Real analysis

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Contents

This notes contains mainly my solutions for Rudin's book Principle in Mathematical Analysis.

Chapter 1

Basic topology

1.1 Finite, countability

exple:uncountable_01 **Example 1.1.0.1** Let A be the set of all sequences whose elements are the digits 0 and 1. Then A is uncountable.

theo:countable_union **Theorem 1.1.0.2** (Union of countable sets in a countable collection) Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of countable sets, and put $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable.

1.2 Open and closed sets

Example 1.2.0.1

Openess of a set is relative to the metric space. See remark 2.29 Rudin baby_rudin [1].

For example the set $[0, 1)$ is an neighborhood of 0 in $[0, 2]$ but it is not a neighborhood of 0 in $\mathbb{R}$.

Definition 1.2.0.2 (Closure)**.** If X is a metric space, if $E \subset X$ and if E' denote the set of all limit points of E in X, then the **closure** of E is the set $\overline{E} = E \cup E'$.

The closure of a set E is understood to be the smallest closed subset of the metric space that contains E. This is indicated by the following theorem

Theorem 1.2.0.3 (Closure of a metric space)
If X is a metric space and $E \subset X$, then

(a) $\overline{E}$ is closed.

(b) $E = \overline{E}$ iff E is closed.

(c) $\overline{E} \subset F$ for every closed set $F \subset X$ that contains E.

Example 1.2.0.4
In this figure, open set is, interior is, closed set is,

Example 1.2.0.5 (Set that is both closed and open)
It is possible for a set of a metric space to be both closed and open. A trivial example is $\varnothing$ and the entire metric space.

Let's take another example: Consider metric space (X, d) where $X = \{x_1, \dots, x_n\}$ contains finitely many points. Hence, for any subset E of X then E does not have a limit point, which implies E is closed. On the other hand, for any $x_i \in E$, we can choose sufficient small $r_i > 0$ so $N_{r_i}(x_i) \cap X = \{x_i\}$, which follows $N_{r_i}(x_i) \subset E$. Thus, E is also open.

For nontrivial example, consider the metric space $(\mathbf{Q}, d)$ with metric $d(p, q) = |p - q|$. Let E be set so $p \in E$ if $p^2 < 2$. Then E is both closed and open in $\mathbf{Q}$. This is from Rudin [1], chapter 2, exercise 16 (16).

1.3 Compactness

1.3.1 Motivation

1.3.2 Compactness

collection of sets

Definition 1.3.2.1.

Theorem 1.3.2.2 If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

For $\{K_\alpha\}$ as collection of nested closed intervals, we know more about the intersection $\bigcap K_\alpha$:

Proposition 1.3.2.3 If $\{I_n\}$ is a sequence of intervals in $\mathbf{R}^1$, such that $I_n = [a_n, b_n]$ and $I_{n+1} \subseteq I_n$ for $n = 1, 2, 3, \ldots$ then $\bigcap_1^\infty I_n = [\sup a_n, \inf b_n]$.

Theorem 1.3.2.4 (Compactness in $\mathbf{R}^n$) If a set E in $\mathbf{R}^n$ has one of the following properties, then it has the other two:

(a) E is closed and bouned.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Theorem 1.3.2.5 (Compactness for metric spaces) Given a metric space X then the following are equivalent:

(a) X is compact.

(b) Every infinite subset of X has a limit point in X.

In particular, if X is compact, we can even show that every infinite subset of X has a limit point **in** X.

For the proof for $(b) \implies (a)$, see exercise 26 from [1].

Example 1.3.2.6
A set is closed but is not compact?

Theorem 1.3.2.7 Every compact metric space has a countable base. See [1, Exercise 25 §2].

1.4 Perfect sets

exple:cantor_set **Example 1.4.0.1** (Cantor set) We will construct a perfect set P in $\mathbf{R}^1$ that contains no line segment.

Let E_0 be the interval $[0,1]$. Remove the segment $(1/3, 2/3)$ and let E_1 be the union of intervals $[0,1/3],[2/3,1]$. Next, remove the middle thirds of each interval and let E_2 be the union of the new intervals $[0,1/9],[2/9,3/9],[6/9,7/9],[8/9,1]$.

Continuing this way, we obtain a sequence of sets E_n so $E_1 \supseteq E_1 \supseteq \cdots$ and each E_n is a union of 2^n intervals, each of length 3^{-n}.

The set $P = \bigcap_{i=1}^{\infty} E_i$ is called the Cantor set. One can show that P is compact and that every point in P is a limit point of P.

exer:rudin_chap2_18

Exercise 1.4.0.2 (Rudin, chapter 2, exercise 18 (18))**.** Is there a nonempty perfect set in $\mathbf{R}^1$ which contains no rational number?

1.5 Connected set

Definition 1.5.0.1 (Separated/Connected set)**.** Two subsets A, B of a metric space X are said to be **separated** if both $A \cap \overline{B}$ and $B \cap \overline{A}$ are empty.

A set $E \subset X$ is said to be **connected** if E is not a union of two nonempty separated sets.

Example 1.5.0.2

Separated sets are disjoint, but disjoint sets need not to be separated. For example, $(0,1)$ and $[1,2)$ are not separated but they are disjoint. However, segment $(0,1)$ and $(1,2)$ are seperated.

theo:connectness_R1 **Theorem 1.5.0.3** (Connected set in $\mathbf{R}^1$) A subset E of the real line $\mathbf{R}^1$ is connected if and only if it has the following property: If $x \in E, y \in E$ and $x < z < y$ then $z \in E$.

baby_rudin exer:rudin_chap2_19

Exercise 1.5.0.4 (Rudin [1], chapter 2, exercise 19 (19))**.** Prove that every connected metric space with at least two points is uncountable.

If we can visualised the metric space, an intuitive way to construct a connected set (as the name suggested) is to make the set look connected! Try this method for metric space $\mathbf{R}^2$.

baby_rudin exer:rudin_chap2_20

Exercise 1.5.0.5 (Rudin [1], chapter 2, exercise 20 (20))**.** Are closures and interiors of connected sets always connected?

1.6 Baby Rudin exercises

exer:rudin_chap2_1 1.

exer:rudin_chap2_2 2.

exer:rudin_chap2_3 3.

exer:rudin_chap2_4 4.

exer:rudin_chap2_5 5.

exer:rudin_chap2_6 6.

exer:rudin_chap2_7 7.

exer:rudin_chap2_8 8.

exer:rudin_chap2_9 9. (a) For any $p \in E^\circ$ then there exists neighborhood $N(p)$ of p such that $N(p) \subseteq E$. Since $N(p)$ is open so for any $q \in N(p)$ is an interior point of $N(p)$ or q is an interior point of E. This follows $N(p) \subseteq E^\circ$. As this is true for any $p \in E^\circ$, we conclude E° is open.

(b) From (a), if $E = E^\circ$ then E is open. Conversely, if E is open then $E \subseteq E^\circ$ since all $p \in E$ are interior points of E. On the other hand, $E^\circ \subseteq E$ since E° is set of interior points of E. Thus, $E = E^\circ$.

(c) G is open then every point in G is interior point of $G \subseteq E$.Therefore, $G \subseteq E^\circ$.

(d) Prove $(E^\circ)^c = \overline{E^c}$.

For any $p \in E^c$ then p is not an interior point of E which means $p \notin E^\circ$. For any limit point p of E^c then for any $\varepsilon > 0$, there exists $q \notin E$ so $d(p,q) < \varepsilon$. Hence, $p \notin E^\circ$. Thus, these follow $\overline{E^c} \subseteq (E^\circ)^c$.

For any $p \notin E^\circ$, i.e. p is not interior point of E then either $p \notin E$ or p is limit point of E^c. Thus, $(E^\circ)^c \subseteq \overline{E^c}$.

Thus, $(E^\circ)^c = \overline{E^c}$.

(e) No, let $E = (0,1) \cup (1,2)$ then $\overline{E} = [0,2]$. The interiors of E is $(0,2) \setminus \{1\}$ while the interiors of $\overline{E}$ is $(0,2)$.

(f) No, for E is set of an isolated point x then $\overline{E} = \{x\}$ but $\overline{E^\circ} = \emptyset$.

exer:rudin_chap2_10 10. Not hard to verify that X is a metric space. No subset of resulting metric space is open. Every subset S of the metric space is closed. Indeed, consider its complement S^c then for any $p \in S^c$, there exists a neighborhood $N_r(p)$ of p with radius $r < 1/2$ such that $N_r(p) \subseteq S^c$, otherwise $N_r(p)$ contains infinitely many points x, y in E, which will give $d(p,x) < r < 1/2$ and $d(p,y) < r < 1/2$ but $d(x,y) \le d(x,p) + d(p,y) < 1$, a contradiction. Thus, S^c is open which means S is closed.

Subset S of finitely many points in X will of course be compact. On the other hand, subset S of infinitely many points in X is not compact, as we can consider open cover $\{N_{1/2}(p)\}_{p\in S}$ of S, where each open set $N_{1/2}(p)$ contains exactly one point in S. This means there does not exist a finite subcover of S.

exer:rudin_chap2_11 11. With distance d_1 then X is not metric space since $d_1(1,0)=1>d_1(1,1/2)+d_1(1/2,0)=1/2$.

$(\mathbf{R},d_2)$ is a metric space because

$$\begin{aligned} d_2(x,y) &= \sqrt{|x-y|}, \\ &\le \sqrt{|x-z|+|z-y|}, \\ &\le \sqrt{|x-z|}+\sqrt{|z-x|}, \\ &= d_2(x,z)+d_2(z,x). \end{aligned}$$

$(\mathbf{R},d_3)$ is not metric space because $d_3(1,-1)=0$.

$(\mathbf{R},d_4)$ is not metric space because $d_4(2,1)=0$.

$(\mathbf{R},d_5)$ is a metric space.

exer:rudin_chap2_12 12. For any open cover of K, then there exists an open set U in the collection such that $0\in U$. This follows, there exists a neighborhood $N_r(0)\subseteq U$. Pick sufficient large M so $r>1/M$ then $1/n\in N_r(0)$ for all $n>M$. This means U contains most of elements in K. Hence, a finitely many open sets is sufficient to cover the rest of elements in K. This implies K is compact.

exer:rudin_chap2_13 13. Set $K=\{0\}\cup\{1/n:n\ge 1\}$ is compact according to exercise 12. The limit points of K is just 0, which is countable.

exer:rudin_chap2_14 14. Consider an open cover $\{U_s\}_{s\in\mathbf{Z}^+}$ of $(0,1)$ where $U_s=(1/s,1)$.

For any finite number of open sets U_s, there always exists a sufficient small $x\in(0,1)$ such that x does not belong to any such open sets. This follows $(0,1)$ have no finite subcover for this open cover.

exer:rudin_chap2_15 15. Consider collection of closed subsets of $\mathbf{R}$, which is $\{(-\infty,n]\}_{n\in\mathbf{Z}}$ where $(-\infty,n]\subseteq(-\infty,n+1]$ but $T=\cup_{n\in\mathbf{Z}}(-\infty,n]$ is empty, as for any $x\in\mathbf{R}$, there exists $n<x,n\in\mathbf{Z}$ so $x\notin(-\infty,n]$.

Consider collection of bounded subsets of $\mathbf{R}$, which is $\{(0,1/n)\}_{n\ge 1}$ then $(0,1/(n+1))\subseteq(0,1/n)$ but $\cup_{n\ge 1}(0,1/n)$ is empty.

exer:rudin_chap2_16 16. Set E is obviously bounded. On the other hand, it is both closed and open. It is closed because any limit point $p\in\mathbf{Q}$ of E must be in E. It is open because for any $p\in E$, there exists a neighborhood $N_r(p)\subseteq E$.

To show E is not compact, we can construct a open cover of E so that E has no finite subcover. This can be achieved by using the fact that the rational numbers in E can be arbitrarily close to $\sqrt{2}$. Indeed, for $p\in A$, put the neighborhood $N_r(p)$ into our collection where $r=|p-\sqrt{2}|/2$. With this, we've obtained an open cover of E, but there will be no finite subcover for E.

exer:rudin_chap2_17 17. E is uncountable. The proof follows example exple:uncountable_01 1.1.0.1. E is not dense in $[0,1]$ since $0.55 \in [0,1]$ is not a limit point of E as $N_{0.01}(0.55) \cap E = \varnothing$.

We prove E is closed by showing E^c is open. For $x \notin E$ there decimal representation of x contains a digit at position k that is neither 4 nor 7 then the neighborhood $N_{10^{-k}}(x) \subseteq E^c$. This follows E^c is open or E is closed.

We prove that every point in E is a limit point of E. For $x = 0.x_1x_2 \cdots \in E$ with $x_i \in \{4,7\}$. For any radius $r = 10^{-k}$ then consider $y = 0.y_1y_2\dots$ so $y_i = x_i$ for $i \le k$ or $y > k+1$ and $y_{k+1} = 4$ if $x_{k+1} = 7$ or $y_{k+1} = 7$ if $x_{k+1} = 4$. This shows $y \in E \cap N_r(x)$.

Thus, E is perfect.

exer:rudin_chap2_18 18. We borrow the idea from the construction of Cantor set exple:cantor_set 1.4.0.1. Consider the interval $E_0 = [\sqrt{2}, 1+\sqrt{2}]$. We see that $2 \in E_0$, which we will choose a small irrational number d so $(2-d, 2+d) \subset E_0$. Take that interval out to create new set E_1 which is a union of $[\sqrt{2}, 2-d], [2+d, 1+\sqrt{2}]$.

Next, find all rational numbers of the form $\frac{m}{2}$ so $\gcd(m,2) = 1$ and $m/2 \in E_1$. For each $m/2 \in E_1$, we choose a small irrational number d so $(m/2-d, m/2+d) \in E_1$. Take that interval out and we obtain E_2 as the union of some intervals whose endpoints are irrational numbers.

Similarly, for E_n, we find all rational $\frac{m}{n+1}$ so $\gcd(m, n+1) = 1$ and $\frac{m}{n+1} \in E_n$. For each such $\frac{m}{n+1} \in E_n$, we choose a sufficient small irrational d_m so $\left(\frac{m}{n+1} - d_m, \frac{m}{n+1} + d_m\right) \subset I_n \subset E_n$ (I_n is some interval that is constructed before). We remove such intervals out of E_n to obtain new interval E_{n+1}.

Continuing doing this, we will obtain a sequence of sets E_n so $E_1 \supset E_1 \supset \cdots$. Let $P = \bigcap_{i=1}^{\infty} E_i$ then P is obviously compact.

It's not hard to see that P does not contain any rational numbers, because $m/n \notin E_k$ for all $k \ge n$, which follows $m/n \notin P$ for any rational number m/n.

Hence, it suffices to show that every point in P is a limit point of P. This is completely similar to the proof for Cantor set being a perfect set. Indeed, pick a point $x \in P$, let S be any segment containing x. Let I_n be that interval of E_n so $x \in I_n$. Choose n large enough so $I_n \subset S$. Let $x_n \neq x$ be an endpoint of I_n then $x_n \in P$ from the construction of P. Therefore, x is a limit point of P, which concludes that P is perfect.

exer:rudin_chap2_19 19. (a) Since A, B are closed sets so $A = \overline{A}, B = \overline{B}$. Since $A \cap B = \varnothing$ so $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. Thus, A and B are separated.

(b) For limit point p of B then $p \notin A$, otherwise there exists a neighborhood $N_r(p)$ of p so $N_r(p) \subset A$ but $N_r(p) \cap B \neq \varnothing$ so $A \cap B \neq \varnothing$, which is a contradiction. Thus, this follows $\overline{B} \cap A = \varnothing$. Similarly, $\overline{A} \cap B = \varnothing$.

(c) So $A = \{q \in X : d(p,q) < \delta\}$ and $B = \{q \in X : d(p,q) > \delta\}$. Since A and B are disjoint open sets so we can apply (b).

(d) (Hint: Read the proof about connected set in $\mathbf{R}^1$)

Let E be a connected metric space and $x, y \in E$. Note that the set $(0, d(x,y))$ is uncountable.

If for each $r \in (0, d(x,y))$, there exists $z_r \in E$ so $d(x, z_r) = r$. Since $(0, d(x,y))$ is uncountable so there exists uncountably many such z_r's. Hence, E is uncountable.

If assume otherwise, there exists $r \in (0, d(x,y))$ so there is no $z \in E$ satisfying $d(x,z) = r$ then that means $E = A_r \cup B_r$ where $A_r = \{z \in E : d(z,x) < r\}$ and $B_r = \{z \in E : d(z,x) > r\}$. Since $x \in A_r$ and $y \in B_r$ so A_r, B_r are nonempty. On the other hand, A_r and B_r are seperated according to (c). Therefore, E cannot be connected, a contradiction.

exer:rudin_chap2_20 20. (Try to intuitively guess the answer before going through the technical details)

Closure of a connected set E is connected. Indeed, if $\overline{E}$ is not connected then there exists nonempty seperated sets A, B such that $\overline{E} = A \cup B$. This implies that p is a limit point of E iff p is either a limit point of A or a limit point of B.

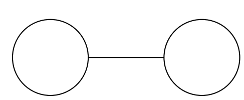
If p is a limit point of A then $p \in \overline{A}$ and since $\overline{A} \cap B = \varnothing$, $p \notin B$. We also know p is a limit point of E so $p \in \overline{E} = A \cup B$ which follows $p \in A$ since $P \notin B$. Thus, any limit point of A is in A, which means $A = \overline{A}$. Similarly, $B = \overline{B}$.

Denote E' as the set of limit points of E that are not in E then $E = \overline{E} \setminus E' = (A \setminus E') \cup (B \setminus E')$ where from the previous observation, we find $\overline{A \setminus E'} \subseteq A$ and $\overline{B \setminus E'} \subseteq B$. Therefore, $A \setminus E'$ and $B \setminus E'$ are seperated sets.

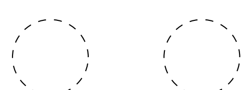
If $A \setminus E'$ is empty then $A \subseteq E'$ which means $A \cap E = \varnothing$. This follows $E \subseteq B$, but that means any $p \in A$ will be a limit point of B, a contradiction since A, B are seperated and that $A = \overline{A}, B = \overline{B}$. Thus, $A \setminus E'$ and $B \setminus E'$ are nonempty.

This follows E is is not connected, a contradiction. Thus, the closure of connected set E must also be connected.

Interior of a connected set is not always connected. Consider the metric space $\mathbf{R}^2$ and consider the following connected set:



Its interior will look like below, which is obviouly not a connected set:



exer:rudin_chap2_21 21. ($\mathbf{p}(t)$ can be visualised as a line segment connecting $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^k$)

(a) Suppose otherwise that A_0 and B_0 are not separated. WLOG, there exists $t \in \overline{A_0} \cap B_0$. Since $t \in B_0$ so $\mathbf{p}(t) \in B$. Since $t \in \overline{A_0}$ so either $t \in A_0$ or t is a limit point of A_0. If $t \in A_0$ then $\mathbf{p}(t) \in A_0$. If t is a limit point of A_0 then $\mathbf{p}(t)$ is a limit point of A so $\mathbf{p}(t) \in \overline{A}$. Thus, we obtain $\mathbf{p}(t) \in \overline{A} \cup B$, a contradiction since A, B are seperated. Thus, A_0 and B_0 are seperated subsets of $\mathbf{R}^1$.

(b) First, we need to restrict t to be in $[0,1]$. Define $A_1 = A_0 \cap [0,1]$ and $B_1 = B_0 \cap [0,1]$. The statement of part (a) still holds for A_1 and B_1, i.e. A_1 and B_1 are seperated on $\mathbf{R}^1$.

Therefore, we find $A_1 \cup B_1$ is not connected on $\mathbf{R}^1$. Since $0 \in A_1, 1 \in B_1$ and that $A_1 \cup B_1 \subset [0,1]$, according to theorem 1.5.0.3 theo:connectness_R1 about connectness in $\mathbf{R}^1$, there exists $t_0 \in (0,1)$ such that $t_0 \notin A_1 \cup B_1$. This follows $t_0 \notin A_0$ and $t_0 \notin B_0$, which means $\mathbf{p}(t_0) \notin A \cup B$.

(c) If there exists a convex subset E of $\mathbf{R}^k$ that is not connected then $E = A \cup B$ where A, B are nonempty seperated sets on $\mathbf{R}^k$. With similar argument as in (a), (b), we find that for $\mathbf{a} \in A, \mathbf{b} \in B$, there exists $t \in (0,1)$ such that $(1-t)\mathbf{a} + t\mathbf{b} \notin E$, which contradicts to the definition of convex set. Therefore, every convex subset of $\mathbf{R}^k$ is connected.

exer:rudin_chap2_22 22. Yes, consider E as the set of points which have only rational coordinates. For any $\mathbf{x} = (x_1, \ldots, x_k) \in X$. We know $\mathbf{Q}$ is dense in $\mathbf{R}^1$ so for any $x_i \in \mathbf{R}^1$, there exists rational $y_i \in \mathbf{Q}$ so $|x_i - y_i| < \varepsilon n^{-1/2}$. This follows with $\mathbf{y} = (y_1, \ldots, y_n) \in E$ then $|\mathbf{y} - \mathbf{x}| < \varepsilon$. Since this is true for any $\varepsilon > 0$ so $\mathbf{x}$ is a limit point of E. Thus, E is dense in $\mathbf{R}^k$.

Now, since $E = \mathbf{Q}^k$ and $\mathbf{Q}$ is countable so E is countable. Thus, $\mathbf{R}^k$ is **seperable**.

exer:rudin_chap2_23 23. Consider a countable dense subset E of the seperable metric space. For any $x \in E$ and any $q \in \mathbf{Q}^+$, we put $N_q(x)$ into our collection $\{V_\alpha\}$. We will show that $\{V_\alpha\}$ is a countable **base** of X.

Indeed, for every $x \in X$ and every open set $G \subset X$ such that $x \in G$, there exists neighborhood $N_r(x)$ of x so $N_r(x) \subset G$. Since E is a dense subset of X, there exists $y \in E$ so $d(x,y) < r/2$. We choose $k \in \mathbf{Q}^+$ so $d(x,y) < k < r - d(x,y)$ and consider the neighborhood $N_k(y) \in \{V_\alpha\}$. For any $p \in N_k(y)$ then $d(p,x) \le d(p,y) + d(x,y) < k + d(x,y) < r$ so $p \in N_r(x)$. Therefore, $x \in N_k(y) \subset N_r(x) \subset G$. Since $N_k(y) = V_\alpha$ for some α so we can conclude that the collection $\{V_\alpha\}$ is a base for X.

Next, we will show that $\{V_\alpha\}$ is countable. From out construction, we find that there is a bijection from $\{V_\alpha\}$ to $E \times \mathbf{Q}^k$. Since $E \times \mathbf{Q}^k$ is countable, we find $\{V_\alpha\}$ is also countable.

Thus, there exists a countable base for every seperable metric space.

exer:rudin_chap2_24 24. Fix $\delta > 0$ and pick $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, we choose x_{j+1} (if possible) so $d(x_{j+1}, x_i) \ge \delta$ for all $1 \le i \le j$.

We will first show that $S_\delta = \{x_i\}$ is finite. Indeed, assume otherwise, there will exist a limit point x of S_δ, which follows there exists $i \neq j$ so $d(x, x_i) < \delta/2$ and $d(x, x_j) < \delta/2$. Hence, $d(x_i, x_j) \le d(x, x_i) + d(x, x_j) < \delta$, a contradiction. Thus, S_δ is finite.

Let $S = \bigcup_{\delta \in \mathbf{Q}^+} S_\delta$ then since S_δ is countable for any $\delta \in \mathbf{Q}^+$ and that $\mathbf{Q}^+$ is countable, according to theorem 1.1.0.2, we find S being countable.

Next, we will show that S is a dense subset of X. Indeed, for any $x \in X$ and for any $\delta > 0$, either $x \in S_\delta \subset S$ or $x \notin S_\delta$, which implies $d(x, y) < \delta$ for some $y \in S_\delta$. This follows either $x \in X$ or x is a limit point of S.

Thus, X is seperable with countable dense subset S.

exer:rudin_chap2_25 25. Since K is compact so according to theorem 1.3.2.5, for any infinite subsets E of K then E has limit point in K. Therefore, from previous exercise 24, K is separable. This means according to exercise 23, K has countable base.

exer:rudin_chap2_26 26. From exercise 23 and 24, we know that X has a countable base $\{T_n\}$. For any open cover $\{U_\alpha\}$ of X then $\bigcup U_\alpha = \bigcup_{k \ge 1} T_{n_k}$. Since $\{T_{n_k}\}_{k \ge 1}$ is from the countable base $\{T_n\}$ so it is countable. Therefore,X has acountable subcover $\{V_n\}$.

If there does not exists a finite subcover of X, then that means for any $i \in \mathbf{Z}^+$, there exists $x_i \in X$ such that $x_i \notin \bigcup_{j \le i} V_j$. On the other hand, since $\{V_n\}$ is an open cover of X, there exists j so $x_i \in V_j$. This follows $x_k \ne x_i$ for all $k > J$. Since this sequence is infinite, there exists a limit point x for this sequence. Since $\{V_n\}$ is a subcover of X, there exists $n \in \mathbf{Z}^+$ so $x \in V_n$, i.e. $x \in N_r(x) \subset V_n$. Since from our construction, $x_i \notin V_n$ for all $i > n$ so that means $d(x, x_i) \ge r$ for all $i > n$. This contradicts the fact that x is the limit point of $\{x_i\}$.

Thus, there must exists a finite subcover $\{V_n\}_{n \le K}$ of X.

exer:rudin_chap2_27 27. We will first show P is perfect. Let p be a limit point of P then for any neighborhood $N_r(p)$ of p, there exists $q \in P$ such that $q \in N_s(q) \subset N_r(p)$. Since $N_s(q)$ contains uncountably many points of E so $N_r(p)$ also contains uncountably many many points of E. This follows $p \in P$. Thus, P is closed.

We will next show that every point in P is a limit point of P. From exercise 22, $\mathbf{R}^k$ is seperable so it has a countable base according to exercise 23. Consider any point $p \in P$ and any neighborhood $N_r(p)$ of p. If $N_r(p)$ does not contain any points from P other than P then there exists an open cover $\{U_\alpha\}$ of $N_r(p)$ so each U_α contains countably many points of E. Since $\mathbf{R}^k$ has a countable base so each $U_\alpha = \bigcup_{k \ge 1} V_{\alpha_k}$ where V_{α_k} is from the countable base. Since $\{U_\alpha\}$ has countably many points of E so each V_{α_k} also has countably many points of E. This follows there exists a countable subcover $\{V_n\}$ of $N_r(p)$ so V_n has countably many points of E, i.e. $V_n \cap E$ is countable. Hence, according to theorem 1.1.0.2, $\bigcup_{n \ge 1}(V_n \cap E) = N_r(p) \cap E$ is countable, a contradiction since $p \in P$. Therefore, there must exist $q \in P$ so $q \in N_r(p)$. Thus, every point $p \in P$ is a limit point of P.

From the two above arguments, we conclude that P is perfect.

Since we're talking about $\mathbf{R}^k$ so according to exercise 22, $\mathbf{R}^k$ has a countable dense subset T. For any $x \in P^c \cap E$, there exists a neighborhood $N_r(x)$ of p so $N_r(x)$ contains countably many points of E. Since T is dense in $\mathbf{R}^k$ so there exists $y \in T$ and a neighborhood $N_s(y)$ of

y so $x \in N_s(y) \subset N_r(x)$. Thus, $x \in N_s(y)$ also contains countably many points of E. With this, we obtain $\{N_s(y)\}_{y\in T}$ as a countable open cover of $P^c \cap E$. On the other hand, we know $N_s(y) \cap E$ is countable so $L = \bigcup_{y\in T}(N_s(y) \cap E)$ is countable according to theorem 1.1.0.2 and the fact that T is countable. We also know that $(P^c \cap E) \subset L$ so $P^c \cap E$ is at most countable.

exer:rudin_chap2_28

28. Yes, use exercise 27, there exists a perfect set P so $P^c \cap E$ is at most countable. Note that every point in P is a limit point of E and since E is closed so $P \subset E$. Therefore, $E = P \cup (P^c \cap E)$.

exer:rudin_chap2_29

29. Consider an open set S so S is equal to union of a collection of disjoint segments $\{(a_\alpha, b_\alpha)\}$. For any α, since $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists a rational number $c_\alpha \in (a_\alpha, b_\alpha)$. Since the segments are disjoint so the rationals c_α's are distinct. This follows that there is a injection from $\{(a_\alpha, b_\alpha)\}$ to $\mathbf{Q}$. Since $\mathbf{Q}$ is countable, the collection is at most countable.

exer:rudin_chap2_30

30. Pick $\mathbf{x}_1 \in G_1$ and consider a neighborhood V_1 of $\mathbf{x}_1$ such that $\overline{V_1} \subset G_1$. This is possible since G_1 is open.

    Suppose we have constructed V_i. Since G_{i+1} is an open dense subset of $\mathbf{R}^k$ so there always exists a neighborhood V_{i+1} such that $\overline{V_{i+1}} \subset V_i$ and that $\overline{V_{i+1}} \subset G_{i+1}$. With this, we obtain $\overline{V_k} \supset \overline{V_{k+1}}$ and $\overline{V_k} \subset G_k$ for all $k \in \mathbf{Z}^+$. On the other hand, since $\overline{V_i}$ is compact so according to theorem 1.3.2.2, $\bigcap \overline{V_n}$ is nonempty. $\bigcap \overline{V_n} \subset \bigcap G_i$ so $\bigcap G_i$ is nonempty.

1.7 Other problems

Open ball $B_\varepsilon(x)$ is a neighborhood of x with radious ε.

For a metric space X, an **ε-net** for X is a subset $S \subseteq X$ such that $A \subset \bigcup_{x\in S} B_\varepsilon(x)$.

A metric space X is **totally bounded** iff for any real number $\varepsilon > 0$, there exists a finite ε-net for X. See here for more information.

Problem 1.7.0.1 (MSE)**.** Suppose X is a compact and connected metric space. Suppose there exists a finite $\varepsilon/2$-net $S = \{x_1, \ldots, x_k\}$. Can we rearrange the order in S into $y_1, \ldots, y_k$ so that $d(y_i, y_{i+1}) < \varepsilon$ for all $1 \le i \le k-1$?

Problem 1.7.0.2 (MSE)**.** Let X be a metric space. If $A, B \in X$ are disjoint, if A is compact, and if B is closed, then there exists $\delta > 0$ such that $|a - b| \ge \delta$ for all $a \in A, b \in B$.

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Chapter 2

Continuous functions

We need to define cluster (limit) point because we want to know the values of the function at points near c....

2.1 Limits of functions

def:limit_function **Definition 2.1.0.1.** Let X, Y be metric spaces. Suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or $\lim_{x\to p} f(x) = q$ if there is a point $q \in Y$ with the following property:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$. The sybols d_X, d_Y refer to the distances in X, Y, respectively.

Remark 2.1.0.2. Note that $p \in X$ but p need not to be a point of E. Even if $p \in E$, we may very well have $f(p) \neq \lim_{x\to p} f(x)$.

example for al mark

We can restate the definition for limit of functions in terms of limits of sequences:

t_function_in_sequence **Theorem 2.1.0.3** Let X, Y, E, f and p be defined in definition 2.1.0.1 (def:limit_function). Then $\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p, \lim_{n\to\infty} p_n = p$.

Remark 2.1.0.4. For the converse of the above theorem 2.1.0.3 (theo:limit_function_in_sequence), as long as we know that $\{f(p_n)\}$ converges (we don't have to know that they all have to converge to q) for every selected $\{p_n\}$ then we can claim $\lim_{x\to p} f(x)$. In particular, the converse of the above theorem can be strengthened as follow:

> If for every such sequence $\{p_n\}$ we have $\{f(p_n)\}$ converges, then $\lim_{x\to p} f(x)$ exists.

Indeed, it suffices to show that for any two sequences $\{x_n\}$ and $\{y_n\}$ converging to p such that $x_n \neq p, y_n \neq p$ then the two sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ converges to the same point. To achieve this, we can create a new sequence containing x_n, y_n: Let $z_{2n} = x_n$ and $x_{2n+1} = y_n$ then $\{z_n\}$ converges to p, which implies $\{f(z_n)\}$ converges to some q. Since $\{z_n\}$ is a refinement of $\{x_n\}$ and $\{y_n\}$ so $\{f(x_n)\}$ and $\{f(y_n)\}$ converges to the same point.

Example 2.1.0.5

We want to show $\lim_{x\to 0} \sin(1/x)$ does not exist. According to theorem 2.1.0.3, is we can find a sequence $\{x_n\}$ converging to 0 such that $\lim_{n\to\infty} f(x_n)$ diverges then we are done.

Indeed, consider the sequence $x_n = \frac{2}{n\pi}$ then $\sin(1/x_n) = \sin(n \cdot \pi/2)$ is 0 when n is even and is ± 1 when n is odd. This follows $\lim_{n\to\infty} \sin(1/x_n)$ does not exist. We also know that $\lim_{n\to\infty} x_n = 0$. Thus, $\lim_{x\to 0} \in (1/x)$ does not exist.

Exercise 2.1.0.6 (Cauchy criterion for functions approaching infinity)**.** Consider a function $f : \mathbf{R} \to \mathbf{R}$. Show that $\lim_{x\to\infty} f(x)$ exists iff for every $\varepsilon > 0$, there exists N such that $|f(x) - f(y)| < \varepsilon$ for any $x, y > N$ and $x, y \in \mathbf{R}$.

We prove the converse using theorem 2.1.0.3. Let $\{x_n\}$ be a sequence of numbers approaching infinity. We find that $\{f(x_n)\}$ is a Cauchy sequence so it converges. We are done.

2.2 Continuous functions

Definition 2.2.0.1. Suppose X and Y are metric spaces, $E \subset X, p \in E$ and f maps E into Y. Then f is said to be **continuous** at p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points $x \in E$ for which $d_X(x, p) < \delta$.

From this definition, we find that f is continuous at any isolated point p in E.

continuity_composition **Theorem 2.2.0.2** (Continuity in compositions of functions) Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by $h(x) = g(f(x))$ for all $x \in E$.

If f is continuous at point $p \in E$ and if g is continuous at point $f(p)$, then h is continuous at p.

Remark 2.2.0.3. We can generalise the result as follow: if $\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r$ then $\lim_{x\to p} g(f(x)) = r$.

Remark 2.2.0.4 (Remark 4.12 [1])**.** In definition of limits of functions, p needs not to be in E, which is why we need to consider subsets of metric space instead of the metric space itself.

The situation is different with continuity. We just may as well talk about continuous mapping from one metric space to another, rather than mappings of subsets

Theorem 2.2.0.5

A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

2.3 Continuity and Compactness

Definition 2.3.0.1. A mapping from f of a set E into $\mathbf{R}^k$ is said to be **bounded** if there is a real number M such that $|f(x)| \le M$ for all $x \in E$.

heo:compact_continuity **Theorem 2.3.0.2** Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

mpact_real_boundedness **Theorem 2.3.0.3** Suppose f is a continuous real function on a compact metric space X, and $M = \sup_{p \in X} f(p), m = \inf_{p \in X} f(p)$ then there exists point $p, q \in X$ such that $f(p) = M, f(q) = m$.

inuity_inverse_compact **Theorem 2.3.0.4** Suppose f is a continuous bijection of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$ $(x \in X)$ is a continuous mapping of Y onto X.

Example 2.3.0.5 (Theorem 4.20, Rudin [1])

Let E be a noncompact set in $\mathbf{R}^1$. By considering two cases of E being bounded or unbounded, we can find:

(a) A continuous function on E which is not bounded.
(b) A continuous and bounded function on E which has no maximum.
(c) If E is bounded: a continuous function on E which is not uniformly continuous.
(d) If E is unbounded: a continuous function on E which is uniformly continuous.

inuity_inverse_compact **Example 2.3.0.6** (Example 4.21, Rudin [1]) Importance of compactness in continuous inverse of continuous mapping 2.3.0.4.

2.3.1 Uniform Continuity

Uniformly continuity of a function depends on the domain (interval to be exact) that it is defined on. Uniformly continuity implies continuity.

inuity_compact_uniform **Theorem 2.3.1.1** Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

2.4 Continuity and Connectedness

heo:continuity_connect **Theorem 2.4.0.1** If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.

continuity_connect_ivt **Theorem 2.4.0.2** Let f be a continuous real-valued function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$ then there exists a point $x \in (a, b)$ such that $f(x) = c$.

2.5 Discontinuity

Example 2.5.0.1

Function $f : \mathbf{R} \to \mathbf{R}$ so $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = a$ is discontinuous at 0 for any a.

Example 2.5.0.2 (Contiuous nowhere function)

The function $f : \mathbf{R} \to \mathbf{R}$ so $f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ -1 & \text{if } x \in \mathbf{Q} \setminus \mathbf{Q} \end{cases}$ is discontinuous everywhere on $\mathbf{R}$.

It seems quite impossible to construct such function without relying on the fact that $\mathbf{Q}$ is dense on $\mathbf{R}$.

Exercise 2.5.0.3. Give an example of a function f for each of following conditions:

(a) f not continuous at any $a \neq 0$.

(b) f is continuous nowhere but $|f|$ is continuous everywhere.

(c) f is continuous at a but not at any other points.

For (a), choose $f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{R} \setminus \mathbf{Q}, \\ 1/p & \text{if } x = p/q; p, q \in \mathbf{Z}, q > 0 \end{cases}$.

For (b), choose $f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ -1 & \text{if } x \in \mathbf{R} \setminus \mathbf{Q} \end{cases}$.

For (c), choose $f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q}, \\ a & \text{if } x \in \mathbf{R} \setminus \mathbf{Q} \end{cases}$.

Example 2.5.0.4 (Removable discontinuity)

Function $g : \mathbf{R} \to \mathbf{R}$ so $g(x) = \sin(x)/x$ with $x \neq 0$ and $g(0) = 2$ has $\lim_{x \to 1} g(x) = 1 \neq f(0)$ so g is not continuous at 0. However, if we redefine g with $g(0) = 1$ then g is continuous at 0. We said g to have **removable discontinuity** at 0.

In particular, for a function $f : \mathbf{R} \to \mathbf{R}$, if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$ then f is said to have removable discontinuity at a.

calculus_spivak

Exercise 2.5.0.5. ([2] chap 6, exer 11) Let f be a function with the property that every point of discontinuity is a removable discontinuity. Define $g(x) = \lim_{y \to x} f(x)$. Prove that g is continuous.

2.6 Monotonic functions

eo:monotonicfunc_limit **Theorem 2.6.0.1** Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exists at every point x of (a, b). More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

Furthermore, if $a < x < y < b$ then $f(x+) \leq f(y-)$.

ty_monotonic_countable **Theorem 2.6.0.2** Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.

2.7 Baby Rudin exercises

exer:rudin_chap4_1

1. No. It may have simple distcontinuity. For example $f(x) = 0$ for all $x \neq 1$ and $f(1) = 1$. Then $\lim_{h\to 0} f(1+h) - f(1-h) = \lim_{h\to 0} f(1+h) - \lim_{h \to 0} f(1-h) = 0$ but f is not continuous at 1.

exer:rudin_chap4_2

2. It suffices to show that for any limit point p of E then $f(p) \in \overline{f(E)}$. Since p is a limit point of E so there exists a sequence $\{q_n\}$ of points in E that converges to p. Since f is continuous at p so this follows that $\{f(q_n)\}$ converges to $f(p)$. Since $\{f(q_n)\}$ is a sequence in $f(E)$ so $f(p)$ is a limit point of $f(E)$ or $f(p) \in \overline{f(E)}$. Thus, $f(\overline{E}) \subset \overline{f(E)}$.

   $f(\overline{E})$ is a proper subset of $\overline{f(E)}$ when there is a limit point p in $\overline{f(E)}$ such that $p \notin f(\overline{E})$. For example, we can create a real-valued function f that approaches 0 as $x \to \infty$, i.e. $0 \in \overline{f(E)}$ but it never reaches 0, i.e. $0 \notin f(E)$ but . Thus, define $f : [1, \infty) \to \mathbf{R}$ so $f(n) = 1/n$ for all $n \in \mathbf{N}$. We then connect two points $(n, 1/n)$ and $(n+1, 1/(n+1))$ to create continuity on $[1,\infty)$. This follows for $E = [1, \infty)$ then $f(\overline{E}) = (0,1]$ but $\overline{f(E)} = [0,1]$.

exer:rudin_chap4_3

3. For any limit point p of $Z(f)$, there exists a sequence $\{p_n\}$ in $Z(f)$ so $\lim_{n\to\infty} p_n = p$. Since $p_n \in Z(f)$ so $f(p_n) = 0$ for all n. This follows $\lim_{n\to\infty} f(p_n) = 0$. Since f is continuous on p so this follows $f(p) = 0$ or $p \in Z(f)$. Thus, $Z(f)$ is closed.

exer:rudin_chap4_4

4. For any $p \in X$ so $p \notin E$ then p is a limit point of E. This follows there exists a sequence $\{p_n\}$ in E that converges to p. We know that $f(p_n) = g(p_n)$ for all n so $\lim f(p_n) = \lim g(p_n)$ and since f, g are continuous at p, $f(p) = \lim f(p_n) = \lim g(p_n) = g(p)$. We are done.

exer:rudin_chap4_5

5. If E is closed then E^c is open and according to exercise 29, E^c is union of a countable collection $\{(a_n, b_n)\}$ of disjoint segments. Since these segments are disjoint so $a_i, b_i \in E$. The function g on each (a_n, b_n) is defined to be a line between $f(a_n)$ and $f(b_n)$.

exer:rudin_chap4_6

6. The **graph** of a function f mapping metric space $E \subseteq X$ to metric space Y is a set S of points $(x, f(x))$ for $x \in E$. This set belongs to the metric space $X \times Y$ with distance function $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, y_1)^2 + d_Y(y_1, y_2)^2}$.

   If f is not continuous at some point $x \in E$ then there exists an infinite sequence $\{x_n\}$ in E that converges to x but $d_Y(f(x_n), f(x)) \geq \varepsilon$. This follows the set $S = \{(x_n, f(x_n) : n \geq 1\}$ is infinite. Note that this set does not have limit point in the graph G of E, otherwise if $(y, f(y))$ is a limit point of S then $\lim x_n = y = x$ which implies $\lim f(x_n) = f(y) = f(x)$, a contradiction. Thus, from theorem 1.3.2.5, G is not compact.

   If f is continuous on E. Define a function $g : E \to X \times Y$ so $g(x) = (x, f(x))$. Since f is continuous on E so g is continuous on E. Since E is compact so according to theorem 2.3.0.2, we find $g(E)$, which is the graph of E, is compact.

exer:rudin_chap4_7

7. $f(0,0) = g(0,0) = 0, f(x,y) = \frac{xy^2}{x^2+y^4}, g(x,y) = \frac{xy^2}{x^2+y^6}$ if $(x,y) \neq (0,0)$.

Note that $f(0,y) = f(x,0) = 0$ and for any $x, y \neq 0$ then $|f(x,y)| \leq \frac{|x|y^2}{2|x|y^2} = \frac{1}{2}$ so f is bounded on $\mathbf{R}^2$. For g, pick $x = 1$ and $y = 1/n$, we have $g(n^{-3}, n^{-1}) = \frac{n^{-5}}{2n^{-6}} = n/2$ so g is unbounded in every neighborhood of $(0,0)$.

Consider the sequence $\{(0, 1/n)\}$ which converges to 0 then $f(0, 1/n) = 0$ for all n so $\{f(0,1/n)\}$ converges to 0. Consider the sequence $\{(n^{-2}, n^{-1})\}$ which converges to 0 and since $f(n^{-2}, n^{-1}) = 1/2$ so $\{f(n^{-2}, n^{-1})\}$ converges to $1/2$. Thus, we find that f is not continuous at $(0,0)$.

