implies

$$\widehat{1}_{\mathcal{O}_{\mathbb{A}}(D)} = p^{\deg D - \deg(K)/2} 1_{\mathcal{O}_A(K-D)},$$

as $\mathcal{O}_{\mathbb{A}} = \prod_v \mathfrak{p}_v^{-n_v}$. On the other hand, as $L(D)$ has $p^{\ell(D)}$ elements so

$$\sum_{x \in \mathbb{F}_p(T)} 1_{\mathcal{O}_{\mathbb{A}}(D)}(x) = p^{\ell(D)}.$$

Similarly, we can also compute

$$\sum_{x \in \mathbb{F}_p(T)} \widehat{1}_{\mathcal{O}_{\mathbb{A}}(D)} = p^{\deg D - \deg(K)/2} \sum_{x \in L(K-D)} 1 = p^{\deg D - \deg(K)/2 + \ell(K-D)}.$$

Thus, by Poisson summation formula, we find

$$\ell(D) = \ell(K - D) + \deg(D) + 1.$$

$\square$

Can one write this in such a way that does not depend on the choice of charts?

4.5. **10/09/2021: The theta correspondence and the Siegel-Weil formula.** I want to take a quick notes on the theta correspondences and the Siegel-Weil formula. At the moment, I just want to sketch the big picture (so some of what I write below may be incorrect), and leave the technical details later in the future. Some references

(1) Proof of a simple case of the Siegel-Weil formula by Paul Garrett, `https://www-users.cse.umn.edu/~garrett/m/v/easy_siegel_weil.pdf`
(2) A brief survey on the theta correspondence by Dipendra Prasad `http://www.math.tifr.res.in/~dprasad/dp.pdf`
(3) On the local theta correspondence `https://www.math.u-bordeaux.fr/~ybilu/algant/documents/theses/Zou.pdf`.

We start with a symplectic vector space W over a global/local field k. Reductive dual pairs (G_1, G_2) are certain paris of subgroups of $\mathrm{Sp}(W)$. For example, if V is an orthogonal vector space then $V \otimes W$ is a symplectic vector space, and we have a pair $(O(V), \mathrm{Sp}(W))$ as a reductive dual pair of $\mathrm{Sp}(V \otimes W)$. So we have a correspondence $O(V) \leftarrow \mathrm{Sp}(V \otimes W) \rightarrow \mathrm{Sp}(W)$ (I think this is true only when we consider two fold covers of these groups but let me ignore this at the moment).

The (lobal) theta correspondence roughly says that there is a one-to-one correspondence between certain irreducible representations of $O(V)$ and certain irreducible representations of $\mathrm{Sp}(W)$ from the Weil representations of $\mathrm{Sp}(V \otimes W)$.

For the global theta correspondence, one can realise Weil representation of $\mathrm{Sp}(V \otimes W)$ as a representation of $\mathrm{Sp}(V \otimes W)$ on the space of nice functions on $V(\mathbb{A})^n$. Then the theta correspondence is roughly obtained by push-pull functions from the correspondence $O(V) \leftarrow \mathrm{Sp}(V \otimes W) \rightarrow \mathrm{Sp}(W)$, i.e. how to get a function on $O(V)$ to a function on $\mathrm{Sp}(W)$ given a function on $\mathrm{Sp}(V \otimes W)$. This is called the *theta lifts* (i.e. it maps automorphic forms on $O(V)$ to automorphic forms on $\mathrm{Sp}(W)$).

The Siegel-Weil formula says that this theta lift at the constant function 1 on $O(V)$ is the Siegel Eisenstein series on $\mathrm{Sp}(W)$.

4.6. **13/09/2021: Mystery point counting.** From my naive point of view, there is something so mysterious of doing counting points over finite fields. Let me try to give out some links this time, as I am unable to explain much

(1) Hausel and Rodriguez-Villegas: `https://arxiv.org/pdf/math/0612668.pdf` (look more into papers of these two), "counting absolutely indecomposable quiver representations, vector bundles with parabolic structure on a projective curve, and irreducible etale local systems", `https://arxiv.org/pdf/1612.01733.pdf`
(2) Motivic integration: A theorem of Weil relates p-adic integration with point counting over finite fields; p-adic integration is used to prove many conjectures
    (a) Birational Calabi–Yau n-folds have equal Betti numbers, `https://arxiv.org/abs/alg-geom/9710020`
    (b) Topological Mirror Symmetry Conjecture by HauselThaddeus for SL_n and PGL_n, `https://arxiv.org/abs/1707.06417`; Later these people use p-adic integration to reprove Ngo's fundamental lemma `https://arxiv.org/abs/1810.06739`
    (c) Igusas theory of local zeta functions. `https://people.math.harvard.edu/~mpopa/571/chapter3.pdf`, `https://link.springer.com/book/10.1007/978-1-4939-7887-8` chapter 0
    (d) What is the relation between p-adic integration with p-adic Hodge theory `https://mat.uab.cat/~masdeu/wp-content/uploads/2017/04/padicint.pdf`?

Motivic integration is is some kind of generalisation of p-adic integration. In one direction, it seems people are trying to use this to define nonabelian Fourier transform (see Kazhdan, Braveman, ...) , also orbital integrals in the Langlands program.

(3) Weil's conjecture and this Weil's conjecture `https://www.math.ias.edu/~lurie/papers/tamagawa-abridged.pdf`, point counting via ℓ-adic cohomology.
(4) Topological Quantum Field Theory: This reminds me of my digging around what's called "geometric function theory" by David Ben-Zvi on 02-06/06/2021 and 16-17/06/2021. My question back then is roughly like this: Given a map $\phi : X \to Y$, it is usually easy to define pullback $\phi^* : Fun(Y) \to Fun(X)$, but what characterise a nice pushforward $\phi_* : Fun(X) \to Fun(Y)$? In the discrete case, 06/06/2021, I think this is doing some sort of weight count. In the other case(?), I think this is some sort of integration along fibers as pullback, something like this `https://golem.ph.utexas.edu/category/2010/11/integral_transforms_and_pullpu.html`. Overall, it certainly relates point counting and integration, and I think it relates to the theory of sheaves with better operations of pushforward (and hence relates to Weil's conjecture?).

4.7. **17/09/2021: Again with left-invariant differential form of SL_2.** On 07/08/2021, I used a map $\mathrm{SL}_2 \times \mathbb{G}_m \to \mathrm{GL}_2$ to find a global left-invariant top form for SL_2 from a global left-invariant top form of GL_2. To be honest, I am not satisfied with this computation as I suspect there is a different way to do this without relying on GL_2 (because the way I found a left-invariant top form for GL_2 is an educated guess, i.e. it should have the form $f(x)dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22}$, and $f(x) = \det(x_{ij})^{-2}$ is guessed from the left-invariant property).

Masoud pointed out a Math Stack Exchange answer `https://math.stackexchange.com/q/2907521/58951` saying that the k-vector space of left-invariant differential forms of SL_2 has basis $x_{22}dx_{11} - x_{12}dx_{21}, x_{22}dx_{12} - x_{12}dx_{22}$ and $-x_{21}dx_{11} + x_{11}dx_{21}$. Upon taking exterior product, this should gives us the global left-invariant top form of SL_2. Today, I want to explain a way to get this basis. In particular, I want to explain a natural way to get from an element of the dual Lie algebra $\mathfrak{sl}_2^*$ to a left-invariant differential form of SL_2, following a proof on page 100 of the book Neron models. This also helps me to digest this proof in the book (although I cannot understand it completely, at least now I can follow it with an example of SL_2 in mind).

I will use the same notation as on 07/08/2021. Let's start. The group scheme SL_2 over $\operatorname{Spec} k$ is a group object in the category of k-schemes (equivalently, it gives $\mathcal{O}(\mathrm{SL}_2)$ the structure of a Hopf algebra). In particular, we have

(1) A unit section $\varepsilon : \operatorname{Spec} k \to \mathrm{SL}_2$ is given by the algebra map $\mathcal{O}(\mathrm{SL}_2) \to k$, defined by sending $x_{ij} \mapsto \delta_{ij}$ (to see this, view x_{ij} as function on $\mathrm{SL}_2(k) \to k$ output the (i,j)-th position, by precomposing with the unit map $1 \to \mathrm{SL}_2(k)$, we get a map $1 \to k$ that either is 0 if $i \neq j$ or 1 if $i = j$).
(2) The multiplication $m : \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SL}_2$ is given by the algebra map $\mathcal{O}(\mathrm{SL}_2) \to \mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathrm{SL}_2)$, sending $f \in \mathcal{O}(\mathrm{SL}_2)$ (viewed as function $\mathrm{SL}_2(k) \to k$) to a function $\mathrm{SL}_2(k) \times \mathrm{SL}_2(k) \to k$, defined by $(a,b) \mapsto f(ab)$. For example, $x_{11} \mapsto (yz)_{11} := y_{11}z_{11} + y_{12}z_{21}$ where (y_{ij}) are generators for the first $\mathcal{O}(\mathrm{SL}_2)$ in $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathrm{SL}_2)$.

We can pullback $\Omega_{\mathrm{SL}_2/k}$ along ε to get a sheaf $\varepsilon^*\Omega_{\mathrm{SL}_2/k}$ of $\mathcal{O}_{\operatorname{Spec} k}$-modules. Its global section is the k-module $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \otimes_{\mathcal{O}(\mathrm{SL}_2)} k$, defined by $\mathcal{O}(\mathrm{SL}_2) \to k$ by ε. In particular, it is a k-module generated by dx_{ij}, modulo the relation $dx_{11} + dx_{22}$ (this is obtained from the relation $x_{22}dx_{11} + x_{11}dx_{22} - x_{12}dx_{21} - x_{21}dx_{12} = 0$ in $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ by letting $x_{ij} \mapsto \delta_{ij}$). This is isomorphic to the dual Lie algebra $\mathfrak{sl}_2^*$.

Now, I want to identify an element in $\Gamma(\operatorname{Spec} k, \varepsilon^*\Omega_{\mathrm{SL}_2/k}) \cong \mathfrak{sl}_2^*$ with a left-invariant differential form of SL_2. Let's choose $dx_{11} \in \Gamma(\operatorname{Spec} k, \varepsilon^*\Omega_{\mathrm{SL}_2/k})$. One can view dx_{11} as an element in $\Gamma(\mathrm{SL}_2, \Omega_{\mathrm{SL}_2/k})$. On the other hand, from the multiplication map $m : \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SL}_2$, we can pullback to obtain a section $m^*(dx_{11})$ of $\Omega_{\mathrm{SL}_2 \times \mathrm{SL}_2/k}$, i.e.

$$m^*(dx_{11}) = d(y_{11}z_{11} + y_{12}z_{21}) = y_{11}dz_{11} + z_{11}dy_{11} + y_{12}dz_{21} + z_{21}dy_{12}.$$

We also have $\Omega_{\mathrm{SL}_2 \times \mathrm{SL}_2 / k} \cong p_y^* \Omega_{\mathrm{SL}_2 / k} \oplus p_z^* \Omega_{\mathrm{SL}_2 / k}$ where p_y, p_z is the projection of $\mathrm{SL}_2 \times \mathrm{SL}_2$ to the first and second factor, respectively. This means

$$m^*(dx_{11}) = \omega_y + \omega_z = (z_{11} dy_{11} + z_{21} dy_{12}) + (y_{11} dz_{11} + y_{12} dz_{21}).$$

Now, consider the twist diagonal map $\delta : \mathrm{SL}_2 \to \mathrm{SL}_2 \times \mathrm{SL}_2$ given by $\delta \mapsto (\delta^{-1}, \delta)$. The claim in the proof is that $\delta^* \omega_z$ is a left-invariant different form for SL_2. Let's describe $\delta^* \omega_z$ explicitly.

First, note that δ corresponds to the map $\delta' : \mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathrm{SL}_2) \to \mathcal{O}(\mathrm{SL}_2)$, defined by sending $y_{ij} \otimes z_{kl}$ to the to the function $\mathrm{SL}_2(k) \to k$, taking $a \mapsto (a^{-1}, a) \mapsto (a^{-1})_{ij} a_{kl}$. In particular, it sends

$$\delta^* \omega_z = \delta^*(y_{11} dz_{11} + y_{12} dz_{21}) = \delta'(y_{11}) d(\delta'(z_{11})) + \delta'(y_{12}) d(\delta'(z_{21})) = x_{22} dx_{11} + (-x_{12}) dx_{21}.$$

This is one of the left-invariant differential form for SL_2 given in the Math Stack Exchange answer. I will stop here for today.

A further nore that in differential geometry, to find left-invariant differential form of a Lie group, there is something called the Maurer-Cartan equations/form that can be used. But I haven't had the time to understand what this means, so I will refer to p.190 of the book Differential Geometry and Lie Groups by Jean Gallier and Jocelyn Quaintance for the future.

4.8. **19/09/2021: Tamagawa measure on function fields.**

4.8.1. *Function field of* $\mathbb{P}^1$. In this section, we aim to describe $\mathbb{P}^1$ over a field k as a smooth, projective, geometrically connected algebraic curves. We then define the function field $k_{\mathbb{P}^1}$ of $\mathbb{P}^1$ and completions of $k_{\mathbb{P}^1}$.

The projective k-scheme $\mathbb{P}^1_k := \operatorname{Proj} k[x_0, x_1]$ can be described as follows:

(1) The points of $\mathbb{P}^1$ consist of homogeneous prime ideals of the $\mathbb{Z}$-graded ring $k[x_0, x_1]$ [16].
(2) For a homogeneous polynomial of positive degree $f \in k[x_1, x_2]$, let $D(f)$ be the set of homogeneous prime ideals of $k[x_0, x_1]$ not containing f. These sets form a basis of open sets for $\mathbb{P}^1$. Furthermore, one can think of $D(f)$ as $\operatorname{Spec}(k[x_0, x_1]_f)_0$, the spectrum of the algebra of elements in $k[x_0, x_1]_f$ having degree 0. For example, one can identify $D(x_i)$ with the affine scheme $\operatorname{Spec} k[x_{0/i}, x_{1/i}]/(x_{i/i} - 1)$, where $x_{i/j}$ is identified with x_i/x_j.
(3) The structure sheaf of $\mathbb{P}^1$ is obtained by giving $D(f)$ the structure sheaf of $\operatorname{Spec}(k[x_0, x_1]_f)_0$. In particular, $\mathcal{O}_{\mathbb{P}^1}(D(x_0)) = k[x_{1/0}], \mathcal{O}_{\mathbb{P}^1}(D(x_1)) = k[x_{0/1}]$ and the glueing of the structure sheaves on $D(x_0) \cap D(x_1) = D(x_0 x_1)$ is obtained by sending $x_{0/1} \mapsto x_{1/0}$.

Being an integral scheme (i.e. $\mathcal{O}_{\mathbb{P}^1}(U)$ is an integral domain for every nonempty subset U of $\mathbb{P}^1$), $\mathbb{P}^1$ has a generic point η (i.e. a point that is dense in $\mathbb{P}^1$), corresponding to the homogeneous prime ideal (0) in $\mathbb{F}_p[x_0, x_1]$ (because every open set $D(f)$ contains (0)). The stalk $\mathcal{O}_{\mathbb{P}^1,\eta}$ of $\mathbb{P}^1$ at η, and hence the residue field $\kappa(\eta)$, is noncanonically isomorphic to $k(T)$ (i.e. if we view η as an element of $\operatorname{Spec} \mathbb{F}_p[x_{0/1}] \hookrightarrow \mathbb{P}^1$, its stalk is then $k[x_{0/1}]_{(0)} = k(x_{0/1})$). We denote this as $k_{\mathbb{P}^1}$ and call it the *function field* of $\mathbb{P}^1$ over k.

A point in $\mathbb{P}^1$ is closed if it is closed in each open set $D(x_i)$ of $\mathbb{P}^1$. Furthermore, a point in an affine scheme $\operatorname{Spec} A$ is closed if it corresponds to a maximal ideal in A, and the maximal ideals of $k[x]$ are in bijection with monic irreducible polynomials in $k[x]$. Thus, closed points of $\mathbb{P}^1$ corresponds to homogeneous polynomials $Q(x_0, x_1) \in k[x_0, x_1]$ so that either $Q(x_{0/1}, 1)$ is monic irreducible in $x_{0/1}$ or $Q(1, x_{1/0})$ is monic irreducible in $x_{1/0}$.

For a closed point $x \in \mathbb{P}^1$ corresponding to a homogeneous polynomial $Q(x_0, x_1) \in k[x_0, x_1]$, the stalk $\mathcal{O}_{\mathbb{P}^1,x}$ of $\mathbb{P}^1$ at x is $k[x_{0/1}]_{(Q(x_{0/1},1))} \cong k[x_{1/0}]_{(Q(1,x_{1/0}))}$. The residue field $\kappa(x)$ at x is then noncanonically isomorphic to $k[t]/(Q(t, 1))$, which is a finite extension k. We denote $\mathcal{O}_x$ to be the completion of the local ring $\mathcal{O}_{\mathbb{P}^1,x}$ then $\mathcal{O}_x$ is a complete discrete valuation ring with residue field $\kappa(x)$, noncanonically isomorphic to the power series ring $\kappa(x)[[t]]$. Let k_x be the fraction field of $\mathcal{O}_x$.

4.8.2. *Group scheme over* $\mathbb{P}^1$. Let $X = \mathbb{P}^1_{\mathbb{F}_p}$. Let G_0 be a linear algebraic group over k_X.

Definition 20. *An integral model of G_0 is an affine and smooth group scheme $\pi : G \to X$ whose generic fiber (i.e. let η be a generic point of X then we have a morphism $\operatorname{Spec} \kappa(\eta) \to \operatorname{Spec} \mathcal{O}_{X,x} \to X$, giving us the generic fiber $G \times_X \operatorname{Spec} \kappa(\eta)$ as a scheme over $\kappa(\eta)$) is isomorphic to G_0.*

Example 21. An integral model $\overline{\mathrm{SL}_2} \to \mathbb{P}^1$ for SL_2 can be obtained by base change $\overline{\mathrm{SL}_2} = \mathrm{SL}_2 \times_{\operatorname{Spec} \mathbb{F}_q} \mathbb{P}^1$.

Example 22. Masoud mentioned that a nontrivial example of an integral model is parahoric group scheme. I couldn't understand that much from googling around, but let me put something down first and leave this for later.

Question: *Give an example of a parahoric group scheme $\mathcal{P}$ over $\mathbb{P}^1$ whose generic fiber is a linear algebraic group G.*

Most references refer to Bruhat Tits papers `http://www.numdam.org/item/PMIHES_1972__41__5_0/`. Benedict Gross wrote an expository paper about parahoric subgroups at `https://abel.`

[16] an ideal of $k[x_0, x_1]$ is homogeneous if it is generated by homogeneous polynomials

math.harvard.edu/~gross/preprints/parahorics.pdf. There are also some answers on MathOverFlow about this. I want to construct one from the suggestions given in https://www.math.lsu.edu/~pramod/selie/10/mcninch.pdf, i.e. from what I read, the $\kappa(x)[[t]] = \mathcal{O}_x$-points of a parahoric group scheme is the stabiliser of x in the Bruhat-Tits building (some geometric gadget that $G(k_X)$ acts).

Given such a group scheme, for every commutative ring R equipped with a map $u : \mathrm{Spec}(R) \to X$, we can associate a group $G(R)$. If u factors through the generic point η of X, we can equipp R is the structure of k_X-algebra via u, and $G(R)$ can be identified with $G_0(R)$. A choice of integral modeal gives additional structures:

(1) For each closed point $x \in X$, we have a morphism $\mathrm{Spec}\, \mathcal{O}_x \to \mathrm{Spec}\, \mathcal{O}_{X,x} \to X$ [17], so we can consider the group $G(\mathcal{O}_x)$ of $\mathcal{O}_x$-valued points of G.
(2) For a closed point $x \in X$, we have a morphism $\mathrm{Spec}\, \kappa(x) \to \mathrm{Spec}\, \mathcal{O}_{X,x} \to X$, so we can consider the group $G(\kappa(x))$ of $\kappa(x)$-valued point of G. We have a surjective map $G(\mathcal{O}_x) \to G(\kappa(x))$ because G is smooth [18].
(3) For a finite set S of closed points of X, we have a morphism $\mathrm{Spec}\, \mathbb{A}_X^S \to X$ so we can consider the group $G(\mathbb{A}_X^S)$ of $\mathbb{A}_X^S$-valued point of G. It is an open subgroup of $G(\mathbb{A}_X) = G_0(\mathbb{A}_X)$ and as $\mathrm{Spec}\, \mathbb{A}_X^S = \prod_{x \in S} \mathrm{Spec}\, k_x \times \prod_{x \notin S} \mathrm{Spec}\, \mathcal{O}_x$, $G(\mathbb{A}_X^S)$ is isomorphic to the direct product $\prod_{x \in S} G(k_x) \times \prod_{x \notin S} G(\mathcal{O}_x)$.

4.8.3. *p-adic integration and point counting over finite fields.* In this section, we will focus on explaining a theorem of Weil that links between p-adic integrations and point counting of algebraic varieties over finite fields. We follow the following reference

(1) p-adic Integration and Birational Calabi-Yau Varieties by Pablo Magni.
(2) First chapter about p-adic integration of the book Motivic Integration by A. Chambert-Loir, J. Nicaise, and J. Sebag.

Let k be a nonarchimedean local field, $\mathcal{O}_k$ be its ring of integers, $\mathbb{F}_q$ be the residue field where q is a prime power of a prime p. Let X be a smooth scheme of relative dimension n over $\mathcal{O}_k$ and $\Omega_{X/\mathcal{O}_k}$ be the sheaf of differentials.

We will define a canonical measure on $X(\mathcal{O}_k)$, called the *Weil measure*. Because X is smooth over $\mathcal{O}_k$, $\Omega^n_{X/\mathcal{O}_k}$ is a locally free sheaf of $\mathcal{O}_X$-modules of rank 1, there exists an affine open cover $\{U_i\}$ of $\mathcal{O}_k$-schemes of X such that we have a trivialisation $\Omega^n_{X/\mathcal{O}_k}|_{U_i} \cong \mathcal{O}_X|_{U_i}$ over each U_i. A trivialisation of $\Omega^n_{X/\mathcal{O}_k}|_{U_i}$ corresponds to a nowhere-vanishing differential form $\omega_i \in \Gamma(U_i, \Omega^n_{X/\mathcal{O}_k})$. From this, we can define a (Radon) measure $d|\omega_i|$ on $U_i(\mathcal{O}_k)$ by integrating with respect to ω_i. We also have $X(\mathcal{O}_k) = \bigcup_i U_i(\mathcal{O}_k)$, so in order to define a (Radon) measure on $X(\mathcal{O}_k)$, the measures $d|\omega_i|$'s must agree on overlaps. This is true because for two nowhere-vanishing differential forms $\omega_i|_{U_i \cap U_j}$ and $\omega_j|_{U_i \cap U_j}$ on $U_i \cap U_j$, there exists a nowhere-vanishing function $f \in \mathcal{O}_X|_{U_i \cap U_j}$ (hence invertible) so that $\omega_i = f\omega_j$ on $U_i \cap U_j$. This gives us the relation $d|\omega_i|(x) = |f(x)|_k d|\omega_j|(x)$ of measures on $U_j(\mathcal{O}_k) \cap U_i(\mathcal{O}_k) = (U_i \cap U_j)(\mathcal{O}_k)$. However, as $f : (U_i \cap U_j)(\mathcal{O}_k) \to \mathcal{O}_k$ is invertible, $|f(x)|_k = 1$ for all $x \in (U_i \cap U_j)(\mathcal{O}_k)$, meaning $d|\omega_i| = d|\omega_j|$ on $(U_i \cap U_j)(\mathcal{O}_k)$.

Remark 23. The Weil measure is canonical in the sense that its construction does not depend on the existence of a global differential form. The main reason for this is because our scheme X is

[17] For example, let $X = \mathrm{Spec}\, \mathbb{Z}$, a prime number in $\mathbb{Z}$ corresponds to a closed point of X, we find $\mathcal{O}_{X,p} = \mathbb{Z}_{(p)}$ is a local ring with maximal ideal $\mathfrak{m}_{X,p} = p\mathbb{Z}_{(p)}$. The completion of $\mathcal{O}_{X,p}$ with respect to this maximal ideal is $\mathcal{O}_p = \mathbb{Z}_p$. Thus, we have $\mathbb{Z} \to \mathbb{Z}_{(p)} \to \mathbb{Z}_p$, giving us $\mathrm{Spec}\, \mathcal{O}_x \to \mathrm{Spec}\, \mathcal{O}_{X,x} \to X$

[18] Smoothness implies a condition on the Jacobian of the local coordinates at a point, and by a generalisation of Hensel's lifting lemma, we have surjectivity. For more details, see p.20 of See Weil's book Adeles and Algebraic Groups

over $\mathcal{O}_k$, hence any invertible function f, defined on an open set U of X, must have $|f(x)|_k = 1$ for all $x \in U(\mathcal{O}_k)$. This also means that one may not be able to repeat this construction to define a measure on $X(k)$. However, if we have a global differential form $\omega \in \Gamma(X, \Omega^n_{X/\mathcal{O}_k})$, we can define a measure on $X(k)$ whose restriction to $X(\mathcal{O}_k)$ is the Weil measure.

In the literature (see Batyrev's paper Birational Calabi-Yau n-folds have equal Betti numbers), it seems that the name Weil measure is given when X has a global nowhere-vanishing differential form, and the measure we have constructed is called the canonical measure, and in fact, the two measures are the same if the Weil measure (as defined in the literature) exists. Thus, for convenience, we will stick with our definition of Weil measure.

Theorem 24 (Weil)**.** *Let X be a smooth scheme of dimension n over $\mathcal{O}_k$. Let μ be the Weil measure on $X(\mathcal{O}_k)$, then*

$$\int_{X(\mathcal{O}_k)} d\mu = \frac{|X(\mathbb{F}_q)|}{q^n}.$$

Sketch. Consider the surjective reduction map $\varphi : X(\mathcal{O}_k) \to X(\mathbb{F}_q)$ sending $x \mapsto \overline{x}$, we then have

$$\int_{X(\mathcal{O}_k)} d\mu = \sum_{\overline{x}\in X(\mathbb{F}_q)} \int_{\varphi^{-1}(\overline{x})} d\mu.$$

It suffices to show $\int_{\varphi^{-1}(\overline{x})} d\mu = q^{-n}$ for all $\overline{x} \in X(\mathbb{F}_q)$. We view $\overline{x} \in X(\mathbb{F}_q)$ as an element of X by taking the value $\overline{x}(\eta)$ at the generic point $\eta \in \operatorname{Spec} \mathbb{F}_q$. Because X is smooth and $\Omega_{X/\mathcal{O}_k}$ is locally free, there exists an affine open set $U \cong \operatorname{Spec} \mathcal{O}_k[x_1, \dots, x_{n+m}]/(f_1, \dots, f_m)$ of $\overline{x}$ such that $\Omega_{X/\mathcal{O}_k}|_U$ is trivialised and the Jacobian matrix $(\partial f_i/\partial x_{n+j})_{1\le i,j\le m}$ is invertible at $\varphi^{-1}(\overline{x}) \subset U(\mathcal{O}_k)$ ($\varphi^{-1}(\overline{x}) \subset U$ as any open set of $\overline{x}$ contains $x \in \varphi^{-1}(\overline{x})$, viewed an element of X by evaluating at the generic point $\eta \in \operatorname{Spec} \mathcal{O}_k$). We consider the map $g : U(\mathcal{O}_k) \to \mathbb{A}^{n+m}_{\mathcal{O}_k}$ defined by $g(x_1, \dots, x_{n+m}) = (x_1, \dots, x_n, f_1(x), \dots, f_m(x))$. Observe that the Jacobian of g at $\varphi^{-1}(\overline{x})$ is a unit in $\mathcal{O}_k$. Therefore, by forgetting the last m coordinates, g induces an etale morphism $h : U \to \mathbb{A}^n_{\mathcal{O}_k}$, which induces a k-analytic isomorphism from $\varphi^{-1}(\overline{x})$ to $\mathfrak{p}^n$ by Hensel's lemma, where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_k$. Furthermore, because $\Omega_{X/\mathcal{O}_k}|_U \cong \mathcal{O}^n_X|_U$, we can find a global nowhere-vanishing differential form $\omega \in \Gamma\left(U, \bigwedge^n \Omega_{X/\mathcal{O}_k}|_U\right)$. We then have $h^*(dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n) = f\omega$, where f is invertible in U, hence has p-adic norm 1 when viewed as a function $f : U(\mathcal{O}_k) \to \mathcal{O}_k$. By definition, $f\omega$ defines a Weil measure on the neighborhood $U(\mathcal{O}_k)$. Thus, we find

$$\int_{\varphi^{-1}(\overline{x})} d\mu = \int_{\mathfrak{p}^n} dt_1 \wedge \cdots dt_n = q^{-n}$$

by the change of variables formula. □

4.8.4. *Tamagawa measure for function fields.* For a linear algebraic group G_0 over k_X, we will define the Tamagawa measure on $G_0(\mathbb{A}_X)$ in this section. We follow §1.3 of Gaitsgory and Lurie's book `https://www.math.ias.edu/~lurie/papers/tamagawa-abridged.pdf`.

Let $\pi : G \to X$ be an integral model of G_0 (such an integral model always exists), let $\Omega_{G/X}$ be the relative cotangent bundle of π. Then $\Omega^n_{G/X} := \bigwedge^n \Omega_{G/X}$ is a line bundle on G, where $n = \dim(G_0)$. Let $\mathcal{L}$ be the pullback of $\Omega^n_{G/X}$ along the identity section $e : X \to G$. Sections of $\mathcal{L}$ can be identified with left-invariant differential forms on G via the canonical isomorphism $\pi^*\mathcal{L} \cong \Omega^n_{G/X}$ (see Neron models p.100). Let $\mathcal{L}_0 := \operatorname{Spec} k_X \times_X \mathcal{L}$ be the generic fiber of $\mathcal{L}$, whose global sections form a 1-dimensional k_X-vector space. A non-zero global section ω of $\mathcal{L}_0$ can be viewed as a global left-invariant nowhere-vanishing algebraic differential form on G_0. For every closed point $x \in X$, ω induces a left-invariant Haar measure $d\mu_{\omega,x}$ on $G(k_x)$.

For an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules and a nonzero global section ω of its generic fiber $\mathcal{L}_0$, one can associate a divisor on X as follows. For a closed point $x \in X$, we consider the stalk $\mathcal{L}_x \subset \mathcal{L}_0$ at x, which is a $\mathcal{O}_x$-module of rank 1 inside a 1-dimensional k_X-vector space. Then $\omega\mathcal{O}_x$ is also a rank 1 $\mathcal{O}_x$-submodule of $\mathcal{L}_0$. Let $t_x \in \mathcal{O}_x$ be a uniformiser element then $\omega\mathcal{O}_x = t_x^{-n_x}\mathcal{L}_x$ for some integer n_x. We define $v_x(\omega) := n_x$ to be the *order of vanishing* of ω at x.

Proposition 25. *For every closed point $x \in X$, We have*

$$\mu_{\omega,x}G(\mathcal{O}_x) = \frac{|G(\kappa(x))|}{|\kappa(x)|^{n+v_x(\omega)}},$$

where $v_x(\omega) \in \mathbb{Z}$ denotes the order of vanishing of ω at x.

Sketch. If we view ω as a left-invariant differential form on $G(k_x)$ via the isomorphism $\pi^*\mathcal{L} \cong \Omega^n_{G/X}$, $v_x(\omega)$ can be described as follows. At the neighborhood U of the identity e of $G(k_x)$, ω can be written as $\omega = f(t)dt_1 \wedge \cdots \wedge dt_n$ where $t_1, \dots, t_n$ are the local coordinates at e, $f : U \to k_x$ is an invertible rational function. Then $f(e)\mathcal{O}_x = t_x^{-v_x(\omega)}\mathcal{O}_x$. In other words, the image of $\omega(e) \in \bigwedge^n T_e^*(G(\mathcal{O}_x))$ under $\bigwedge^n T_e(G(\mathcal{O}_x))$ generates a fractional ideal $\mathfrak{p}^{-v_x(\omega)}$ of k_x. Because ω is left-invariant so for any $g \in G(\mathcal{O}_x)$, the image of $\omega(g) \in \bigwedge^n T_g^*(G(\mathcal{O}_x))$ under $\bigwedge^n T_g(G(\mathcal{O}_x))$ generates the fractional ideal $\mathfrak{p}^{-v_x(\omega)}$ of k_x. In other words, under a new local coordinates $y_1, \dots, y_n$ at the neighborhood of $g \in G(\mathcal{O}_x)$, $\omega = f'(y)dy_1 \wedge \cdots \wedge dy_n$ where f' is a rational function so that $f'(g)\mathcal{O}_x = t_x^{-v_x(\omega)}\mathcal{O}_x$. By the definition of Weil measure, we then find that $t_x^{v_x(\omega)}\omega$ defines the Weil measure on $G(\mathcal{O}_x)$, or ω defines a measure $|\kappa(x)|^{-v_x(\omega)}\mu_{\mathrm{Weil}}$ on $G(\mathcal{O}_x)$. Thus, by Weil's theorem, we find

$$\mu_{\omega,x}(G(\mathcal{O}_x)) = |\kappa(x)|^{-v_x(\omega)}\mu_{\mathrm{Weil}}(G(\mathcal{O}_x)) = \frac{|G(\kappa(x))|}{|\kappa(x)|^{n+v_x(\omega)}}.$$

□

The Tamagawa measure $\mu_{G_0,X}$ of $G(\mathbb{A}_X) = G_0(\mathbb{A}_X)$ is

$$\tau_X(\mathbb{G}_a)^{-n}\prod_{x\in X}{}'\mu_{\omega,x}.$$

We have $\tau_X(\mathbb{G}_a) = q^{g-1}$, where g is the genus of X *need to learn this.* We also have

$$\prod_{x\in X}|\kappa(x)|^{v_x(\omega)} = \prod_{x\in X} q^{\deg(x)v_x(\omega)} = q^{\sum_{x\in X}\deg(x)v_x(\omega)} = q^{\deg\mathcal{L}} = q^{\deg\Omega_{G/X}}.$$

Here $\deg(x) := [\kappa(x) : \mathbb{F}_q]$, $\deg\mathcal{L} := \sum_{x\in X}\deg(x)v_x(\omega)$ where the sum is over all closed points of X [19], $\deg\mathcal{L} = \deg\Omega_{G/X} = n$ because of the isomorphism $\pi^*\mathcal{L} \cong \Omega^n_{G/X}$ (see Stack project `https://stacks.math.columbia.edu/tag/0AYQ`, we have $\deg\mathcal{L} = \deg\pi^*\mathcal{L} = \deg\Omega^n_{G/X} = \deg\Omega_{G/X}$). Thus, we find

$$\mu_{G_0,X}(G(\mathbb{A}_X^{\emptyset})) = q^{n(1-g)-\deg(\Omega_{G/X})}\prod_{x\in X}\frac{|G(\kappa(x)|}{|\kappa(x)|^n}.$$

4.9. **22/09/2021: Cayley transform carries conjugacy classes in G to adjoint orbits in $\mathfrak{g}$.** The classical Cayley transform gives a bijection between the real skew-symmetric matrices ($A^t = -A$) to the orthogonal matrices ($A^tA = I$) by sending

$$A \mapsto \frac{I-A}{I+A}.$$

[19] note that this does not depend on the choice of ω because for any two nonzero global sections ω and ω' of $\mathcal{L}$, there exists a nowhere-vanishing function $f \in \mathcal{O}_X(X)$ so $\omega = f\omega'$, and one can show $\sum_{x\in X} v_x(f)\deg(x) = 0$.