Consider set $E = \{x, ax+b : x \in \mathbf{R}\}$ representing all points on the straight line $y = ax+b$. For $c \neq 0$ then $\lim_{x\to c} x = c$ so

$$\lim_{x\to c} f(x, ax+b) = \lim_{x\to c} \frac{x(ax+b)^2}{x^2 + (ax+b)^4} = f(c, ac+b).$$

Thus, f is continuous at any point $(c, ac+b)$ for $c \neq 0$. On the other hand, if $b \neq 0$ then similarly as above, we obtain f continuous at $(0,b)$. If $b = 0$ then we have $\lim_{x\to 0} f(x, ax+b) = \lim_{x\to 0} \frac{a^2x^3}{x^2+(ax)^4} = 0 = f(0,0)$. Thus, in all cases, the **restriction** of f to every straight line in $\mathbf{R}^2$ is continuous. We apply similar approach for g.

exer:rudin_chap4_8

8. Pick an $\varepsilon > 0$. Since f is uniformly continuous on E so there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in E$ so $|x-y| < \delta$. Since E is bounded so let $p = \inf E$ then there exists $n \in \mathbf{N}$ such that $p < x < p + n\delta$ for all $x \in E$. Let S be the set of all $0 \leq i \leq n-1$ such that $I_i = [p + i\delta, p + (i+1)\delta)$ contains a point p_i in E. Let $L = 2\varepsilon + \max\{|f(p_i) - f(p_j)| : i, j \in S\}$. Hence, for any $x, y \in E$ so $x \in I_i, y \in I_j$ then

   $$|f(x) - f(y)| \leq |f(x) - f(p_i)| + |f(p_i) - f(p_j)| + |f(p_j) - f(y) < L.$$

   Thus, f is bounded on E.

   Pick $f(x) = x$ with $E = (1, \infty)$ then f is uniformly continuous on E as $|f(x) - f(y)| = |x-y|$ and f is unbounded on E.

exer:rudin_chap4_9

9. For any x, y in E so $d_X(x,y) < \delta$ then let $E = \{x, y\}$ we find diam $E = d_X(x,y) < \delta$. Hence, if the requirement of f in terms of diameter is true then f is unifromly continuous.

   Conversely, if f is unifomly continuous, for any set $E \subset X$ so diam $E < \delta$ then $d_X(x,y) < \delta$ for all $x, y \in E$. This follows $d_Y(f(x), f(y)) < \varepsilon$ for all $x, y \in E$ so diam $f(E) < \varepsilon$.

exer:rudin_chap4_10

10. If f is not uniformly continuous, then for some $\varepsilon > 0$, there exists two sequences $\{q_n\}$ and $\{p_n\}$ in X such that $d_X(q_n, p_n) \to 0$ as $n \to \infty$ but $d_Y(q_n, p_n) > \varepsilon$. Since X is compact, there exists a limit point p of $\{p_n\}$, i.e. there exists a subsequence $\{p_{n_k}\}$ that converges to p. Since $d_X(p_{n_k}, q_{n_k}) \to 0$ as $k \to \infty$ so $\{q_{n_k}\}$ also converges to p. Since f is continuous so $\{f(p_{n_k})\}$ and $\{f(q_{n_k})\}$ both converge to $f(p)$, which contradicts to the fact that $d_Y(q_n, p_n) > \varepsilon$ for all n. Thus, f must be uniformly continuous.

exer:rudin_chap4_11 11. Since f is uniformly continuous so for any ε, there exists δ such that $d_Y(f(x), f(y)) < \varepsilon$ for all $d_X(x, y) < \delta$ so $x, y \in X$.

If $\{x_n\}$ is a Cauchy sequence in X so there exists integer N such that $d_X(x_n, x_m) < \delta$ for all $n, m \geq N$.

Thus, we obtain that for all $n, m \geq N$ then $d_X(x_n, x_m) < \delta$ which implies $d_Y(x_m, x_n) < \varepsilon$. Hence, $\{f(x_n)\}$ is a Cauchy sequence.

exer:rudin_chap4_12 12. Let f be a uniformly continuous function mapping metric space X to metric space Y. Let g be a uniformly continuous function mapping metric space Y to metric space Z. We will show that the function h defined by $h = g \circ f$ is uniformly continuous.

Indeed, for any $\varepsilon > 0$, there exists $\delta_Y > 0$ such that $d_Z(g(x), g(y)) < \varepsilon$ for all $x, y \in Y$ so $d_Y(x, y) < \delta_Y$.

For $\delta_Y > 0$, there exists δ_X such that $d_Y(f(x), f(y)) < \delta_Y$ for all $x, y \in X$ so $d_X(x, y) < \delta_X$.

Thus, for all $x, y \in X$ so $d_X(x, y) < \delta_X$ then $d_Y(f(x), f(y)) < \delta_Y$ so $d_Z(g(f(x)), g(f(y))) < \varepsilon$, as desired.

exer:rudin_chap4_13 13. For each $p \in X$ so $p \notin E$ then p is a limit point of E. Choose an arbitrary sequence $\{p_n\}$ in E that converges to p. This follows $\{p_n\}$ is a Cauchy sequence. Since f is uniformly continuous on E so from exercise 11, we find $\{f(p_n)\}$ is a Cauchy sequence in $\mathbf{R}^1$, which follows that $\{f(p_n)\}$ converges to some q. Let $g(p) = q$. From here, we also find that for **any** sequence $\{p_n\}$ in E that converges to p then $\{f(p_n)\}$ converges to q.

After we've defined function g, we need to show that g is continuous on X. Indeed, suppose g is not continuous at a point $p \in X$ then there exists sequence $\{p_n\}$ converging to p such that $|g(p_n) - g(p)| \geq \varepsilon$. For each $p_n \notin E$, from our construction, we can find a $q_n \in E$ such that $d_X(p_n, q_n) < 1/n$ and $|g(p_n) - g(q_n)| < \varepsilon/2$.

On the other hand, since $d_X(p_n, q_n) < 1/n$ for all n and since $\{p_n\}$ converges to p, we can find that $\{q_n\}$ converges to p. Since $\{q_n\}$ is sequence in E so $\{g(q_n)\}$ or $\{f(q_n)\}$ converges to $g(p)$. Hence, there exists N such that $|g(q_n) - g(p)| < \varepsilon/2$ for all $n > N$.

Therefore, from the two above observations, we find that for all $n > N$ then

$$|g(p_n) - g(p)| \leq |g(p_n) - g(q_n)| + |g(q_n) - g(p)| < \varepsilon.$$

This contradicts to the condition $|g(p_n) - g(p)| \geq \varepsilon$ for all n.

Following the hint from the textbook:

Since $f(V_n(p))$ are subsets of $\mathbf{R}^1$ so since $\overline{f(V_n(p))}$ is closed and bounded, we find $\overline{f(V_n(p))}$ is compact. On the other hand, $\overline{f(V_n(p))} \supseteq \overline{f(V_{n+1}(p))}$ for all n so we find that $K = \bigcap_{n \geq 1} \overline{f(V_n(p))}$ contains at least one point. Since $\lim_{n \to \infty} \operatorname{diam} V_n(p) = 0$ so this follows K contains exactly one point $g(p)$...

The problem is still true if the range of space if $\mathbf{R}^k$ instead of $\mathbf{R}^1$. However, it may not be true for any compact metric space because we've used the condition of a set being closed and bounded to imply its compactness and this property does not hold for any compact metric space.

The problem is still true if the range of the space is a complete metric space according to its definition or according to exercise **??** (Chap 3).

The problem may not be true if the range of the space is a metric space.

exer:rudin_chap4_14 14. Consider the continuous function g on I defined by $g(x) = x - f(x)$. Since $g(0) = -f(0) \le 0$ and $g(1) = 1 - f(1) \ge 0$ so from theorem 2.4.0.2, there exists $x \in (0, 1)$ such that $g(x) = 0$ or $f(x) = x$.

exer:rudin_chap4_15 15. Suppose f is not continuous then WLOG, there exists $x < y < z$ such that $f(x) < f(y)$ and $f(y) > f(z)$. Consider the interval $I = [x, z]$ which is compact so $f(I)$ is closed and bounded so from theorem 2.3.0.3, there exists $u \in [x, z]$ such that $f(u) = \sup_{p\in I} f(p)$. Since $y \in [x, z]$ and $f(y) > f(x), f(y) > f(z)$ so $f(u) > f(x), f(u) > f(z)$ which implies $u \in J = (x, z)$.

    Let $v \in [x, z]$ so $f(v) = \inf_{p\in[x,z]} f(p)$ then we find $f(J) \subset (f(v), f(u))$. Since $J = (x, z)$ is connected so from theorem 2.4.0.1, $f(J)$ is connected. We also know that $f(J)$ is open so $f(J) = (f(v), f(u))$. On the other hand, we know $f(u) \in f(J)$ since $u \in (x, z)$, which is a contradiction.

    Thus, f must be monotonic.

exer:rudin_chap4_16 16. Function $\lfloor x \rfloor$ is discontinuous at any $n \in \mathbf{Z}$ and continuous at the remaining points. Since $\{x\} = x - \lfloor x \rfloor$ so the discontinuity of $\{x\}$ is the same as of $\lfloor x \rfloor$.

exer:rudin_chap4_17 17. Let's first show that such p, q, r exist for each $x \in E$. p obviously exists. Let $\varepsilon = |f(x-)|/n$ then there exists δ such that $|f(t) - f(x-)| < |f(x-)|/n$ for all $x - \delta < t < x$. This follows $f(t) < f(x-) + |f(x-)|/n < p$ for sufficiently small n. Hence, any rational q between $x - \delta$ and x is sufficient. We choose r similarly.

    Next, let's show that each triples (p, q, r) is associated with at most one point in E. Suppose the contrary, there exists two points $x_1, x_2 \in E$ that gives the same triples (p, q, r). WLOG, $x_1 < x_2$ and we know $f(t) < p$ for all $t \in (q, x_1)$ or $t \in (q, x_2)$. Hence, $f(t) < p$ for all $t \in (x_1, x_2)$. However, we know $f(t) > p$ for all $t \in (x_1, r)$, a contradiction.

    Finally, let's consider cases for other types of simple discontinuities. For x so $f(x-) > f(x+)$ we proceed similarly.

    For x so $f(x-) = f(x+) \ne f(x)$, let S be the set of such x's on (a, b). Let $S_n = \{x \in S : |f(x) - f(x-)| \ge 1/n\}$ then $S = \bigcup_{n\ge 1} S_n$. To show that S is at most countable, we show that each S_i is at most countable. Assume the contrary, there exists n such that S_n is uncountable. This follows there exists a limit point x of S_n that is in S_n (otherwise we can create a set of disjoint open intervals where each contains at most one point in S_n, this set is injective with $\mathbf{Q}$, which is countable). Let $\{x_m\}$ be a sequence in S_n that converges to x. This follows $f(x_m) \to f(x-) = f(x+)$ as $x_n \to x$ according to the definition of $f(x-), f(x+)$. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x-) - f(x_m)| < \varepsilon$ for all $0 < |x_m - x| < \delta$. This follows $|f(x-) - f(x_m-)| < \varepsilon$ for all $0 < |x_m - x| < \delta$. Hence,

$$|f(x_m) - f(x_m-)| \leq |f(x-) - f(x_m)| + |f(x-) - f(x_m-)| < 2\varepsilon.$$

For sufficient small ε we can have $1/n < 2\varepsilon$, which is a contradiction since $x_m \in S_n$. Thus, S_n must be countable, which follows S is countable.

Remark 2.7.0.1. The idea behind dividing S into subsets is to give a lower bound for $|f(x) - f(x-)|$. Say, if instead of considering S_n, we consider a limit point of S right away, then as $x_m \to x$, $|f(x_m) - f(x_m-)|$ can approach 0 and we can't give any contradiction from that.

Actually, we can use this argument a little bit earlier without using bijection with $\mathbf{Q}^3$. See here for such solution.

exer:rudin_chap4_18 18. For any number $p \in \mathbf{R}^1$, it suffices to show that for any sequence $\{p_n\}$ of rational numbers that converges to p then $\{f(p_n)\}$ converges to 0.

For any positive integer a, there exists $b \in \mathbf{Z}$ so $\frac{b}{a} < p < \frac{b+1}{a}$. Hence, this follows that there are finitely many m such that $p_m = k/a$ for some $k \in \mathbf{Z}$ so $\gcd(k, a) = 1, a \geq 1$. This follows for any $n \in \mathbf{N}$ then there are finitely many m so $f(p_m) = 1/n$. Thus, for any K, there exists N such that $f(p_n) < \frac{1}{K}$ for all $n \geq N$. This concludes $\{f(p_n)\}$ converging to 0.

Thus, $\lim_{x\to p} f(x) = 0$ for any point $p \in \mathbf{R}^1$. This explains why f is continuous at any irrational point and has simple discontinuity at every rational point.

exer:rudin_chap4_19 19. Suppose f is not continuous at point p, then there exists a sequence $\{p_n\}$ in $\mathbf{R}$ converging to p such that, WLOG, $p_n > p$ and $f(p_n) - f(p) > \varepsilon$ for all n. Fix a rational number q between $f(p)$ and $f(p) + \varepsilon < f(p_n)$ then since f has the Intermediate Value Property, there exists $q_n \in (p, p_n)$ so $f(q_n) = q$ for all n.

Since $q_n \in (p, p_n)$ and $p_n \to p$ as $n \to \infty$ so $q_n \to p$ as $n \to \infty$. On the other hand, since $q \in \mathbf{Q}$ so the set $S = \{x \in \mathbf{R} : f(x) = q\}$ is closed, which implies $p \in S$, which is a contradiction since $f(p) \neq q$.

Thus, f is continuous on $\mathbf{R}$.

exer:rudin_chap4_20 20. (a) If $\rho_E(x) = \inf_{x\in E} d(x, z) = 0$ then either $x \in E$ or x is a limit point of E. Hence, $x \in \overline{E}$. The other direction is similar.

(b) For any $x, y \in X$ then

$$\inf_{z\in E} d(y, z) + d(x, y) = \inf_{z\in E}(d(y, z) + d(x, y)) \geq \inf_{z\in E} d(x, z).$$

This follows $d(x, y) \geq \rho_E(x) - \rho_E(y)$. Similarly, we obtain $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$.

exer:rudin_chap4_21 21. Suppose that there does not exist such δ. Hence, there exist two sequences $\{p_n\}$ in K and $\{q_n\}$ in F such that $d(p_n, q_n) < 1/n$ for all n. Since K is compact, according to theorem 1.3.2.5, the sequence has subsequence $\{p_{n_k}\}_{k\geq 1}$ that converges to some point p in K. Therefore, there exists N such that for all $k \geq N$ then $d(p_{n_k}, p) < 1/N$ so $d(p, q_{n_k}) < d(p_{n_k}, p) + d(p_{n_k}, q_{n_k}) < 2/N$ (since $n_k \geq k \geq N$). Thus, we obtain a sequence $\{q_{n_k}\}$ in F that converges to p. Since F is closed so $p \in F$. Thus, $p \in F \cap K$, a contradiction. Thus, there must exists some δ.

The conclusion may fail for two disjoint closed sets if neither is compact. For example, consider set $\mathbf{Q}$ of rational numbers with the metric $d(x, y) = |x - y|$. Consider two disjoint subsets of $\mathbf{Q}$, which is $S = \{p \in \mathbf{Q} : p^2 < 2\}$ and $T = \{p \in \mathbf{Q} : p^2 > 2\}$ then S, T are disjoint closed sets but there does not exist δ such that $d(x, y) > \delta$ for $x \in S, y \in T$.

exer:rudin_chap4_22 22. From exercise 20 (Chap 4), we know ρ_A are continuous function on X so $\lim_{x\to p} \rho_A(x) + \rho_B(x) = \rho_A(p) + \rho_B(p)$. Since A, B are disjoint closed sets so $\rho_A(p) + \rho_B(p) \neq 0$ for all $p \in X$. From these, we can imply that f is continuous on X.

Since $\rho_A(p) \geq 0$ for all $p \in X$ so $f(p) \in [0, 1]$ for all $p \in X$. We have $f(p) = 0$ when $\rho_A(p) = 0$ or $p \in A$ according to exercise 20 (Chap 4). Similarly, we have $f(p) = 1$ when $\rho_B(p) = 1$ or $p \in B$.

Let $V = f^{-1}([0, \frac{1}{2}))$ and $W = f^{-1}((\frac{1}{2}, 1])$. First, we will show that V is open. For any $p \in V$ then we find $f(p) < 1/2$ or $\rho_A(p) < \rho_B(p)$. Let $\rho_B(p) - \rho_A(p) = \varepsilon$. Consider a neighborhood S of p with radius $\varepsilon/2$. For any $q \in S$ then $d(p, q) < \varepsilon/2$. Therefore, according to the inequality from exercise 20, we have

$$\rho_A(q) \leq \rho_A(p) + d(p, q) < \rho_B(p) - d(p, q) \leq \rho_B(q).$$

This follows $f(q) < 1/2$ or $q \in V$ for all $q \in S$. Thus, V is open. Similarly, W is open. On the other hand, it's obvious that V, W are disjoint.

Next, we will show that $A \subset V$. Indeed, for any $p \in A$ then $f(p) = 0$ so $p \in V$. Similarly, $B \subset W$.

Thus, pair of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called **normality**.

exer:rudin_chap4_23 23. This is about **convex** function.

We will show that for any $y \in (a, b)$ then $f(y+) = f(y)$. Indeed, consider a sequence $\{x_n\}$ converging to y so $x_n > y$ for all n.

Let's rewrite our inequality in a more useful form. Let $x = x_n$ and $\lambda = \frac{x_m - y}{x_n - y}$ then the inequality is equivalent to

$$f(x_m) - f(y) \leq \frac{x_m - y}{x_n - y}(f(x_n) - f(y)).$$

Fix n, let $m \to \infty$ or $x_m \to y$ then $\limsup_{m\to\infty}(f(x_m) - f(y)) \leq 0$. Similarly, fix m and let $n \to \infty$ or $x_n \to y$, we obtain $\liminf_{n\to\infty}(f(x_n) - f(y)) \geq 0$. In conclusion, we obtain

$\lim_{n\to\infty} f(x_n) = f(y)$. Since this is true for any sequence $\{x_n\}$ converging to y, we obtain $f(y+) = f(y)$. Similarly, $f(y-) = f(y)$ so f is continuous at y.

Let f be an increasing convex function and g be a convex function. For any $0 < \lambda < 1$, we have $g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda)g(y)$. Since f is increasing so

$$(f \circ g)(\lambda x + (1-\lambda)y) \le f\left(\lambda g(x) + (1-\lambda)g(y)\right).$$

Since f is convex so

$$f\left(\lambda g(x) + (1-\lambda)g(y)\right) \le \lambda (f\circ g)(x) + (1-\lambda)(f\circ g)(y).$$

Combining these two inequalities, we got what we want.

For the last inequality, use the rewritten inequality in previous argument.

exer:rudin_chap4_24 24. First, we prove that for any $\lambda \in \mathbf{Q} \cap (0,1)$ then $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$. Indeed, let $\lambda = m/n$. WLOG $m > n/2$ we have

$$f\left(\frac{m}{n}x + \frac{n-m}{n}y\right) \le \frac{1}{2}\left(f\left(\frac{2m-n}{n}x + \frac{2n-2m}{n}y\right) + f(x)\right).$$

(If $m < n/2$ then take $f(y)$ out instead of $f(x)$ as above). Next, we do the same thing with $f\left(\frac{2m-n}{n}x + \frac{2n-2m}{n}y\right)$ and keep going. Note that this step will repeat because of the following periodic function:

Indeed, consider the function $f : \mathbf{Z}_n \to \mathbf{Z}_n$ so for $1 \le m \le n-1$ then

$$f(m) = \begin{cases} 2m & m < n/2, \\ 2m-n & m > n/2, \\ m/2 & m = n/2. \end{cases}.$$

This follows $f(m) = 2m \pmod n$ for $m \ne n/2$. Since the sequence $m, 2m, 2^2m, \ldots, \pmod n$ is periodic. ... It's a bit harder to take it from here.

See here for a better proof.

Second, using f being continuous, we can show that f is convex. For any $\lambda \in (0,1)$ and any x, y, choose $\varepsilon > 0$. Since f is continuous at $\lambda x + (1-\lambda)y$ so there exists $\delta > 0$ such that $\lambda + \delta \in \mathbf{Q}$ and

$$|f(\lambda x + (1-\lambda)y + \delta(x-y)) - f(\lambda x + (1-\lambda)y)| < \varepsilon.$$

for all $|x-y| < \delta$. On the other hand, since $\lambda + \delta \in \mathbf{Q}$ so

$$f((\lambda+\delta)x + (1-\lambda-\delta)y) \le \lambda f(x) + (1-\lambda)f(y) + \delta(f(x) - f(y)).$$

Combining these two inequalities, we obtain

$$f(\lambda x + (1-\lambda)y) - [\lambda f(x) + (1-\lambda)f(y)] \le \varepsilon + \delta(f(x) - f(y)).$$

By taking $\delta \to 0$, we obtain

$$f(\lambda x + (1-\lambda)y) - [\lambda f(x) + (1-\lambda) f(y)] \leq \varepsilon.$$

Since this is true for any ε so

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

exer:rudin_chap4_25 25. (a) Consider a limit point $p \in K+C$ then there exists sequences $\{k_n\}$ in K and $\{c_n\}$ in C such that $k_n + c_n \to p$ as $n \to \infty$. Since K is compact, this infinite sequence $\{k_n\}$ has a limit point k, i.e. there exists a subsequence $\{k_{n_m}\}_{m\geq 1}$ converging to k. We also have $\{k_{n_m} + c_{n_m}\}_{m \geq 1}$ converging to p so

$$\lim_{m\to\infty} c_{n_m} = \lim_{m\to\infty} (k_{n_m} + c_{n_m}) - \lim_{m\to\infty} k_{n_m} = p - k.$$

Since C is closed so we find $p - k = c \in C$. Thus, $p = k + c \in K + C$, as desired.

There's also another approach given by the hint in [1].

(b) C_1, C_2 are closed because there does not exist a limit point for each of those sets.

We need to prove that $C_1 + C_2 = \{a + \alpha b : a, b \in \mathbf{Z}\}$ is dense in $[0,1]$. It suffices to show that the set $S = \{\{\alpha b\} : b \in \mathbf{Z}\}$ is dense in $[0,1]$, since $\{\alpha b\} = \alpha b - \lfloor \alpha b \rfloor \in C_1 + C_2$.

First, we prove that we can find pairs of elements in S that are arbitrarily close to each other. There are two ways to do that: one uses pigeonhole principle and the other uses compactness 1.3.2.5 for $[0,1]$. The second method is quite simple: first you prove that any two $\{i\alpha\}$ and $\{j\alpha\}$ are distinct, which implies S is an infinite subset of compact set $[0,1]$ so S has limit point in $[0,1]$. For the first method: Fix a $n \in \mathbf{Z}^+$, since α is irrational so $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ are pairwise distinct. The interval $[0,1]$ can be divided into n subintervals so by pigeonhole principle, there exists $0 \leq i, j \leq n$ such that $0 < \{j\alpha\} - \{i\alpha\} < \frac{1}{n}$ [1].

Fix a $n \in \mathbf{Z}^+$, from previous argument, there exists $i, j \in \mathbf{Z}$ such that $0 < \{i\alpha\} - \{j\alpha\} < 1/n$. This follows for any subinterval $I_m = \left[\frac{m}{n}, \frac{m+1}{n}\right]$ of $[0,1]$, there exists $k \in \mathbf{Z}^+$ such that $k(\{i\alpha\} - \{j\alpha\}) \in I_m$. This is because when you starts from 0 and for each step, you jump to a point with distance less than $1/n$ from the previous point, eventually you will land on every I_m. Furthermore, since $\lfloor a + b \rfloor \in \{\lfloor a \rfloor + \lfloor b \rfloor, \lfloor a \rfloor + \lfloor b \rfloor + 1\}$ so

$$\begin{aligned} k(\{i\alpha\} - \{j\alpha\}) &= \{k(\{i\alpha\} - \{j\alpha\})\}, \\ &= \{k(i-j)\alpha + k(\lfloor j\alpha \rfloor - \lfloor i\alpha \rfloor)\}, \\ &= \{k(i-j)\alpha\}. \end{aligned}$$

[1] It is possible to get $< 1/n$ instead of $\leq 1/n$ because when we arrange $n+1$ numbers $\{i\alpha\}$ on $[0,1]$ where the first one appears in 0, if the distance between any two consecutive numbers is $1/n$ then essentially $\{\{i\alpha\} : 0 \leq i \leq n\} = \{i/n : 0 \leq i \leq n\}$, i.e. there exists i such that $\{i\alpha\} = 1$, a contradiction.

Thus, there exists $k \in \mathbf{Z}^+$ such that $\{k(j-i)\alpha\} \in I_m$. In other words, for any $x \in [0,1]$ and any $n \in \mathbf{Z}^+$, there exists $p \in S$ such that $|x-p| < 1/n$.

Since $C_1 + C_2 \neq \mathbf{R}$ and that $C_1 + C_2$ is dense in $\mathbf{R}$, $C_1 + C_2$ is not closed.

exer:rudin_chap4_26

26. Note that g is a continuous 1-1 mapping of a compact metric space Y onto a metric space $g(Y)$ so by theorem 2.3.0.4, g has inverse g^{-1} defined on $g(Y)$ which is also a continuous mapping from $g(Y)$ to Y. Since g is continuous and Y is compact so $g(Y)$ is compact according to theorem 2.3.0.2. Thus, g^{-1} is a continuous mapping from a compact set $g(Y)$ to Y. According to theorem 2.3.1.1, we find g^{-1} is uniformly continuous on $g(Y)$.

    On the other hand, we know that $f(x) = g^{-1}(h(x))$ for all $x \in X$. Since g^{-1}, h are uniformly continuous so f is uniformly continuous on X according to exercise 12.

    Similarly, if h is continuous then f is also continuous.

    For example: $X = Z = \{\mathbf{x} \in \mathbf{R}^2 : |\mathbf{x}| = 1\}$ and $Y = [0, 2\pi)$ with $g : Y \to Z$ so $g(t) = (\sin t, \cos t)$ and $f = g^{-1}$ and $h = I_X$. Then from example 2.3.0.6, h is continuous and is uniformly continuous but f is not even continuous.

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Chapter 3

Sequence and series

3.1 Convergent sequences

3.2 Subsequences

subsequence_convergent **Theorem 3.2.0.1** (a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.

(b) Every bounded sequence in $\mathbf{R}^k$ contains a convergent subsequence

3.3 Cauchy sequences

Definition 3.3.0.1 (Cauchy sequence)**.** A sequence $\{p_n\}$ in a metric space X is said to be **Cauchy sequence** if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n, m \geq N$.

Definition 3.3.0.2 (Diameter of a set)**.** Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$ with $p, q \in E$. The supremum of S is called the **diameter** of E.

From this definition, we have a new way of defining Cauchy sequence

Definition 3.3.0.3. If $\{p_n\}$ is a sequence in X and if E_N consists of points $p_N, p_{N+1}, \ldots$, then $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N\to\infty} \mathrm{diam} E_N = 0.$$

theo:sequence_cauchy **Theorem 3.3.0.4** (a) In a metric space X, every convergent sequence is a Cauchy sequence.

(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X then $\{p_n\}$ converges to some point of X.

(c) In $\mathbf{R}^k$, every Cauchy sequence converges.

Example 3.3.0.5
An example where a sequence is Cauchy but not convergent.

def:metric_complete **Definition 3.3.0.6** (Complete metric space)**.** A metric space in which every Cauchy sequence converges is said to be **complete**.

Previous theorems say that all compact metric spaces and all Euclidean spaces are complete. Theorem 3.3.0.4 implies that every closed subset E of a complete metric space X is complete.

uence_conver_monotonic **Theorem 3.3.0.7** (Convergence of monotonic sequences) Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

3.4 Upper and lower limits

The upper and lower limits allow us gives us a range about the behaviour of real sequence as $n \to \infty$.

ef:sequence_sup_inf **Definition 3.4.0.1.** Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of all numbers x (in the extended real system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. This set E contains all the subsequential limits, plus possibly numbers $+\infty, -\infty$.

We define $\limsup_{n\to\infty} s_n$ to be $\sup E$ and $\liminf_{n\to\infty} s_n$ to be $\inf E$.

theo:sequence_sup_inf **Theorem 3.4.0.2** Let $\{s_n\}$ be a sequence of real numbers. Let E be defined as in definition 3.4.0.1. Then we have the following:

1. $s^* = \limsup_{n\to\infty} s_n \in E$.
2. If $x > s^*$ there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with these two properties. The same can be said about $\liminf_{n\to\infty} s_n$.

3.5 Some special sequences

theo:sequence_special **Theorem 3.5.0.1** These are sequences of real numbers.

(a) If $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) If $p > 0$ then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(p+1)^n} = 0$.

(e) If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$.

3.6 Series

eries_cauchy_criterion **Theorem 3.6.0.1** (Cauchy criterion) $\sum a_n$ converges if and only if for every $\varepsilon > 0$, there is an integer N such that $|\sum_{k=n}^m a_k| < \varepsilon$ if $m \geq n \geq N$.

series_comparison_test **Theorem 3.6.0.2** (Comparison test) (a) If $|a_n| \leq c_n$ for $n \geq N_0$ where N_0 is some fixed integer, and if $\sum c_n$ converges then $\sum a_n$ converges.

(b) (For sequences of real numbers) If $a_n \geq d_n \geq 0$ for $n \geq N_0$ and if $\sum d_n$ diverges then $\sum a_n$ diverges.

3.7 Series of nonnegative terms

Theorem 3.7.0.1

Suppose $a_1 \geq a_2 \geq \ldots \geq 0$. Then the series $\sum_{n=1}^\infty a_n$ converges if and only if the series $\sum_{k=0}^\infty 2^k a_{2^k}$ converges.

From this theorem, we obtain the following useful convergent series:

theo:series_p **Theorem 3.7.0.2** $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

3.8 The number e

def:euler_number **Definition 3.8.0.1.** $e = \sum_{n=0}^\infty \frac{1}{n!}$.

Theorem 3.8.0.2

$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

3.9 The root and ratio tests

theo:series_root_test **Theorem 3.9.0.1** (Root test) Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then

(a) If $\alpha < 1$ then $\sum a_n$ converges;

(b) If $\alpha > 1$ then $\sum a_n$ diverges;

(c) If $\alpha = 1$, the test gives no information.

theo:series_ratio_test **Theorem 3.9.0.2** (Ratio test) The series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

(a) diverges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Example 3.9.0.3
Example shows the application of the two tests. Example shows root test is stronger than ratio test.

3.10 Power series

theo:power_series **Theorem 3.10.0.1** Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, R = 1/\alpha.$$

(If $\alpha = 0, R = +\infty$; if $\alpha = +\infty, R = 0$) Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

3.11 Absolute convergence

The comparison test 3.6.0.2 and the root test 3.9.0.1 are tests for absolute convergence. In particular, from theorem 3.10.0.1, power series converges absolutely in the interior of the circle of convergence.

3.12 Summation by parts

Theorem 3.12.0.1 (Partial summation formula)
Given two sequences $\{a_n\}, \{b_n\}$, put $A_n = \sum_{k=0}^{n} a_k$ if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

From this, we can show that

_partial_sum_monotonic **Theorem 3.12.0.2** Suppose the partial sums A_n of $\sum a_n$ form a bounded sequence and $b_0 \geq b_1 \geq \dots$ and $\lim_{n\to\infty} b_n = 0$.

3.13 Addition and Multiplication of series

:series_product_cauchy **Theorem 3.13.0.1** Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely so $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} = B$. Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for $n = 0, 1, 2, \dots$ then $\sum_{n=0}^{\infty} c_n = AB$.

3.14 Baby Rudin exercises

exer:rudin_chap3_1

1. If $\{s_n\}$ converges to s then $||s_n| - |s|| \leq |s_n - s|$ so we find that $\{|s_n|\}$ converges to $|s|$.

   The converse is not true. Consider $\{(-1)^n\}$ which diverges but $\{1\}$ converges.

exer:rudin_chap3_2

2. We have $\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+1/n}+1}$ and since $\lim_{n\to\infty}\left(\sqrt{1+1/n}+1\right) = 2$ so $\lim_{n\to\infty}\left(\sqrt{n^2+n} - n\right) = \frac{1}{\lim_{n\to\infty}\left(\sqrt{1+1/n}+1\right)} = \frac{1}{2}$.

exer:rudin_chap3_3

3. We prove by induction that $\{s_n\}$ is monotonically increasing and that $1 < s_n < 2$ for $n = 1, 2, \dots$

   Suppose we've shown that for all $i \leq n$. It's obvious that $s_{n+1} > 1$ and $1 < \sqrt{s_n} < \sqrt{s}_n < 2$ so $(\sqrt{s_n} - 1)(\sqrt{s_n} - 2) < 0$ which implies $s_n^2 < 2 + \sqrt{s_n} = s_{n+1}^2$. This follows $s_n < s_{n+1}$. We also have $s_{n+1} < 2$.

   Thus, $\{s_n\}$ is a bounded monotonic sequence so according to theorem 3.3.0.7, it is convergent. theo:sequence_conver_monotonic

exer:rudin_chap3_4

4. We can show by induction that $s_{2m} = \sum_{i=2}^{m} 2^{-i}$ and $s_{2m+1} = \sum_{i=1}^{m} 2^{-i}$.

   Hence, we predict that $\limsup_{n\to\infty} s_n = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i = 1$ and $\liminf_{n\to\infty} s_n = \sum_{i=2}^{\infty}\left(\frac{1}{2}\right)^i = \frac{1}{2}$.

   We can see that $s_{2n} \to \frac{1}{2}$ as $n \to \infty$ and $s_{2n+1} \to 1$ as $n \to \infty$.

For all $n \geq 2m$, we have $s_n \geq \sum_{i=2}^{m} 2^{-i}$. Hence, by taking $n \to \infty$ and keeping m fixed, we obtain $\liminf_{n\to\infty} s_n \geq \sum_{i=2}^{m} 2^{-i}$. Letting $m \to \infty$, we obtain $\liminf_{n\to\infty} s_n \geq 2^{-1}$. We obtain $\liminf_{n\to\infty} s_n = 2^{-1}$.

For all $n \geq m$, we have $s_n < \sum_{i=1}^{n} 2^{-i} = t_n$ so by taking $n \to \infty$, we obtain $\limsup_{n\to\infty} s_n \leq 1$. Thus, $\limsup_{n\to\infty} s_n = 1$.

exer:rudin_chap3_5

5. If either one of $\limsup_{n\to\infty} a_n$ or $\limsup_{n\to\infty} b_n$ equals to $\pm\infty$ then there's nothing to prove.

   If $\limsup_{n\to\infty} a_n = a$ and $\limsup_{n\to\infty} b_n = b$ then for any $x > a + b$, according to theorem 3.4.0.2, there exists N so $a_n + b_n < x$ for all $n \geq N$. This follows that $\limsup_{n\to\infty}(a_n + b_n) \leq x$. Since this is true for any $x > a + b$, we get what we want.

exer:rudin_chap3_6

6. (a) We have $\sum_{k=1}^{n} a_n = \sqrt{n+1} - 1$ so $\sum a_n$ diverges.

   (b) We have
   $$0 \leq a_n = \frac{1}{n\left(\sqrt{n+1} + \sqrt{n}\right)} < \frac{1}{2n^{3/2}}.$$
   From theorem 3.5.0.1, we know that $\lim_{n\to\infty} \frac{1}{2n^{3/2}} = 0$ so $\lim_{n\to\infty} a_n = 0$.

   (c) From theorem 3.5.0.1, we know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ so for any $\varepsilon > 0$, there exists N so $0 < \sqrt[n]{n} - 1 < \varepsilon$ for all $n \geq N$. This follows that $0 < a_n < \varepsilon^n$. Since $\lim_{n\to\infty} \varepsilon^n = 0$ according to theorem 3.5.0.1 so we can conclude that $\lim_{n\to\infty} a_n = 0$.

   (d) Using De Moivre's Theorem, if we write z as $z = |z|(\cos\theta + i\sin\theta)$ then $z^n = |z|^n(\cos(n\theta) + i\sin(n\theta))$. Hence,
   $$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{1}{(1 + |z|^n\cos(n\theta)) + i|z|^n\sin(n\theta)},\\ &= \frac{1}{\lim_{n\to\infty}(1 + |z|^n\cos(n\theta)) + i\lim_{n\to\infty}|z|^n\sin(n\theta)},\\ &= 1.\end{aligned}$$

exer:rudin_chap3_7

7. From theorem 3.7.0.2, we know that $\sum \frac{1}{n^2}$ converges so for any $\varepsilon > 0$ there is N such that $\sum_{k=n}^{m} \frac{1}{k^2} < \varepsilon$ if $m \geq n \geq N$.

   Since $\sum a_k$ converges so there is M such that $\sum_{k=n}^{m} a_k < \varepsilon$ if $m \geq n \geq M$.

   Therefore, for $m \geq n \geq M + N$, we have
   $$\sum_{k=n}^{m} \frac{\sqrt{a_k}}{k} \leq \sqrt{\left(\sum_{k=n}^{m} a_k\right)\left(\sum_{k=n}^{m} \frac{1}{k^2}\right)} \leq \varepsilon.$$
   Thus, the series $\sum \frac{\sqrt{a_k}}{k}$ converges by the Cauchy criterion 3.6.0.1.

exer:rudin_chap3_8

8. Since $\{b_n\}$ is monotonic and is bounded so from theorem 3.3.0.7, $\{b_n\}$ converges to b.

   WLOG, $\{b_n\}$ is monotonically increasing then $\{b - b_n\}$ is monotonically decreasing that converges to 0.

   Hence, according to theorem 3.12.0.2, $\sum (b - b_n)a_n$ is convergent.

   Since $\sum a_n$ is also convergent so $\sum a_n b_n = \sum b a_n - \sum (b - b_n) a_n$ converges.

exer:rudin_chap3_9

9. (a) We have $\lim_{n\to\infty} \sqrt[n]{n^3} = 1$ from theorem 3.5.0.1 so according to theorem 3.10.0.1, the power series $\sum n^3 z^n$ has radius of convergence 1.

   (b) We find that $\limsup_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!} z^{n+1}}{\frac{2^n}{n!} z^n} \right| = \limsup_{n\to\infty} \left| \frac{2}{n+1} z \right| = 0$ for any z so by the Ratio test 3.9.0.2, the power series $\sum \frac{2^n}{n!} z^n$ has radius of convergence $+\infty$.

   (c) We have $\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^2}} = 2$ according to theorem 3.5.0.1.

   Hence, according to theorem 3.10.0.1, the power series has radius of convergence $R = 1/2$.

   (d) We find that $\limsup_{n\to\infty} \sqrt[n]{\left| \frac{n^3}{3^n} \right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n^3}}{3} = \frac{1}{3}$ according to theorem 3.5.0.1.

   Hence, according to theorem 3.10.0.1, the power series has radius of convergence of 3.

exer:rudin_chap3_10

10. Because $\sqrt[n]{|a_n|} \geq 1$ for infinitely many n so $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$.

    Therefore, according to theorem 3.10.0.1, the power series has radius of convergence of at most 1.

exer:rudin_chap3_11

11. (a) Since $a_n > 0$ so the sequence of partial sums of $\{a_n\}$ is monotonic and divergent so it must be unbounded, i.e. $\sum_{i=1}^n a_i \to \infty$ as $n \to \infty$.

    Assume that the series $\sum \frac{a_n}{1+a_n}$ converges then $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$ which follows $\lim_{n\to\infty} \frac{1}{a_n+1} = 1$. Since $\frac{1}{a_n+1} \neq 0$ for all $n \geq 1$ so

    $$\lim_{n\to\infty} \left(\frac{1}{a_n + 1} \right)^{-1} = \frac{1}{\lim_{n\to\infty} \frac{1}{a_n+1}} = 1.$$

    Hence, $\lim_{n\to\infty} (a_n + 1) = 1$ or $\lim_{n\to\infty} a_n = 0$.

    Fix an $\varepsilon > 0$ then there is N such that $a_n < \varepsilon$ for all $n \geq N$. Hence, for all $n \geq N$ we have $\frac{a_n}{1+a_n} \geq \frac{a_n}{1+\varepsilon}$. Since $\sum_{i=1}^n a_i \to \infty$ as $n \to \infty$ so we obtain that $\sum \frac{a_n}{1+a_n}$ is divergent, a contradiction.

    (b) Since $\{s_n\}$ is monotonically increasing so

    $$\frac{a_{N+1}}{s_{N+1}} + \ldots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + \ldots + \frac{a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Since $\{s_n\}$ is unbounded and is monotonically increasing so as $k \to \infty$ then $\frac{s_N}{s_{N+k}} \to 0$. Hence, for any N, we can choose a sufficiently large k so $\frac{s_N}{s_{N+k}} < \frac{1}{2}$ which implies $\sum_{i=1}^{k} \frac{a_{N+i}}{s_{N+i}} > \frac{1}{2}$. This follows $\sum \frac{a_n}{s_n}$ diverges by the Cauchy criterion 3.6.0.1.

(c) Since $\{s_n\}$ is monotonically increasing so $s_n^2 \ge s_{n-1}s_n$ so

$$\frac{a_n}{s_n^2} \le \frac{a_n}{s_{n-1}s_n} = \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$

This follows

$$\sum_{i=m}^{n} \frac{a_n}{s_n^2} \le \frac{1}{s_{m-1}} - \frac{1}{s_n}.$$

Since $\lim_{n\to\infty} s_n = +\infty$ so $\lim_{n\to\infty} \frac{1}{s_n} = 0$, which means $\{1/s_n\}$ is a Cauchy sequence and therefore $\sum \frac{a_n}{a_n^2}$ converges according to the Cauchy criterion 3.6.0.1.

(d) Since $a_n > 0$ so $0 < \frac{a_n}{1+n^2a_n} < \frac{1}{n^2}$ and since $\sum \frac{1}{n^2}$ converges according to theorem 3.7.0.2 so $\sum \frac{a_n}{1+n^2a_n}$ converges according to comparison test 3.6.0.2.

We can't really say anything about $\sum \frac{a_n}{1+na_n}$. For example, if $a_n = 1$ then $\sum \frac{a_n}{1+na_n}$ diverges. However, if $a_{2^k} = 1$ and $a_n = 1/n^2$ for $n \ne 2^k$ then $\sum a_n$ diverges but $\sum \frac{a_n}{1+na_n}$ converges.

exer:rudin_chap3_12 12. (a) By adding extra zeros to the sequence $\{a_n\}_{n\ge m}$ to get $\{b_n\}$ where $b_i = 0$ for $i < m$ and $b_i = a_i$ for $i \ge m$. Therefore, as $\sum_{n=m}^{\infty} a_n$ converges so for $m < n$:

$$r_m - r_n = \sum_{i=m}^{\infty} a_i - \sum_{i=n}^{\infty} a_i = \sum_{i=0}^{\infty} b_i - \sum_{i=0}^{\infty} c_i = \sum_{i=m}^{n-1} a_i > 0.$$

Hence, $\{r_n\}$ is a monotonically decreasing sequence, which leads to

$$\frac{a_m}{r_m} + \ldots + \frac{a_n}{r_n} > \frac{a_m + \ldots + a_n}{r_m} = 1 - \frac{r_n}{r_m}.$$

Since $\sum a_n$ converges so $r_n \to 0$ as $n \to 0$. Hence, fix a N, we can choose sufficient large k so $\frac{r_{N+k}}{r_N} < \frac{1}{2}$, which implies $\sum_{i=0}^{k} \frac{a_{N+i}}{r_{N+i}} > \frac{1}{2}$. Thus, according to the Cauchy criterion 3.6.0.1, $\sum \frac{a_n}{s_n}$ diverges.

(b) Write $a_n = r_n - r_{n+1}$ then the inequality is equivalent to $(\sqrt{r_n} - \sqrt{r_{n+1}})^2 > 0$, which is true. This follows $\sum_{k=m}^{n} \frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_m} - \sqrt{r_{n+1}})$. Since $r_n \to 0$ as $n \to \infty$ so by the Cauchy criterion 3.6.0.1, $\sum \frac{a_n}{\sqrt{s_n}}$ converges.

exer:rudin_chap3_13 13. Consider two absolutely convergent series $\sum a_n$ and $\sum b_n$. Let $\sum c_n$ be the Cauchy product between these two series. Let $\sum |a_n| = a$ and $\sum |b_n| = b$ then from theorem 3.13.0.1, if $d_n = \sum_{k=0}^{n} |a_k| \cdot |b_{n-k}|$ then $\sum_{n=0}^{\infty} d_n$ converges. We also have $0 \le |c_n| \le d_n$ so according to the comparison test 3.6.0.2, $\sum |c_n|$ converges.

Thus, the Cauchy product of two absolutely convergent series converges absolutely.

exer:rudin_chap3_14 14. (a) Since $\lim_{n\to\infty} s_n = s$ so for any $\varepsilon > 0$, there exists N such that $|s_n - s| < \varepsilon$ for all $n \ge N$. This follows

$$\begin{aligned} |\sigma_n - s| &\le \frac{\left|\sum_{i=0}^{N} s_i - (N+1)s\right|}{n+1} + \frac{1}{n+1}\sum_{i=N+1}^{n} |s_i - s|, \\ &\le \frac{A}{n+1} + \frac{(N+1)|s|}{n+1} + \frac{(n-N)\varepsilon}{n+1}, \\ &< \frac{A + (N+1)|s| - N\varepsilon}{n+1} + \varepsilon. \end{aligned}$$

We follow $\limsup_{n\to\infty} |\sigma(n) - s| < \varepsilon$. Since this is true for any $\varepsilon > 0$ so we conclude $\lim_{n\to\infty} |\sigma_n - s| = 0$ so $\lim \sigma_n = s$.

(b) $s_n = (-1)^n$.

(c) Let $s_n = \begin{cases} 1/n & n \ne 0, e^k \\ k & n = e^k, \\ 0 & n = 0. \end{cases}$ then $s_n > 0$ for all n and $\limsup s_n = \infty$. With this, we have

$$\sigma_n \le \frac{1}{n+1}\sum_{i=1}^{n} \frac{1}{i} + \frac{1}{n+1}\sum_{i=1}^{\ln(n)} i \le \frac{1}{n+1}\sum_{i=1}^{n}\frac{1}{i} + \frac{\ln(n)^2}{n}.$$

If we take a sequence $t_n = 1/n$ for $n \ne 0$ then from (a), we find $\lim \frac{1}{n+1}\sum_{i=1}^{n} \frac{1}{i} = 0$. We also have $\lim \frac{\ln(n)^2}{e^{\ln(n)}} = 0$ from theorem theo:sequence_special 3.5.0.1 so $\lim \left(\frac{1}{n+1}\sum_{i=1}^{n}\frac{1}{i} + \frac{\ln(n)^2}{n}\right) = 0$. This concludes $\lim \sigma_n = 0$.

(d) Show that $\sum_{k=1}^{n} ka_k = \sum_{k=1}^{n} k(s_k - s_{k-1}) = (n+1)s_n - \sum_{i=0}^{n} s_i$.

Consider the complex sequence $\{na_n\}$ then since $\lim(na_n) = 0$ so from (a), we have $\lim \frac{1}{n+1}\sum_{k=1}^{n} ka_k = 0$. From the identity, we find $\lim(s_n - \sigma_n) = 0$. Since $\{\sigma_n\}$ converges so $\{s_n\}$ converges.

(e) We have

$$|s_n - s_i| \le \sum_{k=i+1}^{n} |s_k - s_{k-1}| \le M \sum_{k=i+1}^{n} \frac{1}{k} \le \frac{(n-i)M}{i+1} \le \frac{(n-m-1)M}{m+2}.$$

exer:rudin_chap3_15 15. For $\mathbf{R}^k$, we are not sure about convergence of Cauchy product of two series theo:series_product_cauchy 3.13.0.1 because we don't know what the product should be.

exer:rudin_chap3_16 16. (a) Induction shows that $x_n > \sqrt{a}$ and $\{x_n\}$ decreases monotonically. Thus, $\{x_n\}$ is bounded and is monotonically decreasing so from theorem theo:sequence_conver_monotonic 3.3.0.7, $\{x_n\}$ converges to some $x \ne 0$. This follows $\lim \frac{1}{x_n} = \frac{1}{x}$. We find

$$x = \lim x_{n+1} = \lim \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right).$$

This follows $x^2 = \alpha$ so $x = \alpha$ since $x_n > \sqrt{\alpha}$. Thus, $\lim x_n = \alpha$.

(b) We have

$$\varepsilon_n^2 = (x_n - \sqrt{\alpha})^2 = 2x_n \left[\frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \alpha\right] = 2x_n\varepsilon_{n+1}.$$

Since $x_n > \sqrt{\alpha}$ so $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$. Hence,

$$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{\beta} < \frac{1}{\beta}\left(\frac{\varepsilon_{n-1}^2}{\beta}\right)^2 = \frac{\varepsilon_{n-1}^4}{\beta^3} < \ldots < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}.$$

(c) Yep.

exer:rudin_chap3_17 17. (a) and (b): We will prove by induction on n that $x_{2n-1} > x_{2n+1} > \sqrt{\alpha}$ and $x_{2n} < x_{2n+2} < \sqrt{\alpha}$:

Basic cases: $x_1 > \sqrt{\alpha}$ so $x_2 < \sqrt{\alpha}$ which follows $x_3 > x_1 > \sqrt{\alpha}$ and $x_2 < x_4 < \sqrt{\alpha}$.

If $x_{2n-1} > \sqrt{\alpha}$ then

$$x_{2n} = 1 + \frac{\alpha - 1}{1 + x_{2n-1}} < \alpha.$$

Similarly, if $x_{2n} < \alpha$ so $x_{2n+1} > \alpha$ and $x_{2n+2} < \alpha$.

Now, we have

$$\begin{aligned}
x_{n+1} &= x_n + \frac{\alpha - x_n^2}{1 + x_n}, \\
&= x_{n-1} + \frac{\alpha - x_{n-1}^2}{1 + x_{n-1}} + \frac{\alpha - x_n^2}{1 + x_n}, \\
&= x_{n-1} + \frac{\alpha - x_{n-1}^2}{1 + x_{n-1}} + \frac{(x_{n-1}^2 - \alpha)(\alpha - 1)}{(1 + x_{n-1})^2 + (\alpha + x_{n-1})^2}, \\
&= x_{n-1} + (\alpha - x_{n-1}^2)\frac{2x_{n-1}^2 + (\alpha + 3)x_{n-1} + \alpha^2 - \alpha + 2}{(1 + x_{n-1})^2 + (\alpha + x_{n-1})^2}.
\end{aligned}$$

Hence, since $x_{2n-1} > \alpha$ so $x_{2n+1} < x_{2n-1}$ and since $x_{2n} < \alpha$ so $x_{2n+2} > x_{2n}$.

(c) Since $\{x_{2n}\}$ is a monotonically increasing and is bounded so from theorem 3.3.0.7, $\{x_{2n}\}$ converges to x. Similar to part (a), using the identity about x_{n+1} and x_{n-1}, we solve for x to find $x = \sqrt{\alpha}$. Similarly, $x_{2n+1} \to \sqrt{\alpha}$ as $n \to \infty$. Hence, for any ε, there exists N such that $|x_n - \sqrt{\alpha}| < \varepsilon$ for all $n \geq N$. This implies $\lim x_n = \sqrt{\alpha}$.

(d) It not as fast as the one in exercise 16. If $\varepsilon_n = x_n - \sqrt{\alpha}$ then $\varepsilon_{n+1} = \frac{\alpha + x_n}{1 + x_n} - \sqrt{\alpha} = -\frac{\varepsilon_n(\sqrt{\alpha} - 1)}{1 + x_n}$. We know $\{x_n\}$ is bounded so say $|x_n| \leq M$ for all n then $|\varepsilon_{n+1}| > \varepsilon_n \frac{\sqrt{\alpha} - 1}{1 + M} > |\varepsilon_1| \left(\frac{\sqrt{\alpha} - 1}{1 + M}\right)^n$.

exer:rudin_chap3_18 18. If $x_1 = \sqrt[p]{\alpha}$ then the sequence is constant and is equal to $\sqrt[p]{\alpha}$. If $x_1 > \sqrt[p]{\alpha}$ then AM-GM shows that $x_n > \sqrt[p]{\alpha}$ and then by induction we find $\{x_n\}$ is monotonically decreasing. Thus $\{x_n\}$ converges to x. Solve for x we find $x = \sqrt[p]{\alpha}$.

exer:rudin_chap3_19 19. Let P be the Cantor set and S be the set of all $x(a)$. We know that P is a perfect set from example 1.4.0.1.

First, we show that for all n and all sequence $\{\alpha_n\}$, the sum $\sum_{i=1}^n \frac{\alpha_i}{3^i}$ does not belong to the interval $\left(\frac{3m+1}{3^k}, \frac{3m+2}{3^k}\right)$ for any m, k.

Indeed, if there is such n and $\{\alpha_n\}$ so $\frac{3m+1}{3^k} < \sum_{i=1}^n \frac{\alpha_i}{3^i} < \frac{3m+2}{3^k}$. Since $3^n \sum_{i=1}^n \frac{\alpha_i}{3^i}$ is an integer so there must exists an integer between $3^{n-k}(3m+1)$ and $3^{n-k}(3m+2)$. Hence, we obtain $n > k$. Multiplying both sides by 3^n we get

$$3^{n-k+1}m + 3^{n-k} < \sum_{i=1}^n \alpha_i 3^{n-i} < 3^{n-k+1}m + 2 \cdot 3^{n-k}.$$

Note that any integer between $3^{n-k+1}m + 3^{n-k}$ and $3^{n-k+1}m + 2 \cdot 3^{n-k}$ written in base 3 must have 1 in 3^{n-k} but $\sum_{i=1}^n \alpha_i 3^{n-i}$ in base 3 only contains 2 and 0, a contradiction.

Thus, for any n and any $\{\alpha_n\}$, $\sum_{i=1}^n \frac{\alpha_i}{3^i}$ does not belong to interval $\left(\frac{3m+1}{3^k}, \frac{3m+2}{3^k}\right)$ for any m, k. This follows $\sum_{i=1}^n \frac{\alpha_i}{3^i} \in P$ for any n. Since P is perfect and since that $\sum_{i=1}^n \frac{\alpha_i}{3^i} \to x(a)$ as $n \to \infty$ so $x(a) \in P$ for any $a = \{\alpha_n\}$. Hence, $S \subseteq P$.

To show $P \subseteq S$, pick an arbitrary $p \in P$ and we will construct a sequence $a = \{\alpha_n\}$ of $\{0, 2\}$ such that $x(a) = p$. From the construction of the Cantor set, we can obtain a sequence of intervals I_n so $I_1 \supset I_2 \supset I_3 \supset \cdots$ and $p \in I_n$ for all n. For example, if $p = \frac{8}{9}$ then $I_1 = \left[\frac{2}{3}, 1\right], I_2 = \left[\frac{8}{9}, 1\right], I_3 = \left[\frac{24}{27}, \frac{25}{27}\right], \ldots$ With this, $\bigcap_{i=1}^\infty I_i = \{p\}$ according to proposition 1.3.2.3.

Let x_n be the left endpoint of I_n. In fact, we can show by induction that $x_{n+1} - x_n = \frac{a}{3^{n+1}}$ for $a \in \{0, 2\}$. We also know that $x_n \to p$ as $n \to \infty$ so we can construct a sequence $a = \{\alpha_n\}$ of $\{0, 2\}$ based on $\{x_n\}$ so that $x(a) = p$.

Thus, $P = S$.

exer:rudin_chap3_20 20. Since $\{p_{n_i}\}$ converges to p so for any $\varepsilon > 0$, there is an integer N such that $d(p, p_{n_i}) < \varepsilon$ for any $i \ge N$. Since $\{p_n\}$ is a Cauchy sequence so there is an integer M such that $d(p_n, p_m) < \varepsilon$ for all $n, m \ge M$.

Pick $K = \max(n_N, M, N)$ then for any $n \ge K$, we have $d(p_n, p) \le d(p_n, p_{n_K}) + d(p_{n_K}, p) < 2\varepsilon$. Since this is true for any $\varepsilon > 0$ so $\{p_n\}$ converges to p.

exer:rudin_chap3_21 21. Pick an arbitrary sequence $\{p_n\}$ such that $p_n \in E_n$ for all n. Hence, since $\lim_{n\to\infty} \operatorname{diam} E_n = 0$ so $\{p_n\}$ is a Cauchy sequence. Since X is a complete metric space so $\{p_n\}$ converges to p. Since $p_n \in E_n$ for all n so p is a limit point of E_n for all n. On the other hand, since E_n is closed for all n so $p \in E_n$ for all n. Hence, $p \in E = \bigcap E_n$. Since $\operatorname{diam} E_n \to 0$ as $n \to 0$ and that $E \subset E_n$, we find that E contains exactly one point.

exer:rudin_chap3_22 22. (Baire's theorem) Pick a point $x_1 \in G_1$. Since G_1 is open so there exists neighborhood E_1 of x_1 so $x_1 \in E_1$ and $\overline{E_1} \subset G_1$. Suppose we've chosen nonempty $E_1, \ldots, E_n$ so $\overline{E_1} \supset \cdots \supset \overline{E_n}$. and $\overline{E_i} \subset G_i$ and $\text{diam} E_i < 1/i$ for all $i \le n$. Pick a point $x_n \in E_n$ then since G_{n+1} is dense in X, either $x_n \in G_{n+1}$ or x_n is a limit point of G_{n+1}. Either case will show that there exists a neighborhood E_{n+1} so $\overline{E_{n+1}} \subset G_{n+1}$ and $\overline{E_{n+1}} \subseteq E_n$ and $\text{diam} E_{n+1} < 1/(n+1)$.

Thus, we obtain a sequence of closed sets $\{\overline{E_n}\}$ such that $\lim_{n\to\infty} E_n = 0$. Since X is a complete metric space, from excercise 21, we follow $\bigcap_1^\infty \overline{E_n}$ consists of exactly one point p. Since $\overline{E_n} \subseteq G_n$ so $p \in \bigcap_1^\infty G_n$.

exer:rudin_chap3_23 23. The hint shows that $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbf{R}^1$, which implies $\{d(p_n, q_n)\}$ converges according to theorem 3.3.0.4.

exer:rudin_chap3_24 24. (a) Reflecxive and Symmetric property obviously hold. We will prove transitivity. Indeed, for three Cauchy sequence $\{p_n\}, \{q_n\}, \{r_n\}$ so

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, r_n) = 0$$

Since $d(p_n, r_n) \le d(p_n, q_n) + d(q_n, r_n)$ which follows $\lim_{n\to\infty} d(p_n, r_n) = 0$.

(b) Not hard. $\Delta(P, Q) > 0$ when $P \neq Q$ and is 0 when $P = Q$. $\Delta(P, Q) = \Delta(Q, P)$ and

$$\Delta(P, Q) + \Delta(Q, R) = \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) \ge \lim_{n\to\infty} d(p_n, r_n) = \Delta(P, R).$$

Thus, Δ is the distance function in X^*.

(c) Consider a Cauchy sequence $\{P_n\}$ in X^*.

From (b), we know that to find the convergence for $\{P_n\}$ we only need to find **one** Cauchy sequence $\{q_n\}$. To construct this sequence, as long as we put enough points from sequences in P_n into

For each P_n, pick an arbitrary sequence $\{p_{n,m}\}_{m=1}^\infty$ from P_n. Now consider the sequence $\{q_n\}$ whose elements are from the union $\bigcup_{n=1}^\infty \{p_{n,m}\}_{m=1}^\infty$. For convenience, let say $q_n = p_{n,n}$ for all n. Consider the equivalence class Q of this sequence. We will show that $\{P_n\}$ converges to Q.

First, in order for the existence of Q, we need to show that $\{q_n\}$ is a Cauchy sequence. Indeed, since $\{P_n\}$ is a Cauchy sequence, for any $\varepsilon > 0$, there exists integer N such that $\Delta(P_m, P_k) < \varepsilon$ for all $m, k \ge N$. According to (b), in other words, $\lim_{n\to\infty} d(p_{m,n}, p_{k,n}) < \varepsilon$ for all $m, k \ge N$. This follows there exists integer $M > N$ such that $d(p_{m,n}, p_{k,n}) < 2\varepsilon$ for all $n, m, k \ge M$. Since $\{p_{k,n}\}_{n=1}^\infty$ is Cauchy so there exists $L > M$ such that $d(p_{k,n}, p_{k,t}) < \varepsilon$ for all $n, t \ge L$. Combining with the previous argument, we find

$$d(p_{m,n}, p_{k,t}) < d(p_{m,n}, p_{k,n}) + d(p_{k,n}, p_{k,t}) < 3\varepsilon$$

for all $m, n, k, t \ge L$. Thus $d(q_a, q_b) = d(p_{a,a}, p_{b,b}) < 3\varepsilon$ for all $a, b \ge L$. This implies that $\{q_n\}$ is a Cauchy sequence.

Next, we will show that for any $\varepsilon > 0$, there exists N such that $\Delta(P_m, Q) = \lim_{n\to\infty} d(p_{m,n}, q_n) < \varepsilon$ for all $m \geq N$. This is essentially the same as the above argument. Thus, we find $\{P_m\}$ converges to Q.

Thus, since any Cauchy sequence converges so X^* is a complete metric space.

(d) $\Delta(P_p, P_q) = d(p, q)$ follows from (b).

(e) For $P \in X^*$. Consider a Cauchy sequence $\{p_n\}$ of P. We will show that $\{\varphi(p_n)\}$ converges to P. Indeed, note that $\varphi(p_n)$ contains a sequence whose all terms are p_n. This follows $\Delta(\varphi(p_n), P) = \lim_{m\to\infty}(p_n, p_m)$. And since $\{p_n\}$ is Cauchy, we conclude that $\{\varphi(p_n)\}$ converges to P.

X^* is the **completion** of X.

exer:rudin_chap3_25 25. The completion X^* is isomorphic to set of real numbers. Each $P \in X^*$ uniquely identities a point $p \in \mathbf{R}$ since P contains Cauchy sequences of the rationals that must all converge to some $p \in \mathbf{R}$.

Chapter 4

Differentiation

Continue attention to real-valued function defined on intervals or segments.

4.1 The derivative of a real function

Definition 4.1.0.1 (Derivative)**.** Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$, define $f'(x) = \lim_{t \to x} \frac{f(x)-f(t)}{x-t}$ provided that this limit exists.

See chapter 9, Spivak [2] for some great discussions about the derivative.

Theorem 4.1.0.2 (Differentiation $\implies$ Continuity)

Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$ then f is continuous at x.

Note that the reverse is not true. In other words, continuity does not imply differentiation. See example

Theorem 4.1.0.3 (Chain rule) Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at point $f(x)$. If $h(t) = g(f(t))$ for $a \le t \le b$ then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

Proof. See the proof from [1]. $\square$

See Chapter 10, Spivak [2] for some great discussions about the Chain Rule.

Question 4.1.0.4. Is the following one-line proof for the Chain Rule 4.1.0.3 valid?

$$\lim_{t\to x}\frac{g(f(t))-g(f(x))}{t-x}=\lim_{t\to x}\frac{g(f(t))-g(f(x))}{f(t)-f(x)}\cdot\lim_{t\to x}\frac{f(t)-f(x)}{t-x}=g'(f(x))f'(x).$$

If it is false, how would you fix it?

Remark 4.1.0.5 (Answer question 4.1.0.4)**.** Let see, we know that $f'(x)$ exists so we only need to check for $\lim_{t\to x}\frac{g(f(t))-g(f(x))}{f(t)-f(x)}$. We know that f is continuous at t so as $t\to x$ then $f(t)\to f(x)$. However, when $f(t)=f(x)$ for infinitely many $t\to x$ then $\frac{g(f(t))-g(f(x))}{f(t)-f(x)}$ is undefined for infinitely many $t\to x$, which means the limit does not exist. Hence, the one-line proof is not correct, or at least it is not complete.

One way to complete this proof is to fill the hole where $\frac{g(f(t))-g(f(x))}{f(t)-f(x)}$ is undefined by defining a new function

$$\phi(t)=\begin{cases}\frac{g(f(t))-g(f(x))}{f(t)-f(x)}, & \text{if } f(t)\neq f(x),\\ g'(f(x)) & \text{if } f(t)=f(x).\end{cases}$$

With this, we can be confident to write

$$\frac{g(f(t))-g(f(x))}{t-x}=\phi(t)\cdot\frac{f(t)-f(x)}{t-x}$$

which holds for **any** $t\neq x, t\in[a,b]$. Hence, it suffices to show $\phi(t)$ is continuous at x, or $\lim_{t\to x}\phi(t)=g'(f(x))$.

Example 4.1.0.6 (Continuity $\not\Rightarrow$ Diffferentiation) If you want to construct a function f that is not differentiable at 0 but is continuous at 0. According to the definition, we want $\frac{f(x)-f(0)}{x}$ to oscillate as $x\to 0$.

For convenient, let's only consider when $x=1/n$ and we can just connect two points $f\left(\frac{1}{n+1}\right)$ and $f\left(\frac{1}{n}\right)$ together. By doing this, we can guarantee the continuity of f at 0.

For convenient, let's say $f(0)=0$. Hence, we want $f(1/n)\to 0$ as $n\to\infty$ and we want $nf(1/n)$ to osscilate as $n\to\infty$.

There are two simple functions we know of that osillates around a point: $g(x)=(-1)^n$ or $g(x)=\sin(1/n)$. This gives us two constructions: $f(1/n)=(-1)^n/n$ and $f(1/n)=\sin(n)/n$. From here, we are done.

For the second construction, $f(1/n)=\sin(n)/n$, we find that if the result still holds if we use this definition for all $x\neq 0$, i.e. $f(x)=x\sin(1/x)$ for $x\neq 0$ and $f(0)=0$. Furthermore, using the Chain Rule 4.1.0.3, one can check that f is differentiable at any $x\neq 0$.

Example 4.1.0.7

There is function continuous on the whole line without being differentiable at any point!

What if we want to find a function f so $f^{(n)}$ exists but $f^{(n)}$ is not continuous.

e:differential_n_times **Example 4.1.0.8** (Exactly-n-times-differentiable function) From Spivak [2], exercise 31,32, chapter 10. calculus_spivak

Consider $f_k(x) = x^k \sin(1/x)$ if $x \neq 0$ and $f_k(0) = 0$. If $k \in \{2n, 2n+1\}$ then $f_k^{(0)}, \ldots, f_k^{(n)}(0)$ exists but $f_k^{(n)}$ is not continuous at 0.

4.2 Mean Value Theorem

Theorem 4.2.0.1

Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

rential_generalize_mvt **Theorem 4.2.0.2** (Generalized Mean value theorem) If f, g are continuous real functions on $[a, b]$, which are differentiable in (a, b), then there exists a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

theo:differential_mvt **Theorem 4.2.0.3** (Mean Value Theorem) If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which $f(b) - f(a) = (b - a)f'(x)$.

4.3 The continuity of Derivative

Not every function is a derivative. In particular, derivatives which exist at every point of an interval have one important property in common with functions which are continuous on an interval: Intermediate values are assumed (see theorem theo:continuity_connect_ivt 2.4.0.2). However, this does not imply that the derivative has to always be continuous.

Theorem 4.3.0.1

Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

Corollary 4.3.0.2

If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.

Indeed, if f' has simple discontinuity on $x \in (a, b)$. If $f'(x) \neq f'(x+)$ then since f' satisfies the intermediate value theorem, for any $t > x$ sufficient close to x, there exists $y \in (x, t)$ so $f(y)$ is arbitrarily close to $f'(x)$. This contradicts the fact that $f'(x+) \neq f'(x)$.

A function f is said to be **continuously differentiable** if its derivative f' is continuous.

4.4 L' Hospital's Rule

:differential_hospital **Theorem 4.4.0.1** (L' Hospital's Rule) Suppose f and g are real and differentiable in (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \le a < b \le +\infty$. Suppose

$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a. \tag{4.1}$$

eq:differential_hosp

If

$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a, \tag{4.2}$$

eq:differential_hosp

or if

$$g(x) \to +\infty \text{ as } x \to a, \tag{4.3}$$

eq:differential_hosp

then

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a. \tag{4.4}$$

eq:differential_hosp

Sketch + Motivation. We would need something to connect $\frac{f'(x)}{g'(x)}$ with $\frac{f(x)}{g(x)}$, and we find the Generalized Mean Value Theorem 4.2.0.2. Using this, for any $a < x < y < b$, there exists $t \in (x,y)$ such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)}.$$

Note if $x, y \to a$ then $t \to a$ so $\frac{f'(t)}{g'(t)} \to A$. Hence, all we need to do is to show that $\frac{f(x)-f(y)}{g(x)-g(y)}$ is sufficiently close to some $\frac{f(z)}{g(z)}$.

If (4.2) holds, then by fixing y and let $x \to a$, note that $g(x)-g(y) \to g(y)$ so we can make $\frac{f(x)-f(y)}{g(x)-g(y)}$ sufficiently close to $\frac{f(y)}{g(y)}$.

If (4.3) holds then as $x \to a$, $\frac{f(x)-f(y)}{g(x)-g(y)}$ is very close to $\frac{f(x)}{g(x)}$. □

Formalize your argument. A trick is to take limit to avoid lengthy inequality. Say we've chosen y so that $\frac{f'(t)}{g'(t)}$ is in the neighborhood ε of A. This follows $\frac{f(x)-f(y)}{g(x)-g(y)}$ is in the neighborhood ε of A for any $x \in (a,y)$. We then consider two cases as previously mentioned.

Note that A can be $-\infty$ or $+\infty$ so we need to consider seperate cases for A. A quick way is just consider two cases: A is bounded above and A is bounded below. If A is bounded above so it suffices to show that for any $q > 0$ so $A < q$ then there exists c such that $f(x)/g(x) < q$ for all $a < x < c$. □

4.5 Derivatives of higher order

If $f_1, \dots, f_k$ are the components of $\mathbf{f}$, then $\mathbf{f}' = (f_1', \dots, f_k')$ and $\mathbf{f}$ is differentiable at x if and only if each of the functions $f_1, \dots, f_k$ is differentiable at x.

4.6 Taylor's theorem

eo:differential_taylor **Theorem 4.6.0.1** (Taylor's theorem) Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}$ exists for any $t \in (a,b)$. Let α, β be distinct points of $[a,b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n. \tag{4.5}$$ eq:differential_tayl

For $n=1$ we have Mean Value Theorem. The theorem shows that for any α, $f(\beta)$ can be approximated by a polynomial of degree $n-1$ and we can estimate the error from (4.5). eq:differential_taylor_theo

Idea. Let $g(x) = f(x) - P(x) - M(x-\alpha)^n$. It suffices to show $n!M = f^{(n)}(x)$ for some x between α and β. It does seem very tempting to differentiate $g(x)$ so that we get a polynomial with smaller degree, i.e reduce to the case $n=1$, and maybe apply Mean Value Theorem. ☐

4.7 Taylor series

A good read from MSE to see difference between Taylor series, power series and Maclaurin series. It also mentions **analytic function**, which I think should be defined here.

4.8 Differentiation of vector-valued functions

_vector_valued_func **Remark 4.8.0.1.** For complex functions f defined on $[a,b]$, if $f(x) = f_1(x) + if_2(x)$ for $a \le x \le b$ where $f_1(x), f_2(x)$ are real then $f'(x) = f_1'(x) + if_2'(x)$ and f is differentiable at x iff both f_1 and f_2 are differentiable at x.

We're talking about function f from $[a,b]$ to $\mathbf{R}^k$.

Example 4.8.0.2 (Mean Value Theorem does not work for vector-valued functions)
See Rudin [1] example 5.17. baby_rudin

Example 4.8.0.3 (L' Hospital's rule does not work for vector-valued functions) See Rudin [1] example 5.18. On segment $(0,1)$, define $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$. We find that $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$ but $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0$ so L' Hospital's rule fails in this case.

In the above example, we find $|g'(x)| \geq \frac{2}{x} - 1$ so $|g'(x)| \to +\infty$ as $x \to 0$. which means $g'(x)$ does not converge to any $B \in \mathbf{C}$ when $x \to 0$. Interestingly, Rudin [1] exercise 10 chapter 5 shows that if $g'(x) \to B$ for some $B \in \mathbf{C}$ then our limit holds.

However, a consequence of Mean Value Theorem still works with vector-valued functions:

Theorem 4.8.0.4 Supopse $\mathbf{f}$ is a continuous mapping of $[a,b]$ into $\mathbf{R}^k$ and $\mathbf{f}$ is differentiable in $[a,b]$. There exists $x \in (a,b)$ such that

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)|\mathbf{f}'(x)|.$$

It's easier to work with real-valued functions because we already have Mean Value Theorem. The natural is to break $\mathbf{f}$ down into components but it doesn't seem to work for in this case. Instead, you can consider $\varphi(t) = (\mathbf{f}(b) - \mathbf{f}(a)) \cdot \mathbf{f}(t)$ as in [1].

4.9 Baby Rudin exercises

exer:rudin_chap5_1

1. Fix x and take $y \to x$, we find $\lim_{y \to x} \left| \frac{f(x)-f(y)}{x-y} \right| = 0$, which follows $f'(x) = 0$. Since this is true for all x so f is a constant.

exer:rudin_chap5_2

2. Since $f'(x) > 0$ for all $x \in (a,b)$ so according to Mean Value Theorem, f is trictly increasing on (a,b). Since g is inverse function of f so $g(f(x)) = x$ for all $x \in (a,b)$. For any $x \in (a,b)$, we have
$$g'(x) = \lim_{t \to x} \frac{g(f(x)) - g(f(t))}{f(x) - f(t)} = \lim_{t \to x} \frac{x-t}{f(x)-f(t)} = \frac{1}{f'(x)}.$$
Thus, g is differentiable.

exer:rudin_chap5_3

3. We have $f'(x) = 1 + \varepsilon g'(x)$. Hence, if $\varepsilon < 1/M$ then $f'(x) > 0$ for all x, which means f is strictly increasing which implies f is one-to-one.

exer:rudin_chap5_4

4. Consider the polynomial function $g(x) = C_0 x + \frac{C_1}{2} x^2 + \ldots + \frac{C_n}{n+1} x^{n+1}$ then $g'(x) = C_0 + C_1 x + \ldots C_n x^n$ and $g(0) = g(1) = 0$. Hence, by the Mean Value Theorem 4.2.0.3, there exists $x \in (0,1)$ such that $g'(x) = \frac{g(1)-g(0)}{1-0} = 0$.

exer:rudin_chap5_5

5. According to Mean Value Theorem 4.2.0.3, for any $x > 0$, there exists $y \in (x, x+1)$ so $g(x) = f'(y)$. As $x \to +\infty$ then $y \to +\infty$, which implies $f'(y) \to +\infty$ so $g(x) \to +\infty$.

exer:rudin_chap5_6

6. Since f is continuous for $x \geq 0$ and is differentiable for $x > 0$ so for any $x > 0$, according to Mean Value Theorem 4.2.0.3, there exists $y \in (0,x)$ such that $f'(y) = \frac{f(x)-f(0)}{x-0} = g(x)$. Since $y < x$ and f' is monotonically increasing so $f'(y) \leq f'(x)$ which implies $g(x) \leq f'(x)$.

   For $x > 0$, we have $g'(x) = \frac{f'(x)x - f(x)}{x^2} \geq 0$ since $g(x) \leq f'(x)$, which implies that g is monotonically increasing.

exer:rudin_chap5_7

7. Since $f'(x), g'(x)$ exist and $g'(x) \neq 0$ so according to the limit laws, we have
$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}.$$

exer:rudin_chap5_8

8. (Continuity of the derivative) Since f' is continuous on a compact space $[a,b]$, f' is uniformly continuous on $[a,b]$ according to theorem 2.3.1.1. This follows for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f'(y) - f'(x)| < \varepsilon$ for any $y, x \in [a,b]$ so $|y - x| < \delta$.

   Fix an x, by Mean Value Theorem 4.2.0.3, for any t so $|t - x| < \delta$ then there exists y between t and x such that $f'(y) = \frac{f(t)-f(x)}{t-x}$. From previous paragraph, we know $|f'(y) - f'(x)| < \varepsilon$ so $\left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| < \varepsilon$ for any $t \in [a,b]$ so $|t - x| < \delta$.

   Yes, the result holds for vector-valued functions. This can be achieved by breaking $\mathbf{f}$ down into components and bring back to the real-valued function.

This could be expxessed by saying that f is **uniformly differentiable** on $[a, b]$ iff f' is continuous on $[a, b]$.

exer:rudin_chap5_9 9. No. $f'(0)$ does not exists as long as f is not continuous at 0. Hence, a counter example is to define $f(x) = 3x$ for $x \neq 0$ and $f(0) = 1$.

exer:rudin_chap5_10 10. For vector-valued function, a common approach is to break it down to real-valued function. Since the problem connects between $f(x), g(x)$ and $f'(x), g'(x)$ so we may think of applying L' Hospital after this. However, a problem is that it's hard to break down $\frac{f(x)}{g(x)} = \frac{f_1(x)+if_2(x)}{g_1(x)+ig_2(x)}$ into real-valued functions as we have a complex number in the denominator. However, we can flip it up by noticing the property $\lim_{x\to a} \frac{1}{k(x)} = \frac{1}{\lim_{x\to a} k(x)}$. We also want to connect $g(x)$ with $g'(x)$ through L' Hospital so this motivates us to let $k(x) = \frac{g(x)}{x}$. By doing this, we can break g into $g = g_1 + ig_2$ and can apply L' Hospital to $g_1(x)/x$ and $g_2(x)/x$.

Hence, we have

$$\frac{f(x)}{g(x)} = \frac{\frac{f_1(x)}{x} + i\frac{f_2(x)}{x}}{\frac{g_1(x)}{x} + i\frac{g_2(x)}{x}}.$$

By applying L' Hospital's Rule theo:differential_hospital 4.4.0.1, we obtain $\lim_{x\to 0} \frac{f_1(x)}{x} = A_1$ and $\lim_{x\to 0} \frac{f_2(x)}{x} = A_2$ where $A_1 + iA_2 = A$.

See example exple:differential_vector_valued_func_hospital 4.8.0.3 for explanation.

exer:rudin_chap5_11 11. By L' Hospital's rule theo:differential_hospital 4.4.0.1, we have

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h\to 0} \frac{f'(x+h) + f'(x-h) - 2f'(x)}{2h}.$$

Since $f''(x)$ exists so

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{f'(x-h) - f'(x)}{h} = 0.$$

Combining the two limits, we get what we want.

We want to show that the limit $\frac{f'(x+h)+f'(x-h)-2f'(x)}{2h}$ may exists even if $f''(x)$ does not. For simplicity, let $g(x) = f'(x)$ and we back with first derivative. For simplicity, say $g(x+h) + g(x-h) - 2g(x) = 2h$ for all $h > 0$. Say $g(x) = 1/2$ and $g(x+h) = g(x-h) = 1 + h$ for all $h > 0$ then obviously the limit $\frac{g(x+h)-g(x)}{h}$ as $h \to 0$ does not exist.

exer:rudin_chap5_12 12. For $x \neq 0$, we find $f'(x) = 3x^2$ so $f''(x) = 6|x|$.

For $x = 0$, we need to use the definition where we want to find the limit of $\frac{|x|^3}{x}$ as $x \to 0$, which is 0. Hence, $f'(x) = 3x^2$ for all x. From that, we can find $f''(x) = 0$. This implues $f''(x) = 6|x|$ for all x. With this, we can find that $f^{(3)}(0)$ does not exist.

exer:rudin_chap5_13 13. We have $f(x) = \begin{cases} x^a \sin(|x|^{-c}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$.

(a) If $a > 0$ then f is obviously continuous at $x \neq 0$. For $x = 0$ then $|f(x)| \le x^a$ so $|f(x)| \to 0$ as $x \to 0$, which proves that f is continuous at 0.

If $a < 0$, consider the sequence $\{x_n\} = \left\{ \left(\frac{1}{2\pi n + \pi/2} \right)^{1/c} \right\}_{n \ge 1}$ then $f(x_n) = \left(\frac{1}{2\pi n + \pi/2} \right)^{a/c}$ so $f(x_n) \to +\infty$ as $n \to \infty$ since $a/c < 0$. Thus, f is not continuous at $x = 0$.

(b) This is essentially implied from (a). $f'(0)$ exists if and only if $\lim_{x \to 0} x^{a-1} \sin(|x|^{-c})$ exists (and is 0) if and only if $a - 1 > 0$ or $a > 1$.

(c) For $x > 0$, according to the Chain rule 4.1.0.3 and the Product rule, we have

$$f'(x) = ax^{a-1} \sin(x^{-c}) - cx^{a-c-1} \cos(x^{-c}) \le ax^{a-1} \sin(x^{-c}). \tag{4.6}$$ eq:exer_rudin_chap

If $a \ge 1 + c$ then for $x \in (0, 1]$, we have $x^{a-1} \le x^c$. On the other hand, $\sin(x^{-c}) \le x^{-c}$ for sufficiently large x, which implies $f'(x) \le a$ for all $x \in (0, 1]$.

If $a < 1 + c$ then we consider the sequence $\{x_n\} = \{(2\pi n)^{-1/c}\}_n$ which will give $\sin(x_n^{-c}) = 0$ and $\cos(x_n^{-c}) = 1$. Hence, $f'(x_n) = -c(2\pi n)^{(c+1-a)/c}$. Since $c + 1 > a$ so $f'(x_n) \to -\infty$ as $n \to \infty$ so $f's$ is unbounded.

Thus, for $x \in [0, 1]$ then f' is bounded iff $a \ge 1 + c$. We proceed similarly for the case $x \in [-1, 0]$.

(d) From (c), we know that if $a < 1 + c$ then with $\{x_n\} = \{(2\pi n)^{-1/c}\}$, we have $f'(x_n) \to -\infty$ as $x_n \to 0$. This follows f' is not continuous at 0.

If $a > c + 1$ then from (a), we know that $cx^{a-c-1} \cos(x^{-c})$ and $ax^{a-1} \sin(x^{-c})$ are continuous so from (4.6), we find f' is continuous for $x \in (0, 1]$. Similarly, f' is continuous for $x \in [-1, 0)$. f' is also continuous at 0 because $f'(0) = 0$ (from (b)) and from (4.6), we find $f'(x) \le ax^{a-1}$, which indicates $f(x+) \le 0$.

If $a = c + 1$ then we can take $\{x_n\} = \{((-1)^n 2\pi n)^{-1/c}\}_n$ which implies $f'(x_n) = (-1)^n c$ as $n \to \infty$ or $x_n \to 0$. This implies f' is not continuous at $x = 0$.

(e) Since $f'(0) = 0$ from (b) so we can use the similar idea as in (b), $f''(0)$ exists iff $\lim_{x \to 0} f'(x)$ exists (and is 0). According to (d), this happens when $a > 2 + c$.

(f), (g) I think we can proceed similarly as (c),(d).

exer:rudin_chap5_14 14. (Convex function and Differentiation) If f is convex then from the inequality in exercise 23 (Rudin [1], chapter 4), we find $f'(x) \le f'(y)$ for any $x < y$. Hence, f' is monotonically increasing.

If f is monotonically increasing then note that for $x < y < z$ then

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(y)}{z - y} \iff f\left(x \cdot \frac{z - y}{z - x} + z \cdot \frac{y - x}{z - x} \right) \le f(x) \cdot \frac{z - y}{z - x} + f(z) \cdot \frac{y - x}{z - x}.$$

The first inequality can be achieved using Mean Value Theorem 4.2.0.3 so with this, we can show that f is convex.

f is convex iff f' is monotonically increasing iff $f''(x) \geq 0$ for all $x \in (a, b)$.

exer:rudin_chap5_15 15. Applying Taylor's theorem 4.6.0.1, we have for any $x \in (a, \infty)$ and any $h > 0$ then there exists $\alpha \in (x, x + 2h)$ such that

$$f'(x) = \frac{1}{2h}(f(x+2h) - f(x)) - f''(\alpha)h.$$

This follows $|f'(x)| \leq \frac{1}{h}M_0 + M_2 h$. Fix h, since M_1 is least upper bound for $|f'(x)|$ so we can take $|f'(x)| \to M_1$ to get $M_1 \leq \frac{1}{h}M_0 + M_2 h$ for any $h > 0$. Take $h = \sqrt{M_0/M_2}$ then we obtain $M_1 \leq 2\sqrt{M_0 M_2}$ or $M_1^2 \leq 4M_0 M_2$.

The inequality is still true for vector-valued functions. It suffices to break it down into components and then Cauchy-Schwarz.

exer:rudin_chap5_16 16. Yes, take $a \to \infty$ in exercise 15. Since $M_0 \to 0$ as $a \to \infty$ so $M_1 \to 0$ as $a \to \infty$.

exer:rudin_chap5_17 17. Using Taylor's theorem 4.6.0.1, there exists $x \in (-1, 0)$ such that

$$f(-1) = f(0) - f'(0) + \frac{f''(0)}{2} - \frac{f^{(3)}(x)}{6} \implies 3f''(0) = f^{(3)}(x).$$

If $f^{(3)}(x) \leq 3$ then $f''(0) \leq 1$. Again, from Taylor's theorem, there exists $y \in (0, 1)$ such that

$$f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f^{(3)}(y)}{6} \implies 6 = 3f''(0) + f^{(3)}(y).$$

Since $f''(0) \leq 1$ so $f^{(3)}(y) \geq 3$.

exer:rudin_chap5_18 18. Induction on n.

exer:rudin_chap5_19 19. (a) Since $\alpha_n < 0 < \beta_n$ so $|\beta_n - \alpha_n| > |\alpha_n|$ so

$$\begin{aligned}\left|\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - \frac{f(\beta_n) - f(0)}{\beta_n}\right| &= \left|\frac{\alpha_n(f(\beta_n) - f(0)) - \beta_n(f(\alpha_n) - f(0))}{(\beta_n - \alpha_n)\beta_n}\right|, \\ &< \left|\frac{f(\beta_n) - f(0)}{\beta_n} - \frac{f(\alpha_n) - f(0)}{\alpha_n}\right|.\end{aligned}$$

Hence, taking $n \to \infty$ shows that

$$\limsup_{n\to\infty} \left|\frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} - \frac{f(\beta_n) - f(0)}{\beta_n}\right| = 0.$$

We also know that $\limsup_{n\to\infty} \left|\frac{f(\beta_n)-f(0)}{\beta_n} - f'(0)\right| = 0$ so $\limsup_{n\to\infty} \left|\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} - f'(0)\right| = 0$. This implies $\lim D_n = f'(0)$.

(b) Similarly to (a), since $\{\beta_n/(\beta_n-\alpha_n)\}$ is bounded so there exists $M>0$ such that $\frac{\beta_n}{\beta_n-\alpha_n}<M$ for all n or $\beta_n-\alpha_n>\frac{\beta_n}{M}$ since $\beta_n>\alpha_n$. Therefore $(\beta_n-\alpha_n)\beta_n>\frac{1}{M}\beta_n\alpha_n$ for all n.

(c) From exercise 8, since f' is continuous on $(-1,1)$, for any $\varepsilon>0$, since $\alpha_n,\beta_n\to 0$ as $n\to\infty$, there exists M such that

$$\left|\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(\alpha_n)\right|<\varepsilon.$$

for all $n>M$. Since f' is continuous at 0 so there exists $K>M$ such that $|f'(0)-f'(\alpha_n)|<\varepsilon$ for all $n>K$. This follows

$$\left|\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n}-f'(0)\right|<2\varepsilon.$$

for all $n>K$. Thus, we obtain that $\lim D_n=f'(0)$.

(d) There is a function f we know of whose derivative is not continuous at 0. See example 4.1.0.8. We choose $f(x)=x^2\sin(1/x)$ for $x\neq 0$ and $f(0)=0$ then $f'(0)=0$ but f' is not continuous at 0.