One define this Cayley transform for other classical groups (and I think for reductive groups?). In general, a Cayley transform is a bijection from a (reductive) group G to its Lie algebra $\mathfrak{g}$ (see `https://www.mat.univie.ac.at/~michor/kostant.pdf` The Generalized Cayley Map from an Algebraic Group to its Lie Algebra by Bertram Kostant and Peter W. Michor for complex, reductive, which says that we have the Cayley map whenever we have a faithful representation of our group G). One property of this map is that it carries conjugacy classes in G to adjoint orbits in $\mathfrak{g}$, allowing one to study conjugation action of G via the adjoint action in $\mathfrak{g}$.

It seems that this transform plays some role in the theory of Harish-Chandra characters, Fourier analysis on orbital integrals (see `https://arxiv.org/abs/1111.7057`, which is where I first found the term). I am intestered in the application of this to the study of orbital integrals appearing in the Fundamental lemma of the Langlands program. It seems to me that this is a way to reduce the group theoretic version about orbits to the Lie algebra theoretic version about orbits. But I cannot make it more precise than this.

I don't know if this has anything to do with the orbit method of Kirillov (`https://ncatlab.org/nlab/show/orbit+method`) in representation theory?.

4.10. **22/09/2021: Category theory: Adjunctions and String diagrams.** Some notes on what I have learn from the category theory reading group, following Tom Leinster, Basic Category Theory.

There are many ways to describe adjoint functors F, G between two categories $\mathcal{A}, \mathcal{B}$ (where $F : \mathcal{A} \to \mathcal{B}$ is left-adjoint to $G : \mathcal{B} \to \mathcal{A}$):

(1) First way: For any $A \in \mathcal{A}, B \in \mathcal{B}$, we have bijections $\mathrm{Hom}_{\mathcal{B}}(FA, B) \cong \mathrm{Hom}_{\mathcal{A}}(A, GB)$ (called this taking conjugations) satisfying some natural compatibility conditions.
(2) Second way: We have units $\eta : 1_{\mathcal{A}} \Longrightarrow GF$ and counits $\varepsilon : FG \Longrightarrow 1_{\mathcal{B}}$ satisfying triangle identities.
(3) Third way: We have unit η such that for any $A \in \mathcal{A}$, $\eta_A : A \to GF(A)$ is an initial object in the comma category $(1_A \Rightarrow G)$, where $1_A : 1 \to \mathcal{A}$ is a functor defined by A. To define the comma category: For functors $P : \mathcal{A} \to \mathcal{C}$ and $Q : \mathcal{B} \to \mathcal{C}$, the comma category $(P \Rightarrow Q)$ is the 'pullback' of the diagram of categories

$$\begin{tikzcd} (P \Rightarrow Q) \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "Q"] \\ \mathcal{A} \arrow[r, "P"'] \arrow[ur, Rightarrow] & \mathcal{C} \end{tikzcd}$$

Concretely, it consists of a category $(P \Rightarrow Q)$ with projections functors $p_{\mathcal{B}} : (P \Rightarrow Q) \to \mathcal{B}$ and $p_{\mathcal{A}} : (P \Rightarrow Q) \to \mathcal{A}$, together with a natural transformation $Pp_{\mathcal{A}} \Rightarrow Qp_{\mathcal{B}}$ (Search '2-limit' for more of this).

In the category theory reading group, we learn how to draw functors between categories and natural transformations between functors as string diagrams. I am a bit lazy to write down the way to see this but let me link to the videos by TheCatsters `https://youtube.com/playlist?list=PL50ABC4792BD0A086`. The main idea is that one can turn a natural transformation to a string diagram by Poincare duality, treating 2-morphisms (i.e. natural transformations) as points, 1-morphisms (i.e. functors) as a line and 0-morphisms (objects) as 2-dimensional spaces.

Although up to this point, I am still not that convinced on how one can see triangle identities in adjunctions as string diagrams, i.e. `https://ncatlab.org/nlab/show/triangle+identities` with the above rules of drawings?

What is the relation of string diagrams with Feymann diagrams?

I think I refer to the following reference on June 2021 already (related to Topological QFT) but let me link it again: `https://arxiv.org/abs/math/0512103`.

4.11. **22/09/2021: Characters in representation theory.** I read the section about trace formula in representation theory from `https://arxiv.org/abs/1009.1862v3`

Some things I don't understand: For a finite-dimensional representation π V of a finite group G, one can define the character of V by taking the trace of your representation, which can be view as a linear functional on $\mathbb{C}[G]$. One can generalise the definition of character for compact G and the application of characters in representation theory in both cases are similar: irreducible characters are orthonormal basis of the space of class functions; knowing these irreducible characters help to determine decomposition of any G-representation V into irreducibles (see Folland, Abstract harmonic analysis, §5.3 for example). A generalisation of this for semisimple G is Harish-Chandra character, i.e. it is a distribution of G, defined by

$$\Theta_\pi : f \mapsto \operatorname{Tr}\left(\int_G f(x)\pi(x)dx\right).$$

What is the role of this character theory in representation theory of G? Is there an even more general context of character theory? e.g. categorical version, locally compact version, p-adic version? Is there a conceptual explanation of why taking trace is a natural thing to do? Update 28/09/2021: See Terence Tao's comment in `https://mathoverflow.net/a/13527/89665`, where he said trace is a linearised version of dimension-counting operator. So I guess some sort of decategorification functor (if this makes any sense) ... See also the comment on how this idea fails in characteristic $p > 0$.

Maybe this has something to do with the discussion on 13/08/2021, where character theory is some sort of decategorification of the category $\operatorname{Rep}(G)$.

Another related thing is that I don't know too much of the analytic side of representation theory: distributions, Plancherel formula, ... *Hopefully, this will motivate me to read something like `http://math.uchicago.edu/~ngo/Rep-p-adic.pdf` more carefully.*

4.12. **More things to learn.**

(1) From 19/09/2021: What is the definition of divisors, the correspondence between divisors and invertible sheaves, Weil divisors, Cartier divisors ? Divisors of locally free sheaves over a curves? See Vakil's book or Wedhorn's book on AG.
(2) Analogy of $\operatorname{Spec}\mathbb{Z}$ as a 3-manifold? `https://mathoverflow.net/q/4075/89665` See also Ben-Zvi's Relative Geometric Langlands duality from `https://www.msri.org/workshops/918/schedules/28233` and `https://math.ucr.edu/home/baez/week257.html`.
(3) Motivic characters of SL_2: `https://arxiv.org/abs/math/0609260`, see also 22/09/2021.
(4) Just found some links `https://ncatlab.org/nlab/show/functorial+field+theory` to topological QFT, functorial field theory. This is something I managed to learn a bit on 06/2021, hopefully I have later time to revisit this. See also string diagrams on 22/09/2021. The book "Towards the Mathematics of Quantum Field Theories" is also following this direction I believe (probably the best place to learn).
(5) (22/09/2021) I found a very nice notes by David Nadler in `https://arxiv.org/abs/1009.1862v3`, titled 'The Geometric Nature of the Fundamental Lemma'. I only managed to read the first three sections of his notes, but even so, I loved the way he presented the materials on these sections.

    Need to write some notes on what I learnt from this notes. In particular,
    (a) Trace formula in representation theory, starting from finite group example (see also 22/09/2021).
    (b) Chevalley restriction theorem and characteristic polynomials.
    (c) Reductive groups generality.
    (d) A study of orbits under group action.

5. October 2021

5.1. 01/10/2021: Hecke algebra, groupoid version. I learnt how to describe the Hecke algebra as a groupoid from David Ben-Zvi's lectures Between electric-magnetic duality and the Langlands program (p. 122). I remembered trying to understand something like this on 06/06/2021 (what Ben-Zvi called 'geometric function theory'), but at the time I couldn't figure out what pt/G means in Ben-Zvi's notes back then. Now thanks to the category theory reading group (see 22/09/2021), I can make some sense out of this (at least when G is finite).

First, let me try to restate the question in Ben-Zvi's lectures: Let G be a finite group acting on a space X, and let $K \subset G$ be a subgroup.

What is the symmetry of $\mathbb{C}[X/K]$, i.e. what acts on $\mathbb{C}[X/K]$?

One views $\mathbb{C}[X/K]$ as K-invariant functions $\mathbb{C}[X]^K$ on X, and we have a K-invariant functor of G-representations

$$(-)^K : \text{Rep } G \to \text{Vect}.$$

To study the action on $(-)^K$ means to understand its endomorphism $\text{End}\left((-)^K\right)$. By Frobenius reciprocity (or Hom-$\otimes$ adjunction), the functor $(-)^K$ is representable by

$$V_{G,K} = \text{Ind}_K^G \mathbb{C} = \mathbb{C}[G] \otimes_{\mathbb{C}[K]} \mathbb{C} = \mathbb{C}[G/K].$$

The *Hecke algebra* $\mathcal{H}_{G,K}$ is

$$\mathcal{H}_{G,K} := \text{End}\left((-)^K\right) = \text{End}_{\text{Rep}(G)}(V_{G,K}, V_{G,K}) = (V_{G,K})^K = \mathbb{C}[K \backslash G/K].$$

Have I really answered the question of what is the symmetry of $\mathbb{C}[X/K]$*?* The third equal sign follows follows from the fact that a G-homomorphism $\phi : V_{G/K} \to V_{G/K}$ is determined by the image $\phi(1_K)$, where $1_K \in \mathbb{C}[G/K]$. We also know K fixes 1_K when acting on $\mathbb{C}[G/K]$. This description tells us that the double coset has a canonical algera structure via the endomorphism ring.

The functor $(-)^K$ factors through $\mathcal{H}_{G,K}$-mod, which is equivalent to the subcategory of $\text{Rep}(G)$, consisting of spherical G-representations, i.e. its irreducible components V has non-trivial K-invariants $V^K \neq \emptyset$. The word 'spherical' goes back to the study of spherical harmonics, i.e. $L^2(S^2)$ being acted on by SO_3, where noting that $S^2 = \text{SO}_3 / \text{SO}_2$.

One has a concrete way to describe the algebra structure of $\mathcal{H}_{G,K}$ as follows. We first start by describing the algebra structure of the group algebra $\mathbb{C}[G]$.

In fact,

Now, we want to describe a canonical algebra structure on $\mathbb{C}[G]$ from the multiplication map $\mu : G \times G \to \mathbb{C}$. We first write down all the information we know about G: *Technically this is not all information, we are missing something about identity map, associativity, ... but why are they missing when defining the algebra structure on* $\mathbb{C}[G]$*?*

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, "\mu"] & \\ G & & G \\ & G & \end{tikzcd}$$

Does this link to finite Fourier transform?

Remark 26. We start of by discussing pullback and pushforward of a given map $f : X \to Y$ between two spaces X and Y. For convenience, we take X, Y to be finite.

For a map $\Phi : X \to Y$, it is not hard to define pullback $\Phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$, when we view $\mathbb{C}[Y]$ as the space of functions on Y. However, to define pushforward $\Phi_* : \mathbb{C}[X] \to \mathbb{C}[Y]$, a natural definition of $\Phi_*(f)$ for $f \in \mathbb{C}[X]$ is sending

$$y \mapsto \int_{\Phi^{-1}(y)} f dx := \sum_{x \in \Phi^{-1}(y)} f(x).$$

From this, to define pushforward, it is better to view $\mathbb{C}[X]$ as the space of measures on X, i.e. given $f \in \mathbb{C}[X]$, one can (noncanonically) associate a measure on X by sending open $U \subset X$ to $\int_U f dx$. Now $\Phi^*(f)$ is a measure on Y, defined by sending open $V \subset Y$ to $\int_{\Phi^{-1}(V)} f dx \in \mathbb{C}$.

When X is locally compact Hausdorff, by Riez representation theorem, there is a canonical way to view a measure on X as a linear functional $\mathbb{C}[X] \to \mathbb{C}$. In other words, two identify measures and functions on X, we have essentially identify $\mathbb{C}[X]$ with its dual $\mathbb{C}[X]^*$. [20] In our case where X is finite, to see the duality between functions $\mathbb{C}[X]$ on X and measures $\mathbb{C}[X]^*$ on X, we have fixed a measure on X, i.e. a linear functional $\int_X : \mathbb{C}[X] \to \mathbb{C}$, defined by $f \mapsto \sum_{x \in X} f(x)$. From this, recall we have noncanonically identified a function $f \in \mathbb{C}[X]$ with a measure μ_f on X, sending open $U \subset X$ to $\mu_f(U) = \int_U f dx$. One view this measure as a linear functional $\mathbb{C}[X] \to \mathbb{C}$ by $g \mapsto \int_X g(x)\mu_f(x) = \int_X f(x)g(x)dx$. Thus, in other words, the duality between $\mathbb{C}[X]$ and $\mathbb{C}[X]^*$ is obtained by choosing a linear functional $\int_X : \mathbb{C}[X] \to \mathbb{C}$ (i.e. a measure on X) by $f \mapsto \sum_X f(x)$ and then define an inner product on $\mathbb{C}[X]$ by

$$\langle f, g \rangle = \int_X f(x)g(x)dx := \sum_{x \in X} f(x)g(x).$$

I don't see the point where the Hermitian condition of the inner product comes in naturally This inner product also gives the compatibility between pullback and pushforward, where they are adjoint with respect to this inner product.

So the duality allows one to work with one space (instead of with itself and its dual). Usually it is easier to define pullback on the spaces X and easier to define pushforward on the dual spaces. Having a self-duality allows one to define pullback and pushforward on the same space. [21] [22]

This then reminds me of the discussion on 13/08/2021 and 22/09/2021.

Under the identification $\mathbb{C}[X]$ with $\mathbb{C}[X]^*$, we want to check that the pushforward Φ_* is really coming from $\mathbb{C}[X]^* \to \mathbb{C}[Y]^*$. *Need to check this*

A final comment: A lesson I learnt from this computation is that adjointness is like an approximation for duality. Normally it is difficult to determine left-adjoint of some operator/functor, but once we have duality (between $\mathbb{C}[X]$ and $\mathbb{C}[X]^*$) then constructing the adjoint is easier.

Remark 27. We would like to discuss the G-equivariant case of the previous remark, i.e. we would like to describe pullback and pushforward of $\Phi : X \to Y$ when we have an action of G on X and Y. Again, we assume X, Y are finite first, and we consider G-equivariant functions on X and Y, denoted $\mathbb{C}[X/G]$ or $\mathbb{C}[X]^G$. If Φ is G-equivariant then one can define pullback $\Phi^* : \mathbb{C}[Y/G] \to \mathbb{C}[X/G]$.

One should view pushforward as something coming from taking the dual of pullback, i.e. $\mathbb{C}[X/G]^* \to \mathbb{C}[Y/G]^*$. Thus, to define pushforward $\Phi_* : \mathbb{C}[X/G] \to \mathbb{C}[Y/G]$, our task is to find a way to identify $\mathbb{C}[X/G]^*$ with $\mathbb{C}[X/G]$. Similar to the previous remark, we first need to choose a linear functional

[20] In functional analysis language, a linear functional $\mathrm{Fun}(X) \to \mathbb{C}$ is called a distribution (or generalised function)

[21] Another remark: I think another place where we have the duality between measures and functions on X is when X is a smooth manifold. A tangent vector is roughly a linear map $C^\infty(X) \to \mathbb{C}$, a differential form is dual to a tangent vector, so something like $\mathrm{Func}(C^\infty(X)) \to \mathbb{C}$. One can define a measure with respect to a differential form. We can then define pullback and pushforward of differential forms as analogues of pullback and pushforward of measures. *Make precise*

[22] Another place where we have that self-duality is the case of Hilbert space, where we are given a Hermitian inner product that identifies the Hilbert space H with its dual.

$\int_{X/G} : \mathbb{C}[X/G] \to \mathbb{C}$. We define $f \mapsto \sum_{x \in |X/G|} \frac{f(x)}{\#\mathrm{Aut}(x)}$ for the case X is finite. Here $|X/G|$ is the set of isomorphism classes of elements in X, $\#\mathrm{Aut}(x)$ is the size of the isotropy group of x.

After identifying $\mathbb{C}[X/G]$ with $\mathbb{C}[X/G]^*$, we will now define the pushforward map. *Need to do*

The algebra structure on $\mathbb{C}[G]$ (called convolution) can be described in terms of the multiplication map μ as follows.

$$f_1 * f_2 := \mu_*(f_1 \boxtimes f_2) := \mu_*(\pi_1^* f_1 \cdot \pi_2^* f_2).$$

5.2. **05/10/2021: References: Aaron Mazel-Gee, factorisation homology, ∞-categories.** The website of Aaron Mazel-Gee has many good notes about ∞-categories. For example his introductory lecture notes to Higher Algebra `https://etale.site/teaching/w21/math-128-lecture-notes.pdf`, his guidelines for topics in ∞-category `https://etale.site/teaching/f16-infty/seminar-infty-2.pdf`, a syllabus about factorisation homology `https://etale.site/teaching/s19-fact-hlgy/fact-hlgy-outline.pdf` and his lecture notes.

What caught my interest is that he mentioned factorisation homology as *generalised integration theory. I would like to learn on what he meant by that.*

Some more references about factorisation homology:

(1) Lectures on Factorization Homology, Infinity-Categories, and Topological Field Theories: `https://arxiv.org/abs/1907.00066`
(2) Claudia Isabella Scheimbauer PhD thesis `http://www.scheimbauer.at/ScheimbauerThesis.pdf`

5.3. **19/10/2021: Tamagawa number as Pic/Sha.** So for connected reductive group over number field, we have $\tau(G) = \frac{\mathrm{Pic}(G)}{\text{Ш}(G)}$.

See discussion at `https://mathoverflow.net/q/44184/89665`.

I would like to understand this statement for SL_n or GL_n.

5.3.1. *Picard group.* Let G be a (split) connected semisimple linear agebraic group over a (algebraically closed) field k (of characteristic 0). The Picard group $\mathrm{Pic}\, G$ of G is the group of isomorphism classes of line bundles on X. In this section, we will focus on proving the following

Theorem 28. *If G is simply connected then* $\mathrm{Pic}\, G = 0$.

Trying for SL_2. One way to show $\mathrm{Pic}(\mathrm{SL}_2)$ is to show $\mathcal{O}(\mathrm{SL}_2) = k[x_{ij}]/(x_{11}x_{22} - x_{12}x_{21} - 1)$ is a unique factorisation domain, and the fact that the Picard group of $\mathrm{Spec}\, A$ for a unique factorisation domain A is trivial. But proving $\mathcal{O}(\mathrm{SL}_2)$ is a UFD for any k seems to be nontrivial (in fact this is usually considered as a consequence of the theorem proved via other methods). See `https://math.stackexchange.com/q/173021/58951`.

A second way, following Milne p. 69 `https://www.jmilne.org/math/CourseNotes/RG.pdf`, is as follows: Consider $T \subset B \subset \mathrm{SL}_2$. We have a group homomorphism

$$\begin{aligned} X^*(T) &\to \mathrm{Pic}(\mathrm{SL}_2 / B) \\ \chi &\mapsto L(\chi) \end{aligned}$$

A character $\chi : T \to \mathbb{G}_m$ induces a B-equivariant line bundle $\mathrm{SL}_2 \times \mathbb{A}^1$ on SL_2 by letting B acts on $\mathrm{SL}_2 \times \mathbb{A}^1$ by

$$(g, x)b = (gb, \chi(b^{-1})x), g \in G, x \in \mathbb{A}^1, b \in B.$$

This decends to a line bundle $L(\chi) = \mathrm{SL}_2 \times_{\mathrm{SL}_2} \mathbb{A}^1$ on $G/B = \mathbb{P}^1$. In fact, this is an isomorphism $X^*(T) \cong \mathrm{Pic}(G/B) = \mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$.

Milne claimed that the following sequence is exact

$$0 \to X^*(\mathrm{SL}_2) \to X^*(T) \to \mathrm{Pic}(\mathrm{SL}_2 / B) \to \mathrm{Pic}(\mathrm{SL}_2) \to 0$$

Milne didn't prove this. I tried to look elsewhere for the proof of exactness of this sequence and found this `https://www-fourier.ujf-grenoble.fr/~mbrion/enney.pdf`, which I couldn't understand ... For example, is there a way to see $\mathrm{Pic}(\mathrm{SL}_2/B) \to \mathrm{Pic}(\mathrm{SL}_2)$ is surjective?

From the exactness of the above sequence, we find $\mathrm{Pic}(\mathrm{SL}_2) = 0$.

Trying to understand the general proof. I found the proofs to be given in:

(1) Section 4 (Proposition 4.6) of Local properties of algebraic group actions by F. Knop, H. Kraft, D. Luna and T. Vust [in: "Algebraische Transformationsgruppen und Invariantentheorie" (H. Kraft, P. Slodowy, T. Springer eds.) DMV-Seminar 13, Birkhuser Verlag (Basel-Boston) (1989), pp. 63-76]
(2) Another proof by Iversen 19 sketched in p.70 Milne Reductive groups notes. There are some similarities between the two proofs.
(3) Another proof in Voskresenskii's Algebraic Groups and Their Birational Invariants, p. 45 using Kummer exact sequence and etale cohomology.

A sketch of what I understand about the first two proofs:

(1) Show $\mathrm{Pic}\, G$ is finite.

  An 'independent' proof I found in Voskresenskii's Algebraic Groups and Their Birational Invariants, p. 45: By the theory of split semisimple group, there exists $g \in G(k)$ so NgB is open in G (N is a nilpotent group, B is a Borel) and $V = G \setminus NgB$ is the union $\bigcup_{i=1}^{l} F_i$ of closed irreducible subsets of G of codimension 1 (i.e. prime divisors of G), where $l = \mathrm{rank}\, G = \dim T$. Because G is smooth, $\mathrm{Pic}\, G$ can be identified with (Weil?Cartier?) divisors, so the above implies (see p.312 of the book Ulrich Grtz, Torsten Wedhorn, Algebraic Geometry I: Schemes) an exact sequence

$$\bigoplus_{i=1}^{l} \mathbb{Z} \cdot [F_i] \to \mathrm{Pic}(G) \to \mathrm{Pic}(NgB) \to 0$$

  As $NgB \cong N \times T \times N$ so $\mathrm{Pic}(U) = 0$ (I think this goes roughly as $\mathrm{Pic}(\mathbb{G}_a) = \mathrm{Pic}(\mathbb{G}_m) = 0$?). This means $\mathrm{Pic}(G)$ is the quotient of $\mathbb{Z}^l$ by principal divisors on U, i.e. regular invertible functions on U, which is $k(U)^* \cong k^* \times X^*(T)$ (Voskresenskii didn't explain this but I guess it roughly goes like this: N is $\mathbb{G}_a^r$ so invertible functions on $U = N \times T \times N$ are the same as invertible functions on T because invertible function on $\mathbb{G}_a$ is trivial?).

  All of this implies $\mathrm{Pic}(G)$ is finite.
(2) Identify $\mathrm{Pic}(G)$ with central extensions $\mathrm{Ext}(G, \mathbb{G}_m)$ of G by $\mathbb{G}_m$. This implies for any $L \in \mathrm{Pic}(G)$, there exists a finite covering $\pi : G' \to G$ of algebraic groups so $\pi^*(L) = 0$.

  I am still trying to understand this step. For example, in Lemma 4.3 of Knop, Kraft, Luna, I don't understand the definition of $\mu : L \times L \to L$, i.e. how to map $L \times L \to p_1^*(L) \otimes p_2^*(L)$.

  People commented that this step is similar to the abelian variety case. See Serre Algebraic groups and class fields VII. §16.
(3) Because $\mathrm{Pic}(G)$ is finite with elements $L_1, \dots, L_k$. By previous step, there exists finite covering $G_i' \to G$ of G, then $\prod G_i' \to G$ is a finite covering of G such that the induced map $\mathrm{Pic}(G) \to \mathrm{Pic}\left(\prod G_i'\right) = 0$.
(4) If G is simply connected, meaning any finite covering $G' \to G$ is an isomorphism, giving an isomorphism $\mathrm{Pic}(G) \to \mathrm{Pic}(G')$ Hence, we can choose $G' = \prod G_i'$ from previous step to get $\mathrm{Pic}(G) = 0$.

For more general G. There is a formula of $\mathrm{Pic}(G)$ for more general reductive G. See `https://mathoverflow.net/q/296965/89665` or `https://mathoverflow.net/a/297093/89665`.

A discussion of the proof is given in `https://mathoverflow.net/a/273883/89665`.

See also Voskresenskii's p. 45.

5.3.2. *Tate-Shafarevich set.* I would like to mention of historial facts I learnt about this.

For a global field k and algebraic group G, there is a natural map

$$H^1(k, G) \to \prod_v H^1(k_v, G)$$

The kernel of this map is called Tate-Shafarevich set, denoted $\text{Ш}(G)$. Its geometric meaning is all G-torsors X so $X(k_v) \neq 0$ for all places v.

(1) When G is a linear algebraic group, $\text{Ш}(G)$ is shown to be finite in

 J-J. Sansuc, Groupe de Brauer et arithmetique des groupes algebriques lineaires sur un corps de nombres

(2) When G is an abelian variety, this is a group. But it seems that the finiteness of this is still a conjecture, and this directly ties with the Birch and Swinnerton-Dyer conjecture.
(3) When $G = \mathrm{SL}_n$ or GL_n, there is a simple proof that $H^1(k, \mathrm{SL}_n) = 1$ given in Voskresenskii p 28 but I didn't have enough time to learn this.
(4) For simply-connected algebraic group, $\text{Ш}(G) = 0$ and this is the Hasse principle for algebraic groups.

5.4. **21/10/2021: Mednykh's formula, TQFT, character varieties, skein algebra.** The story from 06/06/2021, 13/09/2021 and 01/10/2021 again, about Mednykh's formula and TQFT. Maybe also 05/10/2021.

There is an open question on MathOverflow asking a modular analogue of Mednykh's formula `https://mathoverflow.net/q/313599/89665`.

Until now, I still don't understand the proof of this seminar and historical background (`https://arxiv.org/pdf/math/0703073.pdf`), what kind of generalisations can it get.

Sam Gunningham asked what aspects of character theory that can be simplified using TQFT.

I get a bit more motivated about this topic because I just learned about Madeline's research which is Temperley Lieb algebra, which leads me to related topics like skein theory: `https://arxiv.org/pdf/1908.05233.pdf`.

I wonder if one can always described this sort of algebra (Hecke, Temperley Lieb) as some TQFT? What advantage can this point of view give?

Another link between skein algebra as quantization of algebra of functions on a character varieties: `https://youtu.be/GF5PXRkEn3w`. There are connections to mirror symmetry but I don't understand this.

What is the precise relation here between TQFT, skein, character varieties, Mednykh's formula?

5.5. **23/10/2021: Condensed mathematics and functional analysis.** Peter Scholze and Dustin Clausen are using condensed mathematics to rewrite the theory of functional analysis that have better functional properties. Condensed sets are analogues of compactly generated space (which includes locally compact topological spaces). *I wonder what is the right notion of measures of a (locally comact) topological spaces in this setting, and then Fourier transform? What is adeles in this setting? What is $\mathbb{R}$? What is the difference between $\mathbb{R}$ and $\mathbb{Q}_p$ in this setting?* For a talk, see `https://youtu.be/cDkuhXD7hn0`. See also Scholze's notes `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf` for locally compact abelian groups. Here we have Pontryagin duality.

5.6. **23/10/2021: How to discover Hecke algebras: Knot invariants.** First motivation is through the study of the algebra of B-equivariant functions on $G = \mathrm{GL}_n(\mathbb{F}_q)$. See 01/10/2021. A blog post here explains how to categorify this idea `https://sbseminar.wordpress.com/2009/04/09/interpreting-the-hecke-algebra-ii-the-sheafification/`.

Second motivation is through knot invariants `https://sbseminar.wordpress.com/2009/04/13/how-to-get-an-algebra-from-a-knot-invariant/` and `https://mathoverflow.net/a/12986/89665`. *What exactly does this mean? i.e. how to one sees the quadratic relation of the Hecke algebra from knot invariants? It seems the quadratic relation comes from skein relation? And then what is the motivation for skein relation?* What is the stuff about fundamental groups of a torus mentioned in `https://mathoverflow.net/a/13461/89665`?

5.7. **More things to learn.**

(1) For a Lie group G, let $LG = Map(S^1, G)$ be a Lie group of all morphisms between manifolds $f : S^1 \to G$. The group structure is given by pointwise multiplication. Why the Lie algebra of this is $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$. Why does this group has trivial central extension? Taking the Lie algebra of the central extensions, we should get affine Lie algebras.

(2) Principles between cohomology as classifying obstructions: `https://math.stanford.edu/~conrad/210BPage/handouts/gpext.pdf` (Brian Conrad has so many good notes)

(3) Spectral sequences, computing cohomology. What is a good motivation to do this? Perverse sheaves?

6. More things to learn

(1) Everthing about $\mathrm{SL}_2, \mathrm{GL}_2$: Representations of $\mathrm{SL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{Q}_p), U(\mathfrak{sl}_2)$, quantum affine $\mathfrak{sl}_2$, ... Some links:
   (a) https://jenseberhardt.com/teaching/S21Seminardata/plan.pdf, https://jenseberhardt.com/teaching/W2021Seminardata/plan.pdf, https://www.maths.usyd.edu.au/u/romanova/Talks/TwistSheavessl2.pdf
   (b) Rep of $\mathrm{SL}_2(\mathbb{F}_q)$, book by Cedric Bonnafe (see rep lie group of finite type folder).
   (c) automorphic representations of $\mathrm{SL}_2, \mathrm{GL}_2$ over adeles https://virtualmath1.stanford.edu/~conrad/conversesem/refs/NgoGL2.pdf, trace formula https://www.math.stonybrook.edu/~aknapp/pdf-files/355-405.pdf (see more at folder about automorphic forms), Anton Deitmar book Automorphic forms looks very introductory
   (d) More general refs: https://virtualmath1.stanford.edu/~conrad/conversesem/,
   (e) Langlands fundamental lemma for SL_2 (Bill Casselman Essays on the Fundamental Lemma).

(2) Geometric Langlands and its connection to mathematical physics: https://web.ma.utexas.edu/users/vandyke/notes/langlands_sp21/langlands.pdf; Quantization problem https://arxiv.org/pdf/math/0210466.pdf.

(3) Perverse sheaves, Kazhdan-Lusztig conjectures. See Archar's book, Chriss Ginzburg, Goerdie and Anna's notes, Humphreys book, https://chenhi.github.io/math7390-s21/, Gelfand and Manin books, any many more ...

(4) Algebraic geometry, complex geometry
   (a) do Vakil exercises. Learn sheaf cohomology in more details.
   (b) Moduli space of vector bundles over Riemann surfaces, over $\mathbb{P}^1$ (see Sabin Cautis notes).

(5) Something is myterious to me about "hypergeometric", as it seems appear in many areas. Some references: Zoladek The Monodromy Group (see Geometry folder), Kapranov Hypergeometric functions on reductive groups (Hypergeometric folder), Macdonald hypergeometric functions, Hypergeometric functions over finite fields by Jenny Fuselier, Ling Long, Ravi Ramakrishna, Holly Swisher, Fang-Ting Tu.

   I would like to at least understand all the application in terms of the $._2F_1$ hypergeometric function.

7. November 2021

7.1. **07/11/2021: Coxeter systems.** Today I read the book Introduction to Soergel bimodules, up to §1.1.5. Learnt some interesting ways to describe Coxeter systems type A,B,D via strand diagrams.

7.2. **08/11/2021: Stoke's theorem.** Read Bott Tu Differential forms in Algebraic topology. Learn about Stoke's theorem and its proof.

7.3. **10/11/2021: Valuation theory.** Revise about valuation theory of fields, in particular nonarchimedean valuation from the book Introduction to non-Archimedean Geometry by Piotr Achinger. The goal would be to learn some nonarchimedean geometry and its applications.

7.4. **11/11/2021: Poincare lemmas: De Rham and compactly supported cohomology of $\mathbb{R}^n$.** Today I read some more pages of Bott Tu's book. In particular, I learned on how one computes De Rham and compactly supported cohomology of $\mathbb{R}^n$. Importantly, one needs to prove the Poincare lemma, saying that

$$H^*_{dR}(\mathbb{R}^n \times \mathbb{R}) \cong H^*_{dR}(\mathbb{R}^n)$$

and

$$H^*_c(\mathbb{R}^n \times \mathbb{R}) \cong H^{*-1}_c(\mathbb{R}^n).$$

For the de Rham cohomology, one can show that the projection map $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and the inclusion $\mathbb{R} \hookrightarrow \mathbb{R}^n \times \mathbb{R}$ via $r \mapsto (r, 0)$ induce the isomorphism.in cohomology, i.e. p^*e^* and e^*p^* are homotopic to the identity map.

For the compactly supported cohomology, we have the integration-over-fibers map that induces map $H^*_c(\mathbb{R}^n \times \mathbb{R}) \to H^{*-1}_c(\mathbb{R}^n)$ on cohomology. Together with the map $\Omega^*_c(\mathbb{R}^n) \to \Omega^{*+1}_c(\mathbb{R}^n \times \mathbb{R})$ by adding dt in the $\mathbb{R}$ component of $\mathbb{R}^n \times \mathbb{R}$, these two map induces the isomorphism $H^*_c(\mathbb{R}^n \times \mathbb{R}) \cong H^{*-1}_c(\mathbb{R}^n)$ on cohomology. The compositions of these two maps are homotopic to the identity maps, although the construction of homotopic operators doesn't seem to be trivial to me.

Poincare lemma also holds for any smooth manifold M instead of $\mathbb{R}^n$.

I also learned about degree map: given a proper map of smooth real manifolds $f : \mathbb{R}^n \to \mathbb{R}^n$, one can define pullback of compactly supported cohomology $f^* : \Omega^*_c(\mathbb{R}^n) \to \Omega^*_c(\mathbb{R}^n)$. Choose a compactly supported n-form $\alpha \in \Omega^n_c(\mathbb{R}^n) \cong \mathbb{R}$ with volume 1 when integrating over $\mathbb{R}^n$. The degree map is $\deg f := \int_{\mathbb{R}^n} f^*\alpha$. Surprisingly, this turns out to be an integer. The proof uses Sard's theorem. Actually this situation is very much the same as the compact Riemann surface case, see 05/09/2021.