Next, take $\beta_n=(2n\pi+\pi/2)^{-1}$ and $\alpha_n=(1n\pi)^{-1}$ then $\sin\left(\frac{1}{\beta_n}\right)=1$ and $\sin\left(\frac{1}{\alpha_n}\right)=0$. Hence,

$$D_n=\frac{2\pi n}{2\pi n+\pi/2}\cdot\frac{-2}{\pi}.$$

This follows $\lim D_n=-2/\pi\neq f'(0)$.

exer:rudin_chap5_20 20. (Taylor's theorem for vector-valued function) Let $\mathbf{f}$ be a mapping from $[a,b]$ to $\mathbf{R}^m$ such that $\mathbf{f}^{(n-1)}$ is continuous on $[a,b]$ and $\mathbf{f}^{(n)}(t)$ exists for any $t\in(a,b)$. Let α,β be distinct points of $[a,b]$. Define

$$\mathbf{P}(t)=\sum_{k=0}^{n-1}\frac{\mathbf{f}^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there exists x between α and β such that

$$|\mathbf{f}(\beta)-\mathbf{P}(\beta)|\leq\frac{|\beta-\alpha|^n}{n!}\cdot|\mathbf{f}^{(n)}(x)|.$$

The proof is similar to theorem 4.8.0.4. Indeed, define a real-valued function $\varphi(t)=\mathbf{f}\cdot(\mathbf{f}(\beta)-\mathbf{P}(\beta))$. Applying Taylor's theorem to this function, we find

$$\varphi(\beta)=\sum_{k=0}^{n-1}\frac{\varphi^{(k)}(\alpha)}{k!}(\beta-\alpha)^k+\frac{\varphi^{(n)}(x)}{n!}(\beta-\alpha)^n$$

for some x between α and β. We have

$$\begin{aligned} \textstyle\sum_{k=0}^{n-1} \frac{\varphi^{(k)}(\alpha)}{k!}(t-\alpha)^k &= (\mathbf{f}(\beta)-\mathbf{P}(\beta))\cdot \mathbf{P}(\beta), \\ \tfrac{\varphi^{(n)}(x)}{n!}(\beta-\alpha)^n &= (\mathbf{f}(\beta)-\mathbf{P}(\beta))\cdot \left[\tfrac{\mathbf{f}^{(n)}(x)}{n!}(\beta-\alpha)^n\right]. \end{aligned}$$

This follows

$$|(\mathbf{f}(\beta)-\mathbf{P}(\beta))|^2 = (\mathbf{f}(\beta)-\mathbf{P}(\beta))\cdot \left[\frac{\mathbf{f}^{(n)}(x)}{n!}(\beta-\alpha)^n\right].$$

exer:rudin_chap5_21 21. Given a closed set E on $\mathbf{R}$, we need to find a real-valued function f on $\mathbf{R}$ such that

(a) $f(x)=0$ iff $x \in E$.

(b) f is n times differentiable on $\mathbf{R}$ or f has derivatives of all orders on $\mathbf{R}$.

First, since E is closed so $\mathbf{R}\setminus E$ is open so according to exercise 29 (Rudin [1], chapter 2), $\mathbf{R}\setminus E$ is open so it is the union of at most countable collection of disjoint segments. Say $\mathbf{R}\setminus E = \bigcup (a_n,b_n)$ where $a_n < b_n \le a_{n+1} < b_{n+1}$ for all n. This follows $E=\bigcup[b_n,a_{n+1}]$. Thus, we need $f(x)=0$ for all $x\in[b_n,a_{n+1}]$. This follows $f^{(n)}(x)=0$ for all $x \in E$ and for all $n\in\mathbf{Z}_{\ge 0}$. This obviously holds for $x\in(b_n,a_{n+1})$ so we specifically want $f^{(n)}(a_n) = f^{(n)}(b_n)=0$.

It suffices to construct a nonzero function g on (a_n,b_n) such that $g^{(n)}(a_n)=g^{(n)}(b_n)=0$ (or more precisely, the left-hand nth derivative of g at b_n and the right-hand nth derivative of g at a_n are all 0) and g is n times differentiable on (a_n,b_n).

When we think of a non-zero function that has derivatives of all orders in $\mathbf{R}$, we think of e^x. With this idea, we define $g(x)=\exp\left(-\frac{1}{(x-a_n)(b_n-x)}\right)$ for all $x\in(a_n,b_n)$ and $g(a_n)=g(b_n)=0$. It's obvious that for any $m\in\mathbf{Z}_{\ge 0}$, $g^{(m)}(x)$ exists for $x\in(a_n,b_n)$. It suffices to show $g^{(m)}(a_n)=g^{(m)}(b_n)=0$.

Indeed, we will first prove this by induction on m that $\lim_{x\to a_n+} g^{(m)}(x)=0$ for all $m\in\mathbf{Z}_{\ge 1}$. First, we need to show this for $m=1$. For $x\in(a_n,b_n)$ then according to the Chain Rule, we find

$$\begin{aligned} g'(x) &= [(x-a_n)(b_n-x)]^{-2}(a_n+b_n-2x)\exp\left\{-[(x-a_n)(b_n-x)]^{-1}\right\}, \\ &= (a_n+b_n-2x)\frac{t^2}{e^t}, \qquad (t=[(x-a_n)(b_n-x)]^{-1}). \end{aligned}$$

As $x\to a_n$ so $t\to\infty$ and from theorem 3.5.0.1, we find $\lim_{x\to a}\frac{t^2}{e^t}=\lim_{t\to\infty}\frac{t^2}{e^t}=0$, which follows $\lim_{x\to a} g'(x)=0$. If we differentiate g' using the Chain Rule 4.1.0.3, we find that $g^{(m)}(x)$ is a linear combination of $\frac{t^k}{e^t}(a+b-x)^h$ for some $k,h\in\mathbf{Z}_{\ge 0}$. Hence, it suffices to show $\lim_{x\to a_n}\frac{t^k}{e^t}(a+b-x)^h=0$ for any $k,h\in\mathbf{Z}_{\ge 0}$. This can be proved similarly as when $n=1$ using theorem 3.5.0.1.

We've shown that $\lim_{x\to a_n+} g^{(m)}(x) = 0$ for all $m \in \mathbf{Z}_{\geq 1}$. Therefore, by L' Hospital's rule, we have

$$g^{(m)}(a_n) = \lim_{x\to a_n+} \frac{g^{(m-1)}(x)}{x-a_n} = \lim_{x\to a_n} g^{(m)}(x) = 0.$$

Thus, $g^{(m)}(a_n) = 0$ for all $m \in \mathbf{Z}_{\geq 1}$.

Thus, we can define our function f as:

$$f(x) = \begin{cases} 0 & \text{if } x \in [b_n, a_{n+1}]\ \forall n \in \mathbf{Z}_{\geq 1}, \\ \exp\left(-\frac{1}{(x-a_n)(b_n-x)}\right) & \text{if } x \in (a_n, b_n)\ \forall n \in \mathbf{Z}_{\geq 1}. \end{cases}$$

exer:rudin_chap5_22 22. (a) Let $g(x) = f(x) - x$ then $g'(x) \neq 0$ for all g. If there are at least two fixed points $x_1 < x_2$ then $g(x_1) = g(x_2)$. According to Mean Value Theorem 4.2.0.3, there exists $x \in (x_1, x_2)$ such that $g'(x) = 0$, a contradiction. Thus, there are at most one fixed point.

(b) Because $(1+e^t)^{-1} > 0$ so $f(t) > t$ for all t. Thus, f has no fixed point.

(c) First, we will show that $\{x_n\}$ converges or $\{x_n\}$ is Cauchy. By repeatly applying Mean Value theorem 4.2.0.3, we obtain

$$\begin{aligned} |x_m - x_n| &= |f(x_{m-1}) - f(x_{n-1})|, \\ &= |f'(y_1)(x_{m-1} - x_{n-1})|, \\ &= |f'(y_1)(f(x_{m-2} - f(x_{n-2})|, \\ &\cdots \\ &= |f'(y_1)f'(y_2)\cdots f'(y_{n-1})(x_{m-n+1} - x_1)|, \\ &\leq A^{n-1}|x_{m-n+1} - x_1|, \\ &\leq A^{n-1}\sum_{i=1}^{m-n} |x_{i+1} - x_i|, \\ &\leq A^{n-1}\sum_{i=1}^{m-n} A^{i-1}|x_2 - x_1|, \\ &= |x_2 - x_1| \sum_{i=n-1}^{m-n-1} A^i. \end{aligned}$$

Since $A < 1$ so the series $\sum_{i=1}^{\infty} A^i$ converges, which means the sequence of partial sums of this series is Cauchy, which implies that there exists N such that $\sum_{i=n-1}^{m-n-1} A^i < \varepsilon$ for all $m \geq n > N$. Therefore, $|x_m - x_n| < \varepsilon|x_2 - x_1|$ for all $m, n > N$. Since this is true for any ε, we find $\{x_n\}$ is a Cauchy sequence. Therefore, $\{x_n\}$ converges to x. We find $|f(x) - f(x_n)| < \varepsilon$ or all $n > M$, but since $|f(x_n) - x| = |x_{n+1} - x| < \varepsilon$ so $|f(x) - x| < 2\varepsilon$. Since this is true for any $\varepsilon > 0$, we conclude $f(x) = x$.

(d) From x_1 find x_2 and test if x_2 is a fixed point or not. If not, move on to x_3 and keep going like that.

23. Note that

$$f(x) = \frac{1}{3}(x-\alpha)(x-\beta)(x-\gamma) + x, \qquad f'(x) = x^2. \tag{4.7}$$

(a) If $x_1 < \alpha$ then we find $x_{n+1} < x_n$ for all n. We also have $f'(x) > f'(y)$ for all $|x| > |y|$. Using the same approach as in exercise 22, i.e. Mean Value Theorem, we find

$$\begin{aligned} |x_m - x_n| &= |f'(y_1)\cdots f'(y_{n-1})|x_{m-n+1} - x_1|, \ (y_i < x_1 < 0) \\ &\geq |f'(x_1)|^{n-1}|x_2 - x_1|. \end{aligned}$$

Since $|f'(x_1)| = x_1^2 > 1$ so for any $M > 0$, there exists $N > 0$ such that $|x_m - x_n| > M$ for all $m, n \geq N$. Since $x_m < x_n < 0$ for $m > n$ so we find $x_n \to -\infty$ as $n \to \infty$.

(b) First, we will show that the sequence converges.

From (4.7), we find that if $\alpha < x_1 < \gamma$ then $\alpha < x_n < \gamma$ for all n. This follows that if $x_n < \beta$ then $x_{n+1} > x_n$ and if $x_n > \beta$ then $x_{n+1} < x_n$.

Hence, if there exists N such that $x_n > \beta$ (or $x_n < \beta$) for all $n > N$ then $\{x_n\}_{n>N}$ is a monotonic bounded sequence so it converges to x. From (4.7), we find $x \in \{\alpha, \beta, \gamma\}$. On the other hand, from the observation of the sequence $\{x_n\}$, we find $x = \beta$ or $\{x_n\}$ converges to β.

If there does not exist such N then there exists infinitely many n such that $x_n < \beta, x_{n+1} > \beta$ or $x_n > \beta, x_{n+1} < \beta$. If $x_n < \beta$ and $x_{n+1} > \beta$ then

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{3}|(x_n-\alpha)(x_n-\beta)(x_n-\gamma)|, \\ &\leq \frac{1}{3}|x_n - \beta| \cdot \frac{(\alpha-\gamma)^2}{4}, \\ &\leq \frac{4}{3}(\beta - x_n), \\ x_{n+1} - \beta &\leq \frac{1}{3}(\beta - x_n). \end{aligned}$$

Similarly with the other case $x_n > \beta, x_{n+1} < \beta$. We obtain $|x_{n+1} - \beta| \leq \frac{1}{3}|x_n - \beta|$ for all such n. This follows $x_n \to \beta$ as $n \to \infty$.

(c) If $\gamma < x_1$ then we find $x_{n+1} > x_n > 1$ for all n. We also have $f'(x) > f'(y)$ if $x > y > 1$. Therefore,

$$\begin{aligned} |x_m - x_n| &= |f'(y_1)\cdots f'(y_{n-1})|x_{m-n+1} - x_1|, \ (y_i > x_1 > 1) \\ &\geq |f'(x_1)|^{n-1}|x_2 - x_1|. \end{aligned}$$

Similar to (a), we conclude $x_n \to +\infty$ as $n \to \infty$.

exer:rudin_chap5_24 24. We have $f'(x) = \frac{x^2-\alpha}{2x^2}$ and $g'(x) = \frac{1-\alpha}{(1+x)^2}$. Note that as $x \to \sqrt{\alpha}$ then $f'(x) \to 0$ while $g'(x) = \frac{1-\alpha}{(1+\alpha)^2}$, which explains somehow why convergence with f is so much more rapid than it is with g.

If $0 < \alpha < 1$ then note that $g'(x) > 0$ for all x. This follows that convergence with g will never reach $\sqrt{\alpha}$ in this case, because g is strictly increasing.

If $\alpha > 1$ then g is strictly decreasing. Hence, with $x_1 < \alpha$ the convergence in 17 will not work.

exer:rudin_chap5_25 25. (a) Draw the tangent to f at x_n that intersects the x-axis at x_{n+1}.

(b) Since $f'(x) \geq \delta > 0$ for all $x \in [a,b]$ so f is monotonically increasing on $[a,b]$, which follows $x_{n+1} < x_n$. Next, we will show that $x_{n+1} \geq \xi$ inductively. Indeed, it suffices to show $x_n - x_{n+1} \leq x_n - \xi$ or $\frac{f(x_n)}{f'(x_n)} \leq x_n - \xi$ or $f(x_n) \leq f'(x_n)(x_n - \xi)$ since $f'(x) > 0$ for all $x \in [a,b]$. According to the MVT 4.2.0.2, there exists $t \in (\xi, x_n)$ so $f(x_n) = f(x_n) - f(\xi) = f'(t)(x_n - \xi)$. On the other hand, since $f''(x) \geq 0$ so f' is monotonically increasing on $[a,b]$, which imlies $f'(t) \leq f'(x_n)$ so $f(x_n) \leq f'(x_n)(x_n - \xi)$, as desired. Thus, we obtain $x_{n+1} \geq \xi$ given that $x_n \geq \xi$. By induction, we obtain that $\{x_n\}$ is a monotonically bounded sequence so it converges to some x.

With this, we obtain

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) \implies \lim_{n\to\infty} \frac{f(x_n)}{f'(x_n)} = 0.$$

On the other hand, note that $f'(x) \geq \delta > 0$ for all $x \in [a,b]$ and that f' is continuous on $[a,b]$ so $\lim_{n\to\infty} f'(x_n) = f'(x) > 0$. Similarly, we also find $\lim_{n\to\infty} f(x_n) = f(x)$. Therefore, $\lim_{n\to\infty} \frac{f(x_n)}{f'(x_n)} = \frac{f(x)}{f'(x)} = 0$ which implies $f(x) = 0$. Since $x \in (\xi, \beta)$ so $x = \xi$. Thus, $\lim_{n\to\infty} x_n = \xi$.

(c) According to the Taylor's theorem 4.6.0.1, there exists $t_n \in (\xi, x_n)$ such that

$$0 = f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2.$$

Combining with the fact that $x_{n+1} - \xi = x_n - \xi - \frac{f(x_n)}{f'(x_n)}$, we obtain

$$x_{n+1} - \xi = x_n - \xi - \frac{f(x_n)}{f'(x_n)} = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2.$$

(d) Since $x_n > x_{n+1} \geq \delta$, $f''(x) \leq M$ and $f'(x) \geq \delta$ so

$$\begin{aligned} 0 \leq x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2, \\ &\leq A(x_n - \xi)^2, \\ &\leq \frac{1}{A}[A(x_{n-1} - \xi)]^4, \\ &\leq \cdots \\ &\leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}. \end{aligned}$$

This is a good algorithm for finding root for f since the recursion formula is simple and the convergence is extremely rapid.

(e) Observe $g(x) = x$ then $\frac{f(x)}{f'(x)} = 0$ or $f(x) = 0$ so by finding roots of f through Newton's method, we can find a fixed point for g.

We have $g'(x) = \frac{f''(x)f(x)}{(f'(x))^2}$ so $|g'(x)| \leq \frac{M}{\delta^2}|f(x)|$, which means for x near ξ then $g'(x)$ near 0.

(f) If $f(x) = x^{1/3}$ then $x_{n+1} = -2x_n$, which implies that $\{x_n\}$ is not convergent.

exer:rudin_chap5_26 26. Let $\alpha = \sup\{a \leq x_0 \leq b : f([a, x_0]) = 0\}$. We will show that $\alpha = b$. Indeed, fix $x_1 \in [\alpha, b]$ and let

$$M_0 = \sup\{|f(x)| : x \in [\alpha, x_1]\}, M_1 = \sup\{|f'(x)| : x \in [\alpha, x_1]\}.$$

Since $|f'(x)| \leq A|f(x)|$ for all x so $M_1 \leq AM_0$. On the other hand, from Mean Value Theorem 4.2.0.2, we find $|f(x)| \leq M_1(x_1 - \alpha) \leq AM_0(x_1 - \alpha)$. This implies $M_0 \leq AM_0(x_1 - \alpha)$, which means with x_1 so $\alpha \leq x_1 \leq \min\left\{\frac{1}{A} + \alpha, b\right\}$ then $A(x_1 - \alpha) < 1$ which follows $M_0 = 0$ or $f = 0$ for all $x \in [a, x_1]$. According to the supremum of α, we obtain $\alpha = b$.

The problem is also true for vector-valued functions: Suppose $\mathbf{f} : [a, b] \to \mathbf{R}^k$ is differentiable on $[a, b]$, $\mathbf{f}(a) = 0$ and there exists real number A such that $|\mathbf{f}'(x)| \leq A|\mathbf{f}(x)|$ for all $x \in [a, b]$. Then $\mathbf{f} = 0$. The proof is completely similar, with the application of Mean Value Theorem for vector-valued function 4.8.0.4.

exer:rudin_chap5_27 27. If there are two solutions f, g for the initial-value problem then consider $h = f - g$ then $h'(x) = \phi(x, f(x)) - \phi(x, g(x))$ so $|h'(x)| \leq A|h(x)|$ for all $x \in [a, b]$. We also have $h(a) = f(a) - g(a) = 0$. Therefore, from previous exercise 26, we find $h = 0$ on $[a, b]$. Thus, such problem has at most one solution.

With $y' = y^{1/2}$ then $\frac{1}{2}y^{-1/2}y' = \frac{1}{2}$ or $(y^{1/2})' = \frac{1}{2}$. Therefore, $y^{1/2} = x/2 + c$. From here and $y(0) = 0$, we find two solutions $f(x) = 0$ and $f(x) = x^2/4$. Thus, these are the only two solutions for the initial-value problem.

exer:rudin_chap5_28 28. The initial-value problem $\mathbf{y}' = \Phi(x, \mathbf{y}), \mathbf{y}(a) = \mathbf{c}$ where $\mathbf{y} = (y_1, \ldots, y_k)$ ranges over a k-cell, Φ is the mapping of a $(k+1)$-cell into the Euclidean k-space whose components are functions

$\phi_1, \ldots, \phi_k$ and $\mathbf{c} = (c_1, \ldots, c_k)$. This problem has at most one solution if there exists constant A such that

$$|\Phi(x, \mathbf{y_1}) - \Phi(x, \mathbf{y_2})| \le A|\mathbf{y_1} - \mathbf{y_2}|.$$

when ever $(x, \mathbf{y_1})$ and $(x, \mathbf{y_2})$ belong to the $(k+1)$-cell.

Proof. For any two solutions $\mathbf{f}, \mathbf{g}$ for the initial-value problem, consider $\mathbf{h} = \mathbf{f} - \mathbf{g}$ then $\mathbf{h}'(x) = \Phi(x, \mathbf{f}(x)) - \Phi(x, \mathbf{g}(x))$. By appying exercise 26 for vector-valued functions, we obtain what we want. □

exer:rudin_chap5_29

29. Let $\Phi(x, \mathbf{y}) = \left(y_2, y_3, \ldots, y_k, f(x) - \sum_{j=1}^k g_j(x) y_i\right)$ then $\mathbf{y}' = \Phi(x, \mathbf{y})$. For any two values $\mathbf{z}, \mathbf{t}$, we have

$$|\Phi(x, \mathbf{z}) - \Phi(x, \mathbf{t})| \le |\mathbf{z} - \mathbf{t}| + |(\mathbf{z} - \mathbf{t}) \cdot \mathbf{g}(x)| \le (1 + \max\{|\mathbf{g}|(x)\})|\mathbf{z} - \mathbf{t}|.$$

Since $\mathbf{g} = (g_1, \ldots, g_k)$ is continuous on $[a, b]$ so it is bounded, which implies the existence of $A = 1 + \max\{|\mathbf{g}|(x)\}$. According to exercise 28, we conclude the problem has at most one solution.

By letting $\mathbf{y} = (y, y', \ldots, y^{(k-1)})$, we find that the equation

$$y^{(k)} + g_k(x) y^{(k-1)} + \ldots + g_2(x) y' + g_1(x) y = f(x)$$

with initial condition $\mathbf{y}(a) = (c_1, \ldots, c_k)$ has at most one solution.

4.10 Otherexercises

1. If $P(x)$ is a real polynomial has n distinct real roots in $(1, +\infty)$. Set:

$$Q(x) = (x^2 + 1) P(x) P'(x) + x(P^2(x) + P'^2(x))$$

Prove that $Q(x) = 0$ has at least $2n - 1$ distinct real roots.

See MSE.

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Chapter 5

Riemann integral

5.1 Definition and existence of the integral

Definition 5.1.0.1 (Riemann integral)**.** Let $[a, b]$ be a given interval. By a **partition** P of $[a, b]$ we mean a finite set of points $x_0, \ldots, x_n$ where $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$. We write $\Delta x_i = x_i - x_{i-1}$ $(i = 1, \ldots, n)$.

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$, we put

$$\begin{aligned} M_i &= \sup f(x)\ (x_{i-1} \leq x \leq x_i), \\ m_i &= \inf f(x)\ (x_{i-1} \leq x \leq x_i), \\ U(P, f) &= \sum_{i=1}^{m} M_i \Delta x_i, \\ L(P, f) &= \sum_{i=1}^{m} m_i \Delta x_i, \end{aligned}$$

and finally,

$$\overline{\int_a^b} f dx = \inf U(P, f), \tag{5.1}$$

eq:integral_rieman

$$\underline{\int_a^b} f dx = \sup L(P, f). \tag{5.2}$$

eq:integral_rieman

eq:integral_riemann_def_

where the inf and the sup are taken over all partition P of $[a, b]$. The left members of (5.2) and (5.1) are called the **upper** and **lower Riemann integrals** of f over $[a, b]$, respectively.

eq:integral_riemann_def_upper

If the upper and lower integrals are equal, we say that f is **Riemann-integrable** on $[a, b]$, we write $f \in \mathscr{R}$ (that is, $\mathscr{R}$ denotes the set of Riemann-integrable functions), and we denote the common value of (5.2) and (5.1) by

$$\int_a^b f dx \text{ or } \int_a^b f(x)dx.$$

This is the **Riemann integral** of f over $[a, b]$. Since f is bounded, there exists two numbers, m and M such that $m \le f(x) \le M$ for all $x \in [a, b]$. Hence, for every P,

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$$

so that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set. This shows the upper and lower integrals are defined or every bounded function f.

Definition 5.1.0.2 (Riemann-Stieltjes integral)**.** Let α be a monotonically increasing function on $[a, b]$ (which follows α is bounded on $[a, b]$). Corresponding to each partition P of $[a, b]$, we write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. It is clear that $\Delta\alpha_i \ge 0$. For any real function f which is bounded on $[a, b]$ we put

$$U(P, f, \alpha) = \sum_{i=1}^{m} M_i \Delta\alpha_i, L(P, f, \alpha) = \sum_{i=1}^{m} m_i \Delta\alpha_i,$$

where m_i, M_i have the same meaning as in the previous definition. We define

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha), \tag{5.3}$$

$$\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha), \tag{5.4}$$

the inf and sup are again being taken over all partitions.

If left members of (5.4) and (5.3) are equal, we denote their common value by

$$\int_a^b f d\alpha \text{ or } \int_a^b f(x) d\alpha(x).$$

This is the **Riemann-Stieltjes integral** (or simply the **Stieltjes integral**) of f with respect to α, over $[a, b]$. If that exists, we say f is integrable with respect to α, in the Riemann sense, and write $f \in \mathscr{R}(\alpha)$.

By taking $\alpha(x) = x$, the Riemann integral is seen to be a special case of the Riemann-Stieltjes integral. In general case, α need not to be continuous.

Definition 5.1.0.3 (Refinement)**.** We say that the partition P^* is a **refinement** of P if $P \subset P^*$. Given two partitions P_1 and P_2, we say that P^* is their **common refinement** if $P_1 \cup P_2 = P^*$.

Theorem 5.1.0.4
If P^* is a refinement of P, then $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and $U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

heo:integral_criterion **Theorem 5.1.0.5** For $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

_criterion_consequence **Theorem 5.1.0.6** Consider the criterion theo:integral_criterion 5.1.0.5:

- The inequality holds for any refinement P^* of P.
- For arbitrary points s_i, t_i in $[x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon.$$

- If $f \in \mathscr{R}$ and the criterion holds then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon.$$

heo:integral_continuous **Theorem 5.1.0.7** Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$ and α is continuous on every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

Proof. Let $c_1 < \ldots < c_k$ be the points on $[a, b]$ where f is discontinuous. This follows α is continuous at $c_1, \ldots, c_k$. Hence, there exists δ such that $|\alpha(c_i + \delta) - \alpha(c_i - \delta)| < \varepsilon$. We also know that f is continuous on $I_i = [c_i + \delta, c_{i+1} - \delta]$ so we can partition this interval into Q_i so $U(Q_i, f|_{I_i}, \alpha) - L(Q_i, f|_{I_i}, \alpha) < \varepsilon$. On the other hand, since f is continuous on $[a, c_1 - \delta]$ and $[c_k + \delta, b]$ so there exists paritions A, B of these two intervals such that $U(A, f_{[a, c_1 - \delta]}, \alpha) - L(A, f_{[a, c_1 - \delta]}, \alpha) < \varepsilon$ and similarly for B.

Let P be a common refinement of Q_i's and A, B and let $|f(x)| \leq M$ for all x then we have

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) \leq & \sum_{i=1}^{k-1} [U(Q_i, f|_{I_i}, \alpha) - L(Q_i, f|_{I_i}, \alpha)] \\ & + \sum_{i=1}^{k} 2M(\alpha(c_i + \delta) - \alpha(c_i - \delta)), \\ & + 2\varepsilon, \\ \leq & (k-1)\varepsilon + 2Mk\varepsilon + 2\varepsilon. \end{aligned}$$

With sufficient small ε, from theorem 5.1.0.5, we conclude $f \in \mathscr{R}(\alpha)$. □

Remark 5.1.0.8. A consequence of this theorem 5.1.0.7 is that if f is continuous on $[a, b]$ then $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

If f and α have common point of discontinuity, then f need not to be in $\mathscr{R}(\alpha)$. See [1, Exercise 3,§6] (3) for an example of this.

Question 5.1.0.9. What about function f with infinitely many points of discontinuity? Does there exist such $f \in \mathscr{R}(\alpha)$?

continuous_composition **Theorem 5.1.0.10** Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.

5.2 Properties of the integral

theo:integral_property **Theorem 5.2.0.1** (Properties of integrals) See [1, Theorem 6.12] of Rudin.

egral_property_product **Theorem 5.2.0.2** (Product of two integrable functions, f and $|f|$) If $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a, b]$, then

(a) $fg \in \mathscr{R}(\alpha;$

(b) $|f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f d\alpha\right| \le \int_a^b |f| d\alpha$.

(c) If $g \neq 0$ and g is bounded then f/g is integrable on $[a, b]$.

gral_stieltjes_riemann **Theorem 5.2.0.3** Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ iff $f\alpha \in \mathscr{R}$. In that case

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx.$$

Theorem 5.2.0.4 (Change of variable) Suppose φ a strictly increasing continuous functions that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$\beta(y) = \alpha(\varphi(y)), \; g(y) = f(\varphi(y)).$$

Then $g \in \mathscr{R}(\beta)$ and

$$\int_A^B g d\beta = \int_a^b f d\alpha.$$

Note a special case for this is when $\alpha(x) = x$, which means $\beta = \varphi$ and by applying theorem 5.2.0.3, we find

$$\int_a^b f(x)dx = \int_A^B f(\varphi(y))\varphi'(y)dy.$$

Example 5.2.0.5

Example for the application of change of variables 5.2.0.4.

Question 5.2.0.6. Is there an intuitive explanation for change of variables? (geometric interpretation for example)

Theorem 5.2.0.7 Let f be a bounded function on $[a, b]$. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $a < a_n < b_n < b$ for all n, but $\lim a_n = a$ and $\lim b_n = b$. Suppose $f \in \mathscr{R}$ on $[a_n, b_n]$ for all n. Then $f \in \mathscr{R}$ on $[a, b]$, and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_{a_n}^{b_n} \int f(x)dx.$$

For the proof, see [3, §5.2].

5.2.1 Alternate definitions and Darboux integral

Note that from previous theorems, we know that if $P*$ is a refinement of P then $U(P*, f, \alpha) - L(P*, f, \alpha) < U(P, f, \alpha) - L(P, f, \alpha)$. In other words, this suggests that if $f \in \mathscr{R}(\alpha)$ on $[a, b]$ then if we create a more explicit partition with sufficient small $\Delta\alpha_i$ then our upper/lower Riemann integrals will approach the Riemann integral. This matches with our geometric intuition. However, we have not had a formal theorem for this, which suggests the following question:

Question 5.2.1.1. If $f \in \mathscr{R}(\alpha)$ on $[a,b]$, for any $\varepsilon > 0$, does there exists $\delta > 0$ such that for any partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ satisfying $\Delta x_i < \delta$ then $U(P,f) - L(P,f) < \varepsilon$?

Fortunately, the answer is yes and this is a very useful result. The below theorem is taken from [3, §5, exercise].

Theorem 5.2.1.2 A function f is Riemann integrable with respect to α on $[a,b]$ if and only if there exists an $I \in \mathbf{R}$ such that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $P = \{x_0, \ldots, x_n\}$ is a partition with $\Delta\alpha_i < \delta$ for all i, then $|L(P,f,\alpha) - I| < \varepsilon$ and $|U(P,f,\alpha) - I| < \varepsilon$. If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ then $I = \int_a^b f d\alpha$.

Proof. If the converse is true, this means for any ε, there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < 2\varepsilon$. Therefore, from theorem 5.1.0.5, we find $f \in \mathscr{R}(\alpha)$ on $[a,b]$, so $L(P,f,\alpha) \le \int_a^b f d\alpha \le U(P,f,\alpha)$. On the other hand, we have $|L(P,f,\alpha) - I| < \varepsilon$ and $|U(P,f,\alpha) - I| < \varepsilon$ so $I - \varepsilon < \int_a^b f d\alpha \le I + \varepsilon$. Since this is true for any $\varepsilon > 0$ so we find $I = \int_a^b f d\alpha$.

If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ then pick $I = \int_a^b f d\alpha$. Again from theorem 5.1.0.5, we find that for any $\varepsilon > 0$, there exists a partition $P = \{x_0, \ldots, x_n\}$ such that $0 \le I - L(P,f,\alpha) < \varepsilon$ and $0 \le U(P,f,\alpha) - I < \varepsilon$. Let $Q = \{y_0, \ldots, y_k\}$ be an arbitrary partition of $[a,b]$ such that

$$\min\{\alpha(y_j) - \alpha(y_i) : 1 \le i < j \le k\} < \delta < \min\{\alpha(x_j) - \alpha(x_i) : 1 \le i < j \le k\}.$$

Let $T = P \cup Q$ be a common refinement of P and Q. We find

$$\begin{aligned} 0 \le I - L(T,f,\alpha) \le I - L(P,f,\alpha) \le \varepsilon, \\ 0 \le U(T,f,\alpha) - I \le U(P,f,\alpha) - I \le \varepsilon \end{aligned}$$

Next, we compare $L(T,f,\alpha)$ and $L(Q,f,\alpha)$. Since f is bounded on $[a,b]$ so there exists M such that $|f(x)| < M$ for all $x \in [a,b]$. Therefore, if $x_i \in [y_{j-1}, y_j]$ then

$$\begin{aligned} 0 \le \inf f([y_{j-1}, x_i])(\alpha(x_i) - \alpha(y_{j-1})) + \inf f([x_i, y_j])(\alpha(y_j) - \alpha(x_i)) \\ - \inf f([y_{j-1}, y_j])(\alpha(y_j) - \alpha(y_{j-1})) \le 2M(\alpha(y_j) - \alpha(y_{j-1})) \le 2M\delta \end{aligned}$$

This follows $0 \le L(T,f,\alpha) - L(Q,f,\alpha) \le 2M(n+1)\delta$. Since n is fixed under P, one can choose Q with sufficient small δ and obtain

$$0 \le I - L(Q,f,\alpha) \le \varepsilon + 2M(n+1)\delta.$$

The same argument can be made for $L(Q,f,\alpha)$. In the end, we can choose $\delta > 0$ such that for any partition Q so $\Delta_i < \delta$, we have $|I - L(Q,f,\alpha)| < \varepsilon$ and $|I - U(Q,f,\alpha)| < \varepsilon$. $\square$

From this theorem 5.2.1.2, we obtain the following theorem, which states the equivalence between Darboux integral and Riemann integral:

gral_criterion_darboux **Theorem 5.2.1.3** (Darboux integrals same as Riemann integrals) A function $f \in \mathscr{R}(\alpha)$ on $[a,b]$ if and only if there exists an $I \in \mathbf{R}$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $P = \{x_0, \dots, x_n\}$ is a partition of $[a,b]$ with $\Delta\alpha_i < \delta$ for all i, then

$$\left| \sum_{i=1}^{n} f(c_i)\Delta\alpha_i - I \right| < \varepsilon$$

for any set $\{c_1, \dots, c_n\}$ with $c_i \in [x_{i-1}, x_i]$. If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ then $I = \int_a^b f d\alpha$.

The proof is not hard once we know theorem 5.2.1.2.

Proof. If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ then from previous theorem 5.2.1.2, there exists partition P with $\Delta\alpha_i < \delta$ for all i and $|U(P,f,\alpha) - I| < \varepsilon/2$ and $|L(P,f,\alpha) - I| < \varepsilon/2$. Therefore, we obtain

$$\begin{array}{c} L(P,f,\alpha) \le \sum_{i=1}^{n} f(c_i)\Delta\alpha_i \le U(P,f,\alpha),\ \forall c_i \in [x_{i-1}, x_i], \\ L(P,f,\alpha) \le \int_a^b f d\alpha = I \le U(P,f,\alpha) \end{array}.$$

This implies $|\sum_{i=1}^{n} f(c_i)\Delta\alpha_i - I| < \varepsilon$ for any partition P with $\Delta\alpha_i < \delta$.

Conversely, if there exists $I \in \mathbf{R}$ such that: for any$\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P with $\Delta\alpha_i < \delta$ then $|\sum_{i=1}^{n} f(c_i)\Delta\alpha_i - I| < \varepsilon$ for $c_i \in [x_{i-1}, x_i]$. One can choose c_i such that $0 \le f(c_i) - \inf f([x_{i-1}, x_i]) < \gamma$. Then

$$|L(P,f,\alpha) - I| \le \left| \sum_{i=1}^{n} f(c_i)\Delta\alpha_i - I \right| + \gamma(b-a) < \varepsilon + \gamma(b-a).$$

Therefore, by choosing c_i's to get sufficient small γ, we can obtain $|L(P,f,\alpha) - I| < \varepsilon$. We use similar approach for $U(P,f,\alpha)$. Thus, from theorem 5.2.1.2, we conclude $f \in \mathscr{R}(\alpha)$ on $[a,b]$ with $I = \int_a^b f d\alpha$. ∎

5.2.2 Unit step function and related integrals

Definition 5.2.2.1 (Unit step function)**.** The **unit step function** I is defined by

$$I(x) = \begin{cases} 0 & (x \le 0) \\ 1 & (x > 0) \end{cases}.$$

ntegral_unit_step_func **Theorem 5.2.2.2** If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then $\int_a^b f d\alpha = f(s)$.

Theorem 5.2.2.3

Suppose $c_n \geq 0$ for $n = 1, 2, \dots$ and $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b), and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. Let f be continuous on $[a, b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Thoughts. It suffices to show

$$\lim_{k\to\infty} \sum_{n=1}^{k} c_n f(s_n) = \lim_{k\to\infty} \int_a^b f d\left(\sum_{n=1}^{k} c_n I(x - s_n)\right) = \int_a^b f d\alpha.$$

It's easier if we deal with $U(P, f, \alpha)$ and $U\left(P, f, \sum_{n=1}^{k} c_n I(x - s_n)\right)$. Hence, first thing you need eq:integral_unit_step_func_series to do is write out (5.5). Next, you use the fact that $\lim_{k\to\infty} \sum_{n=1}^{k} c_n I(x - s_n) = \alpha(x)$ and that f is continuous to bound the difference. Here is the final version of the proof:

Since f is continuous on $[a, b]$ so f is uniformly continuous on $[a, b]$, which means there exists partition $P = \{x_0, \dots, x_m\}$ of $[a, b]$ such that $\sum_{i=1}^{m} \sup\{|f(x)| : x_{i-1} \leq x \leq x_i\} = \sum_{i=1}^{m} M_i \leq \varepsilon$ and $\sum_{i=1}^{m} m_i \leq \varepsilon$ where $m_i = \inf\{|f(x)| : x_{i-1} \leq x \leq x_i\}$. On the other hand, since P is finite and that $\lim_{k\to\infty} \sum_{n=1}^{k} c_n I(x - s_n) = \alpha(x)$, there exists K such that $\left|\alpha(x_i) - \sum_{n=1}^{k} c_n I(x_i - s_n)\right| < \varepsilon$ for all $i = 1, \dots, m$ and all $k > K$.

With this, we have for all $k > K$ then

$$\begin{aligned}
\left|U(P, f, \alpha) - U\left(P, f, \sum_{n=1}^{k} c_n I(x - s_n)\right)\right| \leq & \sum_{i=1}^{m} M_i \left|\alpha(x_i) - \sum_{n=1}^{k} c_n I(x_i - s_n) - \right. \\
& \left. - \alpha(x_{i-1}) + \sum_{n=1}^{k} c_n I(x_{i-1} - s_n)\right|, \qquad (5.5) \\
\leq & 2\varepsilon \sum_{i=1}^{m} M_i, \\
\leq & 2\varepsilon^2.
\end{aligned}$$

eq:integral_unit_s

In the end, we obtain for all $k > K$ then

$$\begin{aligned} I(P, f, \alpha) - 2\varepsilon^2 &\le I\left(P, f, \sum_{n=1}^{k} c_n I(x - s_n)\right), \\ &\le \int_a^b f d\left(\sum_{n=1}^{k} c_n I(x - s_n)\right), \\ &\le U\left(P, f, \sum_{n=1}^{k} c_n I(x - s_n)\right), \\ &\le U(P, f, \alpha) + 2\varepsilon^2. \end{aligned}$$

Hence, if we refine P *from the beginning* such that $U(P, f, \alpha) - I(P, f, \alpha) < \varepsilon$ then

$$\left| \int_a^b f d\left(\sum_{n=1}^{k} c_n I(x - s_n)\right) - \int_a^b f d\alpha \right| < 4\varepsilon^2 + \varepsilon, \ \forall k > K$$

This gives what we want. □

5.3 Integration and Differentiation

Given a function $f \in \mathscr{R}$, under what conditions of f can we say that there exists F such that $F' = f$?

Theorem 5.3.0.1 Let $f \in \mathscr{R}$ on $[a, b]$. For $a \le x \le b$, put $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$; furthermore, if f is continuous at x_0 of $[a, b]$, then F is differentiable at x and $F'(x_0) = f(x_0)$.

Remark 5.3.0.2. From the theorem 5.3.0.1, we obtain that for any continuous function f on $[a, b]$, there exists differentiable function F such that $F' = f$. Furthermore, we also have a degree of freedom about F. In particular, $F(x) = \int_a^x f(t)dt + C$ for any constant C works.

Question 5.3.0.3. If f is not continuous, does there exist F such that $F' = f$?

Now, we will turn to the relation between integration and differentiation.

Theorem 5.3.0.4 (Fundamental theorem of calculus) If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Theorem 5.3.0.5 (Integration by parts) Suppose F and G are differentiable functions on $[a,b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Question 5.3.0.6. Intuitive explanation for integration by parts or fundamental theorem of calculus?

Remark 5.3.0.7. An extra condition about boundedness of f can change improper integral to proper integral. In particular, if we know that f is bounded on $[a,b]$ and that $f \in \mathscr{R}$ on $[a.c]$ for any $a < c < b$ then we can imply $f \in \mathscr{R}$ on $[a,b]$ right away. This is followed from theorem 5.2.0.7.

5.4 Improper integrals

See exercises 7, 8, 9, 18 from [1, §6].

Definition 5.4.0.1 (Improper integrals)**.** Suppose $f : [a,b) \to \mathbf{R}$ is a function (not necessarily bounded) that is Riemann integrable on $[a,c]$ for any $c < b$. We define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$$

if the limit exists. Similarly, we can define

$$\int_a^\infty f(x)dx = \lim_{c \to \infty} \int_a^c f(x)dx$$

if f is Riemann integrable on $[a,c]$ for any $c < \infty$.

Question 5.4.0.2. Does the properties for integrals in theorem 5.2.0.1 hold for improper integrals?

Proposition 5.4.0.3

(From [3, §5.5]) Suppose $f : [a,\infty) \to \mathbf{R}$ is nonnegative and $f \in \mathscr{R}$ on $[a,b]$ for all b so $b > a$. Then

$$\int_a^\infty f = \sup\left\{\int_a^x f : x \geq a\right\}$$

and

$$\int_a^\infty f = \lim_{n\to\infty} \int_a^{x_n} f$$

for any sequence $\{x_n\}$ with $\lim x_n = \infty$.

Question 5.4.0.4. What about other types of function f (not necessarily negative)? Is the above proposition still true?

In the previous definition for improper integrals, we only take the limit at one side. Here's how we can take limit at both endpoints:

Definition 5.4.0.5. Suppose $f : (a, b) \to \mathbf{R}$ is Riemann integrable function on $[c, d]$ for all c, d such that $a < c < d < b$, then we define

$$\int_a^b f(x)dx = \lim_{c \to a^+} \lim_{d \to b^-} \int_c^d f(x)dx$$

if the limits exist. Similarly, we can define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{c \to -\infty} \lim_{d \to \infty} \int_c^d f(x)dx$$

if the limits exist.

Remark 5.4.0.6. One may ask that does the order of taking limits in definition 5.4.0.5 change the result, i.e. what if we take the limit $c \to a^+$ before taking $d \to b^-$? The answer is no, as seen in the following proposition.

Proposition 5.4.0.7 (Order if limits for improper integrals can be switched if they exist) Suppose $f : (a, b) \to \mathbf{R}$ is a Riemann integrable function on $[c, d]$ for all c, d such that $a < c < d < b$, we have

$$\lim_{c \to a^+} \lim_{d \to b^+} f(x)dx$$

converges iff

$$\lim_{d \to b^+} \lim_{c \to a^-} f(x)dx$$

converges. In this case, the two expressions are equal to each other.

Proof. There is one proof for infinite limits in Lebl's book [3, Proposition 5.5.10.]. □

Example 5.4.0.8 (sinc function)
Define

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then

$$\int_{-\infty}^{\infty} \operatorname{sinc}(x) dx = \pi \text{ while } \int_{-\infty}^{\infty} |\operatorname{sinc}(x)| dx = \infty.$$

This sinc function is a continuous analogue of the alternating harmonic series $\sum \frac{(-1)^n}{n}$. A proof for this can be seen in Lebl's book [3, Example 5.5.12].

proper_comparison_test **Theorem 5.4.0.9** (Comparison test for improper integrals) Let $f : [a, \infty) \to \mathbf{R}$ and $g : [a, \infty) \to \mathbf{R}$ be functions that are Riemann integrable on $[a, b]$ for all $b > a$. Suppose that for all $x \geq a$ we have $|f(x)| \leq g(x)$. If $g \in \mathscr{R}$ on (a, ∞) then $f \in \mathscr{R}$ on (a, ∞). If $\int_a^\infty f(x)dx$ diverges then $\int_a^\infty g(x)dx$ diverges.

Remark 5.4.0.10. Intuitively, this makes sense. We see f lies between the two curves g and $-g$. If $\int_a^\infty g(x)dx$ exists then that means the area bounded by two curves g and $-g$ is finite, which leads to the area $\int_a^\infty f(x)dx$ being finite.

This may also be true for normal integral $\int_a^b f(x)dx$.

5.5 Integration of vector-valued functions

Definition 5.5.0.1. Let $f_1, \dots, f_k$ be real functions on $[a, b]$, and let $\mathbf{f} = (f_1, \dots, f_k)$ be the corrsponding mapping of $[a, b]$ into $\mathbf{R}^k$. If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathscr{R}(\alpha)$ means that $f_j \in \mathscr{R}(\alpha)$ for $j = 1, \dots, k$. We define

$$\int_a^b \mathbf{f} d\alpha = \left(\int_a^b f_1 d\alpha, \dots, \int_a^b f_k d\alpha \right).$$

Theorems 5.3.0.1, 5.3.0.4 and 5.2.0.3 are still true for vector-valued functions.

vector_valued_func_abs **Theorem 5.5.0.2** If $\mathbf{f}$ maps $[a, b]$ into $\mathbf{R}^k$ and if $f \in \mathscr{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$ then $|\mathbf{f}| \in \mathscr{R}(\alpha)$, and

$$\left| \int_a^b \mathbf{f} d\alpha \right| \leq \int_a^b |\mathbf{f}| d\alpha.$$

5.6 Rectifiable curves

Definition 5.6.0.1. A continuous mapping γ of an interval $[a, b]$ into $\mathbf{R}^k$ is called a **curve** in $\mathbf{R}^k$. We may also say that γ is a curve on $[a, b]$. If γ is one-to-one then γ is called an **arc**, if $\gamma(a) = \gamma(b)$ then γ is said to be a **closed curve**.

We associate each partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(y_i) - \gamma(y_{i-1})|.$$

Here $\Lambda(P, \gamma)$ is the length of a polygonal path with vertices $\gamma(y_0), \dots, \gamma(y_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely. This makes it seem reasonable to define the **length** of γ as $\Lambda(\gamma) = \sup \Lambda(P, \gamma)$ where the supremum is taken over all paritiitions of $[a, b]$. If $\Lambda(\gamma) < \infty$, we say γ is **rectifiable**.

ral_rectifiable_length **Theorem 5.6.0.2** If γ' is continuous on $[a, b]$, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| dt.$$

5.7 Baby Rudin exercises

exer:rudin_chap6_1

1. Since α is continuous at x_0 so for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x_0-\delta)-\alpha(x_0+\delta)| < \varepsilon$ (if $x_0 = a$ then we take $x_0+\delta$ and x_0). Consider a partition $P = \{a, x_0-\delta, x_0+\delta, b\}$ of $[a,b]$ then $U(P,f,\alpha) = \alpha(x_0+\delta)-\alpha(x_0-\delta) < \varepsilon$ and $L(P,f,\alpha) = 0$. This follows $f \in \mathscr{R}$ on $[a,b]$ and that $\int f d\alpha = 0$.

exer:rudin_chap6_2

2. From theorem 5.1.0.5, for any $\varepsilon > 0$, there exists partition P of $[a,b]$

   If $f(x_0) > 0$ for some $x_0 \in [a,b]$ then since f is continuous on $[a,b]$, there exists $\delta > 0$ such that $f(x) > \varepsilon$ for all $x \in [x_0-\delta, x_0+\delta]$ where $\varepsilon < f(x_0)$. Define a new function g on $[a,b]$ such that $g(x) = \varepsilon$ for $x \in [x_0-\delta, x_0+\delta]$ and $g(x) = 0$ otherwise. With this, we can easily find that $g \in \mathscr{R}$ and that $\int_a^b g(x)dx = 2\delta\varepsilon$. On the other hand, since $g(x) \le f(x)$ for all $x \in [a,b]$, we find $\int_a^b f(x)dx = \int_a^b g(x)dx > 0$, a contradiction. Thus, $f(x) = 0$ for all $x \in [a,b]$.

   Comparing with exercise 1, we find that if $f(x) = 0$ for all $x \neq x_0$ and $f(x_0) \neq 0$ then the Riemann integral of f on $[a,b]$ is still 0. This follows that if we change value of f at finite number of points, the Riemann integral remained unchange.

exer:rudin_chap6_3

3. (a) $f \in \mathscr{R}(\beta_1)$ iff for any $\varepsilon > 0$, there exists a partition P of $[-1,1]$ containing 0 such that $U(P,f,\beta_1) - L(P,f,\beta_1) < \varepsilon$. Denote $x_0 > 0$ as the closest number to 0 in P. We then have $U(P,f,\beta_1) = \sup\{f(x) : x \in [0,x_0]\} \ge f(0)$ and $L(P,f,\beta_1) = \inf\{f(x) : x \in [0,x_0]\} \le f(0)$ so $f(0) \le \sup\{f(x) : x \in [0,x_0]\} < f(0)+\varepsilon$. $f(0)-\varepsilon < \inf\{f(x) : x \in [0,x_0]\} \le f(0)$. Since this is true for any $\varepsilon > 0$, we find $f(0+) = f(0)$ and that $\int f d\beta_1 = f(0)$.

   (b) Similarly, $f \in \mathscr{R}(\beta_2)$ iff $f(0-) = f(0)$ and that then $\int f d\beta_2 = f(0)$.

   (c) Similarly, we find $U(P,f,\beta_3) = \frac{1}{2}\left(\sup\{f(x) : x \in [-x_0,0]\} + \sup\{f(x) : x \in [0,x_0]\}\right)$ and $L(P,f,\beta_3) = \frac{1}{2}\left(\inf\{f(x) : x \in [-x_0,0]\} + \inf\{f(x) : x \in [0,x_0]\}\right)$. Hence, $f \in \mathscr{R}(\beta_3)$ iff $U(P,f,\beta_3) - L(P,f,\beta_3) < \varepsilon$ for any $\varepsilon > 0$ and appropriate P iff $f(0+) = f(0-) = f(0)$ iff f is continuous at 0.

   (d) Use the arguments above.

exer:rudin_chap6_4

4. For any interval I of $[a,b]$ then $\sup\{f(x) : x \in I\} = 1$ and $\inf\{f(x) : x \in I\} = 0$, which implies $L(P,f) = 0$ and $U(P,f) = b-a$ for any partitions P of $[a,b]$. Hence, according to theorem 5.1.0.5, $f \notin \mathscr{R}$ on $[a,b]$ for $a < b$.

exer:rudin_chap6_5

5. Consider $f(x) = \begin{cases} 1 & (x \in \mathbf{Q}) \\ -1 & (x \notin \mathbf{Q}) \end{cases}$. We know that $f^2 \in \mathscr{R}$ on $[a,b]$ but with similar argument in 4, we find $f \notin \mathscr{R}$ on $[a,b]$. Therefore, $f^2 \in \mathscr{R}$ does not follow that $f \in \mathscr{R}$.

   If $f^3 \in \mathscr{R}$ then with $\phi(x) = x^{1/3}$, which is continuous, according to theorem 5.1.0.10, we find $f \in \mathscr{R}$.

exer:rudin_chap6_6 6. From 1.4.0.1, we know that $E_n = \bigcup_{2^n}[a_i,b_i]$ is a union of finitely disjoint intervals that covers P where $b_i - a_i = 3^{-n}$. Since f is continuous at $I = \bigcup[b_i, a_{i+1}]$, f is uniformly continuous, which means there exists δ such that $|f(s)-f(t)| < \varepsilon$ for all $|s-t| < \delta$ and $s,t \in I$.

Let Q_n be a partitions of $[a,b]$ that contains all a_i's and b_i's and if $x_i \in Q$ and is not one of a_i's or b_i's then $\Delta x_i < \delta$. With this, with $M = \sup|f(x)|$, we obtain

$$\begin{aligned} U(Q_n,f) - L(Q_n,f) &< \sum_{i=1}^{2^n} \varepsilon(a_i - b_{i-1}) + \sum_{i=1}^{2^n} M(b_i - a_i), \\ &< \varepsilon(2/3)^n + M(2/3)^n. \end{aligned}$$

Since this is true for any ε and any $n \in \mathbf{Z}_{\geq 1}$ and $(2/3)^n \to 0$ as $n \to \infty$ so $f \in \mathscr{R}$ on $[0,1]$.

exer:rudin_chap6_7 7. This is **improper** integral. (a) If $f \in \mathscr{R}$ on $[0,1]$ then we have

$$\left| \int_0^1 f(x)dx - \int_c^1 f(x)dx \right| = \left| \int_0^c f(x)dx \right| \leq Mc.$$

As $c \to 0$, $Mc \to 0$ which is what we want.

(b) Let $f(x) = (-1)^n n$ if $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$. Therefore, with $c \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, we have

$$\begin{aligned} \int_c^1 f(x)dx &= \int_c^{1/n} f(x)dx + \sum_{i=1}^{n-1} \int_{1/(i+1)}^{1/i} f(x)dx, \\ &= \left(\frac{1}{n} - c\right)(-1)^n n + \sum_{i=1}^{n-1} \left(\frac{1}{i} - \frac{1}{i+1}\right)(-1)^i i, \\ &= (1-cn)(-1)^n + \sum_{i=1}^{n-1} \frac{(-1)^i}{i+1}. \end{aligned}$$

Note that $0 < 1 - cn < \frac{1}{n+1}$ and that $\sum_i \frac{(-1)^i}{i+1}$ converges so we find $\lim_{c\to 0} \int_c^1 f(x)dx$ exists. On the other hand, we have

$$\int_c^1 |f(x)|dx = 1 - cn + \sum_{i=1}^{n} \frac{1}{i+1},$$

which diverges since $\sum_{i=1}^n \frac{1}{i} \to \infty$ as $n \to \infty$.

exer:rudin_chap6_8 8. (Integral test for series) Suppose $f \in \mathscr{R}$ on $[a,b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by $|f|$, it is said to converge **absolutely**.

For $f(x) \geq 0$ and f monotonically decreasing on $[1,\infty)$, note that for $b \in [n, n+1)$ then

$$\begin{aligned}\sum_{i=2}^{n} f(i) + (b-n)f(n+1) \leq L(\{1,2,\ldots,n\}, f),&\\ &\leq \int_1^b f(x)dx,\\ &\leq U(\{1,2,\ldots,n\}, f),\\ &\leq \sum_{i=1}^{n-1} f(i) + (b-n)f(n).\end{aligned}$$

Therefore, if $\int_1^\infty f(x)dx$ exists then the sequence of partial sums of $\sum_{n=1}^\infty f(n)$ is monotonically increasing and is bounded by $f(1) + \int_1^\infty f(x)dx$ so it converges. Similarly, if $\sum_{n=1}^\infty f(n)$ exists then any sequence of $\left\{\int_1^{b_n} f(x)dx\right\}_n$ is monotonically bounded which implies convergence. Hence, $\int_1^\infty f(x)dx$ exists.

exer:rudin_chap6_9 9. (Integration by parts with improper integrals) Integration by parts for improper integrals as in exercises 7, 8. Suppose F, G are differentiable functions on $(a, b]$, $F' = f \in \mathscr{R}$ and $G' = g \in \mathscr{R}$ on $[c, b]$ for any $c \in (a, b]$. If $\int_a^b f(x)G(x)dx$ exists (as in exercise 7) then $\int_a^b F(x)g(x)dx$ exists and that

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

The proof is not hard. You simply use integration by parts 5.3.0.5 on $[c, b]$ and take limit $c \to a$.

Similarly, if $\lim_{k\to\infty} F(k)G(k)$ exists then

$$\int_a^\infty F(x)g(x)dx = \lim_{k\to\infty} F(k)G(k) - F(a)G(a) + \int_a^\infty f(x)G(x)dx.$$

exer:rudin_chap6_10 10. (Holder's inequality) (a) From 14 (chap 5, exercise 14), we know that $f(x) = -\ln(x)$ is convex since $f''(x) = x^{-2} > 0$ for all $x \neq 0$. Therefore,

$$\ln((1-\lambda)x + \lambda y) \geq (1-\lambda)\ln(x) + \lambda\ln(y).$$

By choosing $\lambda = 1/p, x = v^q, y = u^p$ then $\ln(x) = q\ln(v), \ln(y) = p\ln(u)$ so when taking exponential both sides, we obtain $\frac{u^p}{p} + \frac{v^q}{q} \geq uv$, as desired.

(b) From (a), we find

$$1 = \frac{1}{p}\int_a^b f^p d\alpha + \frac{1}{q}\int_a^b g^q d\alpha = \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q}\right) d\alpha \geq \int_a^b fg d\alpha.$$

(c) Let $\|f\|_p = \left\{ \int_a^b |f|^p \right\}^{1/p}$ then $\|f\|_p = 0$ iff $\int_a^b f d\alpha = 0$ from theorem theo:integral_vector_valued_func_abs 5.5.0.2 iff $\int_a^b fg d\alpha = 0$, which satisfies the inequality.

If $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$ then consider (b) for two functions $\frac{|f|}{\|f\|_p}$ and $\frac{|g|}{\|g\|_q}$.

(d) Because limits preverse inequality.

exer:rudin_chap6_11 11. It suffices to show $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$. Indeed, we have

$$\begin{aligned}
\|f+g\|_2^2 &= \int_a^b |f+g|^2 d\alpha, \\
&\le \int_a^b |f|^2 d\alpha + \int_a^b |g|^2 d\alpha + 2\int_a^b |fg| d\alpha, \\
&\le \int_a^b |f|^2 d\alpha + \int_a^b |g|^2 d\alpha + 2\left\{ \int_a^b |f|^2 d\alpha \int_a^b |g|^2 d\alpha \right\}^{1/2}, \\
&\le (\|f\|_2 + \|g\|_2)^2.
\end{aligned}$$

exer:rudin_chap6_12 12. Let $M = \sup(f), m = \inf(f)$ and for $\varepsilon > 0$, let P be a partition of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon^2/(M-m)$. We choose g such that for $x_{i-1}, x_i \in P$ then

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i), \; x_{i-1} \le t \le x_i.$$

Intuitively, in $[x_{i-1}, x_i]$, g is a line from $f(x_{i-1})$ to $f(x_i)$. With this, we obtain $|f-g|(x) \le M_i - m_i$ where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i], m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Hence,

$$\begin{aligned}
\|f-g\|_2^2 &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \|f-g\|^2 d\alpha, \\
&\le \sum_{i=1}^n (M_i - m_i)^2 \alpha_i, \\
&\le (M-m) \sum_{i=1}^n (M_i - m_i)\alpha_i, \\
&= (M-m)(U(P,f,\alpha) - L(P,f,\alpha)), \\
&< \varepsilon^2.
\end{aligned}$$

exer:rudin_chap6_13 13. (a) Use the special case of theorem theo:integral_change_variable 5.2.0.4 where $\varphi(t) = t^{1/2}$ (which is possible since $x > 0$), which is a strictly increasing continuous function that maps $[x^2, (x+1)^2]$ onto $[x, x+1]$. We have $f(\varphi(t)) = \sin t$ and $\varphi' = 1/2t^{-1/2}$ so

$$\int_x^{x+1} \sin(t^2) dt = \int_{x^2}^{(x+1)^2} \sin t \cdot 1/2t^{-1/2} dt.$$

Using integration by parts 5.3.0.5, with $f(x) = \sin t$ and $G = 1/2t^{-1/2}$, we have

$$\begin{aligned}
\int_{x^2}^{(x+1)^2} \sin(t)\cdot 1/2t^{-1/2}dt &= \int_{x^2}^{(x+1)^2} f(x)G(x)dx,\\
&=\frac{-\cos[(x+1)^2]}{2(x+1)} - \frac{-\cos(x^2)}{2x},\\
&\quad - \int_{x^2}^{(x+1)^2} (-\cos t)\left(\frac{-1}{4}t^{-3/2}\right)dt,\\
&=\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos t}{4t^{3/2}}dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|f(x)| &\le \frac{1}{2x} + \frac{1}{2(x+1)} + \int_{x^2}^{(x+1)^2} \frac{1}{4t^{3/2}}dt,\\
&= \frac{2x+1}{2x(x+1)} - \left.\frac{1}{2t^{1/2}}\right|_{t=x^2}^{(x+1)^2},\\
&= \frac{2x+1}{2x(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)},\\
&= \frac{1}{x}.
\end{aligned}$$

(b) From (a), we know that

$$r(x) = \frac{1}{x+1}\cos[(x+1)^2] - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du.$$

Hence,

$$|r(x)| \le \frac{1}{1+x} + \frac{1}{2x(x+1)} = \frac{2x+1}{2x(x+1)}.$$

Hence, $|r(x)| < 1/x$.

(c) (From Roger Cooke, University of Vermont) Since $r(x) \to 0$, it suffices to find the upper limit and lower limit of

$$h(x) = \frac{\cos(x^2) - \cos[(x+1)^2]}{2} = \sin\left(x^2 + x + \frac{1}{2}\right)\sin\left(x + \frac{1}{2}\right) = \sin(t)\sin(t^2 + 1/4),$$

where $t = x + 1/2$. We want to show that the upper and lower limits of h is 1 and -1, respectively.

For any $\varepsilon > 0$ and for sufficient large $n > \frac{2-\varepsilon}{8\varepsilon}$, if $t \in I = ((2n+1/2)\pi - \varepsilon, (2n+1/2)\pi + \varepsilon)$ then t^2+4 is in an interval with length greater than 2π, which implies there exists $u, v \in I$ such that $\sin(u^2+1/4)=1$ and $\sin(v^2+1/4)=-1$. On the other hand we have $h(u)=\sin(u)>\sin\left((2n+1/2)\pi-\varepsilon\right)>1-\varepsilon$ and $h(v)=-\sin(v)<-1+\varepsilon$. This is true for any $\varepsilon>0$ and for sufficient large n, we find $xf(x)$ has upper and lower limits 2 and -2, respectively.

(d) For positive integer N, we have

$$\begin{aligned}\int_0^N \sin(t^2)dt &= \sum_{k=0}^N f(k)+\int_N^x \sin(t^2)dt,\\ &\le f(0)+\sum_{k=1}^N \frac{2+r(k)}{2k}.\end{aligned}$$

Since $|r(k)|<c/k$ so the sum $\sum_{k=1}^N \frac{2+r(k)}{2k}$ converges. With similar approach as in (a), one can show that

$$\lim_{x\to\infty}\int_{\lfloor x\rfloor}^x \sin(t^2)dt=0.$$

This implies that $\int_0^\infty \sin(t^2)dt$ converges.

exer:rudin_chap6_14

14. Similarly, we have

$$\begin{aligned}f(x) &= \int_{e^x}^{e^{(x+1)}} \frac{\sin t}{t}dt,\\ &= -\frac{\cos\left(e^{(x+1)}\right)}{e^{x+1}}+\frac{\cos(e^x)}{e^x}-\int_{e^x}^{e^{x+1}}\frac{\cos t}{t^2}dt,\\ |f(x)| &\le \frac{1}{e^x}+\frac{1}{e^{x+1}}+\int_{e^x}^{e^{x+1}}\frac{1}{t^2}dt,\\ &= \frac{e+1}{e^{x+1}}+\frac{1}{e^x}-\frac{1}{e^{x+1}},\\ &= \frac{2}{e^x}\end{aligned}$$

This follows

$$|r(x)|=e^x\left|\int_{e^x}^{e^{x+1}}\frac{\cos t}{t^2}dt\right|\le 1-e^{-1}.$$

exer:rudin_chap6_15 15. Using integration by parts theo:integral_by_parts 5.3.0.5 with $H = f(x), G = xf(x)$ then

$$\begin{aligned}
\int_a^b xf(x)f'(x)dx &= \int_a^b h(x)G(x)dx, \\
&= H(b)G(b) - H(a)G(a) - \int_a^b H(x)g(x)dx, \\
&= bf^2(b) - af^2(a) - \int_a^b f(x)[f(x) + xf'(x)]dx, \\
2\int_a^b xf(x)f'(x)dx &= bf^2(b) - af^2(a) - \int_a^b f^2(x)dx, \\
&= -1.
\end{aligned}$$

According to Holder's inequality exer:rudin_chap6_10 10 for $p = q = 2$, we have

$$\int_a^b [f'(x)]^2 dx \int_a^b x^2 f^2(x)dx \geq \left| \int_a^b xf(x)f'(x)dx \right|^2 = \frac{1}{4}.$$

exer:rudin_chap6_16 16. (Riemann's zeta function) For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(a) We have

$$\begin{aligned}
s\int_1^{\infty} \frac{\lfloor x \rfloor}{x^{s+1}} dx &= s\sum_{n=1}^{\infty} \int_n^{n+1} \frac{\lfloor x \rfloor}{x^{s+1}}, \\
&= s\sum_{n=1}^{\infty} n \int_n^{n+1} \frac{1}{x^{s+1}} dx, \\
&= s\sum_{n=1}^{\infty} n \frac{1}{s} \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right], \\
&= \sum_{n=1}^{\infty} n \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right], \\
&= \zeta(s).
\end{aligned}$$

(b) Because $\frac{s}{s-1} = \int_1^{\infty} \frac{1}{x^s} dx$.

exer:rudin_chap6_17 17. Since g is continuous on $[a, b]$ so g is uniformly continuous on $[a, b]$, which means there exists δ such that for all $x, y \in [a, b], |x - y| < \delta$ then $|g(x) - g(y)| < \varepsilon$.

Since $G \in \mathscr{R}(\alpha)$ so there exists partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, so that $|x_i - x_{i+1}| < \varepsilon$ for any i from $[n]$ and that $U(P, Q, \alpha) - L(P, Q, \alpha) < \varepsilon$.

On the other hand, since $G' = g$ so there exists $y_i \in [x_{i-1}, x_i]$ such that $G(x_i) - G(x_{i-1}) = g(y_i)(x_i - x_{i-1})$. We have

$$\begin{aligned}\sum_{i=1}^{n} g(y_i)\alpha(x_i)(x_i - x_{i-1}) &= \sum_{i=1}^{n} (G(x_i) - G(x_{i-1}))\alpha(x_i), \\ &= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i.\end{aligned}$$

theo:integral_criterion_consequence

From theorem 5.1.0.6, we find

$$\left| \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i - \int_a^b G d\alpha \right| < \varepsilon.$$

Since $[x_{i-1}, x_i]$ has length less than δ so we find that $g(y_i) > \sup\{g(x) : x \in [x_{i-1}, x_i]\} - \varepsilon$ and we also have $\alpha(x_i) \geq \sup \alpha(x) : x \in [x_{i-1}, x_i]\}$ so

$$\begin{aligned}\sum_{i=1}^{n} g(y_i)\alpha(x_i)(x_i - x_{i-1}) &\geq \sum_{i=1}^{n} \sup\{g\alpha : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}) - \varepsilon \sum_{i=1}^{n} \alpha(x_i)(x_i - x_{i-1}), \\ &\geq U(P, g\alpha) - \varepsilon\alpha(b)(b - a).\end{aligned}$$

Combining three above facts, we find

$$U(P, g\alpha) - L(P, \alpha) < \varepsilon\alpha(b)(b - a),$$

and

$$L(P, g\alpha) - \varepsilon \leq G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha \leq U(P, g\alpha) + \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we conclude that $g\alpha \in \mathscr{R}$ and that

$$\int_a^b g(x)\alpha(x) dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

exer:rudin_chap6_18

18. Note that e^{ix} has period 2π so γ_1 and γ_2 have the same range, which is all complex numbers with absolute value 1. To show that γ_3 also has this range, it suffices to show that there is a mapping t $2\pi t \sin(1/t)$ with $0 \leq t \leq 2\pi$ that contains an interval of length 2π, or a mapping $t \to t \sin(1/t)$ that covers an interval of length 1. Choose $t = 6/\pi$ and $t = 2/(3\pi)$ we get what we want.

Since e^{ix} has absolute value 1 so from theorem 5.6.0.2, we find γ_1 and γ_2 has length 2π and 4π, respectively.

To show that γ_3 is not rectifiable, it suffices to show that

$$\int_0^{2\pi} |\gamma_3'(t)| dt = \int_0^{2\pi} \left| \sin\left(\frac{1}{t}\right) - \frac{\cos(1/t)}{t} \right| dt = \infty.$$

Indeed, we have

$$\left| \sin\left(\frac{1}{t}\right) - \frac{\cos(1/t)}{t} \right| \geq \left| \frac{\cos(1/t)}{t} \right| - |\sin(1/t)| \geq \left| \frac{\cos(1/t)}{t} \right| - 1.$$

Therefore,

$$\int_0^{2\pi} \left| \sin\left(\frac{1}{t}\right) - \frac{\cos(1/t)}{t} \right| dt \geq \int_0^{2\pi} \left| \frac{\cos(1/t)}{t} \right| - 2\pi.$$

We have

$$\begin{aligned}
\int_{\frac{1}{2\pi(n+1/2)}}^{\frac{1}{2\pi n}} \left| \frac{\cos(1/t)}{t} \right| dt &= \int_{\frac{1}{2\pi(n+1/2)}}^{\frac{1}{2\pi n}} \frac{\cos(1/t)}{t} dt, \\
&= \int_{\frac{1}{2\pi(n+1/2)}}^{\frac{1}{2\pi n}} \frac{\cos(1/t)}{(1/t)} \cdot \frac{1}{t^2} dt, \\
&= \int_{2\pi(n+1/2)}^{2\pi n} -\frac{\cos(u)}{u} du, \\
&= \int_{2n\pi}^{2\pi(n+1/2)} \frac{\cos u}{u} du, \\
&\geq \int_{2n\pi}^{2\pi(n+1/2)} \frac{\cos u}{2\pi(n+1/2)} du, \\
&= \frac{1}{2\pi(n+1/2)}.
\end{aligned}$$

This follows

$$\int_{\frac{1}{2\pi k}}^{\pi} \left| \frac{\cos(1/t)}{t} \right| dt = \sum_{n=0}^{k} \int_{\frac{1}{2\pi(n+1/2)}}^{\frac{1}{2\pi n}} \left| \frac{\cos(1/t)}{t} \right| dt = \sum_{n=1}^{k} \frac{1}{2\pi(n+1/2)}.$$

We know that the series $\sum \frac{1}{2\pi(n+1/2)}$ diverges so $\int_{\frac{1}{2\pi k}}^{\pi} \left| \frac{\cos(1/t)}{t} \right| dt$ approaches infinity as $n \to \infty$. Therefore,

$$\int_0^{2\pi} \left| \frac{\cos(1/t)}{t} \right| = \int_{\pi}^{2\pi} \left| \frac{\cos(1/t)}{t} \right| + \lim_{n\to\infty} \int_{\frac{1}{2\pi n}}^{\pi} \left| \frac{\cos(1/t)}{t} \right| dt = \infty.$$

Thus, γ_3 is not rectifiable.

exer:rudin_chap6_19 19. Since ϕ is a continuous $1-1$ mapping so it has continuous $1-1$ inverse φ. This follows γ_1 is an arc iff γ_2 is an arc.

If $\phi(k) = b$ for some $k \neq d$ then there exists $u < k < v, u, v \in [c, d]$ such that $\phi(u), \phi(v) < b$. Since ϕ is continuous on $[c, d]$, there exists $u_1 \in [u, k], v_1 \in [k, v]$ such that $\phi(u_1) = \phi(v_1) < b$. This implies ϕ is not $1-1$, a contradiction. Thus, $\phi(d) = b$. Combining with the fact that $\phi(c) = a$, we find γ_1 is a closed curve iff γ_2 is also a closed curve.

From φ and ϕ, there is a one-to-one correspondence between partitions P of $[a, b]$ and Q of $[c, d]$, which makes $\Lambda(P, \gamma_1) = \Lambda(P, \gamma_2)$. Hence, γ_1 is rectifiable iff γ_2 is rectifiable. Furthermore, if they are, they will have the same length.

5.8 Lebl exercises

1. (Exercise 10,§5.1) Let $f : [0,1] \to \mathbf{R}$ be a bounded function. Let $P = \{x_0, \dots, x_n\}$ be a uniform partition of $[0,1]$, that is $x_j = j/n$. Is $\{L(p_n, f)\}_{n=1}^{\infty}$ always monotone?

   Proof. No. Let $f : [0,1] \to \mathbf{R}$ such that $f(x) = 0$ if $x \notin [1/3, 2/3]$ and $f(x) = 1$ if $x \in [1/3, 2/3]$. We have $L(P_3, f) = 1/3$ and $L(P_4, f) = 0$ and $L(P_8, f) = 1/4$. □

2. (Exercise 11, §5.1) For a bounded function $f : [0,1] \to \mathbf{R}$ let $R_n = (1/n)\sum_{j=1}^{n} f(j/n)$. (a) If f is Riemann integrable on $[0,1]$, show $\int_0^1 f dx = \lim R_n$. (b) Find an f that is not Riemann integrable, but $\lim R_n$exists.

   theo:integral_criterion_darboux

   Proof. From theorem 5.2.1.3, for any ε, there exists a partition $Q = \{x_0, \dots, x_n\}$ of $[0,1]$ such that $0 \le \int_0^1 f(x)dx - L(Q, f) < \varepsilon$ and $0 \le U(Q, f) - \int_0^1 f(x)dx < \varepsilon$. Choose a sufficient large N such that $\min\{x_j - x_i : 1 \le i < j \le n\} > 1/n$. Then $P_n = Q \cup \{0, 1/n, \dots, (n-1)/n, 1\}$ for $n > N$ is a refinement of Q. This implies

   $$0 \le \int_0^1 f(x)dx - L(P_n, f) \le \int_0^1 f(x)dx - L(Q, f) < \varepsilon,$$
   $$0 \le U(P_n, f) - \int_0^1 f(x)dx \le U(Q, f) - \int_0^1 f(x)dx < \varepsilon.$$

   For $[(i-1)/n, i/n]$, choose $c = i/n$, for $[x_i, j/n]$, choose $c = j/n$, for $[j/n, x_i]$, choose $c = x_i$. With this, since $\inf I \le c \le \sup I$ for $c \in I$, we have

   $$L(P_n, f) \le R_n + \frac{1}{n}\sum_{i=1}^{n-1} f(x_i) \le U(P_n, f).$$

   Hence, if we choose n sufficiently large, $\frac{1}{n}\sum_{i=1}^{n-1} f(x_i)$ is sufficiently small. Combining with the argument in previous paragraph, we can conclude that $\left|\int_0^1 f(x)dx - R_n\right| < \varepsilon$ for sufficiently large n. Thus, $\lim_{n\to\infty} R_n = \int_0^1 f(x)dx$.

   (b) Consider the Dirichlet fucntion $f : [0,1] \to \mathbf{R}$, where $f(x) = 1$ if $x \in \mathbf{Q}$ and $f(x) = 0$ if $x \notin \mathbf{Q}$. Then $\underline{\int_0^1} f dx = 0$ while $\overline{\int_0^1} f dx = 1$.

   With such f, we find $R_n = 0$ for all n but f is not Riemann integrable. □

3. (Exercise 14, §5.1) Find an example of functions $f : [0,1] \to \mathbf{R}$ which is Riemann integrable and $g : [0,1] \to [0,1]$ which is one-to-one and onto such that the composition $f \circ g$ is not Riemann integrable.

   Proof. Let $f : [0,1] \to \mathbf{R}$ so $f(x) = x$ and $g : [0,1] \to [0,1]$ such that $g(x) = x$ if $x \in \mathbf{Q}$ and $g(x) = 1 - x$ if $x \notin \mathbf{Q}$. It's not hard to check that g is indeed one-to-one and onto. Now $(f \circ g)(x) = g(x)$. Thus, it suffices to show g is not Riemann integrable on $[0,1]$.

Consider the partition $P_n = \left\{\frac{1}{2n}, \frac{2}{2n}, \ldots, \frac{2n}{2n}\right\}$ of $[0,1]$. Observe that

$$\inf g\left(\left[\frac{i}{2n}, \frac{i+1}{2n}\right]\right) = \begin{cases} \frac{i}{2n} & \text{if } 0 \le i \le n-1, \\ 1 - \frac{i+1}{2n} & \text{if } n \le i \le 2n-1 \end{cases}$$

Therefore,

$$L(P_n, g) = \frac{1}{2n}\left[\sum_{i=0}^{n-1} \frac{i}{2n} + \sum_{i=n}^{2n-1}\left(1 - \frac{i+1}{2n}\right)\right] = \frac{n-1}{4n}.$$

On the other hand, we have

$$\sup g\left(\left[\frac{i}{2n}, \frac{i+1}{2n}\right]\right) = \begin{cases} 1 - \frac{i}{2n} & \text{if } 0 \le i \le n-1, \\ \frac{i+1}{2n} & \text{if } n \le i \le 2n-1. \end{cases}$$

Therefore,

$$U(P_n, g) = \frac{1}{2n}\left[\sum_{i=0}^{n-1}\left(1 - \frac{i}{2n}\right) + \sum_{i=n}^{2n-1} \frac{i+1}{2n}\right] = \frac{3n-1}{4n}.$$

theo:integral_criterion_par

Thus, as $n \to \infty$, $L(P_n, f) \to 1/4$ while $U(P_n, f) \to 3/4$. According to theorem 5.2.1.2, g is not Riemann integrable on $[0,1]$. □

4. (Exercise 12, §5.2) Define $f : [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1/k & x = m/k \text{ with } m, k \in \mathbf{Z}_{\ge 1}, \gcd(m,k) = 1, \\ 0 & x \notin \mathbf{Q}. \end{cases}$$

Show that f is Riemann integrable. In particular, f is a discontinuous function at all rational numbers that is Riemann integrable.

Proof. It's not hard to show that $L(P, f) = 0$ for any partition P of $[0,1]$. To show $\int_0^1 f(x)dx = 0$, it suffices to show $U(P, f)$ can be sufficiently small depending on choice of P.

Choose $n + 1 = \sum_{i=1}^{M} \varphi(i)$ and consider the partition $P = \left\{0, \frac{1}{n+1}, \ldots, \frac{n}{n+1}, 1\right\}$ of $[0,1]$. Note that $\sup f\left(\left[\frac{i}{n+1}, \frac{i+1}{n+1}\right]\right) = \frac{i}{k}$ where $1 \le i < k \le n$ and $\gcd(i,k) = 1$ if $\frac{i}{k} \in \left[\frac{i}{n+1}, \frac{i+1}{n+1}\right]$.

On the other hand, for each $k \le n$, there are $\varphi(k)$ numbers of the form $\frac{i}{k}$ so $\gcd(i,k) = 1$ and $1 \le i \le n$. We also have $n + 1 = \sum_{i=1}^{M} \varphi(i)$ so we find

$$\begin{aligned} U(P_M, f) &= \frac{1}{n+1}\sum_{i=0}^{n} \sup f\left(\left[\frac{i}{n+1}, \frac{i+1}{n+1}\right]\right), \\ &\le \frac{1}{n+1}\sum_{i=1}^{M} \frac{\varphi(i)}{i}. \end{aligned}$$

Since $\frac{\varphi(i)}{i} \le 1$ so $\sum_{i=1}^{M} \frac{\varphi(i)}{i} \le M$. On the other hand, $n+1 = \sum_{i=1}^{M} \varphi(i)$. Denote S_i set of all positive integers that are coprime to i and are less than $i+1$. We then have $|S_i| = \varphi(i)$, which means $\sum_{i=1}^{M} \varphi(i) = \sum_{i=1}^{M} |S_i|$. Now 1 appears in all M sets, 2 appears in at least $M - M/2$ sets, 3 appears in at least $M - M/3$ sets, ... This follows

$$\begin{aligned}\sum_{i=1}^{M} |S_i| &\ge \sum_{i=1}^{M} \left(M - \frac{M}{i} \right), \\ &= M^2 - M \sum_{i=1}^{M} \frac{1}{i}, \\ &\ge M^2/2 - M.\end{aligned}$$

With this, we obtain

$$U(P_M, f) \le \frac{1}{n+1} \sum_{i=1}^{M} \frac{\varphi(i)}{i} \le \frac{1}{M/2 - 1}.$$

Therefore, as $M \to \infty$ then $U(P_M, f) \to 0$. This implies $U(P_M, f) - L(P_M, f) = U(P_M, f) \to 0$ as $M \to \infty$. We conclude f is Riemann integrable on $[0, 1]$. Since $\underline{\int_0^1} f(x)dx = \sup\{L(P, f)\} = 0$ so $\int_0^1 f(x)dx = 0$. $\square$

5.9 Spivak's Calculus exercises

5.10 More exercises

http://home.iitk.ac.in/~psraj/mth101/practice-problems/pp17.pdf.

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Chapter 6

Sequence and series of functions

Functions mentioned in this chapter are complex-valued functions unless stated otherwise.

6.1 Discussion of main problem

Example 6.1.0.1

(From [1, §7.5]) Differentiation is not preserved. Let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ for $x \in \mathbf{R}, n = 1, 2, \ldots$, and $f(x) = \lim_{n\to\infty} f_n(x) = 0$ so $f'(x) = 0$, and $f_n'(x) = \sqrt{n}\cos nx$, so that $\{f_n'\}$ does not converge to f'.

Example 6.1.0.2

(From [1, §7.6]) Limit of the integral need not to be equal to the integral of limit. Consider $f_n(x) = n^2x(1-x^2)^n$ for $0 \le x \le 1, n = 1, 2, \ldots$.

6.2 Uniform convergence

Remark 6.2.0.1. A post from [5, MathStackExchange] gives a nice perspective about the difference between pointwisely convergence and uniformly convergence for sequence of functions

> Pointwise convergence means at every point the sequence of functions has its own speed of convergence (that can be very fast at some points and very very very very slow at others).
>
> Uniform convergence means there is an overall speed of convergence.

Theorem 6.2.0.2 (Cauchy criterion for uniform convergence) The sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N, n \geq N, x \in E$ implies $|f_n(x) - f_m(x)| < \varepsilon$.

Proof. We prove the converse. Fix $x \in E$ then for any $\varepsilon > 0$, we have $|f_n(x) - f_m(x)| < \varepsilon$ so $\{f_n(x)\}$ is a Cauchy sequence in E (which is subset of some compact metric space or $\mathbf{R}^k$) so it converges to $f(x)$ according to theorem 3.3.0.4. We conclude that f_n converges pointwise to f on E.

It suffices to show that f_n converges uniformly to f on E. Indeed, fix $\varepsilon > 0$ then there exists N such that $|f_n(x) - f_m(x)| < \varepsilon/2$ for all $m, n \geq N$ and $x \in E$. On the other hand, since $\lim_{n\to\infty} f_n(x) = f(x)$ so there exists $M > N$ such that $|f_m(x) - f(x)| < \varepsilon$ for $m > M$. This follows for any $m > M, n > N$, we have

$$|f_n(x) - f(x)| \leq |f_m(x) - f(x)| + |f(x)_m(x) - f_n(x)| < \varepsilon.$$

Note that N remains unchange for every x so this is what we want. □

Here is another criterion for uniform convergence.

Theorem 6.2.0.3

Suppsose $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$. Put $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on E iff $M_n \to 0$ as $n \to \infty$.

Foe series, there is a convenient way to test for convergence

Theorem 6.2.0.4 Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ for all $x \in E, n = 1, 2, \ldots$ Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges

Example 6.2.0.5 (Pointwise convergence but not uniformly) Consider the sequence of functions $f_n(x) = \sum_{i=0}^n x^i$ which converges to $f(x) = \frac{1}{1-x}$ on $(-1, 1)$. However, this sequence of function does not converge uniformly. Vaguely speaking, as $x \to 1$, the "speed" at which the sequence of functions converges is getting slower and slower (see remark 6.2.0.1). However, we want uniform speed for uniformly convergent sequence of functions.

For a more rigorous argument, note that

$$|f(x) - f_n(x)| = \left|\frac{x^{n+1}}{1-x}\right|,$$

and $\limsup_{x\to 1} \left|\frac{x^{n+1}}{1-x}\right| = \infty$. This implies for any n and any $\varepsilon > 0$, there exists x sufficiently close to 1 such that $|f(x) - f_n(x)| > \varepsilon$. This implies the sequence of functions does not converge uniformly.

6.3 Uniform convergence and continuity

Theorem 6.3.0.1 Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x a limit point of E, and suppose that $\lim_{t\to x} f_n(t) = A_n$ for $n = 1, 2, \ldots$ Then $\{A_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. In other words, the conclusion is that

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

Sketch. $\{A_n\}$ is uniformly convergent from the fact that $|A_m - A_n| = \lim_{t\to x} |f_n(t) - f_m(t)| < \varepsilon$. This follows $\{A_n\}$ converges to A.

To show the latest limit, note the inequality

$$|A - f(t)| \le |A - A_n| + |A_n - f_n(t)| + |f_n(t) - f(t)|.$$

$\square$

From this theorem 6.3.0.1, we obtain an important collorary.

Theorem 6.3.0.2 If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

In particular, if f_n is continuous at $x \in E$ then f is continuous at x.

Proof. From the conditions, we know $\lim_{t\to x} f_n(t) = f_n(x)$. Hence, by applying theorem 6.3.0.1, we find $\lim_{t\to x} f(x) = \lim_{n\to\infty} f_n(x) = f(x)$. $\square$

Remark 6.3.0.3. The converse of this theorem 6.3.0.2 is not true, i.e. if a sequence of continuous functions converges to a continuous function then it does not imply that this convergence is uniform. Example 6.1.0.2 is of this kind. However, there is a case in which we can assert the convergence.

Theorem 6.3.0.4 (Dini's theorem)

Suppose K is compact, and

1. $\{f_n\}$ is a sequence of continuous functions on K,
2. $\{f_n\}$ converges pointwise to a continuous function f on K.
3. $f_n(x) \ge f_{n+1}(x)$ for all $x \in K, n = 1, 2, \ldots$

Then $f_n \to f$ uniformly on K.

Proof. Let $g_n = f_n - f$ then g_n is continuous and $g_n \to 0$ pointwise and $g_n \ge g_{n+1}$. We need to prove that $g_n \to 0$ uniformly on K.

Indeed, let K_n be set of all $x \in K$ such that $g_n(x) \geq \varepsilon$ then from theorem 2.3.0.4, K_n is closed in a compact set K so it is compact. On the other hand, we have $g_n \geq g_{n+1}$ so $K_n \supset K_{n+1}$. Theorem 1.3.2.2 implies that there exists N such that K_N is empty. This means $g_n(x) < \varepsilon$ for all $n > N$ and for all $x \in K$. We obtain that $g_n \to 0$ uniformly on K. $\square$

Remark 6.3.0.5. Compactness is needed for this. Example see [1, §7.13].

Definition 6.3.0.6. If X is a metric space, $\mathscr{C}(X)$ will denote set of all complex-valued, continuous, bounded functions with domain X.

(Note that if X is compact, boundedness is not needed).

We associate each $f \in \mathscr{C}(X)$ its **supremum norm** $\|f\| = \sup_{x \in X} |f(x)|$. Since f is bounded so $\|f\| < \infty$. Note that we have the inequality $\|f + g\| \leq \|f\| + \|g\|$. Hence, if we define the distance between $f \in \mathscr{C}(X)$ and $g \in \mathscr{C}(X)$ to be $\|f - g\|$, then we've defined a metric space $\mathscr{C}(X)$ with this distance.

With this, theorem 6.2.0.2 can be rephrased as follows:

> A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.

Closed subsets of $\mathscr{C}(X)$ are called **uniformly closed**, the closure of a set $\mathscr{A} \subset \mathscr{C}(X)$ is called its **uniform closure**, and so on.

Theorem 6.3.0.7

$\mathscr{C}(X)$ is a complete metric space with metric defined above.

Example 6.3.0.8

Consider sequence of functions $f_n(x) = \frac{n^2 x}{n^2 x^2 + 1}$ for $x \in \mathbf{R}$. We find that $f_n \to \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ pointwisely. Furthermore, f_n is continuous on $\mathbf{R}$ so theorem 6.3.0.2 implies $\{f_n\}$ does not converge uniformly. In fact, the convergence is not uniform for any interval containing $x = 0$. This is beacause as $x \to 0$, the speed at which the sequence of functions converges is getting slower and slower.

Question 6.3.0.9. Say if we know $\{f_n\}$ converges uniformly to f. We also know that f is continuous on E and there exists a subsequence of functions $\{f_{n_k}\}_{k \geq 1}$ that are also continuous. Can we say anything about the continuity of other functions f_n's ?