7.5. **16/11/2021: Tamagawa numbers and unfinished questions.** I finished my honours about Tamagawa numbers today. I still have quite a lot of questions unanswered regarding this topic (see Tamagawa folder). I just want to record some of those questions here and move on first. Hopefully when I get back I can understand more.

(1) Learn Weil's computation of Tamagawa for other groups.
(2) Learn Langlands's proof of computation of Tamagawa number
   - Eisenstein series, then explicit computations in the notes
   - Rewrite Langlands proof in functorial point of view to see the main idea. See Braveman, Kazhdan Representations of affine Kac-Moody groups over local and global fields: a survey of some recent results.
   - geometric Eisenstein series? There is a notion of geometric Eisenstein series, can this be be used to compute Tamagawa `https://web.ma.utexas.edu/users/benzvi/GRASP/lectures/drinfeldEisen.pdf?`
(3) Yang-Mills theory and Tamagawa Numbers: Read Atiyah Bott paper, Harder -¿ What is the connection with Tamagawa?
   - Connections on principal bundles, Yang Mills, `https://www.math.toronto.edu/mein/teaching/moduli.pdf` and http://www.math.columbia.edu/ thaddeus/papers/odense.pdf
   - Equivariant cohomology -¿ Tu's book, Goergie's book.
(4) Lurie Gaitsgory proof:
   - Moduli stack of vector bundles. Weil's conjecture -¿ Frank Neumann; Behrend, The Lefschetz trace formula for algebraic stacks; Algebraic stacks and moduli of vector bundles Frank Neumann; Alper Introduction to Stacks and Moduli.

- Higher algebra: Lurie's lecture notes on Tamagawa. https://web.ma.utexas.edu/users/gregoric/Appendix.pdf

(5) (3.5) More: affine Tamagawa (Braveman, Kazhdan), Tate-Shafarevich groups, condensed mathematics (it seems to define better space to do Haar measure), Tamagawa in Higgs bundles, characteristic varieties, Siegel Weil Formula https://www.math.ucdavis.edu/~mulase/texfiles/charvar.pdf, p-adic integration, motivic integration, height zeta functions https://www.youtube.com/watch?v=ngwax1V1Bgg&t=1529s,

(6) condensed mathematics: how to test this with Tamagawa numbers? Class field theory? Langlands' proof? Adeles? Cohomology using adeles? The point of condensed set is to have good cohomology theory, where do we use cohomology theory in the proof of Tamagawa number? How Haar measures change under exactness of functions? Can we use Lurie Gaitsgory proof for the number field case? Can we use Langlands proof for number field case? What is analogue of Haar measures here? Can we define Haar measure for not just locally compact? For example $G(\mathbb{C}((t)))$? See Frenkel Analytic theory of langlands over complex curves, section 3,4. "The representation theories of the corresponding groups, such as $G(\mathbb{Q}_p)$ and $G(\mathbb{C})$, are known to follow different paths: for the former we have, in the unramified case, the spherical Hecke algebra and the Satake isomorphism. For the latter, instead of a spherical Hecke algebra one usually considers the center of U(g) (or, more generally, the convolution algebra of distributions on G(C) supported on its compact subgroup K"

(7) Relative Langlands: The principle of Tamagawa numbers is that integrating over adelic quotient gives you values of L-functions. This has another name of 'periods in automorphic forms', where we integrate a function instead of just taking the volume. Recently, Yiannis Sakellaridis and Akshay Venkatesh define periods this for spherical varieties, https://arxiv.org/pdf/1203.0039.pdf. The question is whether there is a Tamagawa number formula for general spherical varieties.

(8) There is a general formula $\tau(G) = |Pic(G)|/|Sha(G)|$. Is is valid over function fields? Do we have a description of this in terms of G-bundles? Just like for semisimple simply connected G, we can formulate $\tau(G) = 1$ in terms of G-bundles.

Now, let's learn something different!

7.6. **17/11/2021: Poincare duality, Kunneth formula and Poincare duals.** Last few days I read about Poincare duality, Kunneth formula and Poincare duals from Bott and Tu book. A few technical points:

(1) For oriented manifold (without boundary) M of dimension n, we have a pairing $H^q(M) \otimes H_c^{n-q}(M) \to \mathbb{R}$, obtained by taking wedge product and then integrating over M. Poincare duality states that this pairing induces an isomorphism $H^q(M) \cong (H_c^{n-q}(M))^*$. Bott and Tu only proved this for the case M has finite good cover and use Mayer-Vietoris in §5. Interestingly, one does not always have $H_c^q(M) \cong (H^{n-q}(M))^*$. But in the case of M having finite good cover (cover of M where intersection of any two is $\mathbb{R}^k$), we do have this.

(2) Kunneth formula says that $H^*(M \times F) = H^*(M) \otimes H^*(F)$ as graded $\mathbb{R}$-vector spaces for manifolds M, F where one of them has finite good cover. This can be generalised to Leray-Hirsch theorem about cohomology of fiber bundle where the base space has a finite good cover.

(3) For any closed submanifold S of manifold M of dimension n, via Poincare duality, one can associate to S a Poincare dual $[\eta_S] \in H^{n-k}(M)$ for any $0 \leq k \leq n$. If S is compact, one can associate compact Poincare dual $[\eta'_S] \in H_c^{n-k}(M)$ to S.

7.7. **24/11/2021: Fourier analysis and rep theory.** I read a nice overview from `https://mathoverflow.net/a/37189/89665`, which describes Fourier analysis in a representation theoretic viewpoint.

(1) For locally compact group G, we want to describe $L^2(G)$ as Hilbert space direct integral of matrix coefficients of irreducible unitary reps (not sure what this means), and for this, one has to find the correct measure (Plancherel measure) on set $\widehat{G}$ of irreducible unitary reps.
(2) When G is "Type I", we have such measure of $\widehat{G}$ and a decomposition of $L^2(G)$, but such measure is not explicitly given.
(3) When G is a real semisimple Lie group, Harish-Chandra describes Plancherel measure explicitly. Those irreducible unitary reps appearing in the spectral decomposition of $L^2(G)$ are called *tempered*. Among those tempered, a rep that have positive spectral measure is called *discrete series* rep. The *principal series* representations are non-discrete tempered and account for the spectral decomposition of functions supported on the hyperbolic elements of the group. The *complementary series* are those irreducible unitary reps that are not tempered.

7.8. **28/11/2021: Tate's thesis I: From Dirichlet characters to quasicharacters on $\mathbb{Q}^\times\backslash\mathbb{A}^\times$.** I read this from Stephen S. Kudla notes on Tate's thesis. From Dirichlet characters to quasicharacters on $\mathbb{Q}^\times\backslash\mathbb{A}^\times$:

(1) A classical Dirichlet character $\underline{\chi}_N : \mathbb{Z} \to \mathbb{C}$ modulo N is a function obtained from a character $\chi_N : (\mathbb{Z}/n)^\times \to \mathbb{C}^\times$ by extending to 0 on $\mathbb{Z}/n$ and then pulling back to $\mathbb{Z}$.
(2) Every Dirichlet character $\underline{\chi}_N$ can be viewed as a continuous character χ on $\widehat{\mathbb{Z}}^\times$ by pulling back via $\widehat{\mathbb{Z}}^\times = \varprojlim_n (\mathbb{Z}/n)^\times \to (\mathbb{Z}/n)^\times$.
(3) As $\mathbb{A}^\times \cong \mathbb{Q}^\times \times \mathbb{R}_+^\times \times \widehat{\mathbb{Z}}^\times$ so any Dirichlet character defines a continuous character $\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$, called a quasicharacter. An example of a quasicharacter is $\omega_s(x) = |x|_{\mathbb{A}}^s$ for $s \in \mathbb{C}$.
(4) A character $\omega_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is *unramified* if it is trivial on $\mathbb{Z}_p^\times$, hence can be written in the form $\omega_p(x) = t_p^{\mathrm{ord}_p(x)}$ for some $t_p = t_p(\omega) \in \mathbb{C}^\times$.
(5) A quasicharacter on $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ induces quasicharacters ω_p on $\mathbb{Q}_p^\times$ via the embedding $\mathbb{Q}_p^\times \hookrightarrow \mathbb{A}^\times$. There are finite set of places $S(\omega)$ containing the archimedean places such that ω_v is unramified for all $v \notin S(\omega)$. If ω corresponds to a Dirichlet character $\underline{\chi}_N$ then $S(\omega) = \infty \cup \{p : p \mid N\}$.
(6) It is harder to define Dirichlet character when the number field is not $\mathbb{Q}$, as one does not simply have $\mathbb{A}^\times = \mathbb{Q}^\times \times \mathbb{R}_+^\times \times \widehat{\mathbb{Z}}^\times$ but need to take into account of global units and ideal class group being nontrivial. This is why the adelic point of view is better.
(7) For a quasicharacter ω, one can associate a partial L-function by taking the Euler product
$$L^S(s,\omega) = \prod_{v \notin S(\omega)} (1 - t_v(\omega) q_v^{-s})^{-1},$$
which is absolutely convergent in the half plane $\mathrm{Re}(s) > 1$. The factors
$$L_v(s,\omega_v) := (1 - t_v(\omega) q_v^{-s})^{-1}$$
are called *local L-factors* associated to ω_v.
(8) The goals are to complete L_S to include local L-factors for $v \in S(\omega)$, and then to prove meromorphic analytic continuation and functional equation of the completed L-function.

Some more definition: The *conductor* of $\chi : \widehat{\mathbb{Z}}^\times \to \mathbb{C}^\times$ is the smallest N_0 for which χ is trivial on the kernel of $\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/N_0)^\times$, i.e. χ is the pullback of a unique Dirichlet character χ_{N_0}.

7.9. **28/11/2021: Torus, elliptic curves and double cover of $\mathbb{CP}^1$.** A very rough ideas of what I learned today. We have the correspondences between:

$$A = \{\text{torus}\}$$

with

$$B = \{\text{elliptic curve}\}$$

with

$$C = \{\text{double cover of } \mathbb{P}^1 \text{ ramified at four points}\}.$$

(1) To go between A and B, one uses Weierstrass function: for an elliptic curve $y^2 = (x-a)(x-b)(x-c)$, one can parametrise its points by $(x, y) = (P(t), P'(t))$ for $t \in \mathbb{C}$ where P is Weierstrass function of this curve, which is doubly periodic. Hence, the map $t \mapsto [P(t), P'(t), 1] \in E \cup \{\infty\}$ defines isomorphism between a torus and an (compactified) elliptic curve. See `https://www.math.purdue.edu/~arapura/graph/elliptic.html` for more details.

(2) To go between B and C: Take the projection to x-axis. Then the ramified points are a, b, c and infinity (after compactifying elliptic curve). A more detailed check that this is indeed a ramified covering is at `https://math.stackexchange.com/a/177564/58951`. A rough idea is that the fiber at $x = a, b, c$ gives $y^2 = 0$.

(3) To go between A and C: Quotient the torus $T = \mathbb{C}/\Lambda$ by relation $z \sim -z$ to get $\mathbb{CP}^1$. This gives a double cover $T \to \mathbb{CP}^1$ ramified at 4 points (these 4 coincide at "infinity" in T). A segment between two of these points form a loop in the torus. For the converse, one can use branch-cut constructions, which I still don't quite understand. See for example `https://math.stackexchange.com/q/3474041/58951`.

7.10. **28/11/2021: Construction of derived categories.** The classical way to define derived functors: For a left exact functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ has enough injectives, its i-th derived functor $R^iF : \mathcal{C} \to \mathcal{D}$ applied to $A \in \mathcal{C}$ is the ith cohomology of $0 \to F(I^0) \to F(I^1) \to \cdots$ where $0 \to A \to I^0 \to I^1 \to \cdots$ is an injective resolution of A.

Two fundamental examples of this:

(1) For R-modules A, B, the functor $Tor(\cdot, B)$ is the left derived functor of $\cdot \otimes_R B$ and $Ext(A, \cdot)$ is the right derived functor of $\mathrm{Hom}(A, \cdot)$.

(2) Category $Sh(X)$ of sheaves of vector spaces/abelian groups over X has enough injectives. Since taking global section Γ is left exact, it has a right derived functor $R^i\Gamma$. When X is Hausdorff and paracompact, $R^i\Gamma(\mathcal{F})$ can be seen to be the Cech cohomology $H^i(X, \mathcal{F})$.

The goal of derived categories is to find a better way to talk about derived functors, i.e. we want to compute cohomology in terms of complexes and neglect morphisms between complexes that induce isomorphisms on cohomology.

(1) We start with an abelian category $\mathcal{A}$. We define category $C(\mathcal{A})$ as category of complexes in $\mathcal{A}$. We have cohomology functor $H^i : C(\mathcal{A}) \to \mathcal{A}$.

(2) As two homotopic morphisms in $C(A)$ induce the same cohomology map, and as we only care about cohomology map, we define the *homotopy category* $K(\mathcal{A})$ over $\mathcal{A}$ as: objects same as $C(\mathcal{A})$, but morphisms are morphisms in $C(\mathcal{A})$ modulo $f : A^\bullet \to B^\bullet$ that is homotopic to 0. The cohomology functor descends to $H^i : K(\mathcal{A}) \to \mathcal{A}$.

(3) $K(\mathcal{A})$ is additive but not abelian, so one cannot speak of short exact sequences in the category. However, $K(\mathcal{A})$ is close to being abelian in the sense that it is a *triangulated category*.

(4) $K(\mathcal{A})$ can be seen as a triangulated category as follows:
   (a) We have a *suspension functor* $\Sigma^n : K(\mathcal{A}) \to K(\mathcal{A})$ sending $A^\bullet \mapsto A[n]^\bullet$ which is the shift functor of complexes: $(A[n])^i = A^{i+n}$ and differential $d^i_{A[n]^\bullet} = (-1)^n d^i_{A^\bullet}$.

(b) In general, a triangulated category consists a class of distinguished triangles, i.e. sequences $X \to Y \to Z \to X[1]$ where composition of any two adjacent arrows is 0, that satisfy 4 axioms. If one sees see $Z[-1] \to X$ as homotopy kernel of $X \to Y$ and $Y \to Z$ as homotopy cokernel of $X \to Y$, these four axioms will say that
   (i) The identity map has zero homotopy kernel and cokernel.
   (ii) Every map has a homotopy kernel and cokernel.
   (iii) Any map is the homotopy kernel of its homotopy cokernel and every map is the homotopy cokernel of its kernel.
   (iv) Homotopy kernels and cokernels are 'functorial'.

   A morphism $f : A^\bullet \to B^\bullet$ in $K(\mathcal{A})$, induces a distinguished triangle
$$A^\bullet \xrightarrow{f} B^\bullet \to \mathrm{cone}(f) \to A[1]^\bullet,$$
   hence giving $K(\mathcal{A})$ the structure of a triangulated category.

(5) Lastly, we would like to consider quasi-isomorphism on $K(\mathcal{A})$ to be invertible, as those induces isomorphism on cohomology. This uses localisation construction of category: take S the set of all quasi-isomorphisms in $K(\mathcal{A})$, one can define a unique localisation category $K(\mathcal{A})[S^{-1}]$ of $K(\mathcal{A})$ at S with a functor $Q : K(\mathcal{A}) \to K(\mathcal{A})[S^{-1}]$, that satisfy the similar localisation property for modules. This gives us the *derived category* $D(\mathcal{A})$ of $\mathcal{A}$.

(6) In this case of taking localisation of a triangulated category $K(\mathcal{A})$, we can describe $D(\mathcal{A})$ quite explicitly as follows: Its objects are the same as $C(\mathcal{A})$, but morphisms are given by equivalence classes of *roofs*. A roof from $X^\bullet$ to $Y^\bullet$ is a diagram
$$\begin{tikzcd} & X^\bullet \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$
where $f, g \in K(\mathcal{A})$, f is a quasi-isomorphism. Two roofs $A^\bullet \to B^\bullet$ are equivalent if there is a third roof such that we have a commutative diagram
$$\begin{tikzcd} & Z^\bullet \arrow[dl] \arrow[dr] & \\ X^\bullet \arrow[d] \arrow[drr] & & Y^\bullet \arrow[d] \arrow[dll] \\ A^\bullet & & B^\bullet \end{tikzcd}$$
Note that there is some checking needed for the above description of $D(\mathcal{A})$ to make sense.

7.11. **30/11/2021: Mayer-Vietoris argument.** I noticed many results (Poincare duality, Kunneth formula) have two main ingredients in their proofs: first is to establish the result in the case of $\mathbb{R}^n$ (Poincare lemma), then use the Mayer-Vietoris argument to deal with the general case (given that the manifold has finite good cover).

After realising the importance of the Mayer-Vietoris argument, I want to sketch it here.

First is the case of de Rham cohomology, 8.1. **06/12/2021: Sato's hyperfunctions.** Today I read a bit about Sato's hyperfunctions that somehow generalise distribution theory of Schwarz et al. The hyperfunctions $\mathcal{B}(U)$ on $U \subset \mathbb{R}$ are defined by holomorphic functions on $\tilde{U} \backslash U$ for open $\tilde{U} \subset \mathbb{C}$ satisfying $\tilde{U} \cap \mathbb{R} = U$, modulo holomorphic functions on $\tilde{U}$. This definition does not depend on $\tilde{U}$, so one should think of hyperfunctions as "boundary values" of holomorphic functions.

This $\mathcal{B}$ is actually a sheaf of modules over sheaf of real analytic functions. There is a linear injection from the space of distributions $\mathcal{D}'(\mathbb{R})$ to the space of hyperfunctions $\mathcal{B}(\mathbb{R})$. So hyperfunctions includes Schwartz distributions as a special case and is more explicit in the sense that it is not based on some dual spaces of some smooth functions. Furthermore, with the definition of hypergeometric functions, one can make sense of differentiation and integration using complex analysis. This means one can make sense of integration of distributions, which seem to be of use to physicists.

Some references to read more. What are the applications of these functions to the theory of differential equations? How to define Fourier transform on these? What are the differences between distribution theory and hyperfunctions?

(1) Urs Graf Introduction to Hyperfunctions and Their Integral Transforms
(2) Henrik Schlichtkrull Hyperfunctions and Harmonic Analysis on Symmetric Spaces.

8.2. **15/12/2021: Connection and curvature.** Given a vector bundle $\pi : V \to M$ of (C^∞-real manifolds, complex analytic, smooth algebraic varieties, smooth rigid analytic variety). I learnt of various ways to interpret/define connection and curvature (see wikipedia on connection):

(1) A connection (in the sense of Ehresmann connection) on π tells you how to move between fibers of the bundle in a horizontal direction, i.e. tells you what paths in V are considered 'horizontal'. This is achieved by identifying a 'horizontal' subspace H_x of V_x for each $x \in M$. If one starts with a path on the base M then a horizontal lift of this path on V is not unique, and the failure of uniqueness is measured by curvature. If curvature vanishes, we say the connection is integrable. This interpretation allows one to link integrable connection with monodromy representations. This interpretation of connection is more topological and is in fact valid for any fiber bundle.

    Equivalently, Ehresmann connection is a vector bundle hom $v : TV \to TV$ that is a projection (i.e. $v^2 = v$). The 'horizontal' subbundle can then be identified with $\ker v$. One should view this v as TV-valued 1-form on V, i.e. v as section of $TV \otimes (TV)^* = TV \otimes \Omega(V)^1$.
(2) A connection (in the sense of Koszul/linear connection) as family of differential operators acting on sections of V: by choosing open U of M so $V|_U$ is affine, with local coordinates $x_1, \ldots, x_n$ on U of M, one can define action of $\frac{\partial}{\partial x_i}$ on $\Gamma(U, V)$. Curvature is then the measure of commutativity of these differential operators, i.e. curvature vanishes if $\frac{\partial}{\partial x_i}$ commute with each other.

    A more formal definition: A connection on V is a bundle map $\nabla : V \to V \otimes \Omega^1_M$ which is additive and satisfies Leibniz rule: for any open $U \subset X$, any $f \in \Gamma(U, \mathcal{O}_U)$ and $s \in \Gamma(U, V)$,

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

    A section s is called horizontal if $\nabla(s) = 0$.

    Let T_X be the tangent bundle on X then for $\xi \in T_X(U)$, we can define a (differential) operator

$$\nabla_\xi : \Gamma(U, V) \to \Gamma(U, V), \nabla_\xi(\phi) = \langle \nabla\phi, \xi \rangle.$$

We view ∇_ξ as differentiating sections of V along vector fields ξ. This is the global version of differential operator as defined before. A connection is said to be integrable if the map $\xi \mapsto \nabla_\xi$ is a Lie algebra homomorphism $T_X \to \mathrm{End}(V)$.

Let $\nabla_1 : V \otimes \Omega^1_M \to V \otimes \Omega^2_M$ be the map

$$s \otimes w \mapsto \nabla(s) \wedge \omega + s \otimes d\omega.$$

The curvature is the map $\nabla_1 \circ \nabla$. A connection is integrable is same as saying curvature vanishes (actually I haven't check this).

(3) A connection as vector-valued differential forms (Cartan connection): Note that (Koszul) connection is a map $V \to V \otimes \Omega^1_M$, where one should view $\Omega^p_M(V) := V \otimes \Omega^p_M$ as V-valued p-form on M. Similar to how we define ∇_1 in the definition of curvature, we can define

$$d_\nabla : \Omega^r(V) \to \Omega^{r+1}(V)$$

by

$$d_\nabla(s \otimes \omega) = s \otimes d\omega + (-1)^{\deg \omega} \omega \wedge \nabla s.$$

In this sense, a connection defines a sequence of morphisms $\Omega^*(V)$. It is not a complex as $d_\nabla^2 \neq 0$. In fact, $d_\nabla^2 = 0$ is equivalent to the fact that the curvature vanishes (see more at wikipedia vector bundle). So, *if the connection is integrable, how much de-Rham theory can be applied to flat connection?*

8.3. **17/12/2021: Algebraic de Rham and Hodge-de Rham spectral sequence.** I learned about the algebraic de Rham and the Hodge-de Rham spectral sequence from Kiran S. Kedlaya notes 'p-adic cohomology: from theory to practice' (see more technical notes at the edited pdf file). The main example would be to use the Hodge-de Rham spectral sequence to compute de Rham cohomology of projective spaces.

8.3.1. *Algebraic de Rham.* Let X be a smooth variety over a field k of relative dimension n (smooth scheme of finite type which is reduced and separated). We have the de Rham complex of sheaves on X:

$$0 \to \Omega^0_{X/k} \to \Omega^1_{X/k} \to \cdots \to \Omega^n_{X/k} \to 0.$$

We define the *algebraic de Rham cohomology* of X, denoted $H^i_{dR}(X)$, as the hypercohomology of the de Rham complex.

(1) By definition, take any quasi-isomorphism $\Omega^*_{X/k} \to I^*$ where I^* is a complex of injective elements. The hypercohomology $H^i(\Omega^*_{X/k})$ is defined to be the cohomology of the complex $\Gamma(I^*)$ where Γ is the global section functor.

(2) Instead of using injective resolution to compute de Rham, one can use acyclic resolution. The point is that it is easier to construct acyclic resolution by using Cech complexes.

(3) For a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X, we can construct an acyclic resolution of $\mathcal{F}$ as follows: Let $\{U_i\}_{i \in I}$ be a finite open cover of X by affine open subschemes with I a totally ordered set. Since X is separated, any nonempty finite intersection of U_i's is affine. For $J \subset I$, let $U_J = \bigcap_{i \in J} U_i$. Let $j_J : U_J \to X$ be the open submersion, which is affine and hence, $(i_J)_*$ is exact. Since every quasi-coherent sheaf on affine space is acyclic, $(j_J)_* j_J^* \mathcal{F}$ is acyclic on X. The acyclic resolution $\mathcal{F}^*$ of $\mathcal{F}$ is then
   (a) $\mathcal{F}^i = \bigoplus_J (j_J)_* j_J^* \mathcal{F}$ over all $(i+1)$-elements subset J of I.
   (b) The map $\mathcal{F}^i \to \mathcal{F}^{i+1}$, sends $x_J \in (j_J)_* j_J^* \mathcal{F}$ to element of $\mathcal{F}^{i+1}$ whose component in $\in (j_L)_* j_L^* \mathcal{F}$ with $L = \{j_0 < \cdots < j_{i+1}\}$ is

$$\sum_{h=0}^{i+1} (-1)^h x_{L \setminus \{j_h\}}.$$

In particular, the sheaf cohomology of $\mathcal{F}$ is the given cohomology of this complex after taking global sections. In ith position of this complex is the direct sum of $\Gamma(U_J, j_J^*\mathcal{F})$ over all $(i+1)$-elements subsets J of I.

(c) From the previous point, we have an acyclic resolution $D^{i,\bullet}$ for each $\Omega^i_{X/k}$. This gives us a double complex $D^{p,q} = \bigoplus_J \Gamma(U_J, \Omega^p_{X/k})$ where the sum is over all $(q+1)$-element subsets J of I. To construct an complex of acyclic objects to $\Omega^*_{X/k}$: Let $D^i = \bigoplus_{j+k=i} D^{j,k}$, define $D^i \to D^{i+1}$ by sending an element of D^i with component $x_{j,k} \in D^{j,k}$ to one with component in $D^{j,k}$ being $d_{horizontal}(x_{j-1,k}) + (-1)^j d_{vertical}(x_{j,k-1})$ (here vertical is just restriction maps, horizontal is the differential). Then $D^\bullet$ is an acyclic resolution of $\Omega^\bullet_{X/k}$.

(d) An example of a computation using this method is given by Kedlaya: Suppose $P(x) = x^3 + ax + b \in k[x]$ has no repeated roots. Let $X = \text{Proj}k[X,Y,W]/(Y^2W - X^3 - aXW^2 - bW^3)$ be the complete elliptic curve. Compute de Rham cohomology of X.

8.3.2. *Hodge-de Rham spectral sequence.* We continue our theme of developing methods to compute the de Rham cohomology. Recall we have reduced the task to computing the cohomology of $D^\bullet$.

The complex $D^\bullet$ has a filtration, i.e. a decreasing sequence of complexes:

$$D^\bullet = F^0D^\bullet \supset F^1D^\bullet \supset \cdots F^nD^\bullet \supset F^{n+1}D^\bullet = 0,$$

where $F^pD^i = \bigoplus_{p'+q=i, p'\geq p} D^{p',q}$. From a filtered complex $F^\bullet D^\bullet$, we can construct a *spectral sequence* in the following sense: it is a sequence $\{E_r, d_r\}_{r=r_0}^\infty$ where each E_r is a bigraded group

$$E_r = \bigoplus_{p,q\geq 0} E_r^{p,q}$$

and

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}, d_r^2 = 0$$

is a map such that $E_{r+1}^{p,q} = H^{p,q}(E_r^{p,q}, d_r)$. In particular, from our filtered complex $F^iD^\bullet$, we get

$$E_0^{p,q} = \frac{F^pD^{p+q}}{F^{p+1}D^{p+q}}, E_1^{p,q} = H^{p+q}(\text{Gr}^pD^\bullet), \ldots$$

where $\text{Gr}^pD^\bullet = \frac{F^pD^\bullet}{F^{p+1}D^\bullet}$.

In fact, this spectral sequence converges to E_∞, i.e. there exists $r > 0$ s.t. $E_r = E_{r+1} = \ldots = E_\infty$. And we can describe E_∞ as *associated graded cohomology*

$$E_\infty^{p,q} := \text{Gr}^pH^{p+1}(D^\bullet) := \frac{F^pH^{p+q}(D^\bullet)}{F^{p+1}H^{p+q}(D^\bullet)},$$

where $F^pH^q(D^\bullet)$ is the *filtered cohomology* of $D^\bullet$, defined as

$$F^pH^q(D^\bullet) = \text{image}(H^q(F^pD^\bullet) \to H^q(D^\bullet)).$$

In our case where $D^\bullet$ comes from the de Rham complex, we find

$$E_1^{p,q} = H^q(X, \Omega^p_{X/k})$$

and it is a theorem that this *Hodge-de Rham spectral sequence* degenerates at E_1 (i.e. $E_1 = E_2 = \ldots = E_\infty$). Therefore, if we know the sheaf cohomology of $\Omega^i_{X/k}$, we know E_∞. Then, by using the formula for E_∞ and that $F^{n+1} = 0$, we can compute the filtered cohomology, in particular, we know $F^0H^q(D^\bullet) = H^q(D^\bullet)$.

An exercise of this method would be to compute the de Rham cohomology of $\mathbb{P}^n$.

8.3.3. *More things.*

(1) See more of algebraic de Rham at Stack Project: `https://stacks.math.columbia.edu/tag/0FK4`
(2) A blog post of Alex Youcis: `https://ayoucis.wordpress.com/2015/07/22/algebraic-de-rham-cohomo`
(3) Future exercise: Deduce Picard-Fuchs equations from Gauss-Manin connection. Why do we care about these? A reference `https://virtualmath1.stanford.edu/~conrad/shimsem/2013Notes/Littvhs.pdf`.

8.4. **17/12/2021: Derived functors of sheaves.** Just want to make a summary about the story of derived functors of sheaves. For the reference, say I learned this from Konstanze Rietsch, An introduction to perverse sheaves, Achar Introduction to Perverse sheaves. You can say it is a continuation from 28/11/2021.

We always assume X is a nice topological space (locally compact, Hausdorff, paracompact, with a countable basis, locally simply connected).

(1) The right-derived functor $RF : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ (of bounded derived categories) of a left-exact functor $F : \mathcal{A} \to \mathcal{B}$ is defined as $RF(\mathcal{F}^\bullet) = F(\mathcal{I}^\bullet)$, where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is an injective resolution (i.e. a quasi-isomorphism with $\mathcal{I}^\bullet$ consists of injective elements). For this to make sense, we also need $\mathcal{A}$ to have enough injectives.
(2) The category of sheaves of vector spaces/abelian groups/modules on X has enough injectives. This is done by using Godement resolution. It follows that every complex of sheaves has an injective resolution (Cartan-Eilenberg resolution).
(3) Some example of functors: Let $f : X \to Y$ be a continuous map. We can define *push-forward* (or *direct image*) functor f_*, *push-forward with proper supports* functor $f_!$ by

$$(f_*\mathcal{X})(U) = \mathcal{X}(f^{-1}(U)),$$
$$(f_!\mathcal{X})(U) = \{s \in \mathcal{X}(f^{-1}(U)) : f : \text{supp } (s) \to U \text{ is proper}\}.$$

These are left-exact functors. If $\pi : X \to \{pt\}$, we find $\pi_* = \Gamma$, the global section functor and $\pi_! = \Gamma_c$, compactly supported global section functor. If $j : X \hookrightarrow Y$ is an inclusion of locally closed subset, $j_!$ is the *extension by* 0 functor, and is exact.

We can also define *pull-back* functor $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ (some people denote this f^{-1} in order to distinguish with pull-back of sheaf of $\mathcal{O}_X$-modules) The key property of this functor is that $(f^*\mathcal{Y})_x = \mathcal{Y}_{f(x)}$. It is an exact functor. Achar notes has a table on more of this.
(4) To compute derived functor RF, injective resolutions are not practical tools. Instead, one use a large class of sheaves, called F-*acyclic* (it is a theorem that derived functor can be computed by acyclic resolution): For left-exact functor $F : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. A sheaf $\mathcal{A} \in \mathrm{Sh}(X)$ is F-acyclic if $R^iF(\mathcal{A}) = 0$ for all $i \neq 0$. So, if $\mathcal{F}^\bullet \in D^b(X)$ is quasi-isomorphic to F-acyclic complex $\mathcal{A}$, then $RF(\mathcal{F}^\bullet) = F(\mathcal{A}^\bullet)$.
(5) In the context of sheaves and $F = f_*, f_!, ...$, there are ways to characterise F-acyclic sheaf. For example, injective $\subset$ flabby $\subset$ solf are all f_*-acyclic sheaf. When I mean 'characterise', I mean somthing like: a sheaf $\mathcal{F}$ is flabby (or flasque) if the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for all open $U \subset X$. Now I know why there are so many adjectives to describe sheaves ...

OK, now I want to see some computations ...

8.5. **31/12/2021: Van Kampen theorem.** I learned about some applications of Van Kampen theorem from `https://people.math.harvard.edu/~hirolee/pdfs/231a-35-van-kampen.pdf` notes by Omar Antolin Camarena.

Let $\pi_{\leq 1}(X)$ be the fundamental groupoid of X, i.e. a category whose objects are the points of X and whose morphisms from p to q are the homotopy classes of paths in X from p to q. The classical fundamental group of X at the based point $x_0 \in X$ is $\mathrm{Hom}_{\pi_{\leq 1}(X)}(x_0, x_0)$.

Here is the statement of the theorem, in the language of groupoids:

(1) No based points version: Let X be a topological space and U, V be open subsets of X such that $X = U \cup V$. Then the following diagram is a pushout of groupoids

$$\begin{tikzcd} \pi_{\leq 1}(U \cap V) \arrow[r] \arrow[d] & \pi_{\leq 1}(U) \arrow[d] \\ \pi_{\leq 1}(V) \arrow[r] & \pi_{\leq 1}(X) \end{tikzcd}$$

In other words, $\pi_{\leq 1}$ preserves pushout square. To generalise this over any open cover $\mathcal{U}$ of X where any finite intersection of elements in $\mathcal{U}$ belongs in $\mathcal{U}$, $\pi_{\leq 1} X = \mathrm{colim}_{U \in \mathcal{U}} \pi_{\leq 1}(U)$.

(2) Based points version: For $A \subset X$, let $\pi_{\leq 1}(X, A)$ denotes the full subcategory of $\pi_{\leq 1}(X)$ whose objects are in A. To have a pushout square of $\pi_{\leq 1}(\cdot, A)$, we need an extra condition: A contains at least one point in each connected component of each of $U \cap V, U$ and V. Then we have a pushout of groupoids

$$\begin{tikzcd} \pi_{\leq 1}(U \cap V, A) \arrow[r] \arrow[d] & \pi_{\leq 1}(U, A) \arrow[d] \\ \pi_{\leq 1}(V, A) \arrow[r] & \pi_{\leq 1}(X, A) \end{tikzcd}$$

Some applications

(1) Compute $\pi_1(S^1)$: Take U, V two open intervals s.t. $S^1 = U \cup V$ and $U \cap V$ is a union of two disjoint open intervals in which lies points p, q, respectively. Let $A = \{p, q\}$. As U is contractible, $\pi_{\leq 1}(U, A)$ consists of two points p, q with a unique morphism from p to q and only identity map from p to itself. Similarly for $\pi_{\leq 1}(V, A)$. $\pi_{\leq 1}(U \cap V, A)$ consists of two points p, q with no morphism from p to q or from q to p. By Van Kampen, there are only two morphisms from p to q in $\pi_{\leq 1}(X, A)$, taken from those in $\pi_{\leq 1}(U, A)$ and $\pi_{\leq 1}(V, A)$, denoted as u, v. As $\pi_{\leq 1}(X, A)$ is the initial object in the pushout square, $(v^{-1} \circ u)^n$ are all distinct as all morphisms from p to itself in $\pi_{\leq 1}(X, A)$. Thus, $\pi_1(S^1) = \mathbb{Z}$.

(2) Compute π_1 of Klein bottle: The Klein bottle K is obtained from the square

$$\begin{tikzcd} {} \arrow[r, "a"] \arrow[d, dotted, "b"'] & {} \arrow[d, dotted, "b"] \\ {} & {} \arrow[l, "a"] \end{tikzcd}$$

by glueing paths a and b. To compute $\pi_1(K)$, draw a smaller square inside this square. Let U be the smaller square and its interior, V be the area between two squares. Then $U \cap V$ is the smaller square and is homotopy equivalent to S^1, U is contractible and V is wedge of two circles formed by a and b (V is essentially the larger square without its interior, and when one glues a and b together, we get two circles with a common point). Thus, $\pi_1(V)$ is a free group with generators a, b, the generator of $\pi_1(U \cap V)$ is sent to the loop $b^{-1}aba$ in $\pi_1(V)$. Thus, by Van Kampen, $\pi_1(K)$ is $\langle a, b \rangle *_{\mathbb{Z}} 1 = \langle a, b | b^{-1}aba \rangle$.

(3) Similarly, for the torus T of genus 1, we find $\pi_1(T) = \langle a, b | [a, b] = 1 \rangle$. And for real projective space P, we find $\pi_1(P) = \langle a, | a^2 = 1 \rangle$.

(4) π_1 behaves well with respect to taking wedge of spaces and taking connected sums.

(5) Attaching cells of dimension at least 3 to a CW complex X do not affect the fundamental group of that complex. Attaching a 2-cell to X then the new space is has fundamental group being the quotient of $\pi_1(X)$ the relation formed by the boundary of that 2-cell. See `https://www.homepages.ucl.ac.uk/~ucahjde/tg/html/vkt02.html` for a more detailed explanation.
(6) Compute fundamental group of mapping tori: See `https://www.homepages.ucl.ac.uk/~ucahjde/tg/html/vkt03.html`. I haven't read this. A particular example is fundamental group of Klein bottle (as in the link), or fundamental group of knots complements (see `https://www.homepages.ucl.ac.uk/~ucahjde/tg/html/braids03.html`, key words: braid). *I haven't read this.*

9.1. **06/01/2022: Motivic integration.** I learned something about motivic integration. So far I can only sketch the ideas. The motivation for motivic integration comes from p-adic integration.

Here is a table of comparison:

p-adic integration	motivic integration
$f \in \mathbb{Z}[x_1, \ldots, x_m]$, $X = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_m]/(f)$	$f \in \mathbb{C}[x_1, \ldots, x_m], X = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_m]/(f)$
$X(\mathbb{Z}/p^{n+1}) = \{$solutions of $f = 0$ over $\mathbb{Z}/p^{n+1}\mathbb{Z} \cong \mathbb{Z}_p/p^{n+1}\mathbb{Z}_p\}$	$\mathcal{L}_n(X)(\mathbb{C}) = \{$solutions of $f = 0$ over $\mathbb{C}[t]/(t^{n+1}) \cong \mathbb{C}[[t]]/(t^{n+1})\}$, called *n-jets*
$X(\mathbb{Z}_p) = \{$solution of f over $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^{n+1}\mathbb{Z}\}$ denote $\pi_m : X(\mathbb{Z}_p) \to X(\mathbb{Z}/p^m\mathbb{Z})$	$\mathcal{L}(X)(\mathbb{C}) = \{$solutions of $f = 0$ over $\mathbb{C}[[t]] = \varprojlim \mathbb{C}[t]/(t^{n+1})\}$, called *arcs* on X $\mathcal{L}(X)$ is obtained by taking inverse limit of $\mathcal{L}_n(X)$, denote $\pi_m : \mathcal{L}(X) \to \mathcal{L}_m(X)$
measure space $X(\mathbb{Z}_p)$	measure space $\mathcal{L}(X)$
measurable sets in $\mathbb{Z}_p$ are generated by open sets in $\mathbb{Z}_p$, i.e. union of cosets of $p^n\mathbb{Z}_p$	*cylinders*, i.e. preimages under π_n of constructible sets in $\mathcal{L}_n(X)$ for some n (constructible subsets in a Noetherian top space is the smallest family of subsets that are obtained from open sets by taking finite intersection and taking complements), are measurable *What is a natural def of measurable sets, constructed from cylinders? Need to read Loeser notes*
measure with values in $\mathbb{R}$: $p \in \mathbb{R}$ $\mu_p(p\mathbb{Z}_p) = p^{-1}$ $p^{-n} \to 0$ as $n \to \infty$ in $\mathbb{R}$	*motivic measure* with values related to $K_0(Var_{\mathbb{C}})$, the Grothendieck ring of algebraic varieties over $\mathbb{C}$, i.e. reduced and separated schemes of finite type over $\mathbb{C}$: $\mathbb{L} = [\mathbb{A}^1] \in K_0(Var_{\mathbb{C}})$ $\mu(t\mathbb{C}[[t]]) = \mathbb{L}^{-1} \in \mathcal{M}_{\mathbb{C}} := K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$ $\mathbb{L}^{-n} \to 0$ as $n \to \infty$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$, the completion of $\mathcal{M}_{\mathbb{C}}$ with respect to its filtration by $F^m\mathcal{M}_{\mathbb{C}}$, a subgroup generated by elements $[S]\mathbb{L}^{-i}$ with $\dim S - i \leq -m$
if X is smooth of dimension ℓ, $\mu_p(X(\mathbb{Z}_p)) = \frac{\lvert\pi_n(X)\rvert}{p^{\ell n}}$ for all $n \geq 1$	if X is smooth of dimension ℓ, then for cylinder $A = \pi_m^{-1}(C)$ in $\mathcal{L}(X)$ where C is constructible in $\mathcal{L}_m(X)$, $\mu(A) := \frac{[\pi_n(A)]}{\mathbb{L}^{n\ell}}$ is stable in $\mathcal{M}_{\mathbb{C}}$ for all $n \geq m$
for singular ℓ-dimensional subvariety Z of $\mathbb{Z}_p^m$, $\mu_p(Z) := \lim_{\epsilon \to 0} \mu_p(Z \setminus B_\epsilon(Z_{\text{sing}}))$, where B_ϵ denotes small tubular neighborhood of radius ϵ. Oesterle showed $\mu_p(Z) = \lim_{n\to\infty} \frac{\lvert\pi_n(Z)\rvert}{p^{n\ell}}$	for a complex algebraic variety X of dimension ℓ, if $A \subset \mathcal{L}(X)$ is a cylinder (more generally, *semi-algebraic set*), then the limit $\mu(A) := \lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{n\ell}}$ exists in $\widehat{\mathcal{M}}_{\mathbb{C}}$
Let Y be reduced closed subscheme of X of codimension at least 1, then $Y(\mathbb{Z}_p)$ has measure 0 in $X(\mathbb{Z}_p)$ (see Mihnea Popa notes p-adic integration)	Veys' notes on p.10 mentioned that for closed subvariety Z of codimension at least 1 of X, $\mathcal{L}(Z)$ has measure 0 in $\mathcal{L}(X)$

Let f be a $\mathbb{Q}_p$-analytic function on measurable subset A of $\mathbb{Z}_p^m$, we find $\int_A \lvert f\rvert_p^s \lvert dx\rvert = \sum_{m\in\mathbb{Z}} \mu_p(\mathrm{ord}(f)=m)p^{-ms}$ assuming the RHS converges in $\mathbb{R}$ *This is only just a part of p-adic integration. For example, replace $\mathbb{Z}_p^m$ by $X(\mathbb{Z}_p)$, dx by volume form on X, and $\lvert f\rvert_p^s$ by arbitrary complex-valued φ. Do we have analogues for these in motivic integration?*	Let A be a measurable subset of $\mathcal{L}(X)$, and $\alpha : A \to \mathbb{Z}\cup\{\infty\}$ be a function s.t. all its fibers are measurable. We say $\mathbb{L}^{-\alpha}$ is integrable if the series $\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{m\in\mathbb{Z}\cup\infty} \mu(A\cap\alpha^{-1}(n))\mathbb{L}^{-n}$ converges in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

I am missing a comparison between change-of-variables formula between two sides ...

Some references I read this from:

(1) Mihnea Popa `https://people.math.harvard.edu/~mpopa/571/index.html`
(2) Devlin Mallory notes Motivic integration
(3) Francois Loeser Arizona winter school notes
(4) Willem Veys Arc spaces, motivic integration and stringy invariants.

Interestingly, there seems to be something called "motivic Poisson summation formula" by Ehud Hrushovski and David Kazhdan. *How far have people extended this dictionary?* Can one relate Langlands' proof of Tamagawa number with Gaitsgory and Lurie proof using this dictionary?

What are the applications of motivic integration?

9.2. **07/01/2021: Factorisation algebras.** I have heard this word "factorisation homology" so many times (in Lurie Gaitsgory proof of Tamagawa number conjecture, in David Ben-Zvi notes about topological quantum field theory, in Lurie higher category theory ...) so I tried to learn a bit about it from a notes of Ryan Mickler and Brian Williams about *factorisation algebras.* See the book Kevin Costello and Owen Gwilliam, Factorization algebras in quantum field theory for more details.

Now I want to record some definitions of this and some examples.

9.2.1. *Definition.* A motivation for this definition is that we want an axiomatic way of talking about the algebra of local observables in a field theory, along with the relationships between observables on different open sets.

Definition 29. *A prefactorisation algebra $\mathcal{F}$ on a topological space M, with values in $Vect^{\otimes n}$ (symmetric monoidal category of vector spaces), is an assignment of a vector space $\mathcal{F}(U)$ for each open set $U\subset M$ together with the following data:*

(1) for an inclusion $U\to V$, a map $\mu_U^V : \mathcal{F}(U)\to\mathcal{F}(V)$;
(2) for a finite collection of disjoint opens $\bigsqcup_{i\in I} U_i \subset V$, an $S_{\lvert I\rvert}$-equivariant map $\bigotimes_{i\in I}\mathcal{F}(U_i)\to\mathcal{F}(V)$.

Of course we need compatibility condition with taking composition.

Note that $\mathcal{F}(\emptyset)$ must be a commutative algebra from this definition.

For the precosheaf $\mathcal{F}$ to be a factorisation algebra, we need it to be a cosheaf with respect to a special topology.

Definition 30. *An open cover $\{U_i\}_{i\in I}$ of U is a Weiss cover if for every finite collection of points $\{x_j\}\subset U$, there exists an element U_k of the cover such that $\{x_j\}\subset U_k$.*

Definition 31. *A factorisation algebra $\mathcal{F}$ on X is a prefactorisation algebra $\mathcal{F}$ on X so that for any open set $U \subset X$ and Weiss cover $\{U_i\}$ of U, the sequence*

$$\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \to \bigoplus_k \mathcal{F}(U_k) \to \mathcal{F}(U) \to 0$$

is exact.

Some remarks:

(1) The choice of topology indicates that in a factorisation algebra, observables on an open set U are "determined" by observables in small neighborhoods of finitely many points.
(2) Suppose $\{U_i\}$ is a Weiss cover on U then $\{U_i^{\times n}\}$ is an open cover on $U^{\times n}$. This implies that Weiss cover on X induces a topology on the Ran space, the collection of all finite subsets of X, denoted $\mathrm{Ran}(X)$. An equivalent way to define factorisation algebra: It is a cosheaf on the Ran space.
(3) One can define factorisation algebras with values in any symmetric monoidal category, for example category of chain complexes of certain additive category where we invert weak equivalences.

9.2.2. *Defects/domain walls as factorisation algebras.* In QFT, we consider *defects*, which are certain class of operators attached to submanifolds. One dimesional defects are called domain walls, where the walls/submanifolds are just points. We will construct factorisation algebras on $\mathbb{R}$ with values in the category $dgVect$ of differential graded vector spaces, where we invert weak equivalences.

(1) No wall: For any (unital) associative algebra A, one can construct a prefactorisation algebra $\mathcal{F}_A$ on $\mathbb{R}$ as follows
    - We assign $\mathcal{F}_A((a,b)) = A$ to each open interval (a,b).
    - For any open set $U = \bigsqcup_j I_j$ where each I_j is an open interval, we set $\mathcal{F}(U) = \bigotimes_j A$.
    - The inclusion $U = \bigsqcup_S I_s \hookrightarrow V = \bigsqcup_T I_t$ of open sets where $S \subset T$ induces $\mathcal{F}(U) \to \mathcal{F}(V)$ by tensoring with $1 \in A$ for every $x \in T \notin S$.
    - $\mathcal{F}_A(\emptyset) = \{0\}$ and the inclusion $\emptyset \to (a,b)$ sends 0 to $1 \in A = \mathcal{F}((a,b))$.
    - The structure map $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(S)$ where $U \sqcup V \subset S$ is from the multiplication of A.

    It is a *locally constant* prefactorisation algebra as $\mathcal{F}_A((a,b)) \to \mathcal{F}_A((c,d))$ are isomorphisms for any inclusion $(a,b) \subset (c,d)$.

    One can check the cosheaf condition to find that this is a factorisation algebra (I think the sequence in the cosheaf condition holds for any cover not just Weiss cover).
(2) One wall: Let A, B be associative algebras. Fix a point $p \in \mathbb{R}$ and a $A - B$ bimodule M. We construct a prefactorisation algebra $\mathcal{F}_{A,M,B}$ on $\mathbb{R}$ as follows
    - $\mathcal{F}_{A,M,B}$ restricted to $(-\infty, p)$ is the locally constant prefactorisation algebra $\mathcal{F}_A$ associated to A; $\mathcal{F}_{A,M,B}$ restricted to (p, ∞) is $\mathcal{F}_B$; $\mathcal{F}_{A,M,B}$ on any open interval containing p is M.
    - The structure maps come from the algebra structures of A, B and the bimodule structure of M. For example,

    $$\mathcal{F}((a_1, a_2)) \otimes \mathcal{F}((a,b)) \to \mathcal{F}((a_1, b))$$

    where $a_1 < a_2 < a < p < b$ corresponds to $A \otimes M \to M$.
    - For any open set U of $\mathbb{R}$, the inclusion $\emptyset \to U$ induces a map $\{pt\} \to \mathcal{F}(U)$ that gives rise to an element $m_U \in M$.

    Is this a factorisation algebra?

(3) Two walls: One can also construct prefactorisation algebra $\mathcal{F}_{A,M,B,N,C}$ with two fixed points: $A-B$ bimodule M attached to $p \in \mathbb{R}$, $B-C$ bimodule N attached to $q \in \mathbb{R}$ with $p < q$. The only new information we need to prescribe is for open sets that contain both p and q. Denote V to be the image of $\mathcal{F}_{A,M,B,N,C}$ under these open sets. From the structure map, we know that there is a map $M \otimes N \to V$. We also this map must factorise through the internal B actions $(M \otimes B) \otimes N \to V$ and $M \otimes (B \otimes N) \to V$, hence there is a map $M \otimes_B N \to V$. However, this map may not be surjective. In fact, for $\mathcal{F}$ to be a factorisation algebra, we need $V = M \otimes_B^L N$, the derived tensor product of M and N. *Try to do this?*

(4) *What happen if there are 3 walls? Higher dimension than* 1*?*

9.2.3. *Factorisation envelope.* We construct a factorisation algebra on $\mathbb{R}$ with values in the category $dgVect$ of differential graded vector spaces, where we invert quasi-isomorphisms.

(1) For a Lie algebra $\mathfrak{h}$, we can construct the Chevalley-Eilenberg chain complex of $U\mathfrak{h}$-module M by

$$C_*(\mathfrak{h}, M) := \mathrm{Sym}(\mathfrak{h}[1]) \otimes_k M.$$

Here $\mathrm{Sym}(\mathfrak{h}[1]) = \bigoplus_n \wedge^n \mathfrak{h}$ is a dg vector space with elements in $\mathfrak{h}$ of degree 1. The differential of $C_*(\mathfrak{h}, M)$ is defined to be

$$\begin{aligned} h_1 \wedge \cdots \wedge h_n \otimes x \mapsto & \left(\sum_{1 \le j < k \le n} (-1)^{j+k} [h_j, h_k] \wedge h_1 \wedge \cdots \widehat{h_j} \cdots \widehat{h_k} \cdots \wedge h_n \right) \otimes x \\ & + \sum_i (-1)^{n-i} h_1 \wedge \cdots \widehat{h_i} \cdots \wedge h_n \otimes (h_i m). \end{aligned}$$

The homology of $C_*(\mathfrak{h}, M)$ is precisely Lie algebra homology $H_*(\mathfrak{h}, M) := k \otimes_{U\mathfrak{h}}^L M$. Furthermore, one can construct $C_*(\mathfrak{h}, M)$ when $\mathfrak{h}$ is a dg Lie algebra, by viewing $C_*(\mathfrak{h}, M)$ as a chain complex of chain complexes, then take totalization *(I know how to define the grading of $C_*(\mathfrak{h}, M)$ but no clue how to define the differential ...)*

(2) For open $U \subset \mathbb{R}$, we consider a differential graded Lie algebra given by

$$\mathcal{L}(U) := \Omega_c(U) \otimes_k \mathfrak{h},$$

with the differential comes from the exterior derivative of the differential forms, and the Lie bracket comes from $\mathfrak{h}$. The *factorisation envelop* of $\mathfrak{h}$ is the prefactorisation algebra defined by

$$\mathbb{U}\mathfrak{h} : U \mapsto H_*(C_*(\mathcal{L}(U), k)).$$

(3) One can show $(\mathbb{U}\mathfrak{h})(\mathbb{R}) = U\mathfrak{h}$, i.e. one recovers the universal enveloping algebra from this factorisation algebra (see my edited notes of Mickler and Williams notes).

9.3. **09/01/2021: State-sum TQFTs.** I learned that *state-sum Topological Quantum Field Theory* means one construct TQFT locally by assigning each n-manifold certain algebraic data that depends on its triangulation. One then shows the algebraic construction does not depend on triangulation, hence we get a TQFT. And this independence of triangulation seems to suggest what kind of algebraic data we want.

I read this from Aaron D. Lauda and Joshua Sussan 'An Invitation to Categorification', January 2022 Notices of the AMS.

9.4. **14/01/2022: Derived homs and derived tensor product of sheaves.** The notations are as in 28/11/2021 and 17/12/2021. Let X be nice topological space, $D^b(X)$ be the bounded derived category of sheaves of k-modules on X. I want to record some definitions and properties of three derived functors

$$R\operatorname{Hom}, R\underline{\operatorname{Hom}}, \otimes^L.$$

I learned this from Achar's book on Perverse sheaves and Representation theory, §1.4 and §A.6.

(1) We can extend the Hom functor $\operatorname{Hom} : Sh(X)^{op} \times Sh(X) \to k\text{-mod}$ to $C^b(X)^{op} \times C^b(X) \to C^b(k\text{-mod})$ by sending $A, B \in C^b(X)$ to the chain complex $\operatorname{Hom}(A,B)^n := \bigoplus_{j-i=n} \operatorname{Hom}(A^i, B^j)$ with the differential given by $d(f) := d_B \circ f + (-1)^{j-i+1} f \circ d_A$ for $f \in \operatorname{Hom}(A^i, B^j)$. It induces the derived Hom functor

$$R\operatorname{Hom} : D^b(X)^{op} \times D^b(X) \to D^b(k\text{-mod})$$

that is triangulated in both variables.

(2) There is a natural isomorphism

$$\operatorname{Hom}_{D(X)}(A,B) \cong H^0(R\operatorname{Hom}(A,B))$$

for $A, B \in D^b(X)$.

(3) One can also defined the sheaf Hom functor $\underline{\operatorname{Hom}} : Sh(X)^{op} \times Sh(X) \to Sh(X)$ by $\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})(U) := \operatorname{Hom}_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$. This functor is left-exact in both variables, and once can extend this to chain complexes as in the case of Hom, therefore inducing the derived functor

$$R\underline{\operatorname{Hom}} : D^b(X) \times D^b(X) \to D^b(X).$$

(4) Relation between Hom and $\underline{\operatorname{Hom}}$: There is a natural isomorphism

$$R\Gamma(R\underline{\operatorname{Hom}}(A,B)) \xrightarrow{\sim} R\operatorname{Hom}(A,B)$$

where $A, B \in D^b(X)$.

(5) If $f : X \to Y$ continuous then Rf_* and f^* are adjoint pairs of functors. This follows from the natural isomorphism

$$Rf_* R\underline{\operatorname{Hom}}(f^*A, B) \cong R\underline{\operatorname{Hom}}(A, Rf_*B),$$

where $A \in D^b(Y), B \in D^b(X)$ and the previous two natural isomorphisms. Indeed, take $R\Gamma$ on both sides to get $R\operatorname{Hom}(f^*A, B) \cong R\operatorname{Hom}(A, Rf_*B)$, then take H^0 to get $\operatorname{Hom}_{D(X)}(f^*A, B) \cong \operatorname{Hom}_{D(Y)}(A, Rf_*B)$.

(6) For $\mathcal{F}, \mathcal{G} \in Sh(X)$, the tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of the presheaf $\mathcal{F} \otimes_{pre} \mathcal{G}$ defined by $(\mathcal{F} \otimes_{pre} \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U)$. This functor $\otimes$ is right exact so it induces the left derived functor

$$\otimes^L : D^b(X) \times D^b(X) \to D^b(X)$$

by taking flat resolutions (every sheaf has a flat resolution, and flat resolutions can be used to computed left derived functors).

(7) For $F, G, H \in D^b(X)$, we have natural isomorphisms

$$R\underline{\operatorname{Hom}}(F \otimes^L G, H) \cong R\underline{\operatorname{Hom}}(F, R\underline{\operatorname{Hom}}(G, H)).$$

Similarly, this implies $\otimes^L$ and $R\underline{\operatorname{Hom}}$ are adjoint pairs of functors over $D^b(X)$.

9.5. **17/01/2022: Vakil §5.2 - Reducedness and integrality of schemes.**

5.2.A (Reducedness is a stalk-local property) A scheme X is reduced iff none of the stalks have nonzero nilpotents. Indeed, if one stalk $\mathcal{O}_{X,x}$ has nonzero nilpotent, by definition, there exists open U of x so that $\mathcal{O}_X(U)$ has nonzero nilpotent, implying X is not reduced. If none of the stalks have nonzero nilpotents, as we have $\mathcal{O}_X(U) \hookrightarrow \prod_{x\in U} \mathcal{O}_{X,x}$ from 2.4.A, we find $\mathcal{O}_X(U)$ is reduced.

If f and g are two functions on a reduced scheme X that agree at all points, then $f = g$. Pick any affine open subset $\operatorname{Spec} A$ of X, then $f - g$ vanishes at every point in $\operatorname{Spec} A$, implying $f - g$, viewed as element in A, lies in the intersection of all prime ideals of A, implying $f - g$ is nilpotent in A by 3.2.12. But X is reduced, hence $\operatorname{Spec} A$ is also reduced, meaning $f = g$ in A, or $f = g$ in $\mathcal{O}_X(\operatorname{Spec} A)$. As this holds for every affine open subset of X, we find $f = g$ in $\mathcal{O}_X(X)$.

5.2.B If ring A is reduced then $A_\mathfrak{p}$ is also reduced for any prime ideal $\mathfrak{p}$ of A. Hence, $\operatorname{Spec} A$ is reduced from 5.2.A. Because $k[x_1, \dots, x_n]$ is reduced so $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are reduced for any field k.

5.2.C The scheme $X = \operatorname{Spec} k[x, y]/(y^2, xy)$ is nonreduced because y is nilpotent in $k[x, y]/(y^2, xy)$. We can try to draw this scheme and see that it is an x-axis with a fuzz at the origin in the direction of y-axis (this can be seen by intersecting $xy = 0$ with $y^2 = 0$).

To see the nonreducedness geometrically means to see the fuzz at the origin. To see the fuzz at the origin, we consider the stalk $\big(k[x, y]/(y^2, xy)\big)_{(x,y)}$ at the origin. This has nonzero nilpotent element y.

For other points, i.e. for any other prime ideal $\mathfrak{p}$ of $k[x, y]/(y^2, xy)$, we show that $\big(k[x, y]/(y^2, xy)\big)_\mathfrak{p}$ is reduced. Indeed, prime ideal $\mathfrak{p}$ of $k[x, y]/(y^2, xy)$, viewed as prime ideal of $k[x, y]$ that contains (y^2, xy), must contains y as $y^2 \in \mathfrak{p}$. Thus, $\mathfrak{p}$ is either (y) or $(x - a, y)$ for $a \in k$.

We want to show $\big(k[x, y]/(y^2, xy)\big)_{(x-a,y)}$ is reduced for $a \in k^\times$. Visually, we take the x-axis with a fuzz at the origin and zoom in at point $x = a$. This suggests that the ring is isomorphic to $k[x]_{(x-a)}$. Indeed, because localisation commutes with taking quotients, we find

$$\big(k[x, y]/(y^2, xy)\big)_{(x-a,y)} \cong k[x, y]_{(x-a,y)}/(y^2, xy)_{(x-a,y)}.$$

As $a \neq 0$ so $x \notin (x-a, y)$, meaning x is invertible in $(y^2, xy)_{(x-a,y)}$, implying $y = x^{-1}\cdot(xy) \in (y^2, xy)_{(x-a,y)}$ or $(y^2, xy)_{(x-a,y)} = (y)_{(x-a,y)}$. Therefore, $k[x, y]_{(x-a,y)}/(y^2, xy)_{(x-a,y)} \cong k[x, y]_{(x-a,y)}/(y)_{(x-a,y)} \cong (k[x, y]/(y))_{(x-a,y)} \cong k[x]_{(x-a)}$, which is reduced.

The ring $\big(k[x, y]/(y^2, xy)\big)_x$, understood as $k[x, y]/(y^2, xy)$ localised at $\{1, x, x^2, \dots\}$, is reduced. To visualise this, one sees $k[x, y]/(y^2, xy)$ as x-axis with a fuzz at the origin, then to localise at x means to throw out points that vanish at x, meaning we throw out the fuzz point, i.e. the origin. What we left is just the x-axis minus the origin, i.e. $\operatorname{Spec} k[x]_x$. This suggests we define a map $k[x, y]/(y^2, xy) \to k[x]_x$ sending $y \mapsto 0$. As $x^n \mapsto x^n$ invertible in $k[x]_x$, we obtain an induced map $\big(k[x, y]/(y^2, xy)\big)_x \to k[x]_x$. This is an isomorphism with inverse being the inclusion map $k[x]_x \to \big(k[x, y]/(y^2, xy)\big)_x$.

5.2.D If X is a quasicompact scheme, and if $\mathcal{O}_{X,x}$ is reduced for all closed points x, then X is reduced. It suffices to show every nonreduced point has a nonreduced closed point in its closure. Indeed, consider nonreduced point $p \in X$, i.e. $\mathcal{O}_{X,p}$ is nonreduced. By 5.1.E, as X is quasicompact, there exists a closed point q in the closure of p. We show $\mathcal{O}_{X,q}$ is also nonreduced. Pick an affine open subset $\operatorname{Spec} A$ of q then we have $p \in \operatorname{Spec} A$ as $q \in \overline{\{p\}}$. One can corresponds $p, q \in \operatorname{Spec} A$ with prime ideals $\mathfrak{p}, \mathfrak{q}$ of A. Then we know $\mathcal{O}_{X,p} = A_\mathfrak{p}$ has a nonzero nilpotent element a/b where $a \in A, b \in A - \mathfrak{p}$. It follows that $a \in A$ is a nonzero nilpotent element in $A_{\mathfrak{p}}$. This means $a^n s' = 0$ for some $n \geq 2, s' \in A - \mathfrak{p}$, and $as \neq 0$ for all $s \in A - \mathfrak{p}$. We will show that as' is a nonzero nilpotent in $A_{\mathfrak{q}}$. As $(as')^n = a^n s'^n = 0$ so as' is nilpotent. It suffices to show as' is nonzero in $A_{\mathfrak{q}}$.

On the other hand, note that $q \in \overline{\{p\}}$ and as $q, p \in \operatorname{Spec} A$, q also lies in the closure of p with respect to the topology of $\operatorname{Spec} A$. Indeed, a closed subset of A containing p is $A \backslash U$ for open U of X not containing p. Then the closure of p relative to $\operatorname{Spec} A$ is $\operatorname{Spec} A \backslash \left(\bigcap_{p \notin U} U \right)$ and relative to X is $X \setminus \left(\bigcap_{p \notin U} U \right)$. As $q \in X \setminus \left(\bigcap_{p \notin U} U \right)$, we know $q \notin \bigcap_{p \notin U} U$, and as $q \in \operatorname{Spec} A$, we are done. Now, q lies in the closure of p relative to $\operatorname{Spec} A$ means $\mathfrak{p} \subset \mathfrak{q}$, implying $A - \mathfrak{q} \subset A - \mathfrak{p}$. Therefore, by combining with the previous paragraph, we find $as \neq 0$ for all $s \in A - \mathfrak{q}$, i.e. as' is nonzero in $A_{\mathfrak{q}}$.

5.2.2 Reducedness is not an open condition, i.e. the locus of reduced points is not necessarily open. Indeed, we will show that the locus of reduced points of the scheme

$$X = \operatorname{Spec} \mathbb{C}[x, y_1, y_2, \ldots]/(y_1^2, y_2^2, \ldots, (x-1)y_1, (x-2)y_2, \ldots)$$

is not Zariski open.

A prime ideal of this ring corresponds to a prime ideal $\mathfrak{p}$ in $\mathbb{C}[x, y_1, y_2, \ldots]$ containing $(y_1^2, y_2^2, \ldots, (x-1)y_1, (x-2)y_2, \ldots)$. Therefore, $\mathfrak{p}$ contains $(y_1, y_2, \ldots)$. It follows that all such prime ideals are either $(y_1, y_2, \ldots)$ or $(x - a, y_1, y_2, \ldots)$. Pick $f \in \mathbb{C}[x, y_1, y_2, \ldots]$ then f belongs to finitely many such prime ideals, meaning any open set of X corresponds to $\mathbb{C}$ minus finitely many points. In other words, X is homeomorphic to $\operatorname{Spec} \mathbb{C}[x]$.

Next, we find the nonreduced points. With the same argument as in 5.2.C, the nonreduced points are $(x - n, y_1, y_2, \ldots)$ where $n = 1, 2, \ldots$. Indeed, for $\mathfrak{p} = (x - a, y_1, y_2, \ldots)$ where $a \neq 1, 2, \ldots$ then as $x - n \notin (x - a, y_1, y_2, \ldots)$, $x - n$ is invertible in $\mathbb{C}[x, y_1, y_2, \ldots]_{\mathfrak{p}}$. Therefore, $(y_1^2, y_2^2, \ldots, (x-1)y_1, (x-2)y_2, \ldots) = (y_1, y_2, \ldots)$ as ideals in $\mathbb{C}[x, y_1, y_2, \ldots]_{\mathfrak{p}}$. Thus, an element in the localisation of $\mathbb{C}[x, y_1, y_2, \ldots]/(y_1^2, y_2^2, \ldots, (x-1)y_1, (x-2)y_2, \ldots)$ at $\mathfrak{p}$ can be written as $(x-a)^k u$ for $k \in \mathbb{Z}_{\geq 0}$ and some u invertible. This cannot be nilpotent. Similar argument can be made for nonreduced points.

Taking the complement of nonreduced points in X corresponds to the set $\mathbb{C} - \{1, 2, \ldots\}$ of points in $\operatorname{Spec} \mathbb{C}[x]$, which is not Zariski open.

5.2.3 Ring of global section of a scheme X is reduced does not imply X is reduced.

Indeed, let X be the scheme cut out by $x^2 = 0$ in $\mathbb{P}_k^2$, i.e. $X = \operatorname{Proj} k[x_0, x_1, x_2]/(x_0^2)$ (see 4.5.P). We show X is nonreduced by $\Gamma(X, \mathcal{O}_X) = k$.

Over the distinguished open set

$$\begin{aligned} D(x_1) &= \operatorname{Spec} \left(\left(k[x_0, x_1, x_2]/(x_0^2) \right)_{x_1} \right)_0, \\ &= \operatorname{Spec} \left(k[x_0, x_1^{\pm}, x_2]/(x_0^2, x_1^{-1}) \right)_0, \\ &= \operatorname{Spec} k[x_{0/1}, x_{2/1}]/(x_{0/1}^2), \end{aligned}$$

we find that X is nonreduced.

Next, we show $\Gamma(X, \mathcal{O}_k) = k$. Take a global section s and restricts it to $D(x_1)$ gives $f \in \Gamma(D(x_1), \mathcal{O}_X) = k[x_{0/1}, x_{2/1}]/(x_{0/1}^2)$. Because $\{1, x_2, x_2^2, \ldots\}$ has no zerodivisors in $k[x_{0/1}, x_{2/1}]/(x_{0/1}^2)$, we have an injection $\Gamma(D(x_1), \mathcal{O}_X) \to \Gamma(D(x_1 x_2), \mathcal{O}_X)$. Furthermore, the isomorphism that glues $D(x_2) = \operatorname{Spec} k[x_{0/2}, x_{1/2}]/(x_{0/2}^2)$ and $D(x_1)$ via the 'intersection' $D(x_1 x_2)$ is given by sending $f(x_{0/1}, x_{2/1}) \mapsto f(x_{0/2}/x_{1/2}, 1/x_{1/2})$ (see 4.4.9). Because f comes from a global section, we also have an injection $\Gamma(D(x_2), \mathcal{O}_X) \to \Gamma(D(x_1 x_2), \mathcal{O}_X)$ that sends $g \in k[x_{0/2}, x_{1/2}]/(x_{0/2}^2)$ to $f(x_{0/2}/x_{1/2}, 1/x_{1/2})$, i.e. $f(x_{0/2}/x_{1/2}, 1/x_{1/2}) = g(x_{0/2}, x_{1/2})$ in $k[x_{0/2}, x_{1/2}]/(x_{0/2}^2)$. It implies f is a constant in $D(x_1 x_2)$, hence is also constant in $D(x_1)$. Similar proof shows the global section s is constant on $D(x_2)$. As $\Gamma(D(x_0), \mathcal{O}_X) = k$, s is also constant on $D(x_0)$. Thus, $\Gamma(X, \mathcal{O}_X) = k$.

5.2.E If X is quasicompact and f a function that vanishes at all points of X, then there is some n such that $f^n = 0$. Furthermore, this may fail if X is not quasicompact.

When $X = \operatorname{Spec} A$, the claim follows from 3.2.12. For any quasicompact scheme X, we can cover it by finitely many affine schemes A_i for $1 \le i \le n$. If $f^{a_i}|_{A_i} = 0$ for all i then by taking $m = \operatorname{lcm}(a_i)$, we find $f^m|_{A_i} = 0$ for all i, implying $f^m = 0$.

To show the claim may fail for X not being quasicompact. Take an infinite disjoint union of $\operatorname{Spec} A_i$ where $A_i = k[x_i]/(x_i^n)$. Take $(x_1, x_2, \ldots)$, a function on X whose restriction to $\operatorname{Spec} A_i$ is x_i. This function vanishes at all points of $X = \bigsqcup \operatorname{Spec} A_i$ as x_i vanishes at all points of $\operatorname{Spec} A_i$. But there is no n such that $(x_1^n, x_2^n, \ldots) = 0$.