6.4 Uniform convergence and integration

Theorem 6.4.0.1 Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$, for $n = 1, 2, \ldots$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and

$$\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha.$$

Question 6.4.0.2. Can this theorem 6.4.0.1 be stated for improper integrals? See [1, Exercise 12, §7]

Corollary 6.4.0.3
If $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \qquad (a \le x \le b),$$

the series converges uniformly on $[a,b]$, then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha.$$

Proof. Let $\{g_m\}$ be a sequence of functions such that $g_m(x) = \sum_{n=1}^m f_n(x)$ then $g_m \to f$ uniformly on $[a,b]$. Since $g_m \in \mathscr{R}(\alpha)$ so from theorem 6.4.0.1, we find

$$\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b g_n d\alpha = \lim_{m\to\infty} \sum_{n=1}^{m} \int_a^b f_n d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha.$$

□

6.5 Uniform convergence and differentiation

Example 6.1.0.1 shows that uniform convergence of $\{f_n\}$ implies nothing about the sequence $\{f_n'\}$. Thus, we need stronger hypothesis for the assertion that $f_n' \to f'$ if $f_n \to f$.

Theorem 6.5.0.1 Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f and $f'(x) = \lim_{n\to\infty} f_n'(x)$ $(a \le x \le b)$.

We will first prove an easier version of the problem, where there is an additional condition that the functions f_n's are continuous on $[a,b]$.

Proof when f_n''s are continuous. From the condition, say $\{f_n'\}$ converges uniformly to f on $[a,b]$.

Since f_n''s are continuous so f is continuous on $[a,b]$ according to theorem 6.3.0.2. Furthermore, since $f_n' \in \mathscr{R}$ so according to theorem 6.4.0.1, we find $f \in \mathscr{R}$. Therefore, according to theorem 5.3.0.1, there exists a differentiable function F such that $F' = f$ for all $x \in [a,b]$ and that $F(x_0) = \lim_{n\to\infty} f_n(x_0)$. With this, we have $F'(x) = f(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$.

It suffices to show that $\{f_n\}$ converges uniformly to F on $[a,b]$.

Since $f, f_n \in \mathscr{R}$ so according to theorem 6.4.0.1 and according to the Fundamental Theorem of Calculus 5.3.0.4, we obtain for any $y \in [a,b]$ then

$$F(y) - F(x_0) = \int_{x_0}^{y} f(x)dx = \lim_{n\to\infty} \int_{x_0}^{y} f_n'(x)dx = \lim_{n\to\infty} (f_n(y) - f_n(x_0)).$$

Since $F(x_0) = \lim_{n\to\infty} f_n(x_0)$ so from the above, we find $F(y) = \lim_{n\to\infty} f_n(y)$. As this is true for any $y \in [a,b]$, we follow f_n converges pointwise to F on $[a,b]$.

It suffices to show that f_n converges uniformly on $[a,b]$. Indeed, let $g_n = F - f_n$ then $g_n' \to 0$ uniformly and $g_n(x_0) \to 0$ as $n \to \infty$. We want to show that $g_n \to 0$ uniformly. According to Mean Value Theorem 4.2.0.3, for any $x \in [a,b]$, there exists y between x and x_0 such that $g_n(x) = g_n(x_0) + g_n'(y)(x - x_0)$. If we choose sufficient large n such that $|g_n(x_0)| < \varepsilon$ and $|g_n'(y)| < \varepsilon$, we can obtain $|g_n(x)| \le \varepsilon(1 + \max\{|a - x_0|, |b - x_0|\})$ for all $x \in [a,b]$. Hence, $g_n \to 0$ uniformly. $\square$

Proof of theorem 6.5.0.1. We will first prove that $\{f_n\}$ converges uniformly. Indeed, since $\{f_n'\}$ converges uniformly so from theorem 6.2.0.2, there exists N such that $|f_m'(x) - f_n'(x)| < \varepsilon$ for all $m, n > N$ and for all $x \in [a,b]$. This follows

$$\left| \int_{x_0}^{y} (f_m'(x) - f_n'(x))dx \right| \le \int_{x_0}^{y} |f_m'(x) - f_n'(x)|dx \le \varepsilon(b - a).$$

On the other hand, since f_m and f_n are differentiable so according to the Fundamental Theorem of Calculus 5.3.0.4, we have

$$\begin{aligned} \left| \int_{x_0}^{y} (f_m' - f_n')(x)dx \right| &= \left| f_m(y) - f_n(y) - (f_m(x_0) - f_n(x_0) \right|, \\ &\ge |f_m(y) - f_n(y)| - |f_m(x_0) - f_n(x_0)|, \ \forall y \in [a,b], m, n > N. \end{aligned}$$

We also have $\{f_n(x_0)\}_{n\ge1}$ converges so there exists $M > N$ such that $|f_m(x_0) - f_n(x_0)| < \varepsilon$ for all $m, n > M$. Thus, we obtain $|f_m(y) - f_n(y)| \le \varepsilon(b - a + 1)$ for all $m, n > N$ and for all $y \in [a,b]$. This implies that $\{f_n\}$ converges uniformly to some f according to theorem 6.2.0.2.

Next, we will show that f is differentiable on $[a,b]$ and that $f'(x) = \lim_{n\to\infty} f_n'(x)$. If we want to show that $f'(x)$ exists at some x, it suffices to find $\lim_{t\to x} \frac{f(x)-f(t)}{x-t}$. However, note that with fixed t then $\lim_{n\to\infty} \frac{f_n(x)-f_n(t)}{x-t} = \frac{f(x)-f(t)}{x-t}$. so our wanted value is actually a double limit. This suggests us to interchange the limit using theorem 6.3.0.1. Indeed:

Fix x, let $\phi_n(t) = \begin{cases} \frac{f_n(t)-f_n(x)}{t-x} & t \neq x, \\ f_n'(x) & t = x \end{cases}$ then we find that $\lim_{t\to x} \phi_n(t) = f_n'(x)$. Furthermore, we have

$$\lim_{n\to\infty} \frac{f_n(x)-f_n(t)}{x-t} = \frac{f(x)-f(t)}{t-x}.$$

Hence, it suffices to show that

$$\lim_{t\to x} \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t),$$

theo:seq_func_uniform_

as the LHS is equal to $f'(x)$ while the RHS is exactly $\lim_{n\to\infty} f_n'(x)$. According to theorem 6.3.0.1, we only need to show that $\{\phi_n\}$ converges uniformly on $[a,b]$. Indeed, note the following inequality

$$|\phi_n(t) - \phi_m(t)| \le \left| \frac{f_n(x)-f_n(t)}{x-t} - f_n'(x) \right| + |f_n'(x) - f_m'(x)| + \left| f_m'(x) - \frac{f_m(x)-f_m(t)}{x-t} \right|.$$

Since $\{f_n'\}$ converges uniformly and f_n is differetiable for all n so there exists $\delta > 0$ and $N > 0$ such that $|\phi_n(t) - \phi_m(t)| < \varepsilon$ for all $n, m > N$ and for all t such that $0 < |t-x| < \delta$.

If $|t-x| \ge \delta$ then note that

$$|\phi_n(t) - \phi_m(t)| \le \frac{|f_n(x)-f_m(x)| + |f_n(t)-f_m(t)|}{|x-t|}.$$

Since $\{f_n\}$ converges uniformly so there exists $M > N$ so $|\phi_n(t) - \phi_m(t)| < \varepsilon$ for all $n, m > M$ and for all t so $|t-x| \ge \delta$.

Thus, there always exist N such that $|\phi_n(t) - \phi_m(t)| < \varepsilon$ for all $n, m > N$ and for all $t \in [a,b]$. This concludes that $\{\phi_n\}$ converges uniformly, as desired. □

Theorem 6.5.0.2

There exists a real continuous function on the real line which is nowhere differentiable.

6.6 Equicontinuous families of functions

Every bounded sequence of complex numbers contiains a convergent subsequence. Is this true for sequence of functions?

Definition 6.6.0.1 (Boundedness for sequence of functions)**.** Let $\{f_n\}$ be a sequence of functions defined on a set E. We say that $\{f_n\}$ is **pointwise bounded** on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E such that $|f_n(x)| < \phi(x)$ for all $x \in E, n = 1, 2, \ldots$

We say that $\{f_n\}$ is **uniformly bounded** on E if there exists a number M such that $|f_n(x)| < M$ for all $x \in E, n = 1, 2, \ldots$

convergent_subsequence **Theorem 6.6.0.2** (Pointwisely bounded sequence on a countable set $\implies$ convergent subsequence) If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Example 6.6.0.3 (Uniformly bounded sequence $\not\Rightarrow$ convergent subsequence)
In particular, even if the $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, there need not exist subsequence which converges pointwise on E.

baby_rudin
(From [1, §7.20]) Let $f_n(x) = \sin nx$ for $0 \le x \le \pi, n = 1, 2, \dots$

Example 6.6.0.4 (Convergent sequence $\not\Rightarrow$ uniformly convergent subsequence)
baby_rudin
(From [1, §7.21]) Let $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ for $0 \le x \le 1$ and $n = 1, 2, \dots$.

We define a new concept equicontinuity.

Definition 6.6.0.5. A family $\mathscr{F}$ of continuous functions f defined on a set E in a metric space X is said to be **equicontinuous** on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenevery $d(x, y) < \delta, x \in E, y \in E$ and $f \in \mathscr{F}$. Here d denotes the metric of X.

ty_uniform_convergence **Theorem 6.6.0.6** (Uniformly convergence $\Rightarrow$ equicontinuity) If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, \dots$, and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Sketch. The main inequality is

$$|f_n(x) - f_n(y)| < |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$

The fact that $\{f_n\}$ being uniformly convergent on metric space $\mathscr{C}(K)$ gives us the existence of N such that $\|f_n - f_N\| < \varepsilon$ for all $n > N$.

The fact that f_N is continuous on compact metric space K implies f_N is uniformly continuous on K, which gives us δ such that $|f_N(x) - f_N(y)| < \varepsilon$ whenever $d(x, y) < \delta$.

With this, we've found a δ such that $|f_n(x) - f_n(y)| < 3\varepsilon$ for all $n > N$ and all $x, y \in K$ so $d(x, y) < \delta$. Finally, we need to deal with $n \le N$, which is not hard. $\square$

Theorem 6.6.0.7 (Equicontinuity on compact space $\implies$ Uniformly convergent subsequence) If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, \ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) $\{f_n\}$ is uniformly bounded on K,

(b) $\{f_n\}$ contains a uniformly convergent subsequence.

Sketch. Compactness is like finiteness so we can consider a clever finite subcover of K that gives the desired conclusion.

For (a), since $\{f_n\}$ is equicontinuous, there exists δ such that $|f_n(x) - f_n(y)| < \varepsilon$ for all $d(x, y) < \delta$ and for all $n = 1, 2, \ldots$ This follows if K has finite subcovers of radius 2δ then from that, we can choose our bound.

For (b), the difficult part is to point out the uniform convergent subsequence. However, note that theorem 6.6.0.2 shows it's possible for countable set. Furtheremore, since K is compact so it has countable base E 1.3.2.7 so we can point out a convergent subsequence by using theorem 6.6.0.2 to E. Thus, it remains to prove such sequence is uniform convergent on K, which can be dealt by using the compactness of K and the fact that E is dense. $\square$

6.7 The Stone-Weierstrass theorem

Theorem 6.7.0.1 (Weierstrass theorem) If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that

$$\lim_{n \to \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

Motivation through probability. This is from MathOverflow [4]. The proof motivates from analysing the following game:

Let f be a continuous function on $[0, 1]$, and run n independent yes/no experiments in which the "yes" probability is x. Pay the gambler $f(m/n)$ if the answer "yes" comes up m times. Since the probability of getting "yes" is x, according to the Law of Large numbers, as $n \to \infty$, number of times getting "yes" out of n times is around nx (since the probability is x for getting yes), which means $m \approx nx$ for large n which means the expected payment received is around $f(x)$.

On the other hand, for fixed n, the gambler's expected gain from this is, of course,

$$P_n(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1 - x)^{n-k}.$$

This is known as the Bernstein polynomial. Thus, it is suggested that that $p_n \to f$ as $n \to \infty$. $\square$

Proof 1. It suffices to consider f on $[0,1]$ as we can take $g(x) = f((b-a)x+a)$. Since f continuous on $[0,1]$ so there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \delta$. Let $M = \sup |f(x)|$.

We first need the following identity:

$$\sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n}, \; 0 \le x \le 1.$$

This can be achieved by noticing the identity $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n$ and we differentiate twice both sides of this wrt p. After that, let $p = x, q = 1-x$.

After this, we have

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} - f(x) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \right|, \\
&\le \sum_{k=0}^{n} |f(k/n) - f(x)| \binom{n}{k} x^k (1-x)^{n-k}, \\
&= \sum_{|k/n - x| < \delta} |f(k/n) - f(x)| \binom{n}{k} x^k (1-x)^{n-k}, \\
&+ \sum_{|k/n - x| \ge \delta} |f(k/n) - f(x)| \binom{n}{k} x^k (1-x)^{n-k}, \\
&\le \varepsilon + 2M \sum_{|k/n - x| \ge \delta} \binom{n}{k} x^k (1-x)^{n-k}, \\
&\le \varepsilon + \frac{2M}{\delta^2} \sum_{k=0}^{n} \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}, \\
&= \varepsilon + \frac{2M}{\delta^2} \cdot \frac{x(1-x)}{n}, \\
&\le \varepsilon + \frac{2M}{4n\delta^2}.
\end{aligned}$$

Thus, with sufficient large n, then $|P_n(x) - f(x)| < \varepsilon$ for all x. We conclude $P_n \to f$ uniformly. □

Remark 6.7.0.2. Comparing with Taylor's theorem, now we can approximate a continuous function without needing to know whether it's smooth or not.

ef:seq_func_algebra **Definition 6.7.0.3.** A family $\mathscr{A}$ of complex functions defined on a set E is said to be an **algebra** if $f + g \in \mathscr{A}$, $fg \in \mathscr{A}$ and $cf \in \mathscr{A}$ for all $f, g \in \mathscr{A}$ and for all complex contants c.

We can also consider algebras of real functions; in this case, the third condition is required to hold for all real c.

If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ $(n = 1, 2, \ldots)$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be **uniformly closed**.

Let $\mathscr{B}$ be the set of all functions which are limits of uniform convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the **uniform closure** of $\mathscr{A}$.

The set of polynomials is an algebra and Weierstrass theorem 6.7.0.1, says that the set of all continuous functions on $[a,b]$ is the uniform closure of the set of polynomials on $[a,b]$.

Definition 6.7.0.4. Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to **seperate points** of E is to every pairs of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ **vanishes at no point** of E.

Example 6.7.0.5

The algebra of all polynomials in one variable has these properties. Set of all even polynomials, i.e. $f(x) = f(-x)$, does not seperate points.

Theorem 6.7.0.6

Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ seperates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is real algebra). Then $\mathscr{A}$ contains a function f such that $f(x_1) = c_1, f(x_2) = c_2$.

This is Stone's generalisation of Weierstrass theorem

stone_weierstrass_real **Theorem 6.7.0.7** (Stone-Weierstrass theorem) Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ seperates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.

The theorem is not true for complex algebras. For a counterexample for this, see [1, Exercise 21, §7] (21). We need an extra condition for $\mathscr{A}$ for this to be true: $\mathscr{A}$ must be **self-adjoint**, namely for every $f \in \mathscr{A}$ its complex conjugate $\overline{f}$ must also belong to $\mathscr{A}$ where $\overline{f}$ is defined by $\overline{f}(x) = \overline{f(x)}$.

ne_weierstrass_complex **Theorem 6.7.0.8** Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ seperates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense in $\mathscr{C}(K)$.

6.8 Baby Rudin exercises

exer:rudin_chap7_1

1. Let $\{f_n\}$ be a uniformly convergent sequence of bounded functions defined on E. Suppose $\{f_n\}$ converges uniformly to f then according to the definition, there exists N sucht that $\|f - f_n\| < \varepsilon$ for all $n > N$. This implies two things: one is that f is bounded on E since $\|f\| \le \|f_n\| + \varepsilon$ for some $n > N$ and f_n is bounded; two is that $\{f_n\}_{n>N}$ is uniformly bounded since $\|f_n\| \le \|f\| + \varepsilon$ for all $n > N$. With this, we can conclude that $\{f_n\}$ is uniformly bounded.

exer:rudin_chap7_2

2. For $\{f_n\}$ and $\{g_n\}$ converges uniformly to f and g on E, respectively, there exists N such that $\|f_n - f\| < \varepsilon$ and $\|g_n - g\| < \varepsilon$ for all $n > N$. This implies,
$$\|f_n + g_n - (f+g)\| \le \|f_n - f\| + \|g_n - g\| < 2\varepsilon,$$
for all $n > N$. Since this is true for any ε, we find $\{f_n + g_n\}$ converges uniformly on E.

   If $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, there exists M such that $\|f_n\| < M$ and $\|g_n\| < M$. This follows for sufficient large n then
$$|f_n(x)g_n(x) - f(x)g(x)| \le |f_n(x)(g_n(x) - g(x)| + |g_n(x)(f_n(x) - f(x)| < 2M\varepsilon.$$
Hence, $\{f_ng_n\}$ converges uniformly on E.

exer:rudin_chap7_3

3. Exercise 2 suggests that we need to choose $\{f_n\}$ to be sequence of unbounded functions. Let $f_n(x) = \frac{1}{x} + \frac{1}{n}$ and $g_n(x) = x$ then $\{f_n\}$ and $\{g_n\}$ converges uniformly on $\mathbf{R}$. On the other hand, the sequence $\{f_ng_n\} = \{1 + x/n\}_n$ does not converge uniformly on $\mathbf{R}$.

exer:rudin_chap7_4

4. If $x > 0$ then $\left|\frac{1}{1+n^2x}\right| < \frac{1}{n^2x}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2x}$ converges absolutely so the series $\sum_{n=1}^{\infty} \frac{1}{n^2x+1}$ converges absolutely according to the comparison test 3.6.0.2.

   If $x < 0, x \neq \frac{-1}{n^2}$ for all $n \in \mathbf{Z}^+$ then we have $\sum_{n=1}^{\infty} \frac{1}{n^2x+1} = -\sum_{n=1}^{\infty} \frac{1}{n^2|x|-1}$. For sufficient large n then $\left|\frac{1}{n^2|x|-1}\right| < \frac{2}{n^2|x|}$ so we also find $\sum_{n=1}^{\infty} \frac{1}{n^2x+1}$ converges absolutely.

   If $x = 0$ then the series diverges, while if $x = \frac{-1}{n^2}$ for some $n \in \mathbf{Z}^+$ then $\frac{1}{n^2x+1}$ is undefined.

   From theorem 6.2.0.4, since $g_n(x) = \frac{1}{n^2x+1} < \frac{1}{n^2x}$ for $x > 0$ so $\sum g_n$ converges uniformly on $(0, \infty)$.

   With $(-\infty, -1)$, it is different from previous interval because $g_1(x) = \frac{1}{1+x}$ is unbounded on $(-\infty, -1)$. However, with the similar argument by using theorem 6.2.0.4, $\sum_{n>1} g_n$ converges uniformly to some g on this interval. Hence, $\sum_{n\ge 1} g_n$ converges uniformly to f on this interval. Similar argument can be made for $\left(-\frac{1}{n^2}, -\frac{1}{(n+1)^2}\right)$ where $n \in \mathbf{Z}^+$. The series fails to converge (uniformly) on intervals that contains 0.

   Since g_n is continuous on each of the intervals $(0,\infty), (-\infty, -1)$ and $\left(\frac{-1}{i^2}, \frac{-1}{(i+1)^2}\right)$ where the series converges, according to theorem 6.3.0.2, f is continuous on such intervals.

Observe that g_n is bounded on $(0,\infty)$ so f is bounded on $(0,\infty)$ according to exercise 1. exer:rudin_chap7_1

Since for a fixed n, g_n is unbounded on $\left(\frac{-1}{n^2}, \frac{-1}{(n+1)^2}\right)$ so we can choose x sufficiently close to $\frac{-1}{n^2}$ to make $g_n(x)$ sufficient large and therefore make $f(x)$ sufficient large. Thus, f is not bounded on $\left(\frac{-1}{n^2}, \frac{-1}{(n+1)^2}\right)$ or $(-\infty,-1)$.

exer:rudin_chap7_5 5. It's not hard to see that for any $x \in \mathbf{R}$ then $\lim_{n\to\infty} f_n(x) = 0$ so $\{f_n\}$ converges to 0. It does not converge uniformly because for any n, $f_n\left(\frac{2}{2n+1}\right) = 1$.

The series $\sum f_n$ converges absolutely for all x as there exists N such that $x > 1/N$ (or $x < 1/N$) which implies $f_n(x) = 0$ for $n > N$. However, the series $\sum f_n$ does not converge uniformly since for any m, n, we have $\sum_{i=m}^n f_i\left(\frac{2}{2m+1}\right) > 1$, which contradicts the Cauchy criterion theo:seq_func_uniform_cauchy 6.2.0.2 for uniformly convergence.

exer:rudin_chap7_6 6. Let $f_n(x) = (-1)^n \frac{x^2+n}{n^2}$ and let E be a bounded interval where $|x| < K$ for all $x \in E$. Since the two series $\sum \frac{1}{n^2}$ and $\sum \frac{(-1)^n}{n^2}$ converges so there exists N such that for all $m, n > N, n > m$ then

$$\left|\sum_{i=m}^n (-1)^i \frac{x^2+i}{i^2}\right| \le K^2 \sum_{i=m}^n \frac{1}{i^2} + \left|\sum_{i=m}^n \frac{(-1)^i}{i}\right| < \varepsilon.$$

According to the Cauchy criterion theo:seq_func_uniform_cauchy 6.2.0.2, $\sum (-1)^n \frac{x^2+n}{n}$ converges uniformly. The series obviously does not converge absolutely for any x since $\sum \frac{1}{n}$ diverges and $\left|(-1)^n \frac{x^2+n}{n^2}\right| > \frac{1}{n}$.

exer:rudin_chap7_7 7. By taking $n \to \infty$, one can guess that $f_n \to 0$ uniformly. Indeed, we have $1 + nx^2 \ge 2\sqrt{n}|x|$ so $|f_n - 0| = \left|\frac{x}{1+nx^2}\right| \le \frac{1}{2\sqrt{n}}$. This follows $\{f_n\}$ converges uniformly to 0 according to the definition.

Since f_n is differentiable on $\mathbf{R}$ so

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}, \; x \in \mathbf{R}.$$

One can see that for $x \ne 0$ then $\lim_{n\to\infty} f_n'(x) = 0 = f'(x) = 0$. However, $f_n'(0) = 1$ so $\lim_{n\to\infty} f_n'(0) = 1 \ne f'(0)$.

exer:rudin_chap7_8 8. Since $|c_n I(x - x_n)| \le |c_n|$ and that $\sum |c_n|$ converges so according to theorem theo:seq_func_uniform_series 6.2.0.4, the series $\sum_{n=1}^\infty c_n I(x - c_n)$ converges uniformly on $[a,b]$.

Observe that f is constant on interval (x_i, x_j) for some $i, j \in \mathbf{Z}^+$ so $x_i < x_j$ and there does not exist x_ℓ between x_i and x_j. This follows f is continuous on (x_i, x_j). Thus, f is continuous for every $x \ne x_n$.

exer:rudin_chap7_9 9. Since $\{f_n\}$ is a sequence of continuous functions which converges uniformly to f so according to theorem 6.3.0.2, f is continuous on E. This follows there exists N such that $|f(x)-f(x_n)|<\varepsilon$ for all $n>N$.

Since $f_n \to f$ uniformly so there exists $M>N$ such that $|f(x)-f_n(x)|<\varepsilon$ for all $n>M$. This follows $|f(x_n)-f_n(x_n)|<\varepsilon$ for all $n>M$. Therefore, for all $n>M$ then

$$|f(x)-f_n(x_n)| \le |f(x)-f(x_n)| + |f(x_n)-f_n(x_n)| < 2\varepsilon.$$

Thus, we find $\lim_{n\to\infty} f_n(x_n) = f(x)$.

The converse is not true. We will use example 6.2.0.5. Let $f_n(x)=\sum_{i=0}^n x^i$ then $\{f_n\}$ converge to $f(x)=\frac{1}{1-x}$ on $(-1,1)$. Furthermore, for any sequence $x_n \to x$, we have

$$\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} \frac{1-x_n^{n+1}}{1-x_n} = \frac{1-\lim_{n\to\infty} x_n^{n+1}}{1-\lim_{n\to\infty} x_n} = \frac{1}{1-x} = f(x).$$

However, as we know from example 6.2.0.5, the series does not converge uniformly on $(-1,1)$.

exer:rudin_chap7_10 10. For $\{x\}$ denoting the fractional part of x, let

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2}, \ (x \in \mathbf{R}).$$

The problem mentions continuity of f so this suggests us to apply theorem 6.3.0.2.

Note that $\lfloor nx \rfloor = i$ for $x \in \left[\frac{i}{n}, \frac{i+1}{n}\right)$ where $i \in \mathbf{Z}$. This follows $\{nx\} = nx - \lfloor nx \rfloor$ is continuous everywhere except at $x = i/n$ for any $i \in \mathbf{Z}$. Hence, $f_n(x) = \frac{\{nx\}}{n^2}$ is continuous everywhere except at $x=i/n$ for any $i \in \mathbf{Z}$. Thus, the functions f_n's are continuous at any $x \in \mathbf{R}\setminus\mathbf{Q}$. Since $\{f_n\}$ converges uniformly to f (using theorem 6.2.0.4) so according to theorem 6.3.0.2, we find that f is continuous at any $x \in \mathbf{R}\setminus\mathbf{Q}$.

From above observation, we suspect that f is not continuous at any $x \in \mathbf{Q}$. Observe that $f(x)=f(x+1)$ so it suffices to show f is not continuous at any $x \in \mathbf{Q}\cap[0,1]$. We can let $x=a/b$ where $0 \le a < b; a,b \in \mathbf{Z}$.

From previous observation, for $n \in \mathbf{Z}^+$ such that $b \nmid n$ then f_n is continuous at a/b. According to theorem 6.3.0.2, we conclude that $g(x) = \sum_{b\nmid n} f_n(x)$ is continuous at $x=a/b$. On the other hand, we have $\{(bk)x\}=0$ for any $k \in \mathbf{Z}$ so

$$f(x) = f\left(\frac{a}{b}\right) = \sum_{b\nmid n} f_n\left(\frac{a}{b}\right) = g\left(\frac{a}{b}\right) = g(x).$$

With $y = x - \frac{1}{m}$ for $m>b$, we have $\{yb\} = 1-\frac{b}{m}$. We also know that g is continuous at $x=a/b$ so there exists M such that for all $m>M$ then $|g(y)-g(x)|<\varepsilon$. Hence,

$$f(y) \ge g(y) + \frac{\{yb\}}{b^2} \ge g(x) - \varepsilon + \frac{1-b/m}{b^2} = f(x) - \varepsilon + \frac{1-b/m}{b^2}.$$

We obtain

$$\liminf_{m\to\infty} f(x-1/m)-f(x) \geq \frac{1}{b^2}.$$

This implies that f is not continuous at $x=a/b$. Thus, $\mathbf{Q}$ is the set of all discontinuities of f.

Since $f_n \in \mathscr{R}$ on any interval $[a,b]$ where $a<b$ and as $f_n \to f$ uniformly, according to theorem 6.4.0.1, we find $f \in \mathscr{R}$.

exer:rudin_chap7_11 11. Let $A_n(x)$ be the n-th partial sum for $\sum f_n$. There exists M such that $|A_n(x)| \leq M$ for all n and all x. Since $g_n \to 0$ uniformly on E, there exists N such that $|g_n(x)| < \varepsilon$ for all $x \in E, n > N$. Furthermore, since $g_n(x) \geq g_{n+1}(x)$ so $g_n(x) \geq 0$ for all x and all n. For $N \leq p \leq q$, we have

$$\begin{aligned}\left|\sum_{n=p}^{q} f_n(x)g_n(x)\right| &= \left|\sum_{n=p}^{q-1} A_n(g_n(x)-g_{n+1}(x)) + A_q(x)g_q(x) - A_{p-1}(x)g_p(x)\right|, \\ &\leq M\left|\sum_{n=p}^{q-1}(g_n(x)-g_{n+1}(x)) + g_q(x) - g_p(x)\right|, \\ &= 2Mg_p(x) \leq 2M\varepsilon.\end{aligned}$$

According to the Cauchy criterion 6.2.0.2, $\sum f_n g_n$ converges uniformly on E.

exer:rudin_chap7_12 12. According to the Comparison test for improper integrals 5.4.0.9, since $\int_0^\infty g(x)dx < \infty$ and $|f_n| \leq g$ so $\int_0^\infty f_n(x)dx < \infty$. On the other hand, since $f_n \to f$ uniformly so we can obtain

$$|f(x)| \leq |f(x)-f_n(x)| + |f_n(x)| \leq \varepsilon + g(x).$$

As $\varepsilon \to 0$, we obtain $|f| \leq g$. Again, from the Comparison test, for improper integrals, we find $\int_0^\infty f(x)dx < \infty$.

Fix $a>0$, we will first show that

$$\lim_{n\to\infty}\int_a^\infty f_n(x)dx = \int_a^\infty f(x)dx.$$

To do this, we intend to use the following inequality

$$\begin{aligned}\left|\int_a^\infty f(x)dx - \int_a^\infty f_n(x)dx\right| &\leq \left|\int_a^\infty f(x)dx - \int_a^b f(x)dx\right| + \left|\int_a^b f(x)dx - \int_a^b f_n(x)dx\right|, \\ &+ \left|\int_a^b f_n(x)dx - \int_a^\infty f_n(x)dx\right|, \\ &= \left|\int_b^\infty f(x)dx\right| + \left|\int_b^\infty f_n(x)dx\right| + \left|\int_a^b f(x)dx - \int_a^b f_n(x)dx\right|\end{aligned}$$

We want to evaluate the RHS of this inequality to be small for sufficient large n. What do we know that holds for all large n? That is $|f_n| \le g$ for all n. Combining with $|f| \le g$, we find

$$\left| \int_b^\infty f(x)dx \right| \le \int_b^\infty |f(x)|dx \le \int_b^\infty g(x)dx, \left| \int_b^\infty f(x)dx \right| \le \int_b^\infty g(x)dx.$$

Luckily, since $\int_0^\infty g(x)dx < \infty$ so there exists b such that $\int_b^\infty g(x)dx < \varepsilon$. Hence, we obtain

$$\left| \int_a^\infty f(x)dx - \int_a^\infty f_n(x)dx \right| \le 2\varepsilon + \left| \int_a^b f(x)dx - \int_a^b f_n(x)dx \right|.$$

theo:seq_func_uniform_integration

On the other hand, according to theorem 6.4.0.1, we find that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

This follows we can choose sufficiently large n to obtain

$$\left| \int_a^\infty f(x)dx - \int_a^\infty f_n(x)dx \right| \le 2\varepsilon + \left| \int_a^b f(x)dx - \int_a^b f_n(x)dx \right| \le 3\varepsilon.$$

In the end, we find

$$\lim_{n\to\infty} \int_a^\infty f_n(x)dx = \int_a^\infty f(x)dx.$$

The proof when $a \to 0$ is completely similar.

theo:seq_func_bound_countable_convergent_subsequence

exer:rudin_chap7_13 13. (a) Note that we have theorem 6.6.0.2. In particular, we can say that there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges to $f(x)$ for every $x \in \mathbf{Q}$. We expand f to $\mathbf{R}$ be defining f such that $f(x) = \sup_{r\in\mathbf{Q}, r<x} f(r)$ for all $x \in \mathbf{R}$.

We first show that f is monotonically increasing on $\mathbf{R}$. First, we show that f is monotonically increasing on $\mathbf{Q}$. For any $x, y \in \mathbf{Q}$ such that $x < y$ then for any $\varepsilon > 0$, there exists N such that $|f(x) - f_{n_k}(x)| < \varepsilon$ and $|f(y) - f_{n_k}(y)| < \varepsilon$ for all $k > N$. This follows

$$f(x) < f_{n_k}(x) + \varepsilon \le f_{n_k}(y) + \varepsilon \le f(y) + 2\varepsilon.$$

Since this is true for any $\varepsilon > 0$ so we conclude $f(x) \le f(y)$. Now for $\mathbf{R}$, for any $x, y \in \mathbf{R}$ such that $x < y$, for any $\varepsilon > 0$, there exists $x_1, y_1 \in \mathbf{Q}$ such that $x_1 \le x < y_1 \le y$ and $|f(x_1) - f(x)| < \varepsilon$ and $|f(y_1) - f(y)| < \varepsilon$. Similarly, this follows $f(x) \le 2\varepsilon + f(y)$ which concludes $f(x) \le f(y)$.

For $x \in \mathbf{R}$ such that $f(x)$ is continuous at x, we claim that $f_{n_k}(x) \to f(x)$. Indeed, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for all $t \in \mathbf{R}$ such that $|t - x| < \delta$. By choosing $t \in \mathbf{Q}$, there exists N such that $|f(t) - f_{n_k}(t)| < \varepsilon$ for all $k > N$. This follows $|f(x) - f_{n_k}(t)| < \varepsilon$ for all $k > N$. If $t < x$, we find $f(x) < f_{n_k}(x) + \varepsilon$ for all $k \ge N$. Similarly for $t > x$ then $f(x) > f_{n_k}(t) - \varepsilon \geq f_{n_k}(x) - \varepsilon$. Since this is true for any ε so we conclude $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

Since f is monotonically increasing on $\mathbf{R}$ so according to theorem 2.6.0.2, there are at most countably many such points at which f is discontinuous. Let the set of such points be $X = \{x_i\}$. According to theorem 3.2.0.1, there exists a convergent subsequence $\{f_{1,k}(x_1)\}_{k\geq 1}$ of $\{f_{n_k}(x_1)\}$. We let $f(x_1) = \lim_{k\to\infty} f_{1,k}(x_1)$. Next, we choose a converget subsequence $\{f_{2,k}(x_2)\}$ from $\{f_{1,k}(x_2)\}$ and keep doing that. In the end, there exists a subsequence $\{f_j\}$ of $\{f_{n_k}\}$ that converges for every $x \in \mathbf{R}$.

(b) Since f is continuous on a compact set K so f is uniformly continuous on K according to theorem 2.3.1.1. This means for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in K$ such that $|x - y| < \delta$. If we fix two points $x_1, x_2 \in \mathbf{R}$ such that $x_2 - x_1 = \delta$ then there exists a fix N such that $|f_{n_k}(x_1) - f(x_1)| < \varepsilon$ and $|f_{n_k}(x_2) - f(x_2)| < \varepsilon$ for all $k > N$.

For any $y \in (x_1, x_2)$, we know that $|f(y) - f(x_1)| < \varepsilon$ and $|f(y) - f(x_2)| < \varepsilon$. Combining with two previous inequalities, we obtain $|f(y) - f_{n_k}(x_1)| < 2\varepsilon$ and $|f(y) - f_{n_k}(x_2)| < 2\varepsilon$ for all $k > N$. Since $x_1 < y < x_2$ so for all $k > N$, we have

$$f(y) - 2\varepsilon \leq f_{n_k}(x_2) - 2\varepsilon \leq f_{n_k}(y) \leq f_{n_k}(x_1) + 2\varepsilon \leq f(y) + 2\varepsilon$$

This concludes $|f(y) - f_{n_k}(y)| < \varepsilon$ for all $y \in (x_1, x_2)$ and all $k > N$. Combining with the condition that K is compact, we are done.

exer:rudin_chap7_14

14. Define functions $f_n(x) = 2^{-n} f(3^{2n-1}t)$ then f_n's are continuous on $\mathbf{R}$. Since $0 \leq f(x) \leq 1$ for all $x \in \mathbf{R}$ so $|f_n(x)| \leq 2^{-n}$. Since $\sum_{n=1}^{\infty} 2^{-n}$ converges so according to theorem 6.2.0.4, we find $\{f_n\}$ converges uniformly to Φ. Furthermore, since f_n's are continuous so according to theorem 6.3.0.2, Φ is continuous.

From [1, Exercise 19, §3] (19) we know that the set $C = \left\{x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} : a = \{\alpha_n\} = \{0,2\}^{\mathbf{N}}\right\}$ is the Cantor set. We follow the hint given, for $t_0 \in C$, we can write $t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$ where $a_i \in \{0, 1\}$. We have

$$\begin{aligned} f(3^k t_0) &= f\left(\underbrace{\sum_{i=1}^{k-1} 3^{k-i-1}(2a_i)}_{=2A, A\in\mathbf{Z}} + \frac{2a_k}{3} + \underbrace{\sum_{i=k+1}^{\infty} 3^{k-i-1}(2a_i)}_{\leq 2\sum_{j=2}^{\infty} 3^{-j} = 1/3}\right), \\ &= a_k. \end{aligned}$$

This follows $\Phi(t_0) = (x(t_0), y(t_0))$ where

$$x(t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, y(t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n}.$$

Since any $(x_0, y_0) \in I^2$ can be represented as $(x_0, y_0) = (x(t_0), y(t_0))$ for some sequence $\{a_n\} \in \{0,1\}^{\mathbf{N}}$ so we conclude Φ maps the Cantor set onto I^2.

exer:rudin_chap7_15 15. Since $\{f_n\}$ is equicontinuous on $[0,1]$ so for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| = |f(nx) - f(ny)| < \varepsilon$ whenever $|x-y| < \delta, x, y \in [0,1]$ and $n \in \mathbf{N}$.

We show that $|f(x) - f(y)| < \varepsilon$ for any $x, y \in \mathbf{R}, xy \geq 0$. Indeed, there exists $n \in \mathbf{Z}$ such that $\left|\frac{x}{n} - \frac{y}{n}\right| < \delta, \frac{x}{n}, \frac{y}{n} \in [0,1]$ so from the above statement, we find $\left|f\left(n \cdot \frac{x}{n}\right) - f\left(n \cdot \frac{y}{n}\right)\right| < \varepsilon$ or $|f(x) - f(y)| < \varepsilon$. Since this is true for any $\varepsilon > 0$ so in the end, $f(x) = c$ for all $x \in \mathbf{R}$.

exer:rudin_chap7_16 16. The main inequality is this

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(y) - f_m(x)|.$$

Note that the first and third absolute value on the right-hand side can be arbitrarily small as $d(x,y)$ is small. Thus, one can compare $|f_n(x) - f_m(x)|$ of x with $|f_n(y) - f_m(y)|$ of y. Since K is compact so we can select a finite set of such y's that helps to compare $|f_n(x) - f_m(x)|$ for all x. Below is the final proof version:

Since $\{f_n\}$ is equicontinuous so for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ whenever $x, y \in K, d(x,y) < \delta$ and $n \geq 1$. Since K is compact so there exists finitely many points $p_1, \ldots, p_r$ in K such that any $x \in K$ satisfies $d(x, p_i) < \delta$ for some $i \in [r]$. Since $\{f_n(p_i)\}_{n \geq 1}$ converges for any $i \in [r]$ so there exists N such that $|f_n(p_i) - f_m(p_i)| < \varepsilon$ for all $m, n > N$ and all $i \in [r]$. From this, we find for any $m, n > N$, for any $x \in K$, there exists p_i so $d(x, p_i) < \delta$ then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(p_i)| + |f_n(p_i) - f_m(p_i)| + |f_m(p_i) - f_m(x)| \leq 3\varepsilon.$$

Since this is true for any $\varepsilon > 0$ so we conclude $\{f_n\}$ converges uniformly on K.

exer:rudin_chap7_17 17. (Generalisation of uniformly convergence and equicontinuity for mappings to metric spaces or to $\mathbf{R}^k$)

exer:rudin_chap7_18 18. We aim to apply theorem theo:seq_func_equicontinuity_compact_convergent_subsequence 6.6.0.7.

From theorem theo:integral_differentiation 5.3.0.1, we know that F_n is continuous, or $F_n \in \mathscr{C}([a,b])$. Since $\{f_n\}$ is uniformly bounded so there exists $M > 0$ such that $|f_n(x)| < M$ for all n and all x. This follows

$$|F_n(x) - F_n(y)| = \left|\int_y^x f_n(t)dt\right| \leq M|y-x|,$$

where the last step uses property of integral theo:integral_property 5.2.0.1. This implies that $\{F_n\}$ is equicontinuous.

We have all our conditions satisfied so according to theorem theo:seq_func_equicontinuity_compact_convergent_s 6.6.0.7, there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a,b]$.

exer:rudin_chap7_19 19. If S is compact then according to theorem 1.3.2.5, every infinite subset $\{f_n\}$ of S has limit point in S. Hence, if $\{f_n\}$ converges uniformly to f then $f \in S$. This follows S is uniformly closed.

   Next, we show that S is pointwise bounded. Indeed, assume the contrary, there exists $x \in K$ such that $S(x) = \{f(x) : f \in S\}$ is unbounded, which means there exists a sequence $\{f_n(x)\}$ where $f_n \in S$ such that for all $M > 0$, there is N so $|f_n(x)| > M$ for all $n > N$. On the other hand, as S is compact, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges uniformly. This follows $\{f_{n_k}(x)\}$ converges or $\{f_{n_k}(x)\}$ is unbounded, a contradiction. Thus, $\{f_n(x)\}_{n\ge 1}$ is bounded for any $x \in K$. This implies S is pointwise bounded.

   Finally, we show S is equicontinuous. If it is not, then there exists $\varepsilon > 0$ such that for all $\delta > 0$ then $|f(x) - f(y)| \ge \varepsilon$ for some $x, y \in K, d(x,y) < \delta$ and $f \in S$. We will construct a sequence in S that is not equicontinuous. Indeed, we choose $\delta = 1/n$ to identify $f_n \in S$ such that $|f_n(x_n) - f_n(y_n)| \ge \varepsilon$ for some $x_n, y_n \in K$ such that $d(x_n, y_n) < 1/n$. Since $\{f_n\}$ does not have any equicontinuous subsequence so according to theorem 6.6.0.6, $\{f_n\}$ has no uniformly convergent subsequence on K.

   Conversely, if S is pointwise bounded and S is equicontinuous, according to theorem 6.6.0.7, every infinite subset of S has a limit point and since S is uniformly closed, this limit point is in S. Therefore, from theorem 1.3.2.5, we obtain that S is compact.

exer:rudin_chap7_20 20. Since f is continuous on $[0,1]$ so according to Weierstrass theorem 6.7.0.1, there exists a sequence of polynomial P_n such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a,b]$. This follows $P_n f \to f^2$ uniformly on $[a,b]$ (this is because f is continuous on $[a,b]$ so it is bounded on $[a,b]$, which makes $|P_n(x)f(x) - P_m(x)f(x)| \le M|P_n(x) - P_m(x)|$). Therefore, according to theorem 6.4.0.1, we find

$$\int_0^1 f^2(x)dx = \lim_{n\to\infty} \int_0^1 P_n(x)f(x)dx.$$

   On the other hand, we know that for any polynomial p then $\int_0^1 f(x)p(x)dx = 0$ so $\int_0^1 f(x)P_n(x)dx = 0$ for all $n = 1, 2, \ldots$ This implies $\int_0^1 f^2(x)dx = 0$ and as f is continuous on $[0,1]$, we conclude $f(x) = 0$ on $[0,1]$.

exer:rudin_chap7_21 21. $\mathscr{A}$ is the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}, \ (\theta \in \mathbf{R}).$$

   For any pair of distinct points $x_1 = e^{i\theta_1} \neq x_2 = e^{i\theta_2}$ then we can just choose $f(e^{i\theta}) = e^{i\theta}$ which will give $f(x_1) \neq f(x_2)$. This implies $\mathscr{A}$ seperates points on K. Furthermore, for each point $e^{i\theta_1}$, we can choose $f(e^{i\theta}) = C + e^{i\theta}$ for appropriate constant C such that $f(e^{i\theta_1}) \neq 0$. Thus, $\mathscr{A}$ vanishes at no point of K.