5.2.F A scheme X is integral if and only if it is irreducible and reduced (thus we picture integral scheme as "one piece, no fuzz").

If X is integral then $\mathcal{O}_X(U)$ is an integral domain, hence is reduced.

Suppose X is not irreducible, i.e. there is two nonempty non-intersecting open affine subsets $\operatorname{Spec} A, \operatorname{Spec} B$ of X. Hence, there exists $f \in A, g \in B$ such that two distinguished open subsets $D(f), D(g)$ of $\operatorname{Spec} A, \operatorname{Spec} B$ are nonempty and $D(f) \cap D(g) = \emptyset$. Note $\operatorname{Spec} A \sqcup \operatorname{Spec} B = \operatorname{Spec} A \times B$ so we can view f, g as functions on $\operatorname{Spec} A \sqcup \operatorname{Spec} B$ (by sending $f \mapsto (f, 0) \in A \times B$). Hence, we have $D(f) \cap D(g) = D(fg) = \emptyset$. Therefore, $fg = 0$. As $D(f)$ and $D(g)$ are nonempty, we find $f \neq 0$ and $g \neq 0$ on $\operatorname{Spec} A \sqcup \operatorname{Spec} B$. Thus, the scheme X is not integral.

Conversely, suppose X is irreducible and reduced. For any open U of X, suppose we have $f, g \in \mathcal{O}_X(U)$ so $fg = 0$. We will show that either $f = 0$ or $g = 0$ in $\mathcal{O}_X(U)$.

Let $D(f)$ be the set of points $x \in U$ such that $f(x) \neq 0$. Then 4.3.G says $D(f)$ is open. As $fg = 0$, we find $D(f) \cap D(g) = D(fg) = \emptyset$. As X is irreducible, its open set U is also irreducible, therefore it cannot have two nonempty non-intersecting open subsets. From this, we find, WLOG, $D(f) = \emptyset$, meaning f vanishes at every point in U. Let $\operatorname{Spec} A$ be any affine open subscheme of U. If f vanishes at every point of $\operatorname{Spec} A$ then by 5.2.E, we find f is nilpotent. As X is reduced, we find $f = 0$ on $\operatorname{Spec} A$. It then implies that $f = 0$ on U.

5.2.G If $\operatorname{Spec} A$ is irreducible and reduced then from 5.2.F, we know A is an integral domain.

Conversely, if A is an integral domain then A is reduced, implying $A_{\mathfrak{p}}$ is reduced for any prime ideal $\mathfrak{p}$ of A. Hence, $\operatorname{Spec} A$ is reduced. We also know from 3.6.C that $\operatorname{Spec} A$ is irreducible.

5.2.H Let X be an integral scheme. Being irreducible, X has a generic point η. Let $\operatorname{Spec} A$ be any nonempty affine open subset of X. Then $\mathcal{O}_{X,\eta}$ is naturally identified with $K(A)$, the fraction field of A. Indeed, if η corresponds to the prime ideal $\mathfrak{p}$ in A then $\mathcal{O}_{X,\eta} = A_{\mathfrak{p}}$.

As $X = \overline{\{\eta\}}$, the closure of $\mathfrak{p}$ with respect to the topology of $\operatorname{Spec} A$ is $\operatorname{Spec} A$. This means for any prime ideal $\mathfrak{q}$ then $\mathfrak{p} \subset \mathfrak{q}$. On the other hand, as X is integral $\operatorname{Spec} A$ is also integral, implying A is an integral domain, implying $\mathfrak{p} = (0)$. Thus, $\mathcal{O}_{X,\eta} = A_{(0)} = K(A)$.

5.2.I For an integral scheme X then the restriction map $\operatorname{res}_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is an inclusion so long as $V \neq \emptyset$. Let $f \in \mathcal{O}_X(U)$ so $\operatorname{res}_{U,V}(f) = 0$. Let $D(f) = \{x \in U : f(x) \neq 0\}$ then $D(f)$ is open from 4.3.G. We find $V \cap D(f) = \emptyset$ and $V \neq \emptyset$. Hence, as U is irreducible, we find $D(f) = \emptyset$, i.e. f vanishes at all points in U. With the same argument as in 5.2.F, it follows that $f = 0$ in $\mathcal{O}_X(U)$ as X is reduced.

For any nonempty affine open subset $\operatorname{Spec} A$ of X then the natural map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta} = K(A)$ is an inclusion for any nonempty open set U. Indeed, as $\eta \in U$, pick any affine open $\operatorname{Spec} B$ of U, then we have an inclusion $\mathcal{O}_X(U) \to \mathcal{O}_X(\operatorname{Spec} B) = B$. As B is an integral domain, we have an inclusion $B \to K(B) = \mathcal{O}_{X,\eta} = K(A)$.

9.6. **19/07/2022: Counting** $\operatorname{Hom}(\Gamma, G)/G$**.**

9.6.1. *Frobenius formula.* In Straddie II, we learned the following formula of Frobenius:

Let $\Gamma = \langle x_1, \ldots, x_n | w\rangle$ be a finitely generated one-relator group, G a finite group, the size of the fundamental groupoid $\operatorname{Hom}(\Gamma, G)/G$, where G acts by conjugation, can be written as

$$|\operatorname{Hom}(\Gamma, G)/G| := \sum_{x \in \operatorname{Hom}(\Gamma,G)/G} \frac{1}{\operatorname{Aut}(x)} = \frac{|\operatorname{Hom}(\Gamma, G)|}{|G|} = \frac{1}{|G|}\sum_{\chi \in \widehat{G}} a_\chi^w \chi(1), \tag{7}$$

where the sum is over all irreducible complex characters χ of G, and $a_\chi^w = \frac{1}{|G|}\sum_{x\in G^n} \overline{\chi(w(x))}$. This is achieved by identifying an element $f^w \in \mathbb{C}[G]^G$, defined as

$$f^w(g) = |\{(g_1, \ldots, g_n) \in G^n : w(g_1, \ldots, g_n) = g\}|.$$

This gives $f^w(1) = \operatorname{Hom}(\Gamma, G)$ and by writing f^w in basis $\widehat{G}$, we get our desired formula.

Some comments on the properties of this formula:

(1) It doesn't seem to depend on the presentation of Γ i.e. if we give another presentation $\Gamma = \langle y_1, \ldots, y_m | w'\rangle$, then I think

$$\sum_{\chi\in\widehat{G}} a_\chi^w \chi = \sum_{\chi\in\widehat{G}} a_\chi^{w'}\chi,$$

or equivalently, $f^w = f^{w'}$.

(2) If the word w can be obtained by joining two distict words w_1, w_2, write $w = w_1 * w_2$, then $f^{w_1} * f^{w_2} = f^{w_1 * w_2}$ where $*$ on the left is the convolution product in $\mathbb{C}[G]^G$, i.e. $(\psi * \varphi)(g) = \sum_{xy=g} \varphi(x)\psi(y)$.

(3) However, this formula depends on the fact that Γ is one-relator. *How does one generalise this to any finitely-generated finitely-presented group?*

Some more examples of Γ:

(1) $X = S^1$ then $|\operatorname{Hom}(\pi_1(S^1), G)| = |\operatorname{Hom}(\mathbb{Z}, G)|$, which is $|G|$. One can use Frobenius formula for $\Gamma = \langle x, y | x\rangle$ to show this.

(2) $\Gamma = \mathbb{Z}\times\mathbb{Z} = \langle x, y | xyx^{-1}y^{-1}\rangle$, i.e. fundamental group of $(S^1)^2$, then $a_\chi = \frac{|G|}{\chi(1)}$ and $|\operatorname{Hom}(\mathbb{Z}\times\mathbb{Z}, G)| = |G| \cdot c_G$, where c_G is number of conjugacy classes of G. *Open: When $G = UL_n(\mathbb{F}_q)$, upper triangular matrices with one on the diagonal and with entries over $\mathbb{F}_q$, we don't even know if c_G is a polynomial in q (according to Masoud)? Can we use Frobenius formula to show this?* See also `https://mathoverflow.net/q/376259/89665`

(3) $\Gamma = \mathbb{Z}/2 = \langle x | x^2\rangle$, i.e. fundamental group of $\mathbb{RP}^2$, then a_χ is the Frobenius-Schur indicator of χ, i.e. if χ corresponds to the irreducible complex representation ρ, then

$$a_\chi = \begin{cases} 1 & \text{if } \rho \text{ can be realised over } \mathbb{R} \text{ up to isomorphism} \\ -1 & \text{if } \chi \text{ is real-valued but } \rho \text{ cannot be realised over } \mathbb{R} \\ 0 & \text{if } \chi \text{ is not real-valued} \end{cases}$$

When $a_\chi = 1$, ρ is a real representation. When $a_\chi = -1$, ρ is a quaternion representation. When $a_\chi = 0$, a_χ is a complex representation.

A proof on why $a_\chi \in \{0, \pm 1\}$: for irreup V with character χ_V then $\chi_V(g^2) = \chi_{V\otimes V}(g) - 2\chi_{\wedge^2 V}(g)$, which comes from $V \otimes V = \operatorname{Sym}^2 V \otimes \wedge^2 V$ and $\chi_V = \operatorname{trace}\rho =$ sum of eigenvalues. Then $a_\chi = \langle \chi_{V\otimes V}, \chi_{tri}\rangle - 2\langle\chi_{\wedge^2 V}, \chi_{tri}\rangle$. Note $\langle\chi_{V\otimes V}, \chi_{tri}\rangle = \dim \operatorname{Hom}_G(V \otimes V, 1) = \dim \operatorname{Hom}_G(V, V^*) \leq 1$ as V is irreducible. It also implies $\langle\chi_{\wedge^2 V}, \chi_{tri}\rangle \leq 1$ because $\langle\chi_W, \chi_{tri}\rangle$ counts number of trivial rep in rep W of G, and $\wedge^2 V$ is a subrep of $V \otimes V$. It follows that

$a_\chi \in \{\pm 1, 0\}$. Proving which value of a_χ correspond to ρ being real/complex/quaternion rep seems to be harder.

Example of real rep: take the trivial rep. Example of complex rep: note $\chi(g^{-1}) = \overline{\chi(g)}$ so for χ to not have real-valued, choose G with some $g \in G$ not conjugate to g^{-1}. Take $G = \mathbb{Z}/3, V = \mathbb{C}$ and send $1 \in \mathbb{Z}/3$ to 3rd root of unity, i.e. $1 \mapsto -1/2 + i\sqrt{3}/2$.

Example of quaternion rep: take $G = Q_8$, the quaternion group, $V = \mathbb{C}\{1, j\}$ and in this basis, we send

$$-1 \mapsto \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, i \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}, j \mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, k \mapsto \begin{pmatrix} & -i \\ -i & \end{pmatrix}.$$

This is a matrix presentation of Q_8. To show this is quaternion rep:

(a) V is irreducible because if it is not then there exists $v_1, v_2 \in V$ that are common eigenvectors for all $g \in Q_8$, but this is not possible from that matrix presentation.
(b) V has real-valued character. Indeed, because trace of every matrix is real.
(c) V cannot be realised over $\mathbb{R}$. Indeed, or else there exists a matrix P s.t. conjugate ρ gives us a representation $P\rho P^{-1} : Q_8 \to \mathrm{GL}_2(\mathbb{R})$. This induces an isomorphism $\mathbb{R}[Q_8] \to \mathrm{Mat}_2(\mathbb{R})$ of $\mathbb{R}$-algebras. But every element in $\mathbb{R}[Q_8]$, except for 0, is invertible (i.e. a division algebra). This is not the case for $\mathrm{Mat}_2(\mathbb{R})$.

(4) When $\Gamma = \mathbb{Z}/n = \langle x | x^n \rangle$, *what is representation-theoretic meaning of a_χ?* Higher Frobenius-Schur indicator.
(5) Because of interest, Γ is likely the fundamental group of some spaces. See 31/12/2021 for some examples: Klein bottle, knots complements. *Compute* $\mathrm{Hom}(\Gamma, G)$ *for these cases?*
(6) If word $w(x_1, \ldots, x_n)$ contains an alphabet x_1 that appears exactly once with no power (i.e. x_1 in $x_1 x_2^2 x_4 x_3^5 x_4$) then $|\mathrm{Hom}(\Gamma, G)| = |G|^{n-1}$ because there are $|G|$ choices for each $x_2, \ldots, x_n$.
(7) If $\Gamma = \pi_1(\Sigma_g) = \langle x_1, \ldots, x_n, y_1, \ldots, y_n | [x_1, y_1] \cdots [x_n, y_n] \rangle$ then by using the property $f^{w_1 * w_2} = f^{w_1} * f^{w_2}$ and the case $\Gamma = \mathbb{Z} \times \mathbb{Z}$, we find $a_\chi = \frac{|G|^{2n-1}}{\chi(1)^{2n-1}}$. This is the case of torus with n genus or compact orientable surface with genus n.
(8) This time $w = [x_1, y_1] \cdots [x_n, y_n] z_1 \cdots z_k$, i.e. Γ is fundamental group of compact orientable surface genus n and k punctures. Then $f^w = |G|^{2n+k-1} \chi_1$ where χ_1 is the trivial character.

9.6.2. *Using TQFT.* One can count $\mathrm{Hom}(\Gamma, G)/G$ using topological quantum field theory, in the sense of 06/06/2021, 17/06/2021. But here they only proved for the case Γ is fundamental group of Σ_n. *Some things I would like to try next:*

(1) Learn how to do the computation for $\Gamma = \pi_1(\mathbb{RP}^2)$ ($\mathbb{RP}^2$ is a non-orientable surface). The only place I've seen doing this is Noah Snyder's paper Mednykh's Formula via Lattice Topological Quantum Field Theories. The thing with this is that we cannot seem to visualise $\mathbb{RP}^2$ as we did for Σ_n.
(2) Try to do this for Klein bottle. In terms of cobordism, the Klein bottle is obtained by connecting the cap $\emptyset \to (S^1)^{\sqcup 2}$, the map $(S^1)^{\sqcup 2} \to (S^1)^{\sqcup 2}$ by fixing one and reflecting the other circle, and the cup $(S^1)^{\sqcup 2} \to \emptyset$.
(3) Try more examples: torus with punctures, $S^1 \times S^1$, knots complements, ...
(4) Every finitely-generated group is fundamental group of some closed 4-manifolds. So can we use this approach to compute any $\mathrm{Hom}(\Gamma, G)$?
(5) There is a notion of extended topological field theory that assigns closed 2, 3-manifolds M roughly the $\mathrm{Hom}(\pi_1(M), G)$. Can we use this? See p.19 Lectures on Field Theory and Topology by Daniel S. Freed.

9.7. **20/01/2022: Trace formulas in automorphic representations.** I watched an introductory talk of Rahul Dalal, titled Statistics of Automorphic Representations through Simplified Trace Formulas, see `https://youtu.be/w-dGghBpcoc`. At the beginning, he gave an introductory overview of using trace formulas in automorphic representation theory. See also some terminologies on 24/11/2021.

(1) For a reductive group G over a number field F, we have a unitary representation $L^2(G(F)\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$, consisting of square-integrable functions, where is action is by right translation. We are interested in finding out (discrete) automorphic representation, i.e. irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}))$.
(2) For $G = \mathrm{GL}_2/\mathbb{Q}$, automorphic reps correspond to new, eigen modular/ Mass forms.
(3) Any automorphic representation π factors through π_v over all places of $v \in F$: $\pi = \widehat{\bigoplus}'\pi_v$ for each π_v admissible, unitary representation of $G(F_v)$. When $G = \mathrm{GL}_2/\mathbb{Q}$, π_∞ relates to notions of modular vs Mass, weight, of π, π_p relates to p^nth Fourier coefficients of π.
(4) **Key question:** Which combination of π_v actually appear in L^2?
(5) How to use trace in answering this question: If $L^2([G]) = \bigoplus \pi$ and R an operator on L^2 then $\mathrm{tr}_{L^2} R = \bigoplus \mathrm{tr}_\pi R$. We can choose R cleverly to put restrictions on π_v to answer the key question. In particular, for compactly support smooth function f, we can consider the convolution operator
$$R_f : v \mapsto \int_{G(\mathbb{A})} f(g)gvdg$$
If $G(F)\backslash G(\mathbb{A})$ is compact, we can write $\mathrm{tr}_{L^2} R_f$ as a sum of conjugacy classes. If $[G]$ is not compact, various issues with convergence arises.

9.8. **21/02/2022: Categorical version of Lefschetz fixed point formula.** It seems there is a categorical version of Lefschetz fixed point formula, given in this paper `https://arxiv.org/abs/1607.06345`.

9.9. **22/01/2022: Motivation for the definition of condensed sets.** Peter Scholze and Dustin Clausen defined condensed sets, viewed as a replacement for a large class of nice topological spaces, where doing algebra is much easier.

(1) Why topological spaces are bad? I would like to quote this answer `https://math.stackexchange.com/a/4199337/58951` from Scholze: Topological spaces formalize the idea of spaces with a notion of "nearness" of points. However, they fail to handle the idea of "points that are infinitely near, but distinct" in a useful way. An example is the topological space $\mathbb{R}/\mathbb{Q}$, which has many distinct points, but they all are infinitely close to each other.
(2) Motivation for the definition of condensed sets: I learned this from `https://youtu.be/OT65JC3gKPY`.
    (a) A site is a category that generalises the notion of open subsets of a topological space X. One can consider sheaves on a site similarly to sheaves on X.
    (b) For a field k, the etale site of a scheme $\operatorname{Spec} k$ consists of collection of etale maps $Y \to \operatorname{Spec} k$ (i.e. $Y = \sqcup \operatorname{Spec} k_i$ where k_i/k finite separable extension), viewed as open covers of $\operatorname{Spec} k$. By Galois theory, elate maps $Y \to \operatorname{Spec} k$ corresponds to discrete sets with a continuous $\mathrm{Gal}(\overline{k}/k)$-action *(is it true that the set has to be discrete? try `https://websites.math.leidenuniv.nl/algebra/GSchemes.pdf`).*
    (c) One can generalise the previous example. For G profinite group, we can consider the site where elements being profinite sets with a continuous action of G, where the collection of G-maps $(f_i : S_i \to S)_{i\in I}$ are covers of S if there is a finite cover of S taken from from a finite subset $J \subset I$. We need the finiteness condition of covers because we don't want something like $\sqcup_{x\in S}\{x\} \to S$ to be covers of S if S is infinite.

(d) Choose $G = *$ to be the trivial group in the previous example, we obtain a site $*$-prosets. Taking the category of sheaves on this site, we obtain condensed sets.

(3) *What is the relation between topological spaces and condensed sets?* The above definition of condensed sets are actually motivated from algebraic number theory where in many situations, we need to consider action of a profinite group on a set. And I don't see why condensed sets is a replacement of topological spaces from its definition.

9.10. **22/01/2022: Adjoint $f^!$ of proper push-forward $Rf_!$.** This is a continuation from 14/01/2022. We consider the case of nice topological spaces as in 17/12/2021 and sheaves of k-modules on those spaces. For a continuous map $f : X \to Y$ of nice topological spaces, we have the left-exact functor $f_! : Sh(X) \to Sh(Y)$ by

$$(f_!\mathcal{F})(U) = \{s \in \mathcal{F}(f^{-1}(U)) | f : \operatorname{supp}(s) \to U \text{ is proper}\}.$$

With this, we obtain the right-derived functor $Rf_! : D^b(X) \to D^b(Y)$ (see 17/12/2021). We will construct the adjoint $f^!$ of $Rf_!$ today, i.e. $f^! : D^b(Y) \to D^b(X)$ such that

$$R\underline{\mathrm{Hom}}(Rf_!\mathcal{F}, \mathcal{G}) \cong Rf_* R\underline{\mathrm{Hom}}(\mathcal{F}, f^!\mathcal{G}).$$

This will give us adjointness between $f^!$ and $Rf_!$ (see the argument (5) from 14/01/2022).

(1) To motivate the definition of $f^!$: Suppose we can construct $f^!$ and $f^!\mathcal{G}$ is just a sheaf. Then for open set $j_U : U \hookrightarrow X$, we have

$$(f^!\mathcal{G})(U) = \Gamma(j_U^* f^!\mathcal{G}) = \mathrm{Hom}(\underline{k}_U, j_U^* f^!\mathcal{G}) \cong \mathrm{Hom}(f_!(j_U)_*\underline{k}_U, \mathcal{G}).$$

Note the RHS does not depend on $f^!$. This suggests us to define $f^! : D^b(Y) \to D^b(X)$ by taking a chain complex $\mathcal{G}$ of sheaves on Y to the chain complex of sheaves on X, defined by

$$U \subset X \mapsto \mathrm{Hom}(f_!(j_U)_*\mathcal{K}|_U, \mathcal{G}),$$

where $\mathcal{K}$ is a flat resolution of $\underline{k}$, Hom is a chain complex as in (1) of 14/01/2022. If $\mathcal{G}$ is a complex of injective sheaves then so is the above complex. Thus, we obtain a derived functor $f^! : D^b(Y) \to D^b(X)$. Note that some technical conditions of k and f are required to make the above construction works. For more details, we refer to Achar's book §1.5.

(2) One reason why passage to the derived category is essential is that $f_! : Sh(X) \to Sh(Y)$ does not necessarily have a right adjoint in the category of sheaves. This is because $f_!$ is not always right-exact.

(3) There are many names for $Rf_!$ and $f^!$: proper/exceptional push-forward and pull-back; or f-upper-shriek for $f^!$ and f-lower-shriek for $f_!$.

9.11. **31/01/2022: Reference for Schwartz's distribution theory.**

(1) Schwartz's distribution theory: `https://www.mat.univie.ac.at/~stein/lehre/SoSem09/distrvo.pdf`.
(2) Define distributions on manifolds: `https://webspace.science.uu.nl/~ban00101/anman2009/lecture2.pdf`.
(3) Applications: `https://mathoverflow.net/a/260634/89665`.
(4) For non-Archimedean local fields, distributions are discussed in the paper by Bernstein and Zelevinsky, Representations of the group $\mathrm{GL}(n, F)$, where F is a local non-Archimedean field.

It seems to me that Schwartz's distribution theory has many common theme with Grothendieck's six operations in the derived category (as discussed 17/12/2021, 14/01/2022, 22/01/2022) (for example, see §2.3 of `https://arxiv.org/abs/1212.3630`). I would like to know more about connections between these two languages.

10. February 2022

10.1. **08/02/2022: Local systems and monodromies.** For a topological space X and $x_0 \in X$, we can define a functor

$$\mathrm{Mon}_{x_0} : \mathrm{Loc}(X, k) \to k[\pi_1(X, x_0)] - \mathrm{mod}$$

from the category of local systems on X to the category of modules over the group ring $k[\pi_1(X, x_0)]$. Last few days I learned about this functor, so I would like to describe this functor and compute some examples of it. The reference I used is Achar's book, §1.7.

First, let me describe Mon_{x_0}:

(1) A sheaf $\mathcal{L}$ of k-modules on a locally connected topological space X is a *local system* if there is an open covering $(U_\alpha)_{\alpha \in I}$ of X such that $\mathcal{L}|_{U_\alpha}$ is a constant sheaf for every $\alpha \in I$. Observe that pullback of a local system along a continuous map $f : X \to Y$ is also a local system.

(2) Given a local system $\mathcal{L}$, we can construct an action of the fundamental group $\pi_1(X, x_0)$ on the stalk $\mathcal{L}_{x_0}$, called the monodromy representation, as follows: For a path $\gamma : [0, 1] \to X$, we can defined a map $\rho_\gamma : \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$ via the isomorphism

$$\mathcal{L}_{\gamma(0)} \cong (\gamma^*\mathcal{L})_0 \xleftarrow{\sim} \Gamma([0, 1], \gamma^*\mathcal{L}) \xrightarrow{\sim} (\gamma^*\mathcal{L})_1 \cong \mathcal{L}_{\gamma(1)}.$$

The second and third isomorphisms come from the fact that $\gamma^*\mathcal{L}$ is a local system on $[0, 1]$, hence is a constant sheaf.

If X is locally connected then one can avoid dealing with direct limit when computing the stalk of a constant sheaf, i.e. $\Gamma(U, \underline{M}_X) \xrightarrow{\sim} (\underline{M}_X)_x \cong M$ for $x \in X$ and any connected neighborhood U of x. In this case, one can describe the monodromy action as follows:

By the compactness of $[0, 1]$ and the fact that $\mathcal{L}$ is a local system on X, there exists a finite set of connected open subsets $(U_i)_{i=1}^n$ of X such that $\mathcal{L}|_{U_i}$ is a constant sheaf for every i, and as one goes along γ, one travels along $\gamma(0) \in U_1, \dots, U_n \ni \gamma(1)$. Let the connected component of $\gamma(t_i)$ in $U_i \cap U_{i+1}$ to be V_i. The monodromy action can then be described as compositions of various restrictions map of $\mathcal{L}$:

$$\mathcal{L}_{\gamma(0)} \xleftarrow{\sim} \mathcal{L}(U_1) \xrightarrow{\sim} \mathcal{L}(V_1) \xleftarrow{\sim} \mathcal{L}(U_2) \xrightarrow{\sim} \mathcal{L}(V_2) \xleftarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{L}(U_n) \xrightarrow{\sim} \mathcal{L}_{\gamma(1)}.$$

(3) Under more restrictive condition of X, the functor Mon_{x_0} gives an equivalence of categories. One can also ask about the effect of the Grothendieck operations on $\mathrm{Loc}(X, k)$ and then on the monodromy representations after applying Mon_{x_0}. I have not read this part of Achar §1.7 in details.

10.1.1. *An example of local system.* I learned this from Balazs Elek's notes on Constructible sheaves: `https://chenhi.github.io/math7390-s21/notes/0402-elek-constructiblesheaves.pdf`. This also appear in Achar's exercise 1.7.3.

(1) Let $\mathcal{Q}$ be a sheaf of $\mathbb{C}$-modules on $\mathbb{C}^\times$ defined over every open subset $U \subset \mathbb{C}^\times$ as

$$\mathcal{Q}(U) = \left\{ \text{solutions } g : U \to \mathbb{C} \text{ to } z\frac{dg}{dz} - \frac{1}{2}g = 0 \right\}.$$

(2) We show $\mathcal{Q}$ is a local system. Indeed, for any $x \in \mathbb{C}^\times$ and any connected, simply connected open neighborhood U of x, one can choose a branch of the complex logarithm by a ray I starting at 0 that does not intersect U. Then $g(z) = e^{\frac{1}{2}\log(z)} \neq 0$ is defined on $\mathbb{C}^\times \backslash I$ and hence lies in $\mathcal{Q}(U)$. We show that $\mathcal{Q}(U) = \mathbb{C}g$, hence implying $\mathcal{Q}|_U \cong \underline{\mathbb{C}}_U$. Indeed, suppose $h \in \mathcal{Q}(U)$, we then have $\frac{d}{dz}\left(\frac{h}{g}\right) = \frac{h'g - hg'}{g^2} = 0$, implying h/g is a constant on U, as desired. In particular, this sheaf essentially describe solutions to the equation $g(z)^2 = z$.

(3) However, $\mathcal{Q}$ is not a constant sheaf $\underline{\mathbb{C}}_{\mathbb{C}^\times}$ as it has no global section. Indeed, let $g \in \Gamma(\mathbb{C}^\times, \mathcal{Q})$ and let $\gamma : [0,1] \to \mathbb{C}^\times$ to be a simple loop around the origin defined by $\gamma(t) = e^{2\pi i t}$. We then have

$$\int_{g\circ\gamma} \frac{1}{z} dz = \int_\gamma \frac{g'(z)}{g(z)} dz = \int_\gamma \frac{1}{2z} dz = \frac{1}{2}\int_0^1 \frac{1}{e^{2\pi i t}} \cdot 2\pi i e^{2\pi i t} dt = \pi i,$$

which is a contradiction, as this supposes to be $2\pi i$ times the winding number of $g \circ \gamma$.

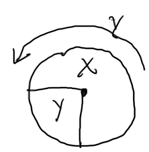
(4) We show that the monodromy representation $\mathbb{Z} = \pi_1(\mathbb{C}^\times, 1) \to \mathrm{GL}_1(\mathbb{C})$ of $\mathcal{Q}$ is given by sending a closed loop γ around the origin to a linear map on $\mathcal{Q}_1 \cong \mathbb{C}$ sending $g \in \mathcal{Q}_1$ to $e^{\pi i} g$.

Indeed, we first define two open sets U, V of $\mathbb{C}^\times$ whose union contains the path γ. Let $U = \mathbb{C}^\times \backslash \mathbb{R}_{\leq 0}$ and $U' = \mathbb{C}^\times \backslash \{iy : y \in \mathbb{R}_{\leq 0}\}$. The two branches $\mathbb{R}_{\leq 0}$ and $\{iy : y \in \mathbb{R}_{\leq 0}\}$ define two complex logarithm functions $\log$ and $\log'$ on U, U', respectively. In particular, $\log(z) := \log|z| + i\arg(z)$ where $-\pi \leq \arg(z) < \pi$ and $\log'(z) := \log|z| + i\arg'(z)$ where $-\pi/2 \leq \arg'(z) < 3\pi/2$. By previous argument, we know $\mathcal{Q}|_U$ and $\mathcal{Q}|_{U'}$ are constant sheaves. In particular, as U and U' are connected and locally connected, their stalks can be identified with the vector spaces $\mathcal{Q}(U) = \mathbb{C}g$ and $\mathcal{Q}(U') = \mathbb{C}g'$, respectively, where $g(z) = e^{\frac{1}{2}\log(z)}$ and $g'(z) = e^{\frac{1}{2}\log'(z)}$. We also have $U \cap U'$ is a disjoint of two connected components $X \cup Y$, where we denote Y to be component below the real axis.

Let γ be a closed loop around 1, traveling counter-clockwise (for example $\gamma(t) = e^{2\pi i}$). The monodromy action is then described by

$$\mathcal{Q}_1 = \mathcal{Q}(U) \xrightarrow{\sim} \mathcal{Q}(X) \xleftarrow{\sim} \mathcal{Q}(U') \xrightarrow{\sim} \mathcal{Q}(Y) \xleftarrow{\sim} \mathcal{Q}(U) = \mathcal{Q}_1.$$

We start with $g \in \mathcal{Q}_1 = \mathcal{Q}(U)$, restricting to X is still g. In order to viewed $g \in \mathcal{Q}(X)$ as element in $\mathcal{Q}(U')$, we write $g = \lambda g'$ over X, then $g\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \lambda g'\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$. It implies $\lambda = 1$. Restricting $g' \in \mathcal{Q}(U')$ to Y is still g', but to go from $\mathcal{Q}(Y)$ to $\mathcal{Q}(U)$, we need to change basis to g in order to define the linear map. Let $g' = \lambda g$ in $\mathcal{Q}(Y)$. Then $g'\left(\frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \lambda g\left(\frac{-\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$. It implies $e^{\frac{i\cdot 5\pi/4}{2}} = \lambda e^{\frac{i\cdot(-3\pi/2)}{2}}$. Thus, $\lambda = e^{i\pi}$. In other words, the monodromy representation sends $[\gamma] \in \pi_1(X, 1) = \mathbb{Z}$ to $e^{\pi i} \in \mathrm{GL}_1(\mathbb{C})$.



10.1.2. *Another example.* Let k be a field. We consider the map $f : \mathbb{C}^\times \to \mathbb{C}^\times$ sending $z \mapsto z^2$. We will describe the sheaf $f_*\underline{\mathbb{C}}_{\mathbb{C}^\times}$.

(1) $f_*\underline{\mathbb{C}}_{\mathbb{C}^\times}$ is a local system. In fact, for every covering map $f : X \to Y$ where Y is locally path-connected and locally simply-connected, then $f_*\mathcal{F}$ is a local system, where $\mathcal{F} \in \mathrm{Loc}(X, k)$. See Achar's lemma 1.7.14.

(2) Furthermore, we will show we will show that $f_*\underline{\mathbb{C}}_{\mathbb{C}^\times} \cong \underline{\mathbb{C}}_{\mathbb{C}^\times} \oplus \mathcal{Q}$. Indeed, we define $\phi : \underline{\mathbb{C}}_{\mathbb{C}^\times} \oplus \mathcal{Q} \to f_*\underline{\mathbb{C}}_{\mathbb{C}^\times}$ as

$$g \in \underline{\mathbb{C}}_{\mathbb{C}^\times}(U) \mapsto g \circ f = g(z^2), g \in \mathcal{Q} \mapsto \frac{g \circ f}{z} = \frac{g(z^2)}{z}.$$

To show ϕ is an isomorphism, we show it is an isomorphism at the level of stalks, i.e. for any $w \in \mathbb{C}^\times$, we show $\phi_w : \underline{\mathbb{C}}_w \oplus \mathcal{Q}_w \to (f_*\underline{\mathbb{C}}_{\mathbb{C}^\times})_w$ is an isomorphism. However, as f is proper, by applying proper base change (proposition 1.2.15 of Achar's book) to the pullback square

$$\begin{tikzcd} f^{-1}(w) \arrow[r] \arrow[d, "f"] & X \arrow[d, "f"] \\ \{w\} \arrow[r, "i_w"] & Y \end{tikzcd}$$

we find $(f_*\underline{\mathbb{C}}_{\mathbb{C}^\times})_w \xrightarrow{\sim} \Gamma(\underline{\mathbb{C}}_{f^{-1}(w)}) = \mathbb{C}_{z_1} \oplus \mathbb{C}_{z_2}$ for $z_1, z_2 \in f^{-1}(w)$, i.e. $z_1^2 = z_2^2 = w$. Thus, it suffices to show ϕ_w is an injection.

If we have $\phi_w(g_1, g_2) = 0$ then there exists a small disc W containing w and small discs V_1, V_2 of $\mathbb{C}^\times$ containing z_1, z_2, respectively, such that

$$g_1(z^2) + \frac{g_2(z^2)}{z} = 0$$

for every $z \in V_1 \cup V_2 = f^{-1}(W)$. We know $g_1(z^2)$ is constant on each V_1 and V_2 as $g_1 \in (\underline{\mathbb{C}}_{\mathbb{C}^\times})(V_1 \cup V_2)$. Furthermore, $g_1(z_1^2) = g_1(z_2^2) = g_1(w)$ so g_1 is constant on $V_1 \cup V_2$. But $z_1 = -z_2$ so

$$g_1(w) = -\frac{g_2(w)}{z_1} = \frac{g_2(w)}{z_2} = -g_1(w),$$

yielding $g_1 = g_2 = 0$ on $V_1 \cup V_2$, as desired.

10.1.3. *Constructible sheaves on* $\mathbb{C}$. For a field k, a sheaf $\mathcal{F}$ of k-modules on $\mathbb{C}$ is said to be *constructible with respect to the stratification* $\mathbb{C} = \mathbb{C}^\times \cup \{0\}$ if $\mathcal{F}|_{\mathbb{C}^\times}$ and $\mathcal{F}|_{\{0\}}$ are local systems. We will describe such sheaves in terms of their monodromy representations.

As $\mathcal{F}|_{\{0\}} = \mathcal{F}_0$ is just a k-module, hence is always a local system. Local systems on $\mathbb{C}^\times$ are determined by the monodromy representation. By choosing a small disc D' of 1 where $0 \notin D'$, this representation can be described by the action of $\mathbb{Z} = \pi_1(\mathbb{C}^\times, 1)$ on $\mathcal{F}(D')$. In particular, it is determined by action $\beta : \mathcal{F}(D') \to \mathcal{F}(D')$ on $\mathcal{F}(D')$ as "it goes around a closed loop of 0". Therefore, for a disc D containing D' and 0, the restriction map $\alpha : \mathcal{F}(D) \to \mathcal{F}(D')$ must satisfy $\beta \circ \alpha = \alpha$. These are all you need to describe constructible sheaves on $\mathbb{C}$ with respect to the stratification $\mathbb{C}^\times \cup \{0\}$.

For example, we will describe $f_*\underline{k}_{\mathbb{C}}$ where $f : \mathbb{C} \to \mathbb{C}$ sending $z \mapsto z^m$.

(1) This is a constructible sheaf as $(f_*\underline{k})_{\mathbb{C}^\times}$ is a local system, as described in the previous example.

(2) With the above notations, we find that $f^{-1}(D)$ is connected (all path-connected to 0), and $f^{-1}(D') = D_0 \sqcup \cdots \sqcup D_{m-1}$, where D_i's are disjoint discs, each D_i contains a nth root of unity ζ_i. Hence, $(f_*\underline{k}_{\mathbb{C}})(D) \cong k$ and $(f_*\underline{k}_{\mathbb{C}})(D') \cong k^{\oplus m}$, where $(\underbrace{0, \ldots, 0}_{i}, 1, 0, \ldots, 0)$ corresponds to the locally constant map $s : D_0 \sqcup \cdots \sqcup D_{m-1} \to k$ sending $s(D_j) = 0$ for all $j \neq i$ and $s(D_i) = 1$.

We also denote $f^{-1}(D_k) = D_{k0} \sqcup \cdots \sqcup D_{k,m-1}$ where $\exp(2\pi i(k/m^2 + j/m)) \in D_{kj}$. In particular, we find $f_*(\underline{k}_{\mathbb{C}})(D_k) \cong k^{\oplus m}$.

(3) We will describe the monodromy action

$$\beta : k^{\oplus m} = (f_*\underline{k}_{\mathbb{C}})(D') \to (f_*\underline{k}_{\mathbb{C}})(D') = k^{\oplus m}.$$

It is given by the isomorphism

$$(f_*\underline{k}_{\mathbb{C}^\times})(D') = (f_*\underline{k}_{\mathbb{C}^\times})(D_0) \xrightarrow{t_0} (f_*\underline{k}_{\mathbb{C}^\times})(D_1) \xrightarrow{t_1} \cdots \xrightarrow{t_{m-1}} (f_*\underline{k}_{\mathbb{C}^\times})(D_{m-1}) \xrightarrow{t_m} (f_*\underline{k}_{\mathbb{C}^\times})(D_0).$$

In particular, t_0 sends $(1, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$, i.e. locally constant map $D_{10} \sqcup \cdots \sqcup D_{1,m-1} \to k$ sending D_{1j} to 0 for $j \neq 0$ and to 1 for otherwise. To see this, pick an open U

containing $D_0 \cup D_1$ as in the figure. We find $f^{-1}(U) = U_0 \sqcup \cdots \sqcup U_{m-1}$ with $D_{0i} \cup D_{1i} \subset U_i$. Then t_0 is the isomorphism

$$(f_*\underline{k}_{\mathbb{C}^\times})(D_0) \xleftarrow{\sim} (f_*\underline{k}_{\mathbb{C}^\times})(U) \xrightarrow{\sim} (f_*\underline{k}_{\mathbb{C}^\times})(D_1).$$

Similarly, we find $(1, 0, \ldots, 0) \in (f_*\underline{k}_{\mathbb{C}^\times})(D')$ is sent to $(1, 0, \ldots, 0) \in (f_*\underline{k}_{\mathbb{C}^\times})(D_{m-1})$, corresponding to the map $D_{m-1,0} \sqcup \cdots \sqcup D_{m-1,m-1} \to k$ sending $D_{m-1,0}$ to 1 and else to 0. Now, we apply t_m, which sends $(1, 0, \ldots, 0) \in (f_*\underline{k}_{\mathbb{C}^\times})(D_{m-1})$ to $(0, 1, \ldots, 0) \in (f_*\underline{k}_{\mathbb{C}^\times})(D_0)$. Thus, $\beta : k^{\oplus m} \to k^{\oplus m}$ is the map $(x_1, \ldots, x_m) \mapsto (x_m, x_1, x_2, \ldots)$.

(4) Finally, the restriction map

$$\alpha : k = (f_*\underline{k}_{\mathbb{C}^\times})(D) \to (f_*\underline{k}_{\mathbb{C}^\times})(D') = k^{\oplus m}$$

is $1 \mapsto (1, \ldots, 1)$ by restricting to each D_i's.

10.1.4. *More examples to do.* What about $f_!\underline{\mathbb{C}}_{\mathbb{C}^\times}$?

Elliptic curves examples: `https://en.wikipedia.org/wiki/Constructible_sheaf`. Consider the family of elliptic curves $\pi : X \to \mathbb{C}$ sending the elliptic curve $y^2 = x(x-1)(x-t)$ to t. Describe $\pi_*\underline{k}_{\mathbb{C}}$ as a constructible sheaf with respect to the stratification $\{0,1\} \cup \mathbb{C}\backslash\{0,1\}$. See also `https://math.stackexchange.com/q/4018140/58951`

What about Hopf fibration $f : S^3 \to S^2$, i.e. describe $f_*\underline{k}_{S^2}$.

`https://math.stackexchange.com/a/179750/58951`

See also Gunningham notes on p.42.

10.2. **14/02/2022: David Baraglia's talk 1: Moduli spaces of Higgs bundles.** My notes for David Baraglia's talk for the seminar Character varieties, E-polynomials and Representation zeta functions, hosted by ANU from 14-18/02/2022. The first talk is about Narasimhan-Seshadri theorem.

Let C be a compact Riemann surface of genus ≥ 2. Let $p : E \to C$ be a complex vector bundle of rank n on C.

A *holomorphic structure* on E is an integrable complex structure on E such that E is locally holomorphically trivial, i.e. exists an open cover $\{U_i\}$ of C and holomorphic trivialisations $E|_{U_i} \cong U_i \times \mathbb{C}^n$. The transition maps $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ are holomorphic.

Defining a holomorphic structure on E is the same as defining an $\overline{\partial}$-operator (Dolbeault) $\Omega^0(E) \to \Omega^{0,1}(E)$. Think of this statement as a generalisation of Cauchy-Riemann equation for a complex manifold.

An example of this holomorphic structure. Let $\nabla : \Omega^0(E) \to \Omega^1(E)$ be a connection on E. Let $\nabla^{0,1}$ denotes the $(0,1)$-part of ∇, i.e. can write $\Omega^1(E) = \Omega^{0,1}(E) + \Omega^{1,0}(E)$. Then $\nabla^{0,1}$ defines a holomorphic structure on E.

Question: Does any $\overline{\nabla}$-operator on E come from a connection on E? Yes, but we would like to find a particular connection.

Given a $\overline{\partial}$-operator $\overline{\partial}_E$ on E, h a Hermitian metric on E, then there exists unique connection ∇_E on E that is compatiblae with h and $\nabla^{0,1} = \overline{\partial}_E$. This connection is called Chern connection.

Question: Can we choose h such that the Chern connection ∇_E satisfies the Yang-Mills equation $d_E(*F_E) = 0$ where $*$ is the Hodge star $* : \Omega^j(E) \to \Omega^{2-j}(E)$ and $F_E : \Omega^0(E) \to \Omega^2(E)$ curvature of ∇_E?

A simple way to satisfy Yang-Mills is to suppose $*F_E = \lambda \mathrm{Id} \in \mathrm{End}(\Omega^0(E))$, equivalently, $F_E = \lambda d\mathrm{vol}_C \otimes \mathrm{Id}_E$.

Recall: smooth complex vector bundles on compact Riemann surfaces C are classified by rank and degree, defined by $\deg(E) = \int_C c_1(E) \in \mathbb{Z}$ where $c_1(E)$ is Chern class of E. By Chern-Weil theory, $\deg(E) = \frac{i}{2\pi}\int_C \mathrm{tr}(F_E)$ for any connection ∇_E on E.

If ∇_E satisfies $F_E = \lambda d\mathrm{vol}_C \otimes \mathrm{Id}_E$, then one can show $\deg(E) = \frac{i}{2\pi}\lambda \mathrm{rank}(E)$. Then $\lambda = \frac{2\pi}{i} \cdot \frac{\deg(E)}{\mathrm{rank}(E)} = \frac{2\pi}{i}\mu(E)$ where $\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}$ is called the slope of E. Thus, λ is completely determined by the topology of E.

A holomorphic vector bundle E on C is *stable* if for all subbundles proper $F \subset E$, we have $\mu(F) < \mu(E)$. We say E is *semistable* if for F as above, we have $\mu(F) \leq \mu(E)$. We say E is *polystable* if it is a direct sum of stable bundles of same slope.

Narasimhan-Seshadri theorem: Let E be a holomorphic vector bundle on C. Then E admits a metric h whose Chern connection ∇_E satisfies $*F_E = \frac{2\pi}{i}\mu(E)\mathrm{Id}_E$ if and only if E is polystable. Note: h is not unique, but the Chern connection ∇_E is unique.

So we have a bijection between algebraio-geometric side

$$\{\text{polystable holo vector bundle rank n, deg d on C}\}/iso$$

with differential geometric side

$$\{\text{rank n unitary connections on C satisfying } *F_\nabla = \frac{2\pi}{i} \cdot \frac{d}{n}\mathrm{Id}\}/iso$$

This correspondence is not just a bijection of sets. The two sides are moduli spaces. LHS, moduli space of polystable vector bundle, can be constructed as a complex projective algebraic variety $\mathcal{M}_{n,d}(C)$ constructed using GIT. The RHS is the moduli space of solutions to a differential geometric equation $*F_E = \frac{2\pi}{i} \cdot \frac{d}{n}\mathrm{Id}_E$. It can be topologised by working in a suitable function space (a Sobolev space of connections) and taking the quotient space by the gauge group (i.e. unitary isomorphisms).

RHs is a topological space using deformation theory, get a real analytic structure. The N-S theorem then gives a real analytic isomorphism of moduli spaces.

If $\deg(E) = 0$, then N-S theorem gives an isomorphism $\mathcal{M}_{n,0}(C)$, rank n deg 0 polystable vector bundles, with rank n unitary connection ∇ satisfying $F_\nabla = 0$, i.e. flat connections. And flat connections are determined by their monodromy/holonomy representation $\rho : \pi_1(C) \to U(n)$, up to conjugations, i.e. up to isomorphic representations. i.e.

$$\mathcal{M}_{n,0}(C) \cong \mathrm{Hom}(\pi_1(C), U(n))/U(n).$$

For the case of general complex connected semisimple Lie group G with maximal compact K. We can define holomorphic principal G-bundles. Can define (semi)-stability/polystable. We can construct via GIT a moduli space $\mathcal{M}_G(C)$ of polystable holomorphic G-bundles on C, a complex projective algebraic variety. The N-S theorem then says $\mathcal{M}_G(C) \cong \mathrm{Hom}(\pi_1(C), K)/K$.

10.3. **14/02/2022: Masoud's talk: Arithmetic of character variety of reductive groups.** For a finite group Γ, the set $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)$ has an algebraic-geometric structure: let $\Gamma = \langle x_1, \dots, x_k | r_1, \dots, r_m \rangle$. Then $\mathrm{Hom}(\Gamma, \mathrm{GL}_n) = (\mathrm{GL}_n)^k/(r_1, \dots, r_m)$ is a variety, called representation variety associated to (Γ, GL_n). This variety structure is independent of presentations of Γ.

Two representations $\Gamma \to \mathrm{GL}_n$ are equivalent if they are conjugate by some $g \in \mathrm{GL}_n$. So we want to study $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)/\mathrm{GL}_n$.

This quotient can be understood as a quotient stack or as a character variety $\mathrm{Hom}(\Gamma, \mathrm{GL}_n)//\mathrm{GL}_n$. By definition of the later, $\mathrm{Spec}\, k[\mathrm{Hom}(\Gamma, \mathrm{GL}_n)]^{\mathrm{GL}_n}$. Its points corresponding to semi-simple representations $\Gamma \to \mathrm{GL}_n$.

The significance of this moduli space: $\Gamma = \pi_1(X)$ where X is Kahler manifold. Then we have correspondences

$$Higgs_n(X) \leftrightarrow \mathrm{Hom}(\Gamma, \mathrm{GL}_n)//\mathrm{GL}_n \leftrightarrow \mathrm{LocSys}_n(X)$$

The first iso is real analytic iso, non-abelian Hodge. The second is complex analytic, Riemann-Hilbert correspondence. The picture appears to be most rich when X is a Riemann surface.

Geometric Langlands relates the two stacks for the case X is a Riemann surface

$$[\mathrm{Hom}(\Gamma, \mathrm{GL}_n)//\mathrm{GL}_n] \leftrightarrow \mathrm{Bun}_n(X).$$

Some results in understanding these moduli spaces: Poincare polynomials, mixed Hodge polynomials, P=W conjectures, mirror symmetry, ... Langlands says we must understand $\mathrm{Hom}(\Gamma, G)/G$ for any reductive G.

The group's on-going project is to generalise Hausel's results to general reductive groups.

Example. Suppose $\Gamma = \mathbb{Z} = \pi_1(S^1) = \pi_1(\mathbb{P}^1 - \{0, 1\})$. G reductive group. Then $\mathrm{Hom}(\Gamma, G) = G$, $[\mathrm{Hom}(\Gamma, G)/G]$ is stack of G-local system on S^1, $\mathrm{Hom}(\Gamma, G)//G = G//G \cong T/W \cong (\mathbb{C}^\times)^n$ where T torus, W Weyl group of G (Chevalley's theorem).

Example. $\Gamma = F_2 = \langle a, b \rangle = \pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$, then $\mathrm{Hom}(F_2, G) = G \times G$, $\mathrm{Hom}(F_2, G)//G = ?$. But $\mathrm{Hom}(F_2, \mathrm{SL}_2)//\mathrm{SL}_2 \cong \mathbb{C}^3$ by Frieke-Klein-Vogt, 1900's.

Example, $\Gamma = \mathbb{Z} \times \mathbb{Z}$, then $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, G) = \{(g, h) \in G \times G : gh = hg\}$. Ngo-Chen 2021 shows $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, G)//G \cong (T/W) \times (T/W)$.

Example, $\Gamma_{g,k} = \pi_1(\Sigma_{g,k}) = \langle x_1, \dots, x_g, y_1, \dots, y_g, z_1, \dots, z_k \rangle / [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_k$. We want to understand the moduli space of rep of $\Gamma_{g,k}$.

Using counting points over finite fields and Weil's conjectures to understand character stacks/varieties. By using Frobenius formula, to count points of character varieties, one needs to understand irreducible complex representations of $G(\mathbb{F}_q)$, which comes from work of Deligne-Lusztig, Lusztig.

Let A be a group. The representation ζ-function of A is $\zeta_A : \mathbb{C} \to \mathbb{C}$ sending $s \mapsto \sum_{\rho \in \widehat{A}} \frac{1}{\dim(\rho)^s}$. Here

Example. If $A = \mathrm{SL}_2(\mathbb{C})$, $\widehat{A}$ as algebraic reps. Then ζ_A is the usual Riemann- Zeta function.

Frobenius theorem is then

$$|\mathrm{Hom}(\Gamma_g, G)/G(\mathbb{F}_q)| = \zeta_{G(\mathbb{F}_q)}(2g-2) \times |G(\mathbb{F}_q)|^{2g-2}.$$

There is a different version of Frobenius theorem in different setting of different moduli spaces. For example, over $\mathbb{R}$ values Riemann-Zeta function is volumes of moduli spaces:

$$\mathrm{vol}\left(\mathrm{Hom}(\Gamma_g, U_n)/U_n\right) = \zeta_{U_n}(2g-2),$$

where LHS is symplectic volume form.

If we replace q with $-q$ in RHS, then the values can be related to Riemann-Zeta function of unitary groups. Some sort of duality with $\mathrm{GL}_n(\mathbb{F}_q)$ with $GU_n(\mathbb{F}_q)$.

If replace q with q^{-1} then these functions are invariant. This comes from Alvis-Curtis duality.

10.4. **15/02/2022: Uri Onn's talk Representation zeta function of arithmetic groups.** If number of irreducible finite-dim representations of dimension n $r_n(G) \le c \cdot n^k$ has polynomial growth for some c then we can define representation zeta function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n n^{-s} = \sum_{\rho \in Irr_f(G)} (\dim\, \rho)^{-s}.$$

This converges on the right-half plane $Re(s) > \alpha_G = \limsup_N \frac{\log \sum_1^n r_n}{\log N}$.

Frobenius (1897) theorem says

$$\mathrm{Hom}(\pi_1(\Sigma_g), G) = |G|^{2g-1}\zeta_G(2g-2).$$

From this formula, $\zeta_{G_1 \times G_2} = \zeta_{G_1} \cdot \zeta_{G_2}$.

Open question: Characterise groups with polynomial representation growths (PRG).

Speaker interested in arithmetic groups: G algebraic groups (semisimple, simply connected) defined over number field K, with rings of integer $\mathcal{O}$. Consider $G(\mathcal{O}_S)$ where $\mathcal{O}_S = \{x \in K | v_p(x) \ge 0 \text{ for all } p \notin S\}$. Examples are $\mathrm{SL}_d(\mathbb{Z}), \mathrm{SL}_d(\mathbb{Z}[i]), \mathrm{SL}_d(\mathbb{Z}[1/p])$ for p prime.

What kind of representations we consider when we define ζ_G when G is arithmetic groups? Need some restrictions.

$\mathrm{SL}_2(\mathbb{Z})$ contains free group F_2 with finite index so r_n infinite. Consider $\mathrm{SL}_d(\mathbb{Z})$ for $d \ge 3$, which is special in the following way:

$G(\mathcal{O}_S)$ has the congruence subgroup property (CSP) if "arithmetic quotients dominate finite quotients". For all normal subgroup N with finite index, exists ideal I of $\mathcal{O}_S$ such that N is a subgroup of $\ker(G(\mathcal{O}_S) \to G(\mathcal{O}_S/I))$. For example,define $N_k = \ker(\mathrm{SL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}/k\mathbb{Z}))$ for $k \in \mathbb{Z}$. For any finite index normal subgroup N of $\mathrm{SL}_d(\mathbb{Z})$, there exists $k \in \mathbb{Z}$ such that $\mathrm{SL}_d(\mathbb{Z}/k\mathbb{Z})$ surjects onto $\mathrm{SL}_d(\mathbb{Z})/N$. A theorem of Bass-Milnor-Serre, $\mathrm{SL}_d(\mathbb{Z})$ has congruent subgroup property if $d \ge 3$.

Lutz-Martin's theorem (2004): $G(\mathcal{O}_S)$ has CSP (congruent subgroup property) iff it has PRG (polynomial representation growths).

Question: What is the relation between representation zeta functions of $\mathrm{SL}_d(k)$ for $k = \mathbb{Z}, \mathbb{Z}_p, \mathbb{C}, \mathbb{Z}/p^k\mathbb{Z}$?

Larsen-Lubotzky JEMS 2008: If $G(\mathcal{O}_S)$ has CSP then exists

$$\zeta_{G(\mathcal{O}_S)}(s) = \zeta_{G(\mathbb{C})}(s)^{[k:\mathbb{Q}]} \cdot \prod_{0 \ne \mathfrak{p} < \mathcal{O}_S} \zeta_{G(\mathcal{O}_\mathfrak{p})}(s).$$

In this formula: $G(\mathcal{O}_S)$ we consdier arbitrary irreps, $G(\mathbb{C})$ algebraic reps, compact $G(\mathcal{O}_p)$ we consider continuous reps.

For example, $\zeta_{\mathrm{SL}_d(\mathbb{Z})}(s) = \zeta_{\mathrm{SL}_d(\mathbb{C})}(s) \prod_{p \text{ prime}} \zeta_{\mathrm{SL}_d(\mathbb{Z}_p)}(s)$.

Ideal of proof: $\mathrm{SL}_d(\mathbb{Z})$ is Zariski dense in $\mathrm{SL}_d(\mathbb{C})$. So if we have irreducible algebraic rep of $\mathrm{SL}_d(\mathbb{C})$ then by precompose, we get irreducible rep of $\mathrm{SL}_d(\mathbb{Z})$.

Similarly, $\mathrm{SL}_d(\mathbb{Z})$ also dense in $\mathrm{SL}_d(\mathbb{Z}_p)$. Then if we consider irreducible continuous rep of $\mathrm{SL}_d(\mathbb{Z}_p)$ then by precompose, we get irreducible rep of $\mathrm{SL}_d(\mathbb{Z})$.

Overall, $\mathrm{SL}_d(\mathbb{Z})$ is dense in $\mathrm{SL}_d(\mathbb{C}) \times \mathrm{SL}_d(\mathbb{Z}_{p_1}) \times \cdots \times \mathrm{SL}_d(\mathbb{Z}_{p_r})$ via the diagonal mapping. *Toan: Looks like strong approximation theorem for adeles for me, but Masoud noticed that there is no* $\mathrm{SL}_2(\mathbb{R})$ *in this picture.*

In fact, all irreps of $\mathrm{SL}_d(\mathbb{Z})$ are obtained in this way (this is not true for SL_2, even though the previous property holds for SL_2). This follows from Margulis super rigidity theorem.

Margulis super rigidity says that ρ finite dim irrep of $\mathrm{SL}_d(\mathbb{Z})$ is either finite image reps or extends to rep of $\mathrm{SL}_d(\mathbb{C})$. By CSP, every finite image reps of $\mathrm{SL}_d(\mathbb{Z})$ factors through $\mathrm{SL}_d(\mathbb{Z}/k\mathbb{Z})$.

Auni 2011: $\alpha_{G(\mathcal{O}_S)} \in \mathbb{Q}$ if $G(\mathcal{O}_S)$ is CSP.

Toan: What about $\mathrm{SL}_d(\mathbb{Q})$*? Its representations? Its zeta functions? Can we discuss this for* $\mathrm{SL}_d(\mathbb{Q}) \leftarrow \mathrm{SL}_d(\mathbb{Q}_p)$ *and etc What relation between* $\zeta_{G(\mathbb{R})}$ *and* $\zeta_{G(\mathbb{C})}$*?*

From what I read about Margulis superridity, why we cannot extend $\mathrm{SL}_2(\mathbb{Z})$ *to* $\mathrm{SL}_2(\mathbb{R})$*?* `http://emis.maths.adelaide.edu.au/journals/Annals/151_3/bass.pdf`*, p 1155, also RIGIDITY AND ARITHMETICITY Marc BURGER, Margulis, Discrete subgroups of semisimple Lie groups*

What is the relation between L-functions and representation zeta functions?

10.5. **15/02/2022: David Baraglia's talk 2: Moduli spaces of Higgs bundles.** Last time: C compact connected Riemann surface of genus $g \geq 2$, G connected complex semisimple Lie group (e.g. $\mathrm{SL}_n(\mathbb{C})$), K compact Lie subgroup (e.g. $SU(n)$). The Narasimhan-Seshadri theorem gives isomorphism between polystable, holomorphic, principal G-bundles/iso and $\mathrm{Hom}(\pi_1(C), K)/K$.

Question: What happens if we instead look at complex representations $\pi_1(C) \to G$?

Unlike compact case, conjugation action of G on $\mathrm{Hom}(\pi_1(C), G)$ is not well-behaved, i.e. some orbits are not closed. Naive group quotient will not be Hausdorff. But note $\mathrm{Hom}(\pi_1(C), G)$ is an affine complex algebraic variety. So can form GIT quotient $\mathrm{Rep}(\pi_1(C), G) = \mathrm{Hom}(\pi_1(C), G)//G$.

Theorem: $\mathrm{Hom}^{red}(\pi_1(C), G)//G = \mathrm{Hom}(\pi_1(C), G)//G$ where $\mathrm{Hom}^{red}(\pi_1(C), G)//G$ is set of of reductive representations, i.e. $\rho : \pi_1(C) \to G$ reductive if adjoint rep $Ad \circ \rho : \pi_1(C) \to \mathrm{GL}(\mathfrak{g})$ is completely reducible. Equivalently, the Zariski closure of ρ is reductive.

N-S theorem extends to complex representations of $\pi_1(C)$ where RHS is $\mathrm{Hom}(\pi_1(C), G)//G$ and the lHS is moduli space of polystable G-Higgs bundles.

Start off with GL case first:

Def: A (general linear) *Higgs bundle* on C is a pair (E, ϕ) where E is a holomorphic vector bundle on C and ϕ is a holomorphic bundle map $\phi : E \to K_C \otimes E$ where $K_C = \wedge^{1,0} T^*C$ is the holomorphic cotangent bundle. ϕ is called *Higgs field*, roughly a "matrix of holomorphic 1-forms").

Masoud said that Higgs bundles are essentially cotangent bundle of moduli space of vector bundles.

Def: (E, ϕ) is stable if for each proper nonzero, ϕ-invariant subbundle $F \subset E$, we have $\mu(F) < \mu(E)$, where $\mu(E) = \deg(E)/\mathrm{rank}(E)$.

Denote $\mathcal{M}^{Higg}_{n,d}(C)$ to be moduli space of polystable, rank n, deg n Higgs bundles, also called Dolbeault moduli space. It can be constructed algebraically, is a complex algebraic quasi-projective algebraic variety.

Every (poly)stable bundle E can be regarded as a (poly)stable Higgs bundle with zero Higgs field. So

$$\mathcal{M}_{n,d}(C) \subset \mathcal{M}^{Higgs}_{n,d}(C)$$

as subvariety.

For example, $\mathcal{M}_{1,0}$ are all holomorphic line bundles of degree 0, which is $Jac(C)$, which is diffeomorphic to torus $(S^1)^{2g}$. And

$$\mathcal{M}^{Higgs}_{1,0}(C) = \{(E, \phi) : \phi \in K_C \otimes E^* \otimes E \cong K_C\} = Jac(C) \times H^0(C, K_C) = T^*Jac(C).$$

By Serre duality, at $[E] \in \mathcal{M}_{n,d}^{st}(C)$ then $T^*\mathcal{M}_{n,d}(C)_{[E]}$ is space of Higgs fields on E. Hence, $T^*\mathcal{M}_{n,d}^{st}(C)$ is dense open in $\mathcal{M}_{n,d}^{Higgs}(C)$. If we work with stacks, this would be an equality.

Can think of $\mathcal{M}_{n,d}^{Higgs}(C)$ as partial compactification of $T^*\mathcal{M}_{n,d}^{st}(C)$.

(Irrelevant: Cotangent bundle of G/B is resolution of singularities of nilpotent cone).

Cotangent bundle $T^*\mathcal{M}_{n,d}^{st}(C)$ has a canonical symplectic form that extends to the smooth locus of $\mathcal{M}_{n,d}^{Higgs}(C)$.

Def (Hitchin equations). Let $(E, \overline{\partial}_E)$ holomorphic vector bundle on C, $\phi : E \to K_C \otimes E$ be (smooth) endomorphism, h be a Hermitian metric on E. The Hitchin equations for (E, ϕ, h) are $\overline{\partial}_E(\phi) = 0$ and $F_{\nabla_h} + [\phi, \phi^*] = \frac{2\pi}{i}\mu(E) d\mathrm{vol}_C \otimes \mathrm{Id}_E$, where ϕ^* is taken wrt h, ∇_h is Chern connection, choose metric g so $\mathrm{vol}(C) = 1$.

Theorem (Hitchin-Simpson). Let $(E, \overline{\partial}_E, \phi)$ be a Higgs bundle. Then there exists Hermitian metric h on E such that $(E, \overline{\partial}_E, \phi, h)$ satisfies the Hitchin equations iff (E, ϕ) is polystable.

This gives a bijection between

$$\{\text{polystable Higgs bundles } (E, \phi)\}/iso$$

and

$$\{\text{solutions of Hitchin equations }\}/\text{unitary gauge trans.}$$

To keep things simple, assume $\deg(E) = 0$, then Hitchin equations are $\overline{\partial}_E\phi = 0$ and $F_{\nabla_h} + [\phi, \phi^*] = 0$. These are equivalent to saying that the connection $\nabla = \nabla_h + \phi + \phi^*$ is flat. So the Higgs fields $\phi + \phi^*$ in some sense contribute to the non-unitary part of the connection. Thus, a solution to Hitchin equations define a flat $\mathrm{GL}_n(\mathbb{C})$-connection.

Question: Given a flat $\mathrm{GL}_n(\mathbb{C})$ connection ∇, does it come from a solution of the Hitchin equations?

Given Hermitian metric h on E, then ∇ can be uniquely split as $\nabla = \nabla_h + \Phi$ where ∇_h is a unitary connection, and $\Phi^* = \Phi$ self-adjoint 1-form valued endo. Hence, $\Phi = \phi + \phi^*$ into $(1,0)$-part and $(0,1)$-part. So $\nabla = \nabla_h + \phi + \phi^*$. Moreover, we get holomorphic structure $\overline{\partial}_E = \nabla_h^{0,1}$, hence ∇_h is the Chern connection of $(E, \overline{\partial}_E, h)$. But this ϕ is not necessarily holomorphic.

Question: Can we find h so that $(E, \overline{\partial}_E, \phi, h)$ satisfies the Hitchin equations?

Yes, this is proved by Donaldson-Corlette. Theorem: Let ∇ flat $\mathrm{GL}_n(\mathbb{C})$ connection on C. Then there exists h s.t. $(E, \overline{\partial}_E, \phi, h)$ satisfies Hitchin equations iff holonomy of ∇ is reductive.

Thus, we have

$$\mathcal{M}_{n,0}^{Higgs}(C) \cong \{\text{sols to Hitchin equations } (E, \overline{\partial}_E, \phi, h)\}$$

by Simpson-Hitchin and

$$\{\text{sols to Hitchin equations } (E, \overline{\partial}_E, \phi, h)\} \cong \{\text{flat } \mathrm{GL}_n(\mathbb{C}) - \text{connections with redutive holonomy }\}$$

by Donaldson-Corlette, and

$$\{\text{flat } \mathrm{GL}_n(\mathbb{C}) - \text{connections with redutive holonomy }\} \cong \mathrm{Hom}(\pi_1(C), G)/G$$

by Riemann-Hilbert correspondence. So the three moduli spaces (Dobeault, de Rham and Betti) are the same.

This works for any complex connected semisimple Lie group G. We still have

$$\mathcal{M}_G^{Higgs}(C) \cong \mathcal{M}_G^{flat}(C) \cong \mathrm{Hom}(\pi_1(C), G)//G.$$

10.6. **16/02/2022: Masoud's talk 2.** Polynomials with residue counting (PORC) if there exists integer d and polynomials $\|Y\|_1, \ldots, \|Y\|_{d-1} \in \mathbb{C}[t]$ such that $|Y(\mathbb{F}_q)\| = \|Y_i\|(q)$ for $i \equiv q \pmod{d}$.

For reductive G with connected center then the stack $X = [\mathrm{Hom}(\Gamma_g, G)/G]$ is PORC count with modulus $d(G^\vee)$.

Consequently, if $q \equiv 1 \pmod{d(G^\vee)}$ then $X_{\mathbb{F}_q}$ is polynomial count with counting polynomial $X\|_1$, so the E-polynomial $E(X_{\mathbb{F}_q}) = \|X\|_1$. In particular, we can compute the dimension of X for general G. This agrees with the dimension of moduli space of semistable Higgs bundles of Σ_g.

10.7. **17/02/2022: Nir Avni's talk Counting points and counting representations.** Here is the paper: `https://arxiv.org/abs/1502.07004`

10.7.1. *Counting points.* Let X be a system of equations. Slogan:

(1) $|X(\mathbb{F}_p)|$ as $p \to \infty$ depends on global invariants, i.e. looks like $X(\mathbb{C})$ as $p \to \infty$, Weil's conjectures.
(2) $|X(\mathbb{Z}/p^n)|$ as $n \to \infty$ depends on local invariants, i.e. singularities of X (Aside: Igusa zeta function is of the same vein in measuring singularities).

X solutions of $x^2+y^2-z^2$. What is $\frac{|X(\mathbb{F}_{q^n})|}{q^{2n}}$? Let $f : \mathbb{Z}_p^3 \to \mathbb{Z}_p$ defined by $f(x,y,z) = x^2+y^2-z^2$. Then

$$\frac{|X(\mathbb{F}_{q^n})|}{q^{2n}} = \frac{(f_*\lambda_{Harr})(p^n\mathbb{Z}_p)}{\lambda_{Haar}(p^n\mathbb{Z}_p)}.$$

These Haar measures come from differential form $\omega = \frac{dx\wedge dy\wedge dz}{f^*dt}$, i.e. $dx \wedge dy \wedge dz = \omega \wedge f^*dt$, which is $\frac{dx\wedge dy}{z}$.

Upshot: Need to compute $\int \frac{dx\wedge dy}{z}$ over the cone $x^2+y^2 = z^2$.

To do this, find a resolution: Over $\mathbb{R}$, use change of coordinates. Let $x = r\cos\theta, y = r\sin\theta, z = r$ then

$$\int dx \wedge dyz = \int |dr \wedge d\theta| < \infty$$

where LHS is over the cone, RHS is over the cylinder.

Over $\mathbb{Z}_p$, use blow up (algebraic equivalent of polar coordinates).

1) In general, for (normal, ..) variety X, then

$$\frac{|X(\mathbb{Z}/p^n)|}{p^{n\dim(X)}} \xrightarrow{n\to\infty} \int_{X^{sm}} |\omega|,$$

where $\omega \in H^0(\Omega_X)$ canonical bundle.

2) If $\pi : \tilde{X} \to X$ is a resolution of singularities, then

$$\int_{X^{sm}} \omega = \int_{\tilde{X}} \pi^*\omega,$$

where $\pi^*\omega$ regular on $\pi^{-1}(X^{sm})$.

$\int_{\tilde{X}} \pi^*\omega < \infty$ iff $\pi^*\omega$ has no poles on $\tilde{X}\backslash\pi^{-1}(X^{sm})$. We say that X has canonical singularities.

10.7.2. *Counting representations.* Γ a group, $r_n(\Gamma)$ number of n-dimen irreps of Γ. Consider zeta function $\zeta_\Gamma(s) = \sum_{n=1}^\infty r_n(\Gamma)n^{-s}$.

Then $\zeta_\Gamma < \infty$ iff $r_n(\Gamma) = O(n^\alpha)$.