First, we show that for any $f \in \mathscr{A}$ then

$$\begin{aligned}\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta &= \int_0^{2\pi}\sum_{n=0}^N c_n e^{i\theta(n+1)}d\theta, \\ &= \sum_{n=0}^N c_n \int_0^{2\pi} (\cos(n+1)\theta + i\sin(n+1)\theta)d\theta, \\ &= 0.\end{aligned}$$

For any $f \in \overline{\mathscr{A}} \setminus \mathscr{A}$ where $\mathscr{A}$ is the uniform closure of $\mathscr{A}$, there exists a sequence of functions $\{f_n\}$ in $\mathscr{A}$ that converges uniformly to f on K. This follows $\{(f_n \circ g)g\}$ converges uniformly to $(f\circ g)g$ on $[0, 2\pi]$ where $g(\theta) = e^{i\theta}$ (this is because $|f_n(g(\theta))g(\theta) - f_m(g(\theta))g(\theta)| \le |f_n(e^{i\theta}) - f_m(e^{i\theta})|$ combining with the fact that $e^{i\theta} \in K$). We know that $(f_n \circ g)g \in \mathscr{R}$ on $[0, 2\pi]$ so according to theorem 6.4.0.1 (for vector-valued functions), we obtain

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta = \lim_{n\to\infty}\int_0^{2\pi} f_n(e^{i\theta})e^{i\theta}d\theta = 0.$$

Thus, any $f \in \overline{\mathscr{A}}$ must satisfy $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta = 0$.

Let $g(e^{i\theta}) = e^{-i\theta/2}$ be a continuous function on K then we have

$$\int_0^{2\pi} g(e^{i\theta})e^{i\theta}d\theta = \int_0^{2\pi} e^{i\theta/2}d\theta = 4i \neq 0.$$

This implies $g \notin \overline{\mathscr{A}}$ while g is continuous on K.

exer:rudin_chap7_22 22. [1, Exercise 12, §6] (12) suggests that there exists sequence of continuous functions $\{f_n\}$ such that $\|f_n - f\|_2 < \varepsilon$. By using Weierstrass' theorem 6.7.0.1, we can find polynomial P_n such that $|P_n(x) - f_n(x)| < 1/n$ for all $x \in [a, b]$ and all n, which implies $\int_a^b |P_n(x) - f_n(x)|^2 d\alpha \le (\alpha(b) - \alpha(a))/n^2$. From here, we find $\|P_n - f\|_2 \le \|P_n - f_n\|_2 + \|f_n - f\|_2 \le \varepsilon + \sqrt{\alpha(b) - \alpha(a)}/n$. This implies $\lim_{n\to\infty}\|P_n - f\|_2^2 = 0$, as desired.

exer:rudin_chap7_23 23. We have the following identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)]\left[1 - \frac{|x| + P_n(x)}{2}\right].$$

By induction on n, for $|x| \le 1$, we find that given $0 \le P_n(x) \le |x|$, one can show $P_n(x) \le$

$P_{n+1}(x) \le |x|$. Furthermore,

$$\begin{aligned} |x| - P_{n+1}(x) &= [|x| - P_n(x)]\left[1 - \frac{|x| + P_n(x)}{2}\right], \\ &\le [|x| - P_n(x)]\left(1 - \frac{|x|}{2}\right), \\ &\le \cdots \le |x|\left(1 - \frac{|x|}{2}\right)^n, \\ &< \frac{2}{n+1}. \end{aligned}$$

The last inequality can be obtained by taking derivative of $|x|\left(1 - \frac{|x|}{2}\right)^n$ to find the maximum (due to symmetry it's maximum on $[-1,1]$ when it's maximal on $[0,1]$, which implies that we can ignore the absolute value when taking derivative). The maximum occurs when $x = \frac{2}{n+1}$ and since $|x|\left(1 - \frac{|x|}{2}\right)^n < |x|$ so we are done. The inequality follows that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on $[-1,1]$.

exer:rudin_chap7_24 24. Since X is a metric space with metric d so $-d(a,p) \le d(x,p) - d(x,a) \le d(a,p)$ which implies $|f_p(x)| \le d(a,p)$. Thus, f_p is bounded for any $p \in X$. Next, we show f_p is continuous on X. Indeed, for any $x, y \in X$ so $d(x,y) < \delta$ we have

$$|f_p(x) - f_p(y)| \le |d(x,p) - d(y,p)| + |d(y,a) - f(x,a)| \le 2d(x,y) \le 2\varepsilon.$$

This follows f_p is continuous on X. Thus, $f_p \in \mathscr{C}(X)$.

We know that $\sup_{x\in X} |f_p(x) - f_q(x)| = \sup_{x \in X} |d(x,p) - d(x,q)| = d(p,q)$ so $\|f_p - f_q\| = d(p,q)$. This follows that $\Phi(p) = f_p$ is an isometry of X onto $\Phi(X) \subset \mathscr{C}(X)$.

def:metric_complete

Since $Y \subset \mathscr{C}(X)$ is closed and $\mathscr{C}(X)$ is complete so according to definition 3.3.0.6 about complete metric space.

exer:rudin_chap7_25 25. We want to construct such function f. The idea is to construct a sequence of functions $\{f_n\}$ that converges uniformly to f. We wish f to satisfy the initial value problem, i.e. $f'(x) = \phi(x, f(x))$ so we wish $\{f_n\}$ to also satisfy such property, i.e. $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \phi(x, f_n(x))$. The simplest way to make f_n integrable for every n is to make f_n' a piecewise-linear function on $[0,1]$. In particular, we can choose f_n so $f_n'(t) = \phi(x_i, f_n(x_i))$ for $t \in (x_i, x_{i+1})$ where $\{x_i\} = \{i/n : i = 0, \ldots, n\}$.

Now, how can we construct such f_n? Observe that since f_n' is a piecewise function so one could easily integrate both side to get $f_n(t) = \phi(0, f_n(0))t$ if $x_0 = 0 \le t \le x_1$ and then $f_n(t) = \phi(x_1, f_n(x_1))(t - x_1) + \phi(0, f_n(0))x_1$ for $x_1 \le t \le x_2$. This suggests us to define f_n recursively as follow:

$$\begin{gathered} f_n(0) = c, \\ f_n(t) = c + \phi(0,c)t, \ \forall t \in [0, x_1], \\ f_n(t) = f_n(x_i) + \phi(x_i, f_n(x_i))(t - x_i), \ (x_i \le t \le x_{i+1}). \end{gathered}$$

In the end, we've constructed a sequence of functions $\{f_n\}$ such that $f_n(0) = c$ and

$$f_n'(t) = \phi(x_i, f_n(x_i)), \text{ if } x_i < t < x_{i+1}.$$

We wish to show that $\{f_n\}$ (or at least a subsequence of this) converges uniformly to some function f that is the solutions for the initial value problem.

(a) We let $\Delta_n(t) = f_n'(t) - \phi(t, f_n(t))$ then for any $t \in [0,1], t \neq x_i$ then $\Delta_n(t) = 0$. except at x_i where $\Delta_n(x_i) = 0$. This implies $f_n'(t) = \Delta_n(t) + f_n'(t)$ for any $t \in [0,1], t \neq x_i$. Therefore, we find

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]dt.$$

Since ϕ is bounded so there exists $0 < M < \infty$ such that $|\phi| < M$. This easily implies that $|f_n'| \le M$ and $|\Delta_n| \le M$. Since f_n is continuous on $[0,1]$ so $g_n(x) = (x, f_n(x))$ is continuous on $[0,1]$. Since ϕ is continuous on the defined strip so $\phi \circ g_n$ is continuous on $[0,1]$. This follows $\phi(t, f_n(t))$ is integrable on $[0,1]$ according to theorem 5.1.0.7. In general, we find $\Delta_n \in \mathscr{R}$. We also find that f_n is bounded, i.e. $|f| \le |c| + M = M_1$ on $[0,1]$ for all n.

(b) Since $|f_n'| \le M$ for all n so

$$|f_n(x) - f_n(y)| = \left| \int_y^x f_n'(t)dt \right| \le M|y - x|.$$

This implies $\{f_n\}$ is equicontinuous on $[0,1]$.

(c) We know that $f_n \in \mathscr{C}([0,1])$, f_n is uniformly bounded and equicontinuous on K, theorem 6.6.0.7 implies that there exists some subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges uniformly to f on $[0,1]$.

(d) Since $|f_n| \le M_1$ for all n, we can restrict to the rectangle R so $0 \le x \le 1, |y| \le M_1$, which is a compact set. Since ϕ is continuous on this compact set so ϕ is uniformly continuous on this rectangle according to theorem 2.3.1.1.

Since $\{f_{n_k}\}$ converges uniformly to f, for any $\varepsilon > 0$, there exists N such that $|f(x) - f_{n_k}(x)| < \varepsilon$ for all $x \in [0,1]$ and all $k > N$. Since ϕ is uniformly continuous on R so $|\phi(t, f_{n_k}(t)) - \phi(t, f(t))| < \varepsilon'$ whenever, $|f_{n_k}(t) - f(t)| < \varepsilon$. This concludes $\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$ uniformly on $[0,1]$.

(e) We show $\Delta_n(t) \to 0$ uniformly on $[0,1]$. From (d), we find $\phi(t, f_n(t))$ is uniformly continuous on $[0,1]$ so there exists $\delta > 0$ such that $|\phi(x, f_n(x)) - \phi(y, f_n(y))| < \varepsilon$ whenever $|x - y| < \delta$. Therefore, there exists N such that $1/N < \delta$, which implies for $t \in (x_i, x_{i+1})$ and $n > N$ then

$$|\Delta_n(t)| = |\phi(x_i, f_n(x_i)) - \phi(t, f_n(t))| < \varepsilon.$$

This concludes that $\Delta_n \to 0$ uniformly on $[0,1]$.

theo:seq_func_uniform_integration

(f) From (d) and (e), using theorem 6.4.0.1, we find

$$f(x) = \lim_{n\to\infty} f_n(x) = c + \lim_{n\to\infty} \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]dt = c + \int_0^x \phi(t, f(t))dt.$$

In (c), we know that f exists. In (f), we know that $f(0) = c$ and $f'(x) = \phi(x, f(x))$. This solves the problem.

exer:rudin_chap7_25

exer:rudin_chap7_26 26. Similar to previous exercise 25. Note that $\Phi(x, \mathbf{y}) = (\phi(x, y_1), \dots, \phi(x, y_k))$. We want to find a function $\mathbf{f} : \mathbf{R} \to \mathbf{R}^k$ where $\mathbf{f} = (f_1, \dots, f_k)$ with $f_1'(x) = \phi(x, f_1(x))$ and $f_1(0) = c_1$. Using exer:rudin_chap7_25 exercise 25, we can find such $f_i : \mathbf{R} \to \mathbf{R}$ which can help us to find $\mathbf{f}$.

6.9 Extra exercises

1. (MSE) Does there exist a continuous function $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 xf(x)\, \mathrm{d}x = 1$ and $\int_0^1 x^n f(x)\, \mathrm{d}x = 0$ for all $n > 1$?

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Chapter 7

Special functions

7.1 Power series

A function f is called **analytic** if it is equal to $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$. If this series converges for all $|x-a| < R$ for some $R > 0$ (can be $+\infty$) then f is said to be expanded in a power series about the point $x = a$. We restrict to real-valued x so we will encounter intervals of convergence instead of circles of convergence.

nc_powerseries_uniform **Theorem 7.1.0.1** Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < R$. Then $\sum c_n x^n$ converges uniformly on $[-R+\varepsilon, R+\varepsilon]$ no matter which $\varepsilon > 0$ is chosen. The function f is continuous and differentiable in $(-R, R)$ and

$$f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \ (|x| < R).$$

The proof is quite straightforward where we try to use theorem theo:seq_func_uniform_differentiation 6.5.0.1 about differentiation for uniformly convergent sequence of functions.

Proof. Let $\varepsilon > 0$ be given then for any $x \in [-R+\varepsilon, R-\varepsilon]$, we have $|x| \le |R-\varepsilon|$. This follows $|c_n x^n| \le |c_n||R-\varepsilon|^n$ and since $\sum c_n (R-\varepsilon)^n$ converges absolutely (in the interval of convergence) so according to theorem theo:seq_func_uniform_series 6.2.0.4, we find $\sum c_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$.

To show f is differentiable on $(-R, R)$, knowing that the series converges uniformly, we aim to apply theorem theo:seq_func_uniform_differentiation 6.5.0.1.

Since $\limsup_{n\to\infty} \sqrt[n]{n} = 1$ so

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \limsup_{n\to\infty} \sqrt[n]{n|c_n|}.$$

According to theorem theo:power_series 3.10.0.1, we find the series $\sum_{n=0}^{\infty} n c_n x^{n-1}$ converges on the same interval $(-R, R)$. Hence, with similar argument, we find $\sum_{n=0}^{\infty} n c_n x^{n-1}$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$.

For any ε, as $\sum_{n=0}^{k} c_n x^n$ is differentiable on $[-R+\varepsilon, R-\varepsilon]$ for any $k \geq 1$. Furthermore, the series converges to c_0 at $x=0$. Combining all the conditions, from theorem 6.5.0.1, we conclude that f is differentiable with the desired differential. □

Corollary 7.1.0.2 Under the hypothesis of theorem 7.1.0.1, f has derivatives of all orders in $(-R, R)$ (**smooth**), which are given by

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-1}.$$

In particular, $f^{(k)}(0) = k! c_k$ for all $k = 0, 1, 2, \ldots$

The corollary shows that if f is analytic and if f is smooth on $(-R, R)$, one can write the coefficients in the power series of f as $c_k = f^{(k)}(0)/k!$. The converse is not true. In particular, a smooth function f (i.e. f has derivatives of all orders) does not imply that f is analytic, (i.e. cannot be expanded in a power series about $x = 0$).

Example 7.1.0.3 (Smooth $\not\Rightarrow$ Analytic)
See [1, Exercise 1, §8].

In previous theorem 7.1.0.1, if the series $\sum c_n x^n$ also converges at $x = R$ then f is continuous at $(-R, R]$. This is shown in below theorem (WLOG, with $R = 1$):

Theorem 7.1.0.4
Suppose $\sum c_n$ converges. Put $f(x) = \sum_{n=0}^{\infty} c_n x^n$ $(-1 < x < 1)$. Then

$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n.$$

Theorem 7.1.0.5 Given double sequence $\{a_{ij}\}, i = 1, 2, 3 \ldots, j = 1, 2, \ldots,$ suppose that $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ for all $i = 1, 2, \ldots$ and $\sum b_i$ converges. Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Theorem 7.1.0.6 (Taylor's theorem) Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$, the series converging in $|x| < R$. If $-R < a < R$, then f can be expanded in a power series about the point $x = a$ which converges in $|x - a| < R - |a|$, and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \ (|x-a| < R - |a|).$$

Sketch. Use corollary 7.1.0.2 to write out $f^{(n)}(a)$ and use theorem 7.1.0.5 to interchange the double sums. □

Corollary 7.1.0.2 shows that if two power series converge to the same function then they are identically the same. The same conclusion can be deduced from much weaker hypothesis:

Theorem 7.1.0.7 Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converge in the segment $S = (-R, R)$. Let E be the set of all $x \in S$ at which

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n.$$

If E has limit point in S, then $a_n = b_n$ for $n = 1, 2, \ldots$

Thoughts and proof. A typical step is to take $f(x) = \sum_{n=0}^{\infty}(a_n - b_n)x^n$ then $f(x) = 0$ for all $x \in E$. It suffices to show $f(x) = 0$ for all $x \in S$ or $E = S$. The condition suggests us to find the connection between f and E and S. We first investigate the properties around $f(x)$ for each types of x (i.e. $x \in E$, x limit point of E, $x \notin E$).

From theorem 7.1.0.1, we know that f is continuous on S. This follows

1. For any point $x \in S \setminus \overline{E}$, i.e. $f(x) \neq 0$, there exists a neighborhood $N(x)$ of x such that $f(y) \neq 0$ for all $y \in N(x)$.

2. For $x \in E$ but x is not a limit point of E, there exists a neighborhood $N(x)$ such that $f(y) \neq 0$ for $y \in N(x)$.

3. Any limit point x of E in S is in E. This is because for any neighborhood $N(x)$ of x, there exists $y \in E \cap N(x)$, i.e. $f(y) = 0$. As f is continuous at x, we find $f(x) = 0$ or $x \in E$.

   In a similar vein to previous cases of x, for this case, what can we say about $f(y)$ for y close to x? Observe that so far, we have not used the fact that f can be expanded in a power series. In particular, for such x, according to Taylor's theorem 7.1.0.6, we have

$$f(y) = \sum_{n=0}^{\infty} d_n (y-x)^n, \ (|y-x| < R - |x|).$$

We claim that $f(y) = 0$ for all y such that $|y - x| < R - |x|$ by showing that $d_n = 0$ for all n. Assume the contrary, if k is the smallest positive integer such that $d_k = 0$ then

$$f(y) = (y - x)^k \sum_{n=k}^{\infty} d_n (y - x)^{n-k} = (y - x)^k g(y).$$

Note that g is continuous at x and $g(x) = d_k \neq 0$ so there exists an open interval $I = (x - \varepsilon, x + \varepsilon)$ such that $f(y) \neq 0$ for all $y \in I$. This contradicts with the fact that $x \in E$.

From these three conditions, we find that: If A is the set of all limit points of E in S and B is the set of other points then A, B are open sets. This follows $A \cap \overline{B} = B \cap \overline{A} = \varnothing$ or A and B are seperated. On the other hand, we know $A \cup B = S$ and S is connected so according to theorem 1.5.0.3, this happens when either A or B is empty. As we know A is nonempty (it contains one limit points of E in S) so B is empty or $A = S$. From the second condition, we also know that $A \subseteq E$ so $E = S$, as desired. $\square$

7.2 The exponential and logarithmic functions

From [1, Exercise 6, §1], we know a definition of exponential number x^y for $x > 1$ and any real y, i.e. $x^y = \sup x^p$ where sup is taken over all rational p such that $p < y$. It's quite difficult to investigate properties of x^y from this definition so here is an equivalent definition (first we define e^z):

$$e^z = E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{7.1}$$

How is this equivalent to previously mentioned definition? Well, from (7.1), we can show that $E(z+w) = E(z)E(w)$ and that E is strictly increasing on R. From the definition of e 3.8.0.1, we know $E(1) = 0$ so $E(n) = (E(1))^n = e^n$ for all positive integer n. For positive integers n, m then with $p = n/m$, we take $e^n = E(n) = E(pm) = (E(p))^m$ so $E(p) = e^p$ for any positive rational p. On the other hand, $E(z)E(-z) = E(0) = 1$ so $E(p) = e^p$ for any rational p. Combining with the fact that E is strictly increasing and E is continuous, we can conclude that the two definitions are equivalent to each other.

We present some properties of the exponential function:

Theorem 7.2.0.1 With e^x defined in (7.1), we find

(a) e^x is continuous and differentiable for all x;

(b) $(e^x)' = e^x$;

(c) e^x is strictly increasing function of x, and $e^x > 0$;

(d) $e^{x+y} = e^x e^y$;

(e) $e^x \to +\infty$ as $x \to +\infty$, $e^x \to 0$ as $x \to -\infty$;

(f) $\lim_{x\to+\infty} x^n e^{-x} = 0$, for every n. In other words, e^x tends to $+\infty$ "faster" than any power of x, as $x \to +\infty$.

(g) $e^{\overline{z}} = \overline{e^z}$ where $\overline{z}$ is the conjugate of z.

The logarithmic function L is essentially the inverse of the exponential function E. Since E maps $\mathbf{R}$ to $\mathbf{R}_{>0}$ so L has domain as set of all positive numbers. The logarithmic function allows us to give a better definition for x^α as $e^{\alpha \ln(x)}$. Due to the continuity and monotonicity of L, this definition agrees with the previously mentioned definition of x^α. Using this definition, it is easier to find the derivative formula and integration formula of x^α (see (f) in below theorem).

Essentially, one find the following properties of logarithmic function:

logarithmic_properties **Theorem 7.2.0.2** With $\ln x$ defined as the inverse function of e^x, we can find:

(a) $\ln x \to +\infty$ as $x \to \infty$ and $\ln x \to -\infty$ as $x \to 0$.

(b) For any $x > 0$ and any $\alpha \in \mathbf{R}$ then $x^\alpha = e^{\alpha \ln x}$.

(c) $\ln(uv) = \ln(u) + \ln(v)$ for any $u, v > 0$.

(d) $\lim_{x\to+\infty} x^{-\alpha} \ln x = 0$.

(e) $\ln'(x) = 1/x$.

(f) $(x^\alpha)' = \alpha x^{\alpha-1}$ and so $\int x^\alpha = \frac{x^{\alpha+1}}{\alpha+1}$ if $\alpha \neq -1$. For $\alpha = -1$ then $\int x^{-1} = \ln(x)$ according to (e).

(g) $\ln(x) = \int_1^x \frac{1}{x} dx$.

theo:differential_chain_real

Proof. For (e), from $L(E(x)) = x$ ($x \in \mathbf{R}$) and using the Chain Rule 4.1.0.3, we find $L'(E(x)) \cdot E(x) = 1$ so $L'(y) = \frac{1}{y}$ where $y = E(x) > 0$. We know $L(1) = L(E(0)) = 0$ and that L' is continuous on $(0, \infty)$ so according to the Fundamental Theorem of Calculus 5.3.0.4, we find

theo:integral_fundamental_theo_cal

$$L(y) = \int_1^y \frac{1}{y} dy.$$

$\square$

7.3 The trigonometric functions

We define

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)],\ S(x) = \frac{1}{2i}[E(ix) - E(-ix)]. \tag{7.2}$$

eq:specialfunc_tri

baby_rudin

Rudin [1, §8] shows that $C(x)$ and $S(x)$ are essentially functions $\cos x$ and $\sin x$ that are geometrically defined.

7.4 The algebraic completeness of the complex field

plexfield_completeness **Theorem 7.4.0.1** Suppose $a_0, \ldots, a_n$ are complex numbers $n \geq 1, a_n \neq 0$, and $P(z) = \sum_{k=0}^{n} a_k z^k$. Then $P(z) = 0$ for some complex number z.

Sketch. Let $\mu = \inf |P(z)|$. We restrict the domain to a closed disk by observing that $|P(z)| \to \infty$ as $|z| \to \infty$. This is because $|a_k z^k|$ can be sufficiently larger than $\sum_{k=0}^{n-1} |a_k z^k|$. All z outside this disk D will have $|P(z)| > \mu$. This implies $\inf |P|_D(z)| = \mu$. Since the disk is closed and $|P|$ is continuous on this disk so there exists z in the disk such that $|P(z)| = \mu$. One can show $\mu = 0$ by contradiction by constructing a z_0 such that $|P(z_0)| < |P(z)|$ if $|P(z)| > 0$. $\square$

7.5 Fourier series

Definition 7.5.0.1. A **trigonometric polynomial** is a finite sum of the form

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx), \; (x \in \mathbf{R}),$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are complex numbers. Using equation (7.2), the above can be written in the form

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}, \; (x \in \mathbf{R}). \tag{7.3}$$

One can see that this polynomial is periodic with period 2π. Furthermore, with the simple integration

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 1, \pm 2, \ldots \end{cases}$$

one can show that

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx. \tag{7.4}$$

A **trigonometric series** is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \; (x \in \mathbf{R}). \tag{7.5}$$

whose N-th partial sum is defined to be the right side of (7.3).

If f is an integrable function on $[-\pi, \pi]$ then the number c_m is defined in (7.4) for all integers m are called the **Fourier coefficients** of f, and the series (7.5) formed with these coefficients is called the **Fourier series** of f.

A natural question raised is that whether the Fourier series of f converges to f, or, more generally, whether f is determined by its Fourier series. If we know the Fourier coefficients of a function, can we find the function, and if so, how?

ier_projection_minimal **Theorem 7.5.0.2** (Least square approximation) Let $\{\phi_n\}$ be orthonormal on $[a, b]$. Let

$$s_n = \sum_{m=1}^{n} c_m \phi_m(x)$$

be the n-th partial sum of the Fourier series of f, and suppose

$$t_n = \sum_{m=1}^{n} \gamma_m \phi_m(x).$$

Then

$$\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx,$$

and equality holds iff $\gamma_m = c_m$ for all $m = 1, 2 \dots, n$.

In language of linear algebra, let $U = \text{span}\{\phi_k\}_{k=1}^n$ be a subspace of vector space V of all complex-valued integrable functions on $[a, b]$ with inner product space

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

We know that $\{\phi_k\}$ is an orthonormal basis of U. Since $f \in V$ then the projection from f onto U is exactly s_n, i.e. $c_m = \langle f, \phi_m \rangle$. We can show that $\langle f - s_n, t_n \rangle = 0$ for any $t_n \in U$ so $f - s_n \in U^\perp$. This follows $\|f - s_n\|_2^2 + \|t_n - s_n\|_2^2 = \|f - t_n\|_2^2$ as $t_n - s_n \in U$. Hence, $\|f - s_n\|_2^2 \le \|f - t_n\|_2^2$, as desired.

unc_fourier_coef_limit **Theorem 7.5.0.3** (Bessel inequality) If $\{\phi_n\}$ is orthonormal on $[a, b]$, then

$$\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \le \|f\|_2^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

In particular, $\lim_{n\to\infty} \langle f, \phi_n \rangle = 0$.

Let $s_n(x) = \sum_{m=1}^{n} \langle f, \phi_m \rangle \phi_n(x)$ as in previous theorem then since $\{\phi_n\}$ is orthonormal so $\|s_n\|_2^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$. It suffices to show $\|s_n\|^2 \le \|f\|_2^2$, which is true since $\|s_n\|^2 \le \|f - s_n\|^2 + \|s_n\|^2 = \|f\|^2$. The last equality holds since $\langle f - s_n, s_n \rangle = 0$ as pointed out in proof of theorem 7.5.0.2. theo:specialfunc_fourier_pr

7.5.1 Trigonometric series

From now on, we will only consider functions f that have period 2π and that are Riemann-integrable on $[-\pi, \pi]$ (and hence on every bounded interval as f is periodic). The Fourier series of f is then the series (7.5) whose coefficients are given by (7.4), and

$$s_N(x) = s_N(f, x) = \sum_{n=-N}^{N} c_n e^{inx}. \tag{7.6}$$

is the N-th partial sum of the Fourier series of f. In order to obtain an expression for s_N that is more managable, this is the **Dirichlet kernel**

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin(N+1/2)x}{\sin(x/2)}. \tag{7.7}$$

The first equality in (7.7) is the definition of D_N, while the second one can be obtained by noticing that

$$\begin{aligned}(e^{ix} - 1)D_N(x) &= e^{(N+1)ix} - e^{-iNx},\\ (e^{ix/2} - e^{-ix/2})D_N(x) &= e^{i(N+1/2)x} - e^{-i(N+1/2)x},\end{aligned}$$

and then uses (7.2).

From the definition (7.6) of s_N and of coefficients c_n in (7.4), we obtain

$$\begin{aligned} s_N(f, x) &= \sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt e^{inx},\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{n=-N}^{N} e^{in(x-t)}dt,\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)dt,\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)dt. \end{aligned} \tag{7.8}$$

The last equality holds because f is periodic with period 2π so the integral is immaterial over any interval length 2π.

Theorem 7.5.1.1 (Pointwise convergence of Fourier series) Given that f has period 2π and f is Riemann-integrable on $[-\pi, \pi]$. For some x, if there is constant $\delta > 0$ and $M < \infty$ such that $|f(x) - f(x+t)| \le M|t|$ for any $t \in (-\delta, \delta)$, then $\lim_{N\to\infty} s_N(f, x) = f(x)$.

Proof. First, we can show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$ by using the definition of D_N in (7.7), i.e. $D_N(x) = \sum_{-N}^{N} e^{inx}$. This follows we can write $f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)D_N(t)dt$. Therefore, combining with the expression for s_N in (7.8), we obtain

$$\begin{aligned} f(x) - s_N(f,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)-f(x-t))D_N(t)dt, \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(x)-f(x-t)}{\sin(t/2)}\sin(N+1/2)tdt. \end{aligned}$$

Using $\sin(N+1/2)t = \cos(t/2)\sin(Nt)+\sin(t/2)\cos(Nt)$ and let $g(x) = \frac{f(x)-f(x-t)}{\sin(t/2)}$, we obtain

$$\begin{aligned} f(x) - s_N(f,x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(x)\cos(t/2)]\sin(Nt)dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}[g(x)\sin(t/2)]\cos(Nt)dt, \\ &= A_N + B_N. \end{aligned}$$

We will show that $g(t)$ is Riemann integrable on $[-\pi,\pi]$. First, we will prove that $g(t)$ is bounded on $[-\pi,\pi]$ using the condition given in the problem. In particular, for $t \in (-\delta,\delta)$ then we have so $|g(t)| \le M\left|\frac{t}{\sin(t/2)}\right|$. If we look at the power series expansion of $\sin(t/2)$, we find that $\sin(t/2) = t/2 + h(t)$ where $\frac{h(t)}{t} \to 0$ as $t \to 0$. This follows limit of $\frac{t}{\sin(t/2)}$ exists at $t = 0$. Hence, this follows $\frac{t}{\sin(t/2)}$ is bounded on $(-\delta,\delta)$ so $g(t)$ is also bounded on $(-\delta,\delta)$. Next, we show $g(t)$ is bounded on $I = [\delta,\varepsilon] \cup [-\varepsilon,-\delta]$, Indeed, for $t \in I$ then $|\sin(t/2)| \ge |\sin(\delta/2)|$; and $f \in \mathscr{R}$ on $[-\pi,\pi]$ so $f(x)-f(x-t)$ is bounded on I. In the end, $g(t)$ is bounded on I. Thus, $g(t)$ is bounded on $[-\pi,\pi]$.

We know that $\sin(t/2) \neq 0$ and $\sin(t/2)$ is bounded on $[a,b]$ for $0 < a < b$ and sufficient small $|b-a|$. On the other hand, $f(x)-f(x-t)$ is also Riemann integrable on any interval so from properties of integral 5.2.0.2, we find $g(t) = \frac{f(x)-f(x-t)}{\sin(t/2)}$ integrable on such $[a,b]$. With sufficient small $\delta > 0$, we can find that $g(t)$ integrable on $[a,\delta]$ for any $a \in (0,\delta)$. Combining with the fact that g is bounded on $[0,\delta]$, from theorem 5.2.0.7, g is integrable on $[0,\delta]$. Similarly, g is integrable on $[-\delta,0]$ so g is integrable on $[-\delta,\delta]$.

From this, we find $g(t)\cos(t/2)$ and $g(t)\sin(t/2)$ are integrable functions on $[-\pi,\pi]$. On the other hand, observe A_N is the imaginary part of Nth Fourier coefficient of $g(t)\cos(t/2)$ so according to Bessel inequality 7.5.0.3, we know this coefficient tends to 0 as $N \to \infty$ so $A_N \to 0$ as $N \to \infty$. Similar argument can be made for B_N. Thus, $\lim_{N\to\infty} f(x) - s_N(f,x) = 0$, as desired. □

Remark 7.5.1.2. From theorem 7.5.1.1, if f is differentiable at x then $s_N(f,x)$ converges to $f(x)$. Generally, the conclusion for theorem 7.5.1.1 is also true if f is continuous piecewise smooth, i.e. f is continuous and periodic and furthermore, there are points $x_0 = -\pi < x_1 < x_2 < \cdots < x_k = \pi$ such that f restriced to to $[x_j, x_{j+1}]$ is continuously differentiable for all j.

From theorem 7.5.1.1, we have the following corollary:

Corollary 7.5.1.3 (Localization theorem) If $f(x) = 0$ for all x in some open neighborhood J, then $\lim s_N(f,x) = 0$ for every $x \in J$. In other words, if $f(t) = g(t)$ for all t in some neighborhood of x, then $s_N(f,x) - s_N(g,x) = s_N(f-g,x) \to 0$ as $N \to \infty$.

This corollary implies that the convergence of $\{s_N(f,x)\}$ depends only on the values of f in some arbitrarily small neighborhood of x. Furthermore, we find that two Fourier series may have the same behaviour in one interval but may behave differently in some other interval. This gives a striking difference between Fourier series and power series, where in power series (see theorem 7.1.0.7), if two power series behave the same in some open interval then the two power series are identical.

Theorem 7.5.1.4 If f is continuous with period 2π and if $\varepsilon > 0$, then there is a trigonometric polynomial P such that $|P(x) - f(x)| < \varepsilon$ for all real x.

Proof. Since f is periodic, it suffices to find trigonometric polynomial P defined on $[-\pi,\pi]$. Let $\mathscr{A}$ be the set of all trigonometric polynomials on $[-\pi,\pi]$ then we can check that $\mathscr{A}$ is an self-adjoint algebra (see 6.7.0.3 for the definition). Indeed, $\mathscr{A}$ is an algebra and since $e^{-inx} = \overline{e^{inx}}$ according to propertiex of exponential function 7.2.0.1 so $\overline{f} \in \mathscr{A}$ if $f \in \mathscr{A}$.

Furthermore, $\mathscr{A}$ seperates points on $[-\pi,\pi]$ as for any $x_1, x_2 \in [-\pi,\pi]$, with $f(x) = \cos(x) + i \sin x$ (points on unit circle) we must have $f(x_1) \neq f(x_2)$. $\mathscr{A}$ also vanishes at no point of $[-\pi,\pi]$. Therefore, according to theorem 6.7.0.8, the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on $[-\pi,\pi]$. In other words, $\mathscr{A}$ is dense in $\mathscr{C}([-\pi,\pi])$. This follows any complex continuous function f on $[-\pi,\pi]$, there exists trigonometric polynomial $P \in \mathscr{A}$ such that $|P(x)-f(x)| < \varepsilon$ for all $x \in [-\pi,\pi]$. Since P, f have period 2π so $|P(x) - f(x)| < \varepsilon$ for all $x \in \mathbf{R}$. □

Theorem 7.5.1.5 (Parseval's theorem) Suppose f and g are Riemann-integrable functions with period 2π, and

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, g(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{inx}.$$

Then

$$\begin{aligned}
\lim_{N\to\infty} \|f - s_N(f)\|_2^2 &= \lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f,x)|^2 dx = 0, \\
\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx &= \int_{-\infty}^{\infty} c_n \overline{\gamma_n}, \\
\|f\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx &= \sum_{-\infty}^{\infty} |c_n|^2.
\end{aligned}$$

Proof. Since f is Riemann integrable on $[-\pi,\pi]$ so from the construction in [1, Exercise 12,§6] (12), there exists a continuous function on $[-\pi,\pi]$ such that $g(-\pi) = g(\pi)$ (with this, we can extend the domain of g to make it periodic) and $\|f - g\|_2 \leq \varepsilon$. Since g is continuous, from theorem 7.5.1.4, there exists a trigonometric polynomial T such that $\|g - T\|_2 \leq \varepsilon$. If the degree of T is N_0 then from theorem 7.5.0.2, we have

$$\|g - s_N(g)\|_2 \leq \|g - T\|_2 \leq \varepsilon.$$

for all $N \geq N_0$. Furthermore, from theorem 7.5.0.3, we have

$$\|s_N(f) - s_N(g)\|_2 = \|s_N(f-g)\|_2 = \sum_{-N}^{N} |\left\langle f-g, e^{inx}\right\rangle|^2 \leq \|f-g\|^2.$$

With these inequality, we obtain for all $n \geq N_0$

$$\|f - s_N(f)\|_2 \leq \|f-g\|_2 + \|g - s_N(g)\| + \|s_N(g) - s_N(f)\| \leq 3\varepsilon.$$

Since this is true for any $\varepsilon > 0$ so this proves that $\|f - s_N(f)\|_2 = 0$.

To prove the next identity, we have

$$\begin{aligned}
\langle s_N(f), g\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{-N}^{N} c_n e^{inx}\overline{g(x)}\right)dx, \\
&= \sum_{-N}^{N} c_n \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{g(x)}dx, \\
&= \sum_{-N}^{N} c_n \overline{\gamma_n}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left|\sum_{-N}^{N} c_n\overline{\gamma_n} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(s_N(f,x) - f(x))g(x)dx\right|, \\
&= \left|\langle s_N(f) - f, g\rangle\right|, \\
&\leq \|s_N(f) - f\|_2 \cdot \|g\|_2.
\end{aligned}$$

The last inequality is true due to Holder's inequality [1, Exercise 10, §6] (10) (this is purely linear algebra, where $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$). Since we know $\lim_{N\to\infty} \|s_N(f) - f\|_2 = 0$ so we are done. The third identity is just a special case of the second one where $g = f$. $\square$

7.6 The gamma function

Definition 7.6.0.1 (Gamma function)**.** For $0 < x < \infty$,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt. \tag{7.9}$$

The integral converges for these x.

Theorem 7.6.0.2

We have the following properties:

(a) The functional equation $\Gamma(x+1) = x\Gamma(x)$ for $0 < x < \infty$.

(b) $\Gamma(n+1) = n!$ for $n = 1, 2, 3, \dots$

(c) $\log \Gamma$ is convex on $(0, \infty)$.

Proof. (a) We use integration by parts for improper integral [1, Exercise 9,§6] (9) with $F(t) = \frac{1}{x}t^x$ and $G(x) = e^{-t}$ to obtain

$$\begin{aligned}
\int_0^\infty t^{x-1}e^{-t}dt &= \lim_{t\to\infty} \frac{1}{x}t^x e^{-t}dt - F(0)G(0) - \int_0^\infty \frac{1}{x}t^x(-e^{-t})dt, \\
&= \frac{1}{x}\int_0^\infty t^x e^{-t}dt, \\
x\Gamma(x) &= \Gamma(x+1).
\end{aligned}$$

(b) Uses (a).

(c) According to [1, Exercise 23,§4] (23), it suffices to show that

$$\log \Gamma(\lambda x + (1-\lambda)y) \le \lambda \log \Gamma(x) + (1-\lambda)\log \Gamma(y),$$

whenever $0 < x, y < \infty, 0 < \lambda < 1$. Since $t^{x-1}e^{-t} > 0$ for $t > 0$ so by taking exponential both sides, it suffices to show that

$$\Gamma(\lambda x + (1-\lambda)y) \le (\Gamma x)^\lambda (\Gamma y)^{1-\lambda},$$

or

$$\int_0^\infty e^{\lambda x + (1-\lambda)y - 1}e^{-t}dt \le \left(\int_0^\infty t^{x-1}e^{-t}dt\right)^\lambda \left(\int_0^\infty t^{y-1}e^{-t}dt\right)^{1-\lambda}.$$

Apply Holder's inequality in [1, Exercise 10, §6] (10) where $f(t) = (t^{x-1}e^{-t})^\lambda$ and $g(t) = (t^{y-1}e^{-t})^{1-\lambda}$ and $1/p = \lambda, 1/q = 1-\lambda$, we obtain the desired inequality. Thus, $\log \Gamma$ is convex on $(0, \infty)$. $\square$

The converse also holds:

ecialfunc_gamma_unique **Theorem 7.6.0.3** If f is positive function on $(0,\infty)$ such that

(a) $f(x+1) = xf(x)$,

(b) $f(1) = 1$,

(c) $\log f$ is convex,

then $f(x) = \Gamma(x)$.

Proof. Since we know that Γ is a solution so it suffices to show that such f is unique. Since $f(x+1) = xf(x)$ so it suffices to determine such f for $x \in (0,1)$. Let $\varphi = \log f$ then $\varphi(x+1) = \log x + \varphi(x), \varphi(1) = 0$ and φ is convex. From this, we can find that $\varphi(n) = \log(n!)$ for positive integer n.

baby_rudin exer:rudin_chap4_23

Proper convexity in [1, Exercise 23,§4] (23) gives us

$$\log(n) = \frac{\varphi(n+1)-\varphi(n)}{1} \le \frac{\varphi(n+1+x)-\varphi(n+1)}{x} \le \frac{\varphi(n+2)-\varphi(n+1)}{1} = \log(n+1).$$

From $\varphi(x+1) = \log x + \varphi(x)$ and $\varphi(n) = \log(n!)$, we know that

$$\varphi(n+1+x) = \log(x) + \log[x(x+1)\cdots(x+n)].$$

Thus,

$$0 \le \varphi(x) - \log\left[\frac{n!x^n}{x(x+1)\cdots(x+n)}\right] \le x\log\left(1+\frac{1}{n}\right).$$

As $n\to\infty$, $\log(1+1/n)\to 0$. Hence, $\varphi(x)$ is uniquely determined. □

Remark 7.6.0.4. From the proof, we obtain the relation

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!x^n}{x(x+1)\cdots(x+n)}.$$

for all $x > 0$.

ialfunc_gamma_betafunc **Theorem 7.6.0.5** If $x > 0$ and $y > 0$, then

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral is called **beta function** $B(x,y)$.

Proof. We want to use the uniqueness of Γ in theorem 7.6.0.3. In particular, fix y, let

$$G(x) = \frac{\Gamma(x+y)}{\Gamma(y)} \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

We want to show $G(x) = \Gamma(x)$ by proving the three properties as in theorem 7.6.0.3. Indeed, we have $G(1) = \frac{\Gamma(y+1)}{\Gamma(y)} \int_0^1 (1-t)^{y-1} dt = 1$. Furthermore, one can show that

$$\begin{aligned} G(x-1) &= \frac{\Gamma(x+y-1)}{\Gamma(y)} \int_0^1 t^{x-2}(t-1)^{y-1} dt, \\ &= \frac{1}{x+y-1} \cdot \frac{\Gamma(x+y)}{\Gamma(y)} \left[\frac{x+y-1}{x-1} \int_0^1 t^{x-1}(1-t)^{y-1} dt \right], \\ &= \frac{\Gamma(x)}{x-1}. \end{aligned}$$

To show that $\log G$ is convex, it suffices to show whenever $1 < p, q < \infty$ so $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\int_0^1 t^{x/p+y/q-1}(1-t)^{k-1} dt \le \left(\int_0^1 t^{x-1}(1-t)^{k-1} dt \right)^{1/p} \left(\int_0^1 t^{y-1}(1-t)^{k-1} dt \right)^{1/q}.$$

This is true according to Holder's inequality [1, Exercise 10, §6] (10). Thus, from theorem 7.6.0.3, we obtain $G(x) = \Gamma(x)$ for all $x > 0$, as desired. □

Example 7.6.0.6 Substitution $t = \sin^2\theta$ in the integral from theorem 7.6.0.5, we find

$$2 \int_0^{\pi/2} (\sin\theta)^{2x-1} (\cos\theta)^{2y-1} d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Taking $x = y = 1/2$ gives us $\Gamma(1/2) = \sqrt{\pi}$.

Theorem 7.6.0.7 (Stirling formula) This gives an approximation for $\Gamma(x+1)$ when x is large (hence for $n! = \Gamma(n+1)$ when $n \in \mathbf{Z}$ is large). The formula

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x \sqrt{2\pi x}} = 1.$$

7.7 Baby Rudin's Exercises

exer:rudin_chap8_1

1. Since $f(x)=e^{-1/x^2}$ for $x \neq 0$ so from property (b) in theorem 7.2.0.1 about exponentials, we have $f'(x)=2x^{-3}e^{-x^{-2}}$ for $x\neq 0$. Let $y=x^{-2}$ then $f(x)/x=e^{-y}y^{1/2}$ so from property (f) of theorem 7.2.0.1, we find $\lim_{x\to 0}\frac{f(x)}{x}=\lim_{y\to\infty}e^{-y}y^{1/2}=0$. This follows $f'(0)=0$. With this, we've defined f' for all x. One can prove inductively that for $x\neq 0$ then $f^{(n)}(x)=Cx^ke^{-x^{-2}}$ for constant C and $k<0$ while $f^{(n)}(0)=0$.

   Remark 7.7.0.1. This exercise shows that a smooth function need not to be analytic.

   Function f is smooth as it has derivatives of all orders. However, it is not analytic as it cannot be expanded in a power series about $x=0$. Otherwise, if $f(x)=\sum a_nx^n$ then $n!a_n=f^{(n)}(0)=0$ so $a_n=0$ for all n, which implies $f(x)=0$ for all x, a contradiction.

exer:rudin_chap8_2

2. We have

$$\sum_i a_{ij}=\sum_{i\geq j}a_{ij}=-1+\sum_{i>j}2^{j-i}=-1+\sum_{k=1}^{\infty}2^{-k}=0.$$

   This follows $\sum_j\sum a_{ij}=0$. On the other hand, we have

$$\sum_j a_{ij}=\sum_{j=1}^{i}a_{ij}=-1+\sum_{k=1}^{i-1}2^{-k}=-2^{-i+1}.$$

   This follows $\sum_i\sum_j a_{ij}=-\sum_i 2^{-i+1}=-2$.

exer:rudin_chap8_3

3. Since $a_{ij}\geq 0$ for all i,j so theorem 7.1.0.5 implies that if either of $\sum_i\sum_j a_{ij}$ or $\sum_j\sum_i a_{ij}$ converges then the two are equal. If both of them diverge then since these series are monotonically increasing so they both diverge to $+\infty$.

exer:rudin_chap8_4

4. (a) We know from property 7.2.0.2 of logarithmic functions that $b^x=e^{x\ln(b)}$ and from property 7.2.0.1 of exponential function that $(e^{x\ln(b)})'=\ln(b)b^x$ so at $x=0$, the derivative equals to $\ln(b)$. On the other hand, according to the definition of derivative, this is also equal to $\lim_{x\to 0}\frac{b^x-1}{x}$, as desired.