Example,

$$\begin{aligned}\zeta_{\mathrm{SL}_d(\mathbb{Z}_p)}(2) &= \lim_{n\to\infty} \zeta_{\mathrm{SL}_d(\mathbb{Z}/p^n)}(2),\\ &= \lim_{n\to\infty} \frac{|\mathrm{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_d(\mathbb{Z}/p^n)|}{|\operatorname{SL}_d(\mathbb{Z}/p^n)|^3},\\ &\approx \lim_{n\to\infty} \frac{|\mathrm{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_d(\mathbb{Z}/p^n))|}{p^{n\cdot 3\dim \mathrm{SL}_d}}.\end{aligned}$$

Note $n \cdot 3\dim \mathrm{SL}_d$ is dimension of representation variety. Hence,

$$r_n(\mathrm{SL}_d(\mathbb{Z}_p)) = O(n^2) \text{ iff vol } (\mathrm{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_d)) < \infty$$

iff $\mathrm{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_d)$ has canonical singularities.

(Aside: There is a differential form on $\mathrm{Hom}(...)//G$ whose pullback is ω of $\mathrm{Hom}(...)$, something about Atiyah-Bott).

There is a one of this variety that rules: if the germ of this point has canonical singularity then all other points have canonical singularities. In particular, $X = \mathrm{Hom}(\pi_1(\Sigma_2), \mathrm{SL}_d)$ has canonical singularity iff $(X, \text{trivial rep})$ has canonical singularity.

Deformation principle: If (X_t, x_t) is a family of pointed affine varieties then (X_0, x_0) is at least as singular as (X_ϵ, x_ϵ) for $0 < \epsilon < 1$.

As a corollary, $r_n(\mathrm{SL}_d(\mathbb{Z}_p)) = O(n^2)$ iff $Y = \{(X_1, Y_1, X_2, Y_2) \in \mathfrak{sl}_d^4 : [X_1, Y_1] + [X_2, Y_2] = 0\}$ has canonical singularities. It is an instance of quiver varieties. and the statement was proved by Budur and generalized by Herbig, Schwartz, Seaton. Y is the tangent cone of $\{(g_1, h_1, g_2, h_2) \in \mathrm{SL}_d^4 : (g_1, h_1) \cdot (g_2, h_2) = 1\}$ where (a, b) is the commutator. This is the identity (trivial rep) of the representation variety X. To get the tangent cone, look at $(1+\epsilon X_1, 1+\epsilon Y_1) \cdot (1+\epsilon X_2, 1+\epsilon Y_2) = 1$.

(Aside: With $G = \prod \mathrm{SL}_d(\mathbb{Z}_p) \times \mathrm{SU}_d$ then $\zeta_G(\chi) = \mathrm{vol}\ (\mathrm{Hom}\ (\pi_1(\Sigma_2), G))$, where this is an adelic manifold and has a top form).

10.8. **17/02/2022: David Baraglia's talk 3.** C compact, connected Riemann surface $g \geq 3$, G complex connected semisimple group. We have three moduli spaces $\mathcal{M}_G^{Higgs} = \mathcal{M}_G^{Dol}$ Dolbault moduli space, $\mathcal{G}^{dR}(C) =$ flat G bundles with reductive holonomy de Rham moduli space, $\mathrm{Rep}(\pi_1(C), G) = \mathcal{M}_G(C)$ Betti moduli space.

Non-abelian Hodge says there are isomorphisms

$$\mathcal{M}_G^{Dol}(C) \cong \mathcal{M}_G^{dR}(C) \cong \mathcal{M}_G^{Betti}(C)$$

Question: Why this is called non-abelian?

Consider the abelian case $G = \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$, then $\mathcal{M}_{1,0}^{Higgs} = T^*\mathrm{Jac}(C) = H^1(C, \mathcal{O}^\times) \oplus H^0(C, K_C)$, de Rham moduli space: flat connections on trivial line bundle, $\nabla = d + a, F_\nabla = da = 0$, modulo gauge symmetry, get $H^1_{dR}(C, \mathbb{C})/H^1(C, \mathbb{Z})$. And Betti moduli space is $\mathrm{Hom}(\pi_1(C), \mathbb{C}^\times) = \mathrm{Hom}(H_1(C, \mathbb{Z}), \mathbb{C}^\times) = H^1(C, \mathbb{C}^\times)$. Thus, the abelian version of non-abelian Hodge is isomorphims

$$H^1(C, \mathcal{O}^\times) \oplus H^0(C, K_C) \cong \frac{H^1_{dR}(C, \mathbb{C})}{H^1(C, \mathbb{Z})} \cong H^1(\mathbb{C}, \mathbb{C}^\times)$$

This follows from isomorphisms

$$H^1(C, \mathcal{O}) \oplus H^0(C, K) \cong H_{dR}(C, \mathbb{C}) \cong H^1(C, \mathbb{C}),$$

which is a special case of Hodge theory.

Look at the geometry of these moduli spaces:

Restrict to locus of stable Higgs bundles, where the moduli space is smooth.

Regard the moduli space as solutions to the Hitchin equations on a fixed smooth principal G-bundle E with a fixed Hermitian metric h. The Hitchin equations are $\begin{cases} \nabla_E^{0,1}\phi = 0 \\ F_E + [\phi, \phi^*] = 0 \end{cases}$. The tangent vector to (E, ϕ) linearise Hitchin equatinos modulo linearisation of gauge group. Consider first order deformations $(\alpha, \psi) \in \Omega^{0,1}(C, \mathrm{Ad}(E)) \oplus \Omega^{1,0}(C, \mathrm{Ad}(E))$, plug $(\nabla_E^{0,1} + t\alpha, \phi + t\psi)$ into Hitchin equations, get first order in t, which is

$$\begin{cases} \nabla_E^{0,1}(\psi) + [\alpha, \phi] = 0, \\ \nabla_E^{0,1}\alpha - \nabla_E^{0,1}(\alpha^*) + [\phi, \psi^*] + [\psi, \psi^*] = 0. \end{cases}$$

To account for the gauge transformations, restrict to (α, ψ) orthogonal to the gauge orbit through (E, ϕ).

There is a natural L^2-metric on tangent space

$$g_{(E,\psi)}((\alpha,\psi),(\alpha,\psi)) = i\int_C k(\alpha^*,\alpha) - k(\psi^*,\psi),$$

where k is the Killing form. This is a hyperKahler metric, i.e. there are integrable complex structure I, J, K satisfying the quaternion relations $IJ = K$, etc, such that g is Kahler wrt I, J, K, i.e. there is $\omega_I = g(I-,-)$, ω_J, ω_K are closed symplectic forms. In particular, $I(\alpha,\psi) = (i\alpha, i\psi), J(\alpha,\psi) = (i\psi^*, -i\alpha^*)$ and $K(\alpha,\psi) = (-\psi^*, \alpha^*)$.

Then I is the natural complex structure on $\mathcal{M}_C^{Higgs}(C)$, J is the natural complex structure on $\mathcal{M}_G^{dR}(C)$ and $\mathcal{M}_G^{Betti}(C)$.

Zero locus of moment maps $\mu_I, \mu_J\mu_K$ are the Hitchin equations, i.e. can see moduli space as hyperkahler quotient $\mathcal{M}_G^{Hitchin}(C) = \pi_I^{-1}(0)\cap\mu_J^{-1}(0)\cap\mu_K^{-1}(0)/\mathcal{G}$ modulo gauge group, hence inherits a hyperkahler structure.

$\mathbb{C}^\times$*-action:*

$\mathbb{C}^\times$ acts holomorphically on $\mathcal{G}^{Higgs}(C)$ by $\lambda(E,\phi) = (E,\lambda\phi), \lambda\in\mathbb{C}^\times$ but is not holomorphic in J, K.

Morevoer, $S^1\subset\mathbb{C}^\times$ acts by isometries and is Hamiltonian wrt ω_I. The moment map $\mu(E,\phi) = \|\phi\|_{L^2}^2\int_C k(\phi,\phi^*)$ is a Morse-Bott function, called Hitchin functional.

It is also a proper map. The critical submanifolds are fixed by the $\mathbb{C}^\times$-action. Hitchin and others have used μ to study topology of $\mathcal{M}^{Higgs}(C)$ (for example, compute Poincare polynomials).

Integrable systems:

Moduli space $\mathcal{M}^{Higgs}$ is a torus fibration f over $B_G(C)$. In particular, there is a singular locus that is very singular, called nilpotent cone. Hitchin defined a map $f : \mathcal{M}^{Higgs}(C)\to B_G(C)$, which is an affine space $\mathbb{C}^n$ where $n = \dim(\mathcal{M}_G^{Higgs}(C))/2$. The coordinates of f are Poisson commuting wrt the holomorphic symplectic structure $\Omega = \omega_J + i\omega_K$. f is proper map. So we get an integrable system. It implies that the fibers are coisotropic, in fact are Lagrangian. Can then use Liouville's theorem on integrable systems to say that the fibers are tori.

As $\mathcal{G}^{Higgs}(C)$ is algebraic, so is f, so fibers are algebraic, hence are abelian varieties.

Langlands duality:

Take Langlands dual group G and ${}^LG$ then $B_G(C)\cong B_{{}^LG}(C)$. Then we get two fibrations $\mathcal{M}_G^{Higgs}(C)$ and $\mathcal{M}_{{}^LG}^{Higgs}(C)$ over the same space, hence gives rise to dual torus fibration. This is an example SYZ mirror symmetry.

Describe fibration $f : \mathcal{M}^{Higgs}(C)\to B_G(C)$: Consider $\mathbb{C}[\mathfrak{g}]^G$, ring of ad-invariant polynomials in $\mathfrak{g}$, by Chevalley, is $\mathbb{C}[p_1,\ldots,p_r]^r$ where p_i homogeneous of degrees $d_1,\ldots,d_r$, $r = \mathrm{rank}(G)$. Then $f(E,\phi) = (p_1(\phi), p_2(\phi),\ldots,p_r(\phi))$. In GL_n case, $p_j = \mathrm{tr}(\phi^j)$.

10.9. **17/02/2022: Masoud's talk 3:** Previous two lectures: counting points over representation variety and character stack.

Some open questions:

(1) Counting points over character variety ? Only $\mathrm{GL}_2, \mathrm{GL}_3$ known.
(2) We can deduce E-polynomials from counting points. But what about mixed Hodge polynomials?
(3) Understand mirror symmetry between G and $G^\vee$?
(4) $P = W$ conjecture.

Consider punctured Riemann surface: $\Gamma = \Gamma_{g,k} = \langle x_1,\ldots,x_g,y_1,\ldots,y_g,z_1,\ldots,z_k\rangle/\prod[x_i,y_i]\prod z_i$.

$\mathrm{Hom}_C(\Gamma,G) = \{f:\Gamma\to G, f(z_i)\in C_i\}$ and $X = [\mathrm{Hom}_C(\Gamma,G)/G]$.

For $C = (C_i,\ldots,C_k)$ conjugacy classes. If $G = \mathrm{GL}_n$ and C is generic then $[\mathrm{Hom}_C(\Gamma,G)/G] = \mathrm{Hom}_\Gamma(\Gamma,G)//G$, proved by Hausel, ...

Goal: Try to prove this for general type.

10.9.1. *Counting points.* Consider character variety with regular monodromy. And C consists of regular semisimple or regular unipotent conjugacy classes. (regular = stabiliser in Weyl group is trivial) Let S, N be set of regular semisimple and regular nilpotent conjugacy classes.

Frobenius formula

$$|X]_{(S,N)}(\mathbb{F}_q)| = \sum_{\chi \in Irr(G(\mathbb{F}_q))} \left(\frac{|G(\mathbb{F}_q)|}{\chi(1)}\right)^{2g-2} \frac{\chi(S)}{\chi(1)}|C_S| \times \frac{\chi(N)}{\chi(1)}|C_N|,$$

where $|C_S|, |C_N|$ the number of elements in orbits of S, N under conjugation.

Use Deligne-Lusztig theorem: $\chi(S) = 0$ unless χ irreducible constituent of principal series representations $B(\theta) = \mathrm{ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\theta)$ for $\theta \in T(\mathbb{F}_q)^\vee = \mathrm{Hom}(T(\mathbb{F}_q), \mathbb{C}^\times)$. Hence, can rewrite $Irr(G(\mathbb{F}_q))$ as $\sum_{[\theta] \in T(\mathbb{F}_q)^\vee/W} \sum_{\chi \in B(\theta)} \cdots$.

By Green(?)-Lehrer-Lusztig theorem, there exists unique constituent $\chi_\theta \in B(\theta)$ so $\chi_\theta(N) \neq 0$. Rewrite the sum $\sum_{[\theta] \in T(\mathbb{F}_q)^\vee/W} \cdots$

Can compute $\chi_\theta(N), \chi_\theta(S), \chi_\theta(1)$ quite explicitly. We also know $|C_N|, |C_S|$.

Want to show that the result is polynomial that does not depend on q. The trouble here is that we are summing over $T(\mathbb{F}_q)^\vee/W$, which depends on q.

First, write it as sum over $T(\mathbb{F}_q)^\vee$: $\sum_{[\theta] \in T(\mathbb{F}_q)^\vee} \frac{|W|}{|W_\theta|} \cdot ...$

We have a map $\Phi : T(\mathbb{F}_q)^\vee \to \pi(W)$ where $\pi(W)$= reflection subgroups of W, sending $\theta \mapsto W_\theta = \mathrm{Stab}_W(\theta)$.

Can rewrite the sum over reflection subgroups L of W: $\alpha_L(q) = \sum_{L \in \pi(W)} \frac{|W|}{|W_L|} \times ...$ except the term $\sum_{\theta \in \Phi^{-1}(L), w \in W} \theta(w \cdot s)$. It remains to compute this sum and demonstrate that this is polynomial in q.

Example for GL_n $L = S_{n_1} \times \cdots S_{n_m} \subset S_n$, the sum is $\sum_{\theta \in T(\mathbb{F}_q), W_\theta \cong L}$. What it is follows from Frobenius inversion.

Open: What happens for other groups? Problem is we don't know what does "generic" mean in other groups.

Generic of C in GL_n: $(C_1, \ldots, C_k)$ is generic if whenever exists nonempty subspace $V \subset k^n$ and $X_i \in C_i$ s.t. $X_i V \subset V$ then $\det(\prod x_i|_V) \neq 1$.

10.10. **18/02/2022: Uri On talk 3: Local, global factors.** For $d \geq 3$, recall we have

$$\zeta_{\mathrm{SL}_d(\mathbb{Z})}(s) = \zeta_{\mathrm{SL}_d(\mathbb{C})}(s) \prod_{p \text{ prime}} \zeta_{\mathrm{SL}_d(\mathbb{Z}_p)}(s).$$

Continuous representation $\mathrm{SL}_d(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{C})$ factors through some finite quotient $\mathrm{SL}_d(\mathbb{Z}/p^k\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{C})$.

Theorem (Jaikin-Zapirain): p odd prime, G Fab p-adic analytic group (technical condition guarantees that p-adic zeta function converges). Then there exists $N, n_1, \ldots, n_N, f_1, \ldots, f_N \in \mathbb{Q}(t)$ such that $\zeta_G(s) = \sum_{i=1}^N n_i^{-s} f_i(p^{-s})$. Note that these n_i, f_i depend on p. For pro-p-adic $\zeta_G(s) = \int_{\mathbb{Z}_p^d} |f_1(x)|_p^s |f_2(x)|_p d\mu$ using Orbit method where $f_i : \mathbb{Z}_p^d \to \mathbb{Z}_p$ definable functions.

Theorem (Avni, Klopsch, Voll, Duke 2013): G semisimple simply connected algebraic group over number field K with ring of integers $\mathcal{O}$. There exists finite set S of places of K, $r \in \mathbb{Z}_{\geq 0}$, $R(X_1, X_2, \ldots, X_r; Y) \in \mathbb{Q}(X_1, \ldots, X_r; Y)$ s.t. for every non-archimedean place $\mathfrak{p} \notin S$, the following holds: exists algebraic integers $\lambda_1 = \lambda_1(\mathfrak{p}), \cdots, \lambda_r(\mathfrak{p})$ s.t. for all $\mathcal{O}|o_\mathfrak{p}$, let inertia degree f of $\mathcal{O}|o_\mathfrak{p}$, $d = \dim G$, $m >> 0$:

$$\zeta_{G(\mathcal{O})^m}(s) = q^{fdm} R(\lambda_1^f, \ldots, \lambda_r^f; q_\mathfrak{p}^{-fs}),$$

where $G(\mathcal{O})^m$ is the mth congruent subgroup.

Problem: Does $\zeta_{\mathrm{SL}_d(\mathbb{Z})}$ admit meromorphic continuity?

Look at $\zeta_{G(\mathbb{F}_q)}$ where $G(\mathbb{F}_q)$ finite group of Lie type. Let $\mathfrak{g}$ Lie algebra of G.

Define $\tilde{G}(\mathbb{F}_q)$ as $\mathfrak{g}(\mathbb{F}_q) \times \mathfrak{g}(\mathbb{F}_q)$ as a set, with multiplication $(X_1, Y_1) * (X_2, Y_2) := (X_1 + X_2, Y_1 + Y_2 + \frac{1}{2}[X_1, X_2])$. It has $q^{2\dim G}$. It is a unipotent group. Another way to realise this group is

$$\left\{ \exp \begin{pmatrix} 0 & X & Y \\ & 0 & X \\ & & 0 \end{pmatrix} | X, Y \in \mathfrak{q}(\mathbb{F}_q) \right\} \subset \mathrm{GL}_{3d}(\mathbb{F}_q)$$

If $G = \mathrm{GL}_1$ then $\tilde{G}$ is the Heisenberg group.

Question: Find $\zeta_{\tilde{G}(\mathbb{F}_q)}$? How is it relate to $\zeta_{G(\mathbb{F}_q)}$? Not hard to find $\zeta_{\tilde{G}(\mathbb{F}_q)}$?

Open problem: Given arithmetic group, there are only finitely many other arithmetic groups with the same zeta function?

Pick a group, pick fields K, L over $\mathbb{Q}$ with same Dedekind zeta function. Consider congruent subgroups wrt K, L then they have the same representation zeta function?

10.11. **21/02/2022: Branched cover of $\mathbb{P}^1$ from algebraic curves.** Today I learned the following process: Given an affine variety defined by a polynomial $f(x,y) \in \mathbb{C}[x,y]$, we can obtain a branch cover $\overline{X} \to \mathbb{P}^1$ for some compact Riemann surface $\overline{X}$. I will explain this along with an example $f(x,y) = y^2 - x^4 + x^2$. See also the notes 28/11/2022 where I have roughly described this process for the case of elliptic curves.

I learned this from Akhil Mathew's notes `https://math.uchicago.edu/~amathew/287y.pdf` for the course Geometry of Algebraic Curves by Joe Harris, lecture 2.

(1) We find singular points of X, which $p \in \mathbb{C}^2$ of f satisfies $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$. In this case $f(x,y) = y^2 - x^4 + x^2$, there is $(0,0)$.

(2) Define $\varphi : X \to \mathbb{C}$ to be the projection to x-axis. The map is holomorphic at nonsingular points of X so locally it looks like $z \mapsto z^m$ at those points, i.e. a finite covering map.

(3) By removing a singularity and consider φ over the punctured disk centered at this point, the map over this punctured disk is still a covering map. Specifically, if $q = \varphi(p)$ for singular point p, there is a small disk D containing q such that $\varphi : \varphi^{-1}(D^*) \to D^*$ (for D^* a punctured disk) is a covering map.

(4) Any covering map of a punctured disk D^*, say $D = S^1 \subset \mathbb{C}$ is of the form $z \mapsto z^m$. This means $\varphi^{-1}(D^*)$ is disjoint union of punctured disks. By adding new points to these punctured disks, we obtain a Riemann surface X' and a holomorphic map $X' \to \mathbb{C}$.

   So in our example, for singular point $(0,0)$ of $y^2 - x^4 + x^2$, if we take preimage of a punctured disk of $x = 0$ under the projection φ, we get two punctured disk. Hence, we obtain a Riemann surface by removing the singularity $(0,0)$ over $0 \in \mathbb{C}$ and adding two new points over 0. This gives a Riemann surface X'.

(5) We want to extend $X' \to \mathbb{C}$ to $\overline{X} \to \mathbb{P}^1$ where $\overline{X}$ is a compact Riemann surface. Similarly, there exists a punctured disk $U_R := \{x : |x| > R\}$ of ∞ in $\mathbb{P}^1$ so that $\varphi^{-1}(U_R) \to U_R$ is a covering map. We can then complete X' to a compact Riemann surface $\overline{X}$ by adding a point to each connected component of $\varphi^{-1}(U_R)$ to X'.

   In our example of $f = y^2 - x^4 + x^2$, we need to determine how many points to add over infinity to X'. Because $\varphi : X' \to \mathbb{C}$ in this case is either locally two-to-one or one-to-one, it suffices to check if $\varphi^{-1}(U_R)$ is connected or not.

   It is not connected. Indeed, let $g(z) = z^4 - z^2$. that given a choice of square root of $y_0 = g(R)$, we can analytically continued square-root function $\sqrt{x}$ along $g(R)e^{8\pi it}$. We also know that for sufficiently large R, we can also analytically continued $\sqrt{x}$ along $g(Re^{2\pi it})$ (i.e. R is large so $g(Re^{2\pi it})$ does not intersect 0, as $\sqrt{x}$ can be analytically along any path not containing 0) . Thus, by the monodromy theorem, as $g(Re^{2\pi it})$ and $g(R)e^{8\pi it}$ are homotopic, analytically continuation of $\sqrt{x}$ along these two paths are the same. As along $g(R)e^{8\pi it}$, we get the same value for $\sqrt{x}$ at $t = 0$ and $t = 1$, we also get the same value at $t = 1$ and $t = 0$ of $\sqrt{x}$ along $g(Re^{2\pi it})$.

   This example can be used to show the following fact: the double cover of $\mathbb{P}^1$ corresponding to of $y = p(x)$ has a branched point at ∞ iff $\deg p$ is odd.

One can also construct $\overline{X}$ algebraically by "normalization" or "blow ups". I have not learned this, and refer to `https://www.math.wustl.edu/~matkerr/436/ch11.pdf` for more details in another day.

11. March 2022

11.1. **05/03/2022: Venkatesh's lecture: Automorphic forms in arithmetic topology.** Venkatesh's Arizona Winter School lecture 2022 in `https://www.math.arizona.edu/~swc/index.html`. There is an analogy in arithmetic topology between number fields and 3 manifolds.

Question: Where are "automorphic forms" in this analogy?

Q: Historical development of arithmetic topology:

Mazur (63/64) said that "$\operatorname{Spec}\mathbb{Z}/p$ is like $\operatorname{Spec}\mathbb{Z}$, which is a simply connected 3-manifold". Where this idea came from?

The first starts with Weil's paper in 1948, which suggests that there is an "algebraic" cohomology $H^*(X)$ theory for varieties X over algebraically closed field K such that when $K = \mathbb{C}$, it recovers singular cohomology of $X(\mathbb{C})$ (algebraic means you get this without knowing the topology of $\mathbb{C}$). This is a crazy idea. For example, take $X = \{x^3 + y^3 + z^3 = 0\}/\mathbb{C}$. The symmetry $x \leftrightarrow y$ acts on $H^*(X)$, $x \mapsto \overline{x}, y \mapsto \overline{y}, z \mapsto \overline{z}$ acts on $H^*(X)$. Furthermore, if you have a discontinuous automorphism $x \mapsto \sigma(x), y \mapsto \sigma(y), z \mapsto (z)$, if you believe Weil's proposal where $H^*(X)$ is completely algebraic, this automorphism should also act on $H^*(X)$!? Weil's proposal was realised by Artin and Grothendieck (for finite coefficients), called etale cohomology $H^*(X)$.

There's another work by Tate (1962) and Paton (1961) which showed etale cohomology of $\operatorname{Spec}\mathbb{Z}$ (or other number rings) has a duality H^i with H^{3-i}, which says $\operatorname{Spec}\mathbb{Z}$ behaves like a 3-manifold. This comes from class field theory.

Q: Why is this a striking phenomenon?

To compute singular cohomology H^* of a manifold, one tries to triangulate it, and then write down a complex $0 \to$ vertices $\to$ edges $\to$ faces ... The nature of this computation is linear algebra.

However, the nature of etale cohomology feels very different to linear algebra. Consider $H^*(\operatorname{Spec}\mathbb{Z}[1/2], \mathbb{Z}/2\mathbb{Z})$, which has H^1 iso to units in $\mathbb{Z}[1/2]$ modulo squares, i.e. $\{\pm 1, \pm 2\}$. So $H^1 \cong (\mathbb{Z}/2\mathbb{Z})^2$ H^2 classifies quatenion algebras on $\mathbb{Z}[1/2]$, which has two classes, so $H^2 \cong \mathbb{Z}/2$.

Q: Compare duality for number rings and 3-manifolds.

Number rings $\mathcal{O}$ means either S-integers in a number field, or functions on smooth curve over $\mathbb{F}_q$. For simplicity, take $\mathbb{Z}[1/p]$ and $H^i(\operatorname{Spec}\mathbb{Z}[1/p], M)$ where M is p-torsion abelian group, i.e. $\mathbb{Z}/p^n\mathbb{Z}$.

Tate duality (drop the notation Spec for convenience) for number rings:

$$(i-1\ deg) \to H^{3-i}(\mathbb{Z}[1/p], M^*)^* \to H^i(\mathbb{Z}[1/p], M) \to H^i(\mathbb{Q}_p, M) \to (i+1\ deg),$$

where $M^* = \{homM \to S^1\}$.

For 3-manifold X with boundary ∂X

$$(i-1\ deg) \to H^i(X, \partial X; M) \to H^i(X, M) \to H^i(\partial X, M) \to (i+1\ deg)$$

For X being manifold, we have its relative cohomology $H^i(X, \partial X; M)$ iso to $H^{3-i}(X, M^*)^*$ from some duality statement.

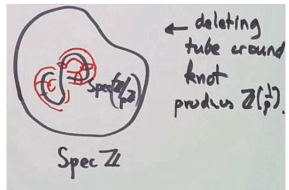
Thus, from these two exact sequences, $\mathbb{Z}[1/p]$ is like a (non-orientable) 3-manifold with boundary $\mathbb{Q}_p$, which is like a 2-manifold.

In Mazur's picture, with $\operatorname{Spec}\mathbb{Z}$ as a 3-manifold and a knot inside representing $\operatorname{Spec}\mathbb{Z}/p\mathbb{Z}$. To get $\operatorname{Spec}\mathbb{Z}[1/p]$ from $\operatorname{Spec}\mathbb{Z}$, one can delete a cube around this knot, then one would get a boundary which represents $\operatorname{Spec}\mathbb{Q}_p$. In other words, $\operatorname{Spec}\mathbb{Z}$ is like $\operatorname{Spec}\mathbb{Z}[1/p]$ glued along a boundary $\operatorname{Spec}\mathbb{Q}_p$ a tube $\operatorname{Spec}\mathbb{Z}_p$.

Examples of 3-dim rings (i.e. those behave like 3-manifolds): $\mathbb{Z}, \mathbb{Z}[1/p], \mathbb{Z}[\sqrt{2}], \mathbb{F}_p(t)$, proj smooth curve over $\mathbb{F}_q$, $\mathbb{Z}_p$

Example of 2-dim objects: $\mathbb{Q}_p, \mathbb{F}_q((t))$, smooth proj curve over $\overline{\mathbb{F}_p}$.

Q: Back to original question, where are automorphic forms in this analogy?



Consider automorphic forms on G, i.e. over $\mathbb{Z}$, it refers to vector space $A_{\mathbb{Z}}$ of functions on $G_{\mathbb{Z}}\backslash G_{\mathbb{R}}$. Over $\mathbb{Z}[1/p]$, $A_{\mathbb{Z}[1/p]}$ is vector space of functions on $G_{\mathbb{Z}[1/p]}\backslash G_{\mathbb{R}} \times G_{\mathbb{Q}_p}$.

We would like an association from 3-manifolds M to vector spaces A_M which behaves similarly to the automorphic forms story.

A non-example is $M \to H^*(M, \mathbb{C})$, which behaves nothing like $\mathbb{Z} \to A_{\mathbb{Z}}$. As it doesn't have the right functoriality. E.g. the 'double cover' $\mathbb{Z} \to \mathbb{Z}[\sqrt{2}]$ doesn't have the corresponding map on automorphic forms. Another example, $H^*(M \cup N) = H^*(M) \oplus H^*(N)$, but $A_{\mathbb{Z}\oplus\mathbb{Z}} = A_{\mathbb{Z}} \otimes A_{\mathbb{Z}}$.

The proposal is that *automorphic forms should correspond to topological quantum field theory* (TQFT) (which has $\oplus$ corresponds to tensor product). This suggestion comes from the work of Kapustin-Witten in 2006.

TQFT in dimension 4 is a functor from (3-manifolds, bordisms) to (vector spaces), which take disjoint union to tensor product. See Atiyah TQFT sec 2 book for more reference.

11.2. **16/03/2022: Venkatesh's lecture 2: Automorphic forms as extended TQFT.** This is a continuation of 05/03/2022. Here is my notes for Venkatesh's second lecture for the Arizona Winter School 2022.

Q: Define TQFT in dimension 4 (Atiyah §2 book).

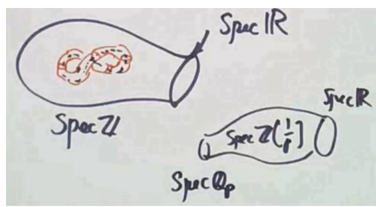
Take 3-dim manifold M to vector space A_M, disjoint union of 3-dim manifolds to tensor product of vector spaces, bordisms to linear maps.

A 4TQFT gives invariants of 4-manifolds. For example, a 4-dim manifold Z with boundary M is a bordism from $\emptyset$ to M, then by TQFT this gives a linear map $\mathbb{C} \to A_M$, i.e. a vector in A_M. Similarly, a 4-manifold Z without boundary gives a complex number.

An extended 4-TQFT assigns 4-manifolds to $\mathbb{C}$, 3-manifolds to vector spaces, 2-manifolds to categories with Hom being vector spaces. In particular, if a (no boundary) 3-dim manifold N is glued by two 3-manifolds N_l, N_r via a 2-dim boundary M, then A_{N_l} and A_{N_r} correspond to an elements a_l, a_r in the category of A_M, and $A_N = \mathrm{Hom}(a_l, a_r)$, which is a vector space.

Q: Back to number theory

Last time, $\operatorname{Spec}\mathbb{Z}[1/p]$ is like a 3-manifold with 3-manifold boundary $\mathbb{Q}_p$. However, the correct statement should be $\operatorname{Spec}\mathbb{Z}[1/p]$ has boundary $\operatorname{Spec}\mathbb{R}$ and $\operatorname{Spec}\mathbb{Q}_p$, and $\operatorname{Spec}\mathbb{Z}$ has boundary $\operatorname{Spec}\mathbb{R}$. (from the point of view that local fields should be on the same footing). But it is harder to convince one self that $\operatorname{Spec}\mathbb{R}$ should be a boundary of $\operatorname{Spec}\mathbb{Z}[1/p]$ ...



Q: Automorphic forms as extended TQFT_4

3-dim	vector spaces
$\mathbb{Z}$	$A_{\mathbb{Z}}$ functions on $G_{\mathbb{Z}}\backslash G_{\mathbb{R}}$
$\mathbb{Z}[1/p]$	$A_{\mathbb{Z}[1/p]}$ functions on $G_{\mathbb{Z}[1/p]}\backslash G_{\mathbb{Q}_p} \times G_{\mathbb{R}}$
X smooth curve over $\mathbb{F}_p$	functions on G-bundles on X
$\mathbb{Z}_p$	functions on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$
2-dim:	categories
$\mathbb{Q}_p$	category of $G(\mathbb{Q}_p)$-representations
$\mathbb{R}$	category of $G(\mathbb{R})$-representations
$\overline{X}$ proj smooth curve over $\overline{\mathbb{F}_p}$	category of sheaves on G-bundles on $\overline{X}$.

Interestingly, $\mathbb{Q}_p$ and $\mathbb{R}$ are local objects but of dimension 2, while $\mathbb{Z}_p$ is a local object but of dimension 3.

A quick check with our intuition above: $\operatorname{Spec}\mathbb{Z}$ has boundary $\operatorname{Spec}\mathbb{R}$, and $A_{\mathbb{Z}}$ is indeed an element of $A_{\mathbb{R}}$, i.e. a representation of $G(\mathbb{R})$. Similarly story with $A_{\mathbb{Z}[1/p]}$ which is a $G(\mathbb{Q}_p) \times G(\mathbb{R})$-representation.

Recall last time, $\operatorname{Spec}\mathbb{Z}$ is obtained by glueing $\operatorname{Spec}\mathbb{Z}_p$ to $\operatorname{Spec}\mathbb{Z}[1/p]$ along $\operatorname{Spec}\mathbb{Q}_p$. We want to check that the appropriate glueing property holds, i.e. $\mathrm{Hom}_{G(\mathbb{Q}_p)}(A_{\mathbb{Z}_p}, A_{\mathbb{Z}[1/p]}) = A_{\mathbb{Z}}$. Note $A_{\mathbb{Z}}$ correspond to elements of $A_{\mathbb{Z}[1/p]}$ that are unramified at p, i.e. invariant by $G(\mathbb{Z}_p)$, i.e. $A_{\mathbb{Z}} = \mathrm{Hom}_{G(\mathbb{Q}_p)}(\text{functions } G(\mathbb{Q}_p)/G(\mathbb{Z}_p), A_{\mathbb{Z}[1/p]})$. This hom encodes Hecke operator at p, i.e. endomorphisms of functions on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ acts on $A_{\mathbb{Z}}$. So $A_{\mathbb{Z}_p}$ is then functions on $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$.

Q: What is Langlands correspondence from this point of view?

Let's call the map $O \to A_O$ from an arithmetic rings to vector spaces/category an "arithmetic field theory". Suppose we have this theory.

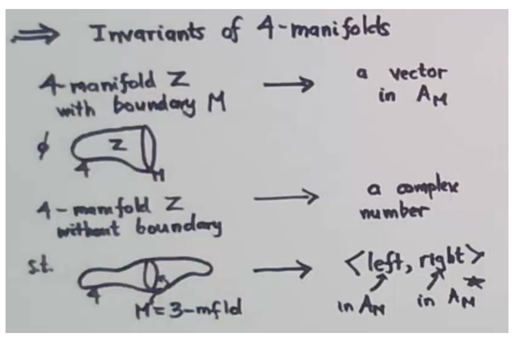
For example, take X smooth proj curve over $\mathbb{F}_p$ and $G = \mathrm{GL}_n$. Langlands correspondence gives an isomorphism between cuspidal functions on n-dim vector bundles on X and functions (i.e. take linear spans) on n-dimensional irreducible Galois reps. This is compatible with the Hecke operators on the left and the Frobenius operators on the right.

This suggests the following viewpoint on the Langlands correspondence: there is a second arithmetic field theory which is built out of Galois reps to $G^\vee$=Langlands dual groups, called $B^{G^\vee}$ and an equivalence of arithmetic field theory

$$A^G \cong B^{G^\vee}$$

Q: There are matching invariants between two sides of the Langlands correspondence (Fourier coef, Rankin-Selberg integral, doubling, theta ... for automorphic side; L-functions for Galois side). Where do these matching invariants come from in QFT?

From TQFT discussion, these invariants should correspond to some 4-manifolds ... Denote O to be 3-dim rings of integers or curves over finite field. Then from TQFT discussion, numerical invariants of automorphic forms live in A^G_O, numerical invariants of Galois reps live in $B^{G^\vee}_O$ (i.e. think of 4-dim manifolds with boundary O, which corresponds to an element in $B^{G^\vee}_O$). One can also think of invariants in automorphic sides dually, to each automorphic function P (i.e. a 4-manifold with boundary O), one associates a function $\varphi \to \langle P, \varphi \rangle$ by glueing two 4-dim manifolds P with φ via O.



Then, to find matching invariants, we want to find matching elements of A_O and B_O.

A *boundary condition* in a TQFT_4 is (informal def) is a consistent assignment, to every 3-manifold M, a distinguished vector in A_M; and to each 2-manifold S, a distinguished object in A_S. See Kapustin 2010 ICM address. Thus, to match invariants, we want matching boundary conditions in A^G and $B^{G^\vee}$.

In joint work with Ben-Zvi and Sakellaridis, informal summary:

(1) G-variety Y gives boundary condition for A^G and B^G.
(2) For suitable Y, this recovers all the familiar invariants of automorphic forms and L-functions.
(3) Propose a class of dual pairs (G, Y) and $(G^\vee, Y^\vee)$ which give matching boundary condition.

11.3. **21/03/2022: Wee Teck Gan's lectures: Local theta correspondence.** I attended Arizona Winter School 2022 virtually from 05/03 till 09/03, where I mostly followed Prof. Wee Teck Gan's lectures on the theta correspondence. Today I will try to give a summary of what I have learned. More technical details can be found in his lecture notes `https://www.math.arizona.edu/~swc/aws/2022/2022GanNotes.pdf` (should be somewhere in my laptop with extensive comments). Many thanks to Jialiang Zou for answering many questions of mine regarding this subject. A reminder to myself is that I have not actually gone through all the exercises for Prof. Gan's lectures.

The goal for today would be to sketch some ingredients of the local theta correspondence. Our first task is to define the local theta lift, i.e. a map between irreducible representations of unitary groups:

(1) Define unitary groups over local fields: Let F be a local field, E be a quadratic extension of F. Let V, W be finite-dimensional Hermitian and skew-Hermitian vector spaces over E, respectively. This means that E-vector spaces V, W are equipped with a nondegenerate E-sesquilinear form $\langle -, - \rangle$ s.t. $\langle v_1, v_2\rangle^c = \epsilon\langle v_2, v_1\rangle$ and $\langle \lambda v_1, v_2\rangle = \lambda\langle v_1, v_2\rangle$ for all $v_1, v_2 \in V, \lambda \in E$. Here the conjugation map $x \mapsto x^c$ on E comes from $\mathrm{Aut}(E/F) = \mathbb{Z}/2$; ϵ can be either 1 (for Hermitian space) or -1 (for skew-Hermitian space).

   Let $U(V)$ be the isometry group wrt to the above product $\langle -, - \rangle$ on V. Denote $\mathrm{Irr}(U(V))$ to be the set of equivalence classes of irreducible smooth representations of $U(V)$.

(2) Define *the local theta lift* $\theta : \mathrm{Irr}(U(V)) \to \mathrm{Irr}(U(W)) \cup \{0\}$.
   (a) By regarding $V \otimes_E W$ as a symplectic space over F via the symplectic form $\mathrm{Tr}(\langle -, - \rangle_V \otimes \langle -, - \rangle_W)$, one has a natural map $U(V) \times U(W) \to \mathrm{Sp}(V \otimes_E W)$. The images of $U(V)$ and $U(W)$ commute with each other, hence we call $(U(V), U(W))$ a *reductive dual pair*. This natural map can be lifted to the metaplectic group level

$$\iota_{\chi_V,\chi_W} : U(V) \times U(W) \to \mathrm{Mp}(V \otimes_E W)$$

   but this lifting is not canonical (see p.14 of Gan's notes), so it depends on a pair (χ_V, χ_W) of characters of $E^\times$ satisfying certain conditions.
   (b) For any symplectic space S over F (in particular $S = V \otimes_E W$ in our situation) and any nontrivial character ψ of F, we can construct the Weil representation ω_ψ of $\mathrm{Mp}(S)$, which is in some sense one of "the smallest" infinite-dim representations (p.11 of Gan's notes) of $\mathrm{Mp}(S)$:
      (i) Define the Heisenberg group $H(S) = S \oplus F$ of S, which has center F. Then $H(S)$ has a unique smooth irreducible representation ω_ψ with central character ψ by Stone-von Neumann theorem. One can construct ω_ψ explicitly as follows: Let $S = X \oplus Y$ be the Witt decomposition of S where X and Y are maximal isotropic subspaces (i.e. $\langle x, x\rangle = 0$ for all $x \in X$). Then $H(X) = X \oplus F$ is an abelian subgroup of $H(W)$. One extends ψ trivially to $H(X)$, then define ω_ψ to be compact induction $\mathrm{ind}_{H(X)}^{H(S)}\psi$. In particular, ω_ψ can be realised on $C_c^\infty(Y)$ with the action of $H(S)$ written down explicitly on p.12.
      (ii) Symplectic group $\mathrm{Sp}(S)$ acts on $H(S)$ by $g(s, t) = (gs, t)$. Then the representation $\omega_\psi^g = \omega_\psi \circ g^{-1}$ is also irreducible and has the same nontrivial central character ω_ψ. Hence, the two ω_ψ^g and ω_ψ are isomorphic via the isomorphism $A_\psi(g)$ on $\mathcal{S} = C_c^\infty(Y)$. By Schur's lemma, $A_\psi(g)$ is well-defined up to $\mathbb{C}^\times$. As ω_ψ is unitary, we can insist that $A_g(\psi)$ is unitary, then it is well-defined up to $S^1 \subset \mathbb{C}^\times$. Then we have a map $A_\psi : \mathrm{Sp}(S) \to \mathrm{GL}(\mathcal{S})/S^1$. Pulling back to $\mathrm{GL}(\mathcal{S})$, we obtain the Weil representation $\omega_\psi : \mathrm{Mp}(S) \to \mathrm{GL}(\mathcal{S})$.

(c) Pullback the Weil representation of $\mathrm{Mp}(V \otimes_E W)$ via the lift ι_{χ_V,χ_W} to get the Weil representation $\Omega = \Omega_{\chi_V,\chi_W,\psi}$ of $U(V) \times U(W)$.

(d) For which $\pi \otimes \sigma \in \mathrm{Irr}(U(V)) \times \mathrm{Irr}(U(W))$ is $\pi \otimes \sigma$ a quotient of Ω? To answer this (see p.15), one can define, for $\pi \in \mathrm{Irr}(U(V))$, the big theta lift

$$\Theta(\pi) = (\Omega \otimes \pi^\vee)_{U(V)}$$

as the $U(V)$-coinvariant space of $\Omega \otimes \pi^\vee$. Because $(U(V), U(W))$ is a reductive dual pair, $\Theta(\pi)$ is a representation of $U(W)$. One should think about the role of $\Theta(\pi)$ as decomposing "$\Omega = \bigoplus_{\pi \in \mathrm{Irr}(U(V))} \pi \otimes \Theta(\pi)$", or more precisely via its functoriality

$$\mathrm{Hom}_{U(V)\times U(W)}(\Omega, \pi \otimes \sigma) \cong \mathrm{Hom}_{U(W)}(\Theta(\pi), \sigma)$$

for any smooth representation σ of $U(W)$.

(e) Howe duality theorem (p.16 for the statement, exercise 8 lecture 2 for a proof sketch) says that if $\Theta(\pi)$ is nonzero, then it has a unique irreducible quotient $\theta(\pi)$. And if $\theta(\pi) \cong \theta(\pi') \neq 0$, then $\pi \cong \pi'$. In other words, we have a map

$$\theta : \mathrm{Irr}(U(V)) \to \mathrm{Irr}(U(W)) \cup \{0\}$$

which is injective on the subset of $\mathrm{Irr}(U(V))$ not sent to 0.

Our next question is when $\Theta(\pi) = 0$?

(1) This is answered in terms of Rallis's tower (p.16), i.e. essentially one consider a family of skew-Hermitian spaces $(W_r)_{r\geq 0}$, called *Witt towers*, with $\dim W_0 = 1$, $W_i = W_{i-1} \oplus \mathbb{H}$, where $\mathbb{H}$ denotes the hyperbolic plane, which is a Hermitian space of dimension 2 (see p.7). Kudla showed for $\pi \in \mathrm{Irr}(U(V))$, there is a smallest $0 \leq r_0 \leq \dim V$ such that $\Theta_{V,W_r,\psi}(\pi) \neq 0$ for any $r \geq r_0$. Furthermore, if π is supercuspidal then $\Theta_{V,W_{r_0},\psi}(\pi)$ is irreducible supercuspidal while for $r > r_0$, $\Theta_{V,W_r,\psi}(\pi)$ is irreducible but is not cuspidal. We call r_0 the *first occurrence index* of π in the Witt tower (W_r).

(2) In the non-Archimedean case, with a fixed dimension, there are only two nonisomorphic skew-Hermitian spaces (p.16), implying that there are only two Witt towers of skew-Hermitian spaces with a fixed dimension modulo 2. Let's call them (W_r) and (W_r'). For $\pi \in \mathrm{Irr}(U(V))$, there is a relation about the first occurrence r_0 and r_0' of these two towers with respect to π by $\dim W_{r_0} + \dim W_{r_0'} = 2\dim V + 2$. Thus, if we know r_0, we know r_0', meaning we know which skew-Hermitian W s.t. $\Theta_{V,W,\psi}(\pi) \neq 0$.

Another question is how to describe $\theta_{\chi_V,\chi_W,\psi}$ explicitly? I think in Gan's notes, this is answered for the case $\dim V = 1, \dim W = 3$ (p.19, 20).

(1) First, we want to describe the Weil representation Ω of $U(V) \times U(W)$ a bit more explicitly. The Schrodinger model provides a way to write down the actions of some elements in $\mathrm{Mp}(S)$ of the Weil representation (see p.13 of Gan's notes). From this, one can describe the action of some elements in $U(V) \times U(W)$ via via the lift ι_{χ_V,χ_W} (see exercise 6, lecture 2 on how to do this when V is split Hermitian of dimension 2, W is skew-Hermitian of dimension 1; see p.19 for our case where $\dim W = 3, \dim V = 1$).

(2) Consider the Witt tower (W_r) where $W_1 = W$. One then can show that if $\chi \in \mathrm{Irr}(U(V)) = \mathrm{Irr}(U_1)$ has nonzero theta lift to $U(W_0)$, then 0 is the first occurrence, meaning $\Theta(\chi)$ is nonzero and non-supercuspidal. Furthermore, $\Theta(\chi)$ is part of a non-tempered principal series representation, obtained by compactly inducting certain character of the Borel group of $U(W)$. On the other hand, if χ has zero theta lift to $U(W_0)$, then $\Theta(\chi)$ is supercuspidal.

(3) Regarding nonvanishing question, in fact $\Theta(\chi) \neq 0$ for any $\chi \in \mathrm{Irr}(U(V))$.

(4) For general V, W, I think someone said that once you are in the stable range $r \geq \dim V$ of the Witt tower, essentially $\Theta_{V,W_{r+1},\psi}(\pi)$ can be obtained from $\Theta_{V,W_r,\psi}(\pi)$ by parabolic induction.

Last question, how is this related to local Arthur packets?

(1) First, let's roughly explain local Arthur packets (p. 30). For split group G, a local A-parameter is $\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to G^\vee$, where L_F is the Weil-Deligne group of non-Archimedean local field F, $G^\vee$ is the complex Langlands dual group of G. One can define $S_\psi = \pi_0\left(Z_{G^\vee}(\psi)/Z(G^\vee)\right)$. To each irreducible representation η of S_ψ, Arthur conjectured that one can attach a unitarizable representation π_η of $G(F)$. The set

$$A_\psi = \{\pi_\eta : \eta \in \mathrm{Irr}(S_\psi)\}$$

is called a local A-packet.

(2) In the case $\dim V = 1, \dim W = 3$ of our local theta correspondence, for a given $\chi \in \mathrm{Irr}(U(V))$, there is a local A-parameter ψ such that the local A-packet A_ψ contains all local theta lifts from $\chi \in \mathrm{Irr}(U_1)$ to $\mathrm{Irr}(U_3)$. In this case, $S_\psi = \mathbb{Z}/2$, and $|A_\psi| = 2$, corresponding to the two theta lifts described above. See p.35 of Gan's notes for more details.

12.1. **06/04/2022: Picard-Lefschetz formula.** This is a continuation of the unfinished business on 08/02/2022. It concerns a family of elliptic curves $\pi : X \to \mathbb{C}$ sending the elliptic curve $y^2 = x(x-1)(x-t)$ to t. The goal back then is to describe $\pi_* \underline{k}_{\mathbb{C}}$ as a constructible sheaf with respect to the stratification $\{0,1\} \cup \mathbb{C}\backslash\{0,1\}$. To do this, it seems to me that the first step should be to understand the monodromy representation induced from π. So this is what we are going to focus on. For today, we will first start by describing the local monodromies, i.e. instead of $y^2 = x(x-1)(x-t)$, we will focus on the fibration $(z,w) \mapsto z^2 + w^2$. This is where we have the Picard-Lefschetz formula that describes the monodromy action.

The reference I used are §2, chapter 4 of Zoladek's book The Monodromy Groups, a short notes about Picard-Lefschetz theory by Jean-Philippe Chasse `https://dms.umontreal.ca/~chasseje/projets/alg-geo2.pdf` and chapter 1 of Carlson, Muller-Stach and Peters book titled Period Mappings and Period Domains.

12.1.1. *Monodromy representation from a fibre bundle.* On 08/02/2022, from a locally constant sheaf (or local system) on a topological space X, we can obtain a (monodromy) representation of its fundamental group $\pi_1(X)$. When X is locally connected, the category of local systems on X is equivalent to the category of covering spaces on X. For instance, starting from a covering map $Y \to X$ of X, one can take its sheaf of sections to get a local system (p. 104 Mac Lane, Saunders Sheaves in geometry and logic : a first introduction to topos theory). Thus, from a covering map of X, one can get a monodromy action of $\pi_1(X)$ on the fiber of the covering map.

More generally, given a fiber bundle $f : Y \to X$ with fiber F, one can deduce an action of $\pi_1(X)$ on F, and the monodromy action is the induced representation of $\pi_1(X)$ on $H_*(F)$. We will describe this process in this section.

Again, let $f : Y \to X$ be a fibre bundle of differential manifolds/topological spaces with fiber $F = f^{-1}(x_0)$ where $x_0 \in X$. For a path $[\alpha] \in \pi_1(X, x_0)$, one can obtain a diffeomorphism/homeomorphism on F. Indeed, pullback f along α, we obtain a fiber bundle α^*Y over $[0,1]$ with fiber F, hence is trivial.

$$\begin{tikzcd} \alpha^*Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ {[0,1]} \arrow[r, "\alpha"'] & X \end{tikzcd}$$

We choose a trivialisation $\varphi : [0,1] \times F \xrightarrow{\sim} \alpha^*Y$ such that its restriction to $\{0\} \times F$ is the identity map. Then we obtain a family of diffeomorphism/homeomorphisms $\varphi_t := \varphi|_{\{t\}\times F} : F \to F_{\alpha(t)}$. Different such trivialisations give rise to different homeomorphisms φ_1 on F that are all homotopic to each other. Hence, one can take homologies of these homeomorphisms to obtain a well-defined map $\rho([\alpha]) : H_*(F) \to H_*(F)$. Then ρ is a representation of $\pi_1(X, x_0)$ on $H_*(F)$, called the monodromy representation.

In geometry, a common way to get a locally trivial fibration is that if you have a proper submersive (or proper smooth in algebraic geometry language) map of smooth manifolds (or smooth schemes) $f : Y \to X$. This result is called Ehresmann's theorem.

Now, we would like to focus on a particular fibre bundle and describe its monodromy action explicitly. Denote $B_2 = \{(x,y) \in \mathbb{C}^2 : |x|^2 + |y|^2 \le 2\}$ and $D_2 = \{z \in \mathbb{C} : |z| \le 2\}$ then we can define a map $f : B_2 \to D_2$ by $(z,w) \mapsto z^2 + w^2$. Note that we have a smooth submersion $f : B_2\backslash f^{-1}(0) \to D_2\backslash\{0\}$ so this is a fibre bundle. As a side note, f is a Morse function with a critical point at $(0,0)$ and this is how every complex Morse function (with domain being a 2-dimensional complex manifold) looks like locally at a critical point.

12.1.2. *Fibers of f and their homologies.* Given f defined in the previous section, we would like to describe its fibers and the homologies of the fibers. For $\lambda \in D_2 \backslash \{0\}$, denote $F_\lambda = f^{-1}(\lambda)$ then F_λ is the Riemann surface associated with the holomorphic function $w = \sqrt{\lambda - z^2}$ over D_2.

First, we will describe F_λ as a Riemann surface. Its branch points are $\pm\sqrt{\lambda}$. By removing a branch cut that connects $-\sqrt{\lambda}$ and $\sqrt{\lambda}$, we obtain a domain D of z so that two copies of D correspond to two single-valued functions on D coming from the function $w = \sqrt{\lambda - z^2}$, where one is negative of the other at the same input z. These two copies of D are open subsets of F_λ.

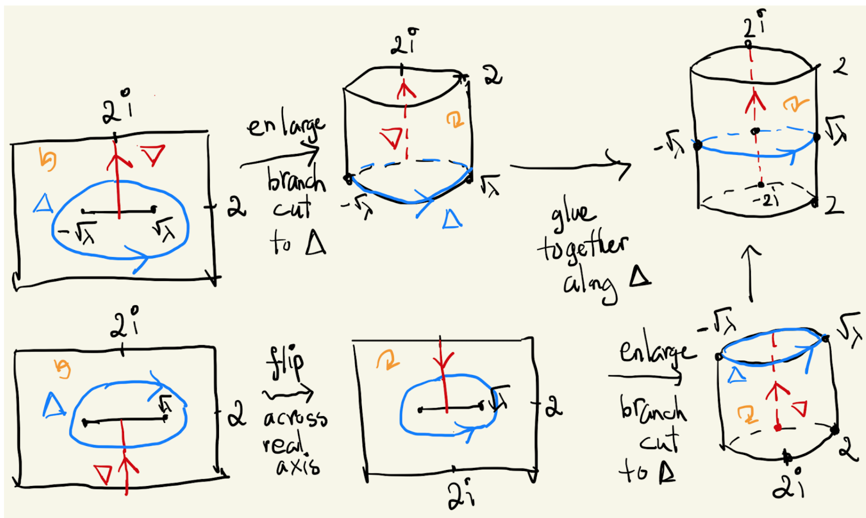


FIGURE 3. F_λ as a Riemann surface

To glue these two copies (drawn as two squares on the left of figure 12.1.2), observe that if we analytically continued w on one copy of D along a closed path Δ that encloses the branch cut, we still stay in the same D. However, if analytically continued w along ∇ that passes through the branch cut (e.g. we let z runs from $-2i$ to $2i$ in the figure), we will move from one D to another as soon as we pass the branch cut. Therefore, if we open up the branch cut into Δ, the two copies can be glued along Δ to create a cylinder as in figure 12.1.2. Note that in order to match ∇ correctly and to match the branch points $\pm\sqrt{\lambda}$ when glueing two copies, we first need to flip one copy of D across the real axis, as indicated in the figure.

From this, we find that the homology group $H_1(F_\lambda, \mathbb{Z}) = \mathbb{Z} \cdot \Delta$. Furthermore, observe that the special fiber F_0 is just a cone, and the degeneration $F_\lambda \to F_0$ to the special fiber sending Δ to 0 in homology. In this sense, Δ is called a *vanishing cycle* corresponding to the special fiber F_0 of $f : B_2 \to D_2$.

We also have ∇ is a generator of $H_1(F_\lambda, \partial F_\lambda) = \mathbb{Z}$, i.e. cycles in F_λ whose boundary lies in ∂F_λ.

For the later purpose of stating the Picard-Lefschetz formula, we will compute the intersection number (Δ, ∇) of Δ and ∇ as they intersect at one point. To do this, we choose an orientation of the cylinder F_λ, which will induce an orientation on D as drawn. Then $(\Delta, \nabla) = \pm 1$ depends on whether the two (ordered) vectors tangent to Δ and ∇ at their intersection give an orientation of F_λ the same as the initial orientation of F_λ or give the opposite orientation (see p.41 of Zoladek's Monodromy groups book). In our case, by simply looking at the orientation of Δ and ∇ in any diagram in the above figure, we find $(\Delta, \nabla) = -1$.

12.1.3. *Monodromy action.* Now, we would like to describe the monodromy action of $\pi_1(D_2 \backslash \{0\})$ on $H_1(F_1)$ and on $H_1(F_1, \partial F_1)$ from the fibre bundle $f : B_2 \backslash f^{-1}(0) \to D_2 \backslash \{0\}$.

We consider the loop $\gamma(t) = e^{2\pi i t}$ in $D_2 \setminus \{0\}$, note that $[\gamma]$ generates $\pi_1(D_2 \setminus \{0\}, 1)$. Then we can construct family of diffeomorphisms $\varphi_t : F_1 \to F_{\gamma(t)}$ of the trivial bundle $\gamma^*(B_2 \setminus f^{-1}(0))$ on $[0,1]$ by sending the z-variables to $z \in F_1 \mapsto z(t) = e^{\pi i t \chi(|z|)} z$, where χ is a smooth bump function on $\mathbb{R}$ so $\chi(r) = 1$ for $0 \le r \le 2$ and 0 for $r \ge 3$. The w-variables change continuously accordingly to the formula $w(t) = \pm\sqrt{\gamma(t) - z^2(t)}$. So $\varphi_t(z,w) = (z(t), w(t))$.

Observe that for large $(z,w) \in F_1$ then φ_t stays constant, while for small $(z,w) \in F_1$ then $\varphi_t(z,w) = e^{\pi i t}(z,w)$. In view of F_λ as two copies of $\mathbb{C}$ minus the branch cut, the points near the cut are rotated by φ_t with velocity two times smaller than the velocity of rotation of γ. In particular, we can see how Δ and ∇ are deformed as in the below picture (from left to right)

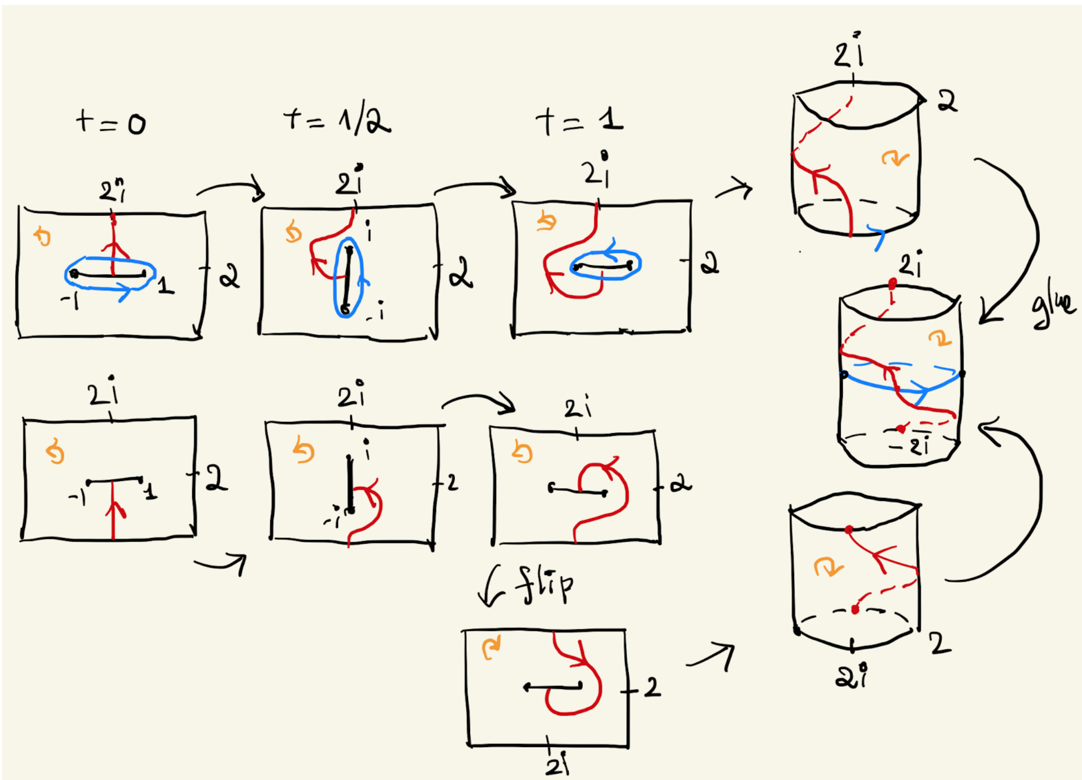


FIGURE 4. Monodromy action around critial point $0 \in D_2$ on ∇ and Δ

We find that Δ stays unchanged, while ∇ gets a twisted in the opposite direction of Δ. We can describe the action homologically as

$$\rho([\gamma])\Delta = \Delta, \rho([\gamma])\nabla = \nabla - \Delta.$$

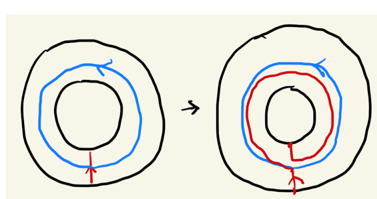


FIGURE 5. ∇ is twisted in the opposite direction of Δ

This is the Picard-Lefschetz formula for our example. I find it is easier to see the action homologically through the picture section 12.1.3.

Let me restate the Picard-Lefschetz formula by putting all the information together: Given a locally trivial fibration $f : B_2 \to D_2$ with a degeneration at $0 \in D_2$. Let $\Delta \in H_1(F_\lambda, \mathbb{Z})$ be a vanishing cycle of the singular fiber F_0. The monodromy action around the singular fiber is obtained by twisting along the vanishing cycles. In particular, if the cycle ∇ intersects Δ then the action can be described as

$$\nabla \mapsto \nabla + (\Delta, \nabla)\Delta,$$

where (Δ, ∇) is the intersection number of Δ and ∇. In our case, we know $(\Delta, \nabla) = -1$. Observe that this formula is independent of the choice of orientation of Δ (but it seems to me that it depends on the choice of orientation of F_λ, which is something I don't quite understand ...).

12.1.4. *Some more links.* Some links that have interesting things I haven't read, i.e. the general Picard-Lefschetz, local system corresponding to fiber bundle, etale setting, ... `https://ayoucis.wordpress.com/2015/07/24/another-basic-viewpoint-on-etale-cohomology/` `https://www.ma.imperial.ac.uk/~skdona/MCGROUP.PDF` and `https://math.stackexchange.com/q/2772531/58951`.

12.2. **06/04/2022: Peter Scholze's talk: Cohomology of algebraic varieties.** Just want to write a summary of what I learned from this talk `https://youtu.be/5NPFQvdav90` of Peter Scholze.

Consider smooth projective variety X over $\operatorname{Spec}\mathbb{Z}$ (or $\operatorname{Spec}\mathbb{Z}[1/N]$ ...). The goal is to understand "the" cohomology of X. One can take cohomology with coefficients in $\mathbb{Z}, \mathbb{F}_\ell, \mathbb{Z}_\ell$ and cohomology of special fibers $X_{\mathbb{F}_p}$ or $X(\mathbb{C})$ of $X \to \operatorname{Spec}\mathbb{Z}$, e.g. $H^*(X_{\mathbb{F}_p}, \mathbb{F}_\ell)$ is cohomology of fiber at p mod ℓ coefficient. So we have two parameters (p, ℓ) for this cohomology of X.

(1) Singular cohomology $H^*_{sing}(X(\mathbb{C}), \mathbb{Z})$ deals with fiber $p = \infty$ and coefficients in $\mathbb{Z}$, i.e. this cohomology theory is defined for $(p, \ell) \in \{(\infty, 2), (\infty, 3), \ldots\}$.
(2) Etale cohomology $H^*_{et}(X_{\overline{\mathbb{F}_p}}, \mathbb{Z}_\ell)$ is defined at points (p, ℓ) for $p \neq \infty$ and $p \neq \ell$.
(3) For algebraic de Rham cohomology $H^i_{dR}(X)$, taking coefficient $H^i_{dR}(X) \otimes_{\mathbb{Z}} \mathbb{F}_p$ at $\mathbb{F}_p$ is the same as taking cohomology $H^i_{dR}(X_{\mathbb{F}_p})$ over fiber with coefficient mod p. So this theory is defined for points $p = \ell$ (including ∞).
(4) Crystalline cohomology $H^*_{crys}(X/\mathbb{Z}_p)$ lives in (p, ℓ) where ℓ is in a neighborhood of p (imagine $\operatorname{Spec}\mathbb{Z}_p$ as a open neighborhood of $\operatorname{Spec}\mathbb{F}_p$).
(5) Can one fill more of this picture by more explicit cohomology theories? Bhatt-Scholze-Morrow showed that one can fill in this diagram by a new cohomology theory with values in "2-dim complete local ring", i.e. the theory lives in a neighborhood of each point (p, p).

See the picture in `https://terrytao.wordpress.com/2019/03/19/prismatic-cohomology/` to see this picture a bit more clearly.

13. June 2022

13.1. **29/06/2022: Classification of algebraic tori.** The goal is to learn how to classify tori by combinatorial data.

The main result is that there is an equivalence of categories between k-tori and finitely generated free abelian groups with a continuous action of $G_k = \mathrm{Gal}(k^s/k)$. The references I used are Milne's book Affine group schemes, `https://personal.math.ubc.ca/~cass/research/pdf/Red.pdf`, `http://www.martinorr.name/blog/2010/01/24/character-groups-of-algebraic-tori/`. Here is what I learned so far today:

(1) We first have an equivalence of categories

$$\{\text{tori over } F \text{ that split over } E\} \to \{\text{split tori over } E \text{ with a compatible } \mathrm{Gal}(E/F) \text{ action}\}$$

Here, compatibility is in the sense that for an E-torus T, the action sending $\sigma \in \mathrm{Gal}(E/F) \mapsto \overline{\sigma} \in \mathrm{Aut}(T)$ must satisfy the commutative diagram

$$\begin{tikzcd} T \arrow[r, "\overline{\sigma}"] \arrow[d] & T \arrow[d] \\ \operatorname{Spec} E \arrow[r, "\sigma"] & \operatorname{Spec} E \end{tikzcd}$$

The functor above is defined by sending an F-torus T to $T \times_{\operatorname{Spec} F} \operatorname{Spec} E$ where the action of $\mathrm{Gal}(E/F)$ on $T \times_{\operatorname{Spec} F} \operatorname{Spec} E$ is by acting on $\operatorname{Spec} E$.

To define the inverse functor: Given a split torus $\mathbb{G}_m^r$ over E with a compatible $\mathrm{Gal}(E/F)$ action σ. The corresponding torus X over F, for a F-algebra B, has its B-points $X(B)$ is the set of $\mathrm{Gal}(E/F)$-equivariant morphisms $\operatorname{Spec}(B \otimes_F E) \to \mathbb{G}_m^r$.

(2) We have an equivalence of categories between

$$\text{split tori over } E \text{ with a compatible } \mathrm{Gal}(E/F) \text{ action}$$

and

$$\text{finitely generated free abelian group with an action of } \mathrm{Gal}(E/F)$$

The functor is defined from one to the other by sending a split E-torus T to $X^*(T)$, the group of characters of T. For $\sigma \in \mathrm{Gal}(E/F)$, we have an action $\sigma_{\mathbb{G}_m}$ on $\mathbb{G}_m$. We are also given an action $\overline{\sigma}$ on T from σ, then σ acts on $\chi \in X^*(T)$ by

$$\sigma * \chi = \sigma_{\mathbb{G}_m} \circ \chi \circ \overline{\sigma}^{-1}$$

(3) Now, we would like to describe the above equivalence of categories for the real group X defined by $x^2 + y^2 = 1$. Over $\mathbb{C}$, this is isomorphic to $\mathbb{G}_m$. Indeed, we have an isomorphism $\mathbb{C}[z^{\pm 1}] \to \mathbb{C}[x,y]/(x^2+y^2-1)$ sending $z \mapsto x + iy$.

Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$ be the conjugation on $\mathbb{C}$. The action of σ on $X_{\mathbb{C}}$ is by $\overline{\sigma} : \mathbb{C}[x,y]/(x^2+y^2-1) \to \mathbb{C}[x,y]/(x^2+y^2-1)$, sending x to x, y to y, 1 to 1 and i to $-i$. Similarly for the action of σ on $\mathbb{G}_m$, i.e. $\mathbb{C}[z^{\pm 1}] \to \mathbb{C}[z^{\pm 1}]$ by just taking conjugation on $\mathbb{C}$.

We will now describe the action of σ on $X^*(X_{\mathbb{C}}) = \mathbb{Z}$. As $X_{\mathbb{C}} \cong \mathbb{G}_m$, we can consider the character $1 : \mathbb{C}[z^{\pm 1}] \to \mathbb{C}[z^{\pm 1}]$ sending $z \mapsto z$. Then by the above formula, we find σ sends 1 to $-1 \in \mathbb{Z}$, i.e. the character on $X_{\mathbb{C}} \cong \mathbb{G}_m$ sending $z \mapsto z^{-1}$ on $\mathbb{C}[z^{\pm 1}]$.

Conversely, if we are given an action of $\mathrm{Gal}(E/F) = \mathbb{Z}/2$ on $\mathbb{Z}$ where conjugation $\sigma \in \mathrm{Gal}(E/F)$ sends 1 to -1, we would like to find the corresponding torus T over F. First, we will find the action of $\mathrm{Gal}(E/F)$ on the split torus $\mathbb{G}_m$ over E. Let $X^*(\mathbb{G}_m) = \mathbb{Z}$ generated by $\chi : E[z^{\pm}] \to E[z^{\pm}]$ that sends $z \mapsto z$. Then we know $\sigma * \chi = \chi^{-1}$ as σ sends $1 \in \mathbb{Z}$ to -1. From here, we find that σ acts on $\mathbb{G}_m$ by $\overline{\sigma}(z) = z^{-1}$ and $\overline{\sigma}(e) = \sigma(e)$ for $e \in E$. Now, the torus T over F that corresponds to this torus $\mathbb{G}_m$ has its F-points being the set of all $\mathbb{Z}/2$-equivariant morphisms $E[z^{\pm}] \to E$. This morphism $E[z^{\pm}] \to E$ is determined by the image of z in $E^{\times}$, the conjugation action σ tells us that $T(F) = \{t \in E^{\times} : t^{-1} = \sigma(t)\}$.