   (b) We know from property 7.2.0.2 of logarithmic functions that $\ln'(x+1)=\frac{1}{x+1}$ so derivative of $\ln'(x+1)$ at $x=0$ is 1, which is also equal to $\lim_{x\to 0}\frac{\ln(x+1)}{x}$ according to the definition.

   (c) We have $(x+1)^{1/x}=e^{\ln(x+1)/x}$. From (b), we know that $\lim_{x\to 0}(1+x)^{1/x}=1$ so $\lim_{x\to 0}(x+1)^{1/x}=e$.

   (d) From (c), we have

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=\lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^{n/x}\right]^x=\left[\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n/x}\right]^x=e^x.$$

exer:rudin_chap8_5

5. (a) We use L'Hospital's Rule. Using Chain Rule, we know that $(1+x)^{1/x}$ is differentiable on $\mathbf{R} \setminus \{0\}$ with derivative

$$\frac{d}{dx}(1+x)^{1/x} = \frac{d}{dx}e^{\ln(1+x)/x} = (1+x)^{1/x}\frac{d}{dx}\frac{\ln(x+1)}{x} = (1+x)^{1/x}\left[\frac{1}{(x+1)x} - \frac{\ln(x+1)}{x^2}\right].$$

Therefore, according to the L'Hospital's Rule 4.4.0.1 multiple times, we obtain

$$\begin{aligned}\lim_{x\to 0}\frac{e-(x+1)^{1/x}}{x} &= -\lim_{x\to 0}(1+x)^{1/x}\left[\frac{1}{(x+1)x} - \frac{\ln(x+1)}{x^2}\right],\\ &= \lim_{x\to 0}(1+x)^{1/x}\lim_{x\to 0}\left[\frac{\ln(x+1)}{x^2} - \frac{1}{(x+1)x}\right],\\ &= e\lim_{x\to 0}\frac{(x+1)\ln(x+1)-x}{x^2(x+1)},\\ &= e\lim_{x\to 0}\frac{[\ln(x+1)+1]-1}{3x^2+2x},\\ &= e\lim_{x\to 0}\frac{1}{(x+1)(6x+2)},\\ &= e/2.\end{aligned}$$

(b) Again with L'Hospital Rule. First, we have $\frac{d}{dx}x^{1/x} = \frac{d}{dx}x^{\ln(x)/x} = x^{1/x}\frac{d}{dx}\frac{\ln(x)}{x}$. Therefore, according to L'Hospital Rule 4.4.0.1, we obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{x^{1/x}-1}{\ln(x)/x} &= \lim_{x\to\infty}\frac{x^{1/x}\frac{d}{dx}\frac{\ln(x)}{x}}{\frac{d}{dx}\frac{\ln(x)}{x}},\\ &= \lim_{x\to\infty}x^{1/x},\\ &= e^{\lim x\to\infty \ln(x)/x},\\ &= e^{\lim_{x\to\infty}1/x} = 1.\end{aligned}$$

(c) Again, L'Hospital 4.4.0.1 we have

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(x)-x}{x(1-\cos x)} &= \lim_{x\to 0}\frac{\tan^2(x)}{1-\cos x + x\sin x},\\ &= \lim_{x\to 0}\frac{2\tan(x)/\cos^2(x)}{x\cos x + 2\sin(x)},\\ &= \lim_{x\to 0}\frac{2\tan(x)}{x\cos^3(x)+2\sin(x)\cos(x)},\\ &= \lim_{x\to 0}\frac{2/\cos^2(x)}{-3x\cos^2(x)\sin(x)+3\cos^3(x)-4\sin^2(x)\cos(x)},\\ &= \frac{2}{3}.\end{aligned}$$

(d) Again, L'Hospital theo:differential_hospital 4.4.0.1 we have

$$\begin{aligned}
\lim_{x\to 0}\frac{x-\sin x}{\tan(x)-x} &= \lim_{x\to 0}\frac{1-\cos x}{\tan^2(x)}, \\
&= \lim_{x\to 0}\frac{\sin(x)}{2\tan(x)/\cos^2(x)}, \\
&= \lim_{x\to 0}\frac{\sin(x)\cos^2(x)}{2\tan(x)}, \\
&= \lim_{x\to 0}\frac{\cos^3(x)+2\cos(x)\sin^2(x)}{2/\cos^2(x)}, \\
&= 1/2.
\end{aligned}$$

exer:rudin_chap8_6

6. Let f be a continuous nonzero function such that $f(x)f(y) = f(x+y)$ for all $x, y \in \mathbf{R}$. Since $f(x) = [f(x/2)]^2 > 0$ so $f(x) > 0$ for all x. Since $f(0) = [f(0)]^2$ so $f(0) = 1$. Furthermore, one can easily show that $f(q) = [f(1)]^q$ for all $q \in \mathbf{Q}$ using the same approach when we prove the properties of our exponential function. If $f(1) = 1$ then $f(q) = 1$ for all $q \in \mathbf{Q}$, which implies $f(x) = 1$ for all x since f is continuous.

   If $f(1) \neq 1$, we show that f is either strictly increasing (i.e. show $f(x) = e^{cx}$ with $c > 0$) or strictly decreasing ($c < 0$). Since $f(1)f(-1) = f(0) = 1$ so either $f(1) > 1$ or $f(1) < 1$. WLOG, if $f(1) > 1$ then since $f(q) = [f(1)]^q$ for $q \in \mathbf{Q}$ so $f(q) > 1$ for all $q > 0, q \in \mathbf{Q}$. Since f is continuous so we find $f(x) > 1$ for all $x > 0$.This follows $f(x) = f(x-y)f(y) > f(y)$ if $x > y$ or f is strictly increasing. Since f is strictly increasing and f is continuous, we find for any real x then $f(x) = \sup f(q)$ $(q < x, q \in \mathbf{Q})$.

   Thus, since $f(x) = \sup f(q) = \sup[f(1)]^q = \sup E(\ln(f(1))q)$ for all x, we obtain $f(x) = E(\ln(f(1))x) = e^{\ln(f(1))x}$.

exer:rudin_chap8_7

7. Let $f(x) = \sin x - 2x/\pi$ then $f(0) = f(\pi/2) = 0$ so according to Mean Value Theorem theo:differential_mvt 4.2.0.3, there exists $x_0 \in (0, \pi/2)$ such that $f'(x_0) = 0$.

   On the other hand, since $f''(x) = -\sin(x) < 0$ for $x \in (0, \pi/2)$ so f' is strictly decreasing. This follows x_0 is the only point in $(0, \pi/2)$ such that $f'(x_0) = 0$. As $f''(x_0) < 0$ so f reaches maximal at $x = x_0$ and does not have critical points elsewhere. Combining with $f(0) = f(\pi/2) = 0$, we obtain $f(x) > 0$ for all $x \in (0, \pi/2)$ or $\sin(x)/x > 2/\pi$.

   For the other side of the inequality, let $g(x) = x - \sin(x)$ then $g'(x) = 1 - \cos(x) > 0$ so g is strictly increasing on $(0, \pi/2$. Since $g(0) = 0$ so $g(x) > 0$ for all $x \in (0, \pi/2)$ or $\sin(x) < x$ for all $x \in (0, \pi/2)$.

exer:rudin_chap8_8

8. Note that $e^{ix} = \cos x + i \sin x$ so $|e^{ix}| = 1$.

$$\begin{aligned}\left|\frac{\sin(nx)}{\sin x}\right| &= \left|\frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}}\right|, \\ &= \left|e^{ix(n-1)}\frac{e^{2inx} - 1}{e^{2ix} - 1}\right|, \\ &= \left|\sum_{k=0}^{n-1} e^{2ixk}\right|, \\ &\le \sum_{k=0}^{n-1}\left|e^{2ixk}\right| \le n.\end{aligned}$$

exer:rudin_chap8_9

9. (a) Note that $\ln(N+1) - \ln(N) = \int_N^{N+1}\frac{1}{x}dx$. Therefore,

$$(s_N - \ln(N)) - (s_{N+1} - \ln(N+1)) = \int_N^{N+1}\frac{1}{x}dx - \frac{1}{N+1} > 0.$$

Thus, the sequence $\{s_N - \log(N)\}$ is strictly decreasing. Furthermore, the sequence is bounded as in

$$s_N - \ln(N) > \sum_{i=1}^{N}\int_i^{i+1}\frac{1}{x}dx - \ln N = \int_1^N \frac{1}{x}dx - \ln N = 0.$$

Thus, the sequence converges to a positive number γ.

(b) Since $s_N > \log N$ so $m \log 10 = 100$ is a good estimate of m.

exer:rudin_chap8_10

10. We proceed as in the given hint. First, we show that $(1-x)^{-1} \le e^{2x}$ for $0 \le x \le 1/2$. Indeed, consider function $f(x) = (1-x)e^{2x} - 1$ then $f'(x) = (1-2x)e^{2x} \ge 0$ so f is increasing. Since $f(0) = 0$ so $f(x) \ge 0$ for all $0 \le x \le 1/2$, as desired.

Fix N, let $p_1, \dots, p_k$ be the primes that divide at least once integer $\le N$. We have

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n} &\le \prod_{j=1}^{k}\left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right), \\ &= \prod_{j=1}^{k}\left(1 - \frac{1}{p_j}\right)^{-1}, \\ &\le \exp\sum_{j=1}^{k}\frac{2}{p_j}.\end{aligned}$$

This follows $\sum_{p<p_k}\frac{1}{p} > \frac{1}{2}\ln\sum_{n=1}^{N}\frac{1}{n}$ so $\sum p^{-1}$ over all the primes diverges.

exer:rudin_chap8_11 11. Since $f \in \mathscr{R}$ on $[0,1]$ and $f(x) \to 1$ as $x \to \infty$ so f is bounded on $\mathbf{R}$, i.e. $|f(x)| < M$ for all $x \in \mathbf{R}$. Since $f(x) \to 1$ as $x \to \infty$ so there exists $A > 0$ such that $|f(x) - 1| < \varepsilon$ for all $x > A$. This follows for any t then

$$\begin{aligned}
\left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| &= \left| t \int_0^A e^{-tx} f(x) dx + t \int_A^\infty e^{-tx} f(x) dx - 1 \right|, \\
&\le \left| Mt \int_0^A e^{-tx} dx + t(1+\varepsilon) \int_A^\infty e^{-tx} dx - 1 \right|, \\
&= \left| Mt \frac{-1}{t} \cdot (e^{-tA} - 1) + t(1+\varepsilon) \cdot \frac{-1}{t} (0 - e^{-tA}) - 1 \right|, \\
&= \left| M(1 - e^{-tA}) + (1+\varepsilon) e^{-tA} - 1 \right|, \\
&\le \left| (M-1)(1 - e^{-tA}) \right| + \varepsilon e^{-tA}.
\end{aligned}$$

Since $\lim_{t \to 0} e^{-tA} = 1$ so there exists $B > 0$ such that $|(M-1)(1-e^{-tA})| < \varepsilon$ and $e^{-tA} < 2$ for all $0 < t < B$. This follows $\left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| < 3\varepsilon$ for all $0 < t < B$. By taking $A \to \infty$, we can make $\varepsilon \to 0$, as desired.

eq:specialfunc_fourier_coef

exer:rudin_chap8_12 12. (a) From 7.4, we find for $m \neq 0$ then

$$\begin{aligned}
c_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx, \\
&= \frac{1}{2\pi} \left[\int_0^\delta e^{-imx} dx + \int_{-\delta}^0 e^{-imx} dx \right], \\
&= \frac{1}{2\pi i m} (e^{im\delta} - e^{-im\delta}), \\
&= \frac{\sin(m\delta)}{\pi m}.
\end{aligned}$$

theo:specialfunc_fourier_converge_pointwise

For $m = 0$, we have $c_m = \frac{\delta}{\pi}$. (b) From theorem 7.5.1.1, with $x = 0$, we find that $|f(0+t) - f(0)| = 0$ for all $t \in (-\delta, \delta)$ so $\lim s_N(f,1) = \lim \sum_{-N}^N c_n = f(0) = 1$.

Note that $c_{-m} = c_m$ since $\sin(-m\delta) = \sin(m\delta)$ so

$$\sum_{-N}^{N} c_n = 2 \sum_{n=1}^{N} c_n + c_0 = 2 \sum_{n=1}^{N} \frac{\sin(n\delta)}{\pi n} + \frac{\delta}{\pi}.$$

This follows

$$\lim_{N \to \infty} 2 \sum_{n=1}^{N} \frac{\sin(n\delta)}{\pi n} + \frac{\delta}{\pi} = 1.$$

In other words, $\sum_{n=1}^\infty \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}$.

theo:specialfunc_fourier_parseval

(c) From Parseval's theorem 7.5.1.5, we know that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx=\sum_{-\infty}^{\infty}|c_n|^2.$$

The left-hand side is equal to $\frac{\delta}{\pi}$ while the right-hand side is equal to

$$\sum_{-\infty}^{\infty}|c_n|^2=2\sum_{n=1}^{\infty}|c_n|^2+|c_0|^2=2\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\pi^2}+\frac{\delta^2}{\pi^2}.$$

In the end, we obtain

$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\pi^2}=\delta\frac{\pi-\delta}{2\pi^2}.$$

(d) Let $f(x)=\left(\frac{\sin(x)}{x}\right)^2$.

The integral $\int_0^{\infty}f(x)dx$ exists since $f(x)\le x^{-2}$ for all $x>0$. This follows there exists $0<B<\infty$ such that for any $B<T$ then

$$\left|\int_0^{\infty}\frac{\sin^2(x)}{x^2}dx-\int_0^{T}\frac{\sin^2(x)}{x^2}dx\right|<\varepsilon.$$

theo:integral_criterion_darboux

With such interval $[0,T]$, from theorem 5.2.1.3, for any $\varepsilon>0$, there exists $\gamma>0$ such that the partition $P=\{\delta i: 0\le i\le T/\delta\}$ of $[0,T]$ with any $0<\delta<\gamma$ gives

$$\left|\sum_{i=1}^{T/\delta}f(\delta i)\Delta x_i-\int_0^{T}f(x)dx\right|=\left|\delta\sum_{i=1}^{T/\delta}\frac{\sin^2(\delta i)}{(\delta i)^2}-\int_0^{T}f(x)dx\right|<\varepsilon.$$

One can choose sufficient small δ to make T/δ big, which implies

$$\left|\frac{\pi-\delta}{2}-\delta\sum_{i=1}^{T/\delta}\frac{\sin^2(\delta i)}{(\delta i)^2}\right|=\left|\sum_{n=1}^{\infty}\frac{\sin^2(\delta n)}{n^2\delta}-\delta\sum_{i=1}^{T/\delta}\frac{\sin^2(\delta i)}{(\delta i)^2}\right|<\varepsilon.$$

From these three inequalities, we obtain for sufficient small δ:

$$\left|\int_0^{\infty}\frac{\sin^2(x)}{x^2}dx-\frac{\pi-\delta}{2}\right|<\varepsilon$$

Taking $\delta\to 0$ gives $\int_0^{\infty}\frac{\sin^2(x)}{x^2}dx=\pi/2$, as desired.

exer:rudin_chap8_13 13. For $f(x) = x$ with $0 \le x < 2\pi$ then for $-\pi \le x < 0$, we have $f(x) = f(x+2\pi) = x + 2\pi$. Therefore, the Fourier coefficients c_m for $m \neq 0$ of f is

$$\begin{aligned} c_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx, \\ &= \frac{1}{2\pi}\left(\int_0^{\pi} xe^{-imx}dx + \int_{-\pi}^{0}(x+2\pi)e^{-imx}dx\right), \\ &= i/m. \end{aligned}$$

For $m = 0$ then $c_0 = \pi$. This follows

$$\sum_{-\infty}^{\infty} |c_n|^2 = \pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

On the other hand, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{1}{2\pi}\left(\int_0^{\pi} x^2dx + \int_{-\pi}^{0}(x+2\pi)^2dx\right) = \frac{4\pi^2}{3}.$$

theo:specialfunc_fourier_parseval

Paserval's theorem 7.5.1.5 implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \sum_{-\infty}^{\infty}|c_n|^2 \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

exer:rudin_chap8_14 14. Let $\sum c_n e^{inx}$ be the Fourier series of f with $s_N(f,x) = \sum_{-N}^{N} c_n e^{inx}$. We will first show that theo:specialfunc_fourier_converge_pointwise $s_N(f,x)$ converges pointwise to $f(x)$ using theorem 7.5.1.1. Indeed, for a fixed x, consider $t \in (-\delta, \delta)$ for some $\delta > 0$, we have for all $t \in (-\delta,\delta)$ then

$$\begin{aligned} |f(x+t) - f(x)| &= |(|x+t| - \pi)^2 - (|x| - \pi)^2|, \\ &= |(|x+t| - |x|)(|x+t| + |x| - 2\pi)|, \\ &\le |t| \cdot (|x+t| + |x| + 2\pi), \\ &\le |t|(2|x| + \delta + 2\pi) = M|t|. \end{aligned}$$

This implies $\lim_{N\to\infty} s_N(f,x) = f(x)$.

Hence, it suffices to find $s_N(f,x)$. The Fourier coefficient of f is $c_m = \frac{2}{m^2}$ for $m \neq 0$ and $c_0 = \pi^2/3$. This follows $s_N(f,x) = 2\sum_{n=1}^{N}\frac{2}{n^2}\cos(nx) + \frac{\pi^2}{3}$. Thus, $\lim s_N(f,x) = \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(nx) + \frac{\pi^2}{3}$, which is also equal to $f(x)$, as desired.

We have $\pi^2 = f(0) = \frac{\pi^3}{3} + 4\sum_{n=1}^{\infty} n^{-2}$. This implies $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$.

theo:specialfunc_fourier_parseval

To show $\sum n^{-4} = \pi^4/90$, we use Parseval's theorem 7.5.1.5. We have

$$\sum_{-\infty}^{\infty}|c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^2.$$

On the other hand,

$$\int_{-\pi}^{\pi}|f(x)|^2dx=\int_{-\pi}^{0}(\pi+x)^4dx+\int_0^{\pi}(\pi-x)^4dx=\frac{\pi^4}{5}.$$

Using Parseval's theorem, we obtain

$$\int_{-\pi}^{\pi}|f(x)|^2dx=\sum_{-\infty}^{\infty}|c_n|^2 \implies \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

exer:rudin_chap8_15 15. (Fejer's theorem, uniformly convergence with Fourier series) Using the exponential definition of trignomonetric functions, we have

$$\begin{aligned}
\sum_{n=0}^{N}D_n(x)&=\sum_{n=0}^{N}\frac{\sin(n+1/2)x}{\sin(x/2)},\\
&=\frac{1}{\sin(x/2)}\sum_{n=0}^{N}\frac{1}{2i}\left(\exp(i(n+1/2)x)-\exp(-i(n+1/2)x)\right),\\
&=\frac{1}{e^{ix/2}-e^{-ix/2}}\left[e^{ix/2}\sum_{n=0}^{N}e^{inx}-e^{-ix/2}\sum_{n=0}^{N}e^{-inx}\right],\\
&=\frac{1}{e^{ix/2}-e^{-ix/2}}\left(e^{ix/2}\frac{e^{i(N+1)x}-1}{e^{ix}-1}-e^{-ix/2}\frac{e^{-i(N+1)x}-1}{e^{-ix}-1}\right),\\
&=\frac{e^{i(N+1)x}+e^{-i(N+1)x}-2}{(e^{ix/2}-e^{-ix/2})^2},\\
&=\frac{e^{i(N+1)x}+e^{-i(N+1)x}-2}{e^{ix}+e^{-ix}-2},\\
&=\frac{1-\cos(N+1)x}{1-\cos x}.
\end{aligned}$$

(a) From this expression of K_N, we find $K_N\ge 0$.

(b) For $n\neq 0$, we have

$$\int_{-\pi}^{\pi}e^{inx}dx=\frac{1}{in}(e^{in\pi}-e^{-in\pi})=0$$

so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(x)dx=\int_{-\pi}^{\pi}\sum_{n=-N}^{N}e^{inx}dx=1.$$

This follows

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}K_N(x)dx=\frac{1}{(N+1)}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}D_n(x)dx=1.$$

(c) For $0 < \delta \le |x| \le \pi$ then $\cos x \le \cos\theta$. We also have $\cos(N+1)x \ge -1$ so

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x} \le \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}.$$

From the formula (7.8) for the n-th partial sum of Fourier series of f, we have

$$\begin{aligned}
\sigma_N(f,x) &= \frac{1}{N+1}\sum_{n=0}^{N} s_n(f,x), \\
&= \frac{1}{N+1}\sum_{n=0}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\left(\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)\right)dt, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt.
\end{aligned}$$

Next, we will prove Fejer's theorem: If f is continuous, with period 2π, then $\sigma_N(f,x) \to f(x)$ uniformly on $[-\pi,\pi]$.

From (c), we find that $K_N \to 0$ uniformly on $\delta \le x \le \pi$ for any $\delta > 0$.

Since f is continuous so f is uniformly continuous on $[-\pi,\pi]$. This follows for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|y-x| < \delta$ implies $|f(x)-f(y)| < \varepsilon$.

From (b), we know that for a fixed x then

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)K_N(t)dt.$$

Denote $M = \sup|f(x)|$. Combining all of these with the bounds for K_N in (a),(c), we obtain

$$\begin{aligned}
|\sigma_N(f,x) - f(x)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(f(x-t)-f(x))K_N(t)dt\right|, \\
&\le \frac{1}{2\pi}\left(2M\int_{-\pi}^{-\delta} K_N(t)dt + \varepsilon\int_{-\delta}^{\delta} K_N(t)dt + 2M\int_{\delta}^{\pi} K_N(t)dt\right), \\
&\le \frac{1}{2\pi}\left(4M\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta} + \varepsilon\int_{-\pi}^{\pi} K_N(t)dt\right), \\
&\le \frac{1}{2\pi}\left(4M\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta} + \varepsilon\cdot 2\pi.\right).
\end{aligned}$$

This is true for any x so for sufficient large N then $|\sigma_N(f,x) - f(x)| < \varepsilon$ for any x. Thus, $\sigma_N(f,x) \to f(x)$ uniformly on $[-\pi,\pi]$.

exer:rudin_chap8_16 16. (Pointwise version Fejer's theorem) Fix an x then there exists δ such that $|f(x-t)-f(x+)| < \varepsilon$ for all $-\delta < t < 0$ and $|f(x-t)-f(x-)| < \varepsilon$ for all $0 < t < \delta$.

Since K_N is an even function so

$$\frac{1}{2\pi}\int_0^{\pi} K_N(x)dx = \frac{1}{2\pi}\int_{-\pi}^{0} K_N(x)dx = \frac{1}{2}.$$

With $\sup|f(x)| = M$, we find

$$\begin{aligned}
&\left|\sigma_N(f,x) - \frac{1}{2}(f(x+)+f(x-)\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x-t)K_N(t)dt - \int_0^{\pi} f(x-)K_N(t)dt - \int_{-\pi}^{0} f(x+)K_N(t)dt\right|, \\
&\le \int_0^{\pi} |f(x-t)-f(x-)|K_N(t)dt + \int_{-\pi}^{0} |f(x-t)-f(x+)|K_N(t)dt, \\
&\le \frac{\varepsilon}{2\pi}\left(\int_0^{\delta} K_N(t)dt + \int_{-\delta}^{0} K_N(t)dt\right) + \frac{M}{\pi}\left(\int_{-\pi}^{-\delta} K_N(t)dt + \int_{\delta}^{\pi} K_N(t)dt\right), \\
&\le \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} K_N(t)dt + \frac{2M(\pi-\delta)}{\pi}\cdot\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}, \\
&= \varepsilon + \frac{4M(\pi-\delta)}{\pi(N+1)(1-\cos\delta)}.
\end{aligned}$$

By choosing sufficiently large N, we can get what we want.

exer:rudin_chap8_17 17. (a) Use [1, Exercise 17,§6] (17) with $G(x) = -e^{-inx}/in$ we have

$$\begin{aligned}
nc_n &= \frac{n}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \\
&= \frac{1}{2\pi i}\left(e^{-i\pi n}f(\pi) - e^{i\pi n}f(-\pi)\right) + \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{inx}df, \\
|nc_n| &\le \frac{1}{\pi}(|f(\pi)| + |f(-\pi)|).
\end{aligned}$$

Thus, the sequence $\{nc_n\}$ is bounded.

(b) Since f is monotonic so $f(x-), f(x+)$ exists for every x according to theorem 2.6.0.1.

We use [1, Exercise 14(e), §3] (17), since $\{nc_n\}$ is bounded and $\lim \sigma_N(f,x) = \frac{1}{2}[f(x+) + f(x-)]$ from previous exercise 16, we obtain for every x then

$$\lim_{N\to\infty} s_N(f,x) = \frac{1}{2}[f(x+)+f(x-)].$$

(c) Define a monotonic function g on $[-\pi, \pi]$ as follow:

$$g(x) = \begin{cases} f(\alpha) & x \le \alpha, \\ f(x) & a < x < \beta, \\ f(\beta) & \beta \le x. \end{cases}$$

Since g is monotonic and bounded on $[-\pi, \pi]$ so from (c), we obtain $\lim s_N(g, x) = \frac{1}{2}[g(x+) + g(x-)] = \frac{1}{2}[f(x+) + f(x-)]$ for all $x \in (\alpha, \beta)$. Since $f(x) = g(x)$ for all $x \in (\alpha, \beta)$ so from theo:specialfunc_fourier_converge_localization localization lemma 7.5.1.3, we find that $\lim s_N(f, x) = \lim s_N(g, x) = \frac{1}{2}[f(x+) + f(x-)]$ for all $x \in (\alpha, \beta)$, as desired.

exer:rudin_chap8_18 18. Take sixth/fifth derivative and go from there.

exer:rudin_chap8_19 19. We first prove this for $f(x) = e^{ikx}$. Indeed, for $k \neq 0$, we have $\int_{-\pi}^{\pi} e^{ikx} dx = 0$ and

$$\frac{1}{N} \sum_{n=1}^{N} e^{ik(x+n\alpha)} = \frac{e^{ikx}}{N} \left(\frac{e^{ij(N+1)x} - 1}{e^{ikx} - 1} - 1 \right).$$

The above expression approaches 0 when $N \to 0$, as desired. When $k = 0$ then $\frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx =$ and $\frac{1}{N} \sum_{n=1}^{N} 1 = 1$, as desired.

Since f is continuous and has period 2π so from theo:specialfunc_fourier_poly_approx theorem 7.5.1.4, for any $\varepsilon > 0$, there exists a trigonometric polynomial $P(x)$ such that $|P(x) - f(x)| < \varepsilon$ for all x. This follows for any N then

$$\left| \frac{1}{N} \sum_{n=1}^{N} [f(x + n\alpha) - P(x + n\alpha)] \right| < \varepsilon.$$

Since P is of the form $P(x) = \sum_{-M}^{M} c_m e^{imx}$ so from previous paragraph, we know that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} P(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt.$$

Therefore, one can choose sufficient large N such that

$$\left| \frac{1}{N} \sum_{n=1}^{N} P(x + n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt \right| < \varepsilon.$$

On the other hand, since $|P(x) - f(x)| < \varepsilon$ or all x, we find

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t) dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \right| \le \varepsilon.$$

Overall, for sufficient large N, we obtain

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt \right| < 3\varepsilon.$$

This implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt.$$

exer:rudin_chap8_20 20. We have

$$\begin{aligned}
\int_1^n f(x)dx &= \sum_{m=1}^{n-1} \int_m^{m+1} f(x)dx, \\
&= \sum_{m=1}^{n-1} \int_m^{m+1} [(m+1-x)\ln(m) + (x-m)\ln(m+1)]\, dx, \\
&= \sum_{m=1}^{n-1} (m+1)\ln(m) - m\ln(m+1) + \frac{2m+1}{2}[\ln(m+1) - \ln(m)], \\
&= \sum_{m=1}^{n-1} \frac{1}{2}(\ln(m) + \ln(m+1)), \\
&= \sum_{m=1}^{n-1} \ln(m) + \frac{1}{2}\ln(n), \\
&= \ln(n!) - \frac{1}{2}\ln(n).
\end{aligned}$$

Similarly, one can find $\int_1^n g(x)dx$ and obtain the following:

$$\int_1^n f(x)dx = \ln(n!) - \frac{1}{2}\ln(x) > \frac{-1}{8} + \int_1^n g(x)dx.$$

On the other hand, since $\int x(\ln(x)-1) = \ln(x)$ so we obtain $\int_1^n \ln(x)dx = n(\ln(n)-1)+1$. Since $f(x) \le \ln(x) \le g(x)$ so $\int_1^n f(x)dx \le \int_1^n \ln(x)dx \le \int_1^n g(x)dx$ which implies

$$0 < \int_1^n \ln(x)dx - \int_1^n f(x)dx \le \int_1^n g(x)dx - \int_1^n f(x)dx < \frac{1}{8}.$$

Or

$$0 < (n+1/2)\ln(n) - n + 1 - \ln(n!) < \frac{1}{8} \implies \frac{7}{8} < \ln(n!) - (n+1/2)\ln(n) + n < 1.$$

Taking the exponents both sides gives us

$$e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e.$$

exer:rudin_chap8_21 21. Note that D_n is an even function and that $\sin(x) > 0$ on $(0, \pi)$ so

$$L_N = \frac{1}{\pi}\int_0^\pi |D_N(x)|dx = \frac{1}{\pi}\int_0^\pi \frac{|\sin(N+1/2)x|}{\sin(x/2)}dx$$

We partition the interval $[0, \pi]$ into $I_k = \left[\frac{\pi(k-1)}{2N+1}, \frac{\pi(k+1)}{2N+1}\right]$ for $k = 1, 3, \dots, 2N-1$. Observe that

(a) If $k \equiv 1 \pmod 4$ then for $x \in I_k$, we have $|\sin(N+1/2)x| = \sin(N+1/2)x$. If $k \equiv 3 \pmod 4$ then for $x \in I_k$, we have $|\sin(N+1/x)x| = -\sin(N+1/2)x$.

(b) $\sin(x/2)$ is increasing on $[0, \pi]$ and that $\sin(x/2) < x/2$.

For convenience, let $A = \frac{\pi}{2N+1}$ then we have

$$\begin{aligned}
\int_{(4k)A}^{(4k+2)A} \frac{|\sin(N+1/2)x|}{\sin x}dx &\geq \frac{1}{\sin kA}\int_{4kA}^{(4k+2)A} \sin(N+1/2)x dx, \\
&\geq \frac{1}{kA}\left[\frac{-2}{2N+1}\cos(N+1/2)x\right]_{4kA}^{(4k+2)A}, \\
&\geq \frac{1}{kA}\cdot\frac{-2}{2N+1}(\cos(2k+1)\pi - \cos(2k)\pi), \\
&= \frac{4}{kA(2N+1)} = \frac{4}{\pi k}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\int_{(4k+2)A}^{(4k+4)A} \frac{|\sin(N+1/2)x|}{\sin x}dx, &\geq \frac{1}{\sin(k+1)A}\int_{(4k+2)A}^{(4k+4)A} -\sin(N+1/2)x dx, \\
&\geq \frac{1}{(k+1)A}\cdot\frac{2}{2N+1}\left[\cos(N+1/2)x\right]_{(4k+2)A}^{(4k+4)A}, \\
&\geq \frac{2}{(k+1)\pi}(\cos(2k+2)\pi - \cos(2k+1)\pi), \\
&= \frac{4}{(k+1)\pi}.
\end{aligned}$$

In the end, we obtain

$$L_N \geq \frac{4}{\pi^2}\sum_{k=1}^N \frac{1}{k} > \frac{4}{\pi^2}\ln n.$$

The last inequality is true according to exercise 9. The upper bound is proven similarly.

exer:rudin_chap8_22 22. We first show that the series $\sum \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$ converges absolutely on $(-1,1)$. We use the ratio test 3.9.0.2. We have

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}\left|x\cdot\frac{\alpha-n}{n+1}\right| = |x|\limsup_{n\to\infty}\left|\frac{\alpha/n-1}{1+1/n}\right| = |x|.$$

This follows for $|x|<1$ then the series converges absolutely.

Next, we will show that $(1+x)f'(x) = \alpha(1+x)$. Let $f(x) = 1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$. Since the series within f converges absolutely for $|x|<1$ so according to theorem 7.1.0.1, we find

$$f'(x) = \sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^{n-1}.$$

From this, we can show that $(1+x)f'(x) = \alpha f(x)$. We also have $f(0)=1$ so this is a initial-value problem so according to [1, Exercise 27, §5] (27), there exists a unique solution f. One can check that $(1+x)^\alpha$ is such solution. Thus, $f(x) = (1+x)^\alpha$.

For $\alpha>0$ and from the property 7.6.0.3 of gamma function $\Gamma(x+1) = x\Gamma(x)$, we have

$$\frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} = \frac{(n-1+\alpha)(n-2+\alpha)\cdots(1+\alpha)\alpha\Gamma(\alpha)}{n!\Gamma(\alpha)} = \frac{(\alpha+n-1)\cdots\alpha}{n!}.$$

Therefore, it suffices to show

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha+n-1)\cdots\alpha}{n!}x^n,$$

which is just the generalised binomial theorem with a little modification.

exer:rudin_chap8_23 23. Since γ is continuously differentiable so γ' is continuous. This follows γ'/γ is continuous on $[a,b]$ so γ'/γ is integrable on $[a,b]$. Let $\varphi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)}dt$ then from the Fundamental Theorem of Calculus 5.3.0.1, we have $\varphi' = \frac{\gamma'}{\gamma}$ on $[a,b]$ and $\varphi(a)=0$. Consider $\gamma\exp(-\varphi)$, using the Chain Rule, we find

$$\frac{d}{dt}\gamma\exp(-\varphi) = \gamma'\exp(-\varphi) + \gamma\cdot\frac{-\gamma'}{\gamma}\exp(-\varphi) = 0.$$

This follows $\gamma\exp(-\varphi)$ is a constant. Hence, $\gamma(a)\exp(-\varphi(a)) = \gamma(b)\exp(-\varphi(b))$. On the other hand, we know that $\gamma(a)=\gamma(b)\neq 0$ since γ is a closed curve so $\exp(\varphi(b)) = \exp(\varphi(a)) = \exp(0) = 1$. This follows $\varphi(b) = 2\pi k i$ for some integer $k\in\mathbf{Z}$. Therefore, $\mathrm{Ind}(\gamma) = \frac{1}{2\pi i}\varphi(b) = k$ is an integer.

When $\gamma = e^{int}, a=0, b=2\pi$ then $\mathrm{Ind}(\gamma) = \int_0^{2\pi}\frac{ine^{int}}{e^{int}}dt = n$. The curve winds counterclockwise about 0 a total of n times, hence the name winding number for $\mathrm{Ind}(\gamma)$.

exer:rudin_chap8_24 24. If the range of γ does not intersect the negative real axis then $\gamma(t)+c$ is nonzero for any constant $0 \le c < \infty$ and any t. Hence, we can define the index for $\gamma + c$.

Let $f(c) = \frac{1}{2\pi}\int_a^b \frac{\gamma'(t)}{\gamma(t)+c}dt$. We first show that f is continuous. Indeed, we have

$$\begin{aligned} |f(c_1)-f(c_2)| &= \left| \frac{c_1-c_2}{2\pi}\int_a^b \frac{\gamma'(t)}{(\gamma(t)+c_1)(\gamma(t)+c_2)}dt \right|, \\ &\le \frac{|c_1-c_2|}{2\pi}\int_a^b \left| \frac{\gamma'(t)}{(\gamma(t)+c_2)(\gamma(t)+c_2)} \right| dt. \end{aligned}$$

Note that if $\gamma(t) = x(t)+iy(t)$ then since $\gamma(t)$ is continuous on $[a,b]$ so $x(t), y(t)$ are bounded. This follows $|\gamma(t)+c| = \sqrt{(x(t)+c)^2+y(t)^2} \ge |y(t)| > C$ for any c. Furthermore, we know that $|\gamma'(t)| < M$ so $\int_a^b \left| \frac{\gamma'(t)}{(\gamma(t)+c_1)(\gamma(t)+c_2} \right| dt \le K$. This follows $|f(c_1)-f(c_2)| \le K|c_1-c_2|$ for any $0 \le c_1, c_2 < \infty$. Thus, $f(c)$ is continuous.

With similar argument, we have $|\gamma(t)+c| = \sqrt{(x(t)+c)^2+y(t)^2} \ge |c-K|$ for some constant K, which implies

$$|f(c)| = \frac{1}{2\pi}\left| \int_a^b \frac{\gamma'(t)}{\gamma(t)+c}dt \right| \le \frac{1}{2\pi}\int_a^b \frac{|\gamma'(t)|}{|c-K|}dt.$$

Thus $f(c) \to 0$ as $c \to \infty$.

From previous exercise 23, we know that $f(c)$ is always integer. However, since f is continuous so $f(c) = K$ is a constant integer. On the other hand, since $f(c) \to 0$ as $c \to \infty$ so $f(c) = 0$ for all $c \in [0,\infty)$. This follows $\text{Ind}(\gamma) = 0$.

exer:rudin_chap8_25 25. Put $\gamma = \gamma_1/\gamma_2$ then $|1-\gamma| < 1$ so the range of γ does not intersect the negative real axis. Furthermore, γ is a continuously differentiable closed curve so from previous exercise 24, we find that $\text{Ind}(\gamma) = 0$. On the other hand, we have

$$\frac{\gamma'}{\gamma} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.$$

Thus, we obtain $\text{Ind}(\gamma_1) = \text{Ind}(\gamma_2)$.

exer:rudin_chap8_26 26. We have $|P_1(t)-P_2(t)| \le |P_1(t)-\gamma(t)| + |\gamma(t)-P_2(t)| < \delta/2$. On the other hand, we have

$$\delta - |P_1(t)| < |\gamma(t)| - |P_1(t)| \le |\gamma(t)-P_1(t)| < \delta/4.$$

This follows $|P_1(t)| > 3\delta/4$ so $|P_1(t)-P_2(t)| < |P_1(t)|$. Since P_1, P_2 are trigonometric polynomials so they are continuously differentiable on $[0,2\pi]$. The previous inequality $|P_1(t)| > 3\delta/4$ shows that $|P_j(t)| \ne 0$ for all $t \in [0,2\pi]$. Since P_1, P_2 are periodic with period 2π so $P_j(0) = P_j(2\pi)$. Thus, all conditions for exercise 25 are satisfied, and therefore we can conclude that $\text{Ind}(P_1) = \text{Ind}(P_2)$.

With only the condition that γ is a closed continuous curve in the complex plane with the parameter $[0,2\pi]$ such that $\gamma(t) \ne 0$ for all $t \in [0,2\pi]$:

Exercise 24. We show that there exists $\delta > 0$ such that $|\gamma(t) - k| > \delta$ for all $t \in [0, 2\pi], k < 0$. We know that the range of γ does not intersect the negative real axis, i.e. $|\gamma(t) - k| > 0$ for all $k \in \mathbf{R}, k < 0, t \in [0, 2\pi]$. Since $\gamma(t) \neq 0$ so we can also include $k = 0$ for the statement to make it $k \leq 0$. Note that $|\gamma(t) - k| = \sqrt{(x(t) - k)^2 + y(t)^2}$ and since γ is continuous so $x(t)$ is bounded, which means for any δ, there exists $M < 0$ such that $|\gamma(t) - k| > \delta$ for all $t \in [0, 2\pi], k < M$. Hence, it suffices to find a lower bound for $|\gamma(t) - k|$ where $k \in [M, 0], t \in [0, 2\pi]$. Let $f(t, k) = |\gamma(t) - k|$ then since γ is continuous so f is continuous on a compact set $[0, 2\pi] \times [M, 0]$. This follows from theorem 2.3.0.3, $f(t, k)$ has minimum at some t^*, k^* where $f(t^*, k^*) = \delta > 0$. Thus, one can choose a $\delta > 0$ such that $|\gamma(t) - k| > \delta$ for all $t \in [0, 2\pi], k \leq 0$.

Since γ is continuous on $[0, 2\pi]$, there exists a trigonometric polynomial P such that $|P(t) - \gamma(t)| < \delta/4$ for all $t \in [0, 2\pi]$. This follows $\text{Ind}(P) = \text{Ind}(\gamma)$. It suffices to show that $\text{Ind}(P) = 0$.

For all $k \leq 0$ then

$$\delta - |P_n(t) - k| \leq |\gamma(t) - k| - |P_n(t) - k| \leq |\gamma(t) - P_n(t)| < \frac{\delta}{4n}.$$

Therefore, $|P_n(t) - k| > 0$ for all $t \in [0, 2\pi], k \leq 0$ or in other words, the range of $P_n(t)$ does not intersect the negative real axis. On the other hand, note that P_n is a continuously differentiable closed curve so from exercise 24, we find $\text{Ind}(P_n) = 0$, as desired. □

Exercise 25. Let $\gamma = \gamma_1/\gamma_2$ and it suffices to show that γ is a closed continuous curve in the complex plane with the parameter $[0, 2\pi]$ such that $\gamma(t) \neq 0$ for all $t \in [0, 2\pi]$.

After that, we apply the new version of exercise 24 and proceed similar to the proof of the old version of exercise 24. □

exer:rudin_chap8_27 27. (Generalization of theorem 7.4.0.1 about completness of complex field) We follow the hint. Assume $f(z) \neq 0$ for all z, define $\gamma_r(t) = f(re^{it})$ for $0 \leq r < \infty, 0 \leq t \leq 2\pi$. Note that γ_r is a continuous closed curve in the complex plane with the parameter $[0, 2\pi]$ and that $\gamma_r(t) \neq 0$ for all $t \in [0, 2\pi], r = 0, 1, \ldots$

(a) We have $\gamma_0 = f(0) \neq 0$ which is a constant so from exercise 23, we find $\text{Ind}(\gamma_0) = 0$.

(b) Since $\lim_{|z| \to \infty} z^{-n} f(z) = c$ so there exists N such that $|z^{-n} f(z) - c| < \varepsilon$ for all $|z| > N$. Hence, we have $\left|r^{-n} e^{-int} f(re^{it}) - c\right| < \varepsilon$ for all $t \in [0, 2\pi]$ and all $r > N$. This follows for all $r > N, t \in [0, 2\pi]$ then

$$|c| - r^{-n}|f(re^{it})| \leq \left|r^{-n} e^{-int} f(re^{it}) - c\right| < \varepsilon \implies |f(re^{it})| > (|c| - \varepsilon) r^n.$$

Furthermore, we also have for sufficient large r then

$$|f(re^{it}) - cr^n e^{it}| < \varepsilon r^n < (|c| - \varepsilon) r^n / 4.$$

Therefore, according to exercise 26, for each $r > N$, we found a trigonometric polynomial $P_r(x) = cr^n e^{it}$ such that $|\gamma_r(t) - P_r(t)| < \delta/4$ while $|\gamma_r| > \delta$ for all $t \in [0, 2\pi]$. Therefore, we find $\text{Ind}(\gamma_r) = \text{Ind}(P_r)$. On the other hand, we've shown in exercise 23 that $\text{Ind}(P_r) = n$ so $\text{Ind}(\gamma_r) = n$ for all $r > N$.

(c) We will use exercise 25 to prove the continuity of γ_r as a function of r. Indeed, we have

$$\begin{aligned}
|\gamma_{r_1}(t) - \gamma_{r_2}(t)| &= \left|f(r_1e^{it}) - f(r_2e^{it})\right|, \\
&\le \left|f(r_1e^{it}) - cr_1^n e^{int}\right| + |\left|cr_1^n e^{int} - cr_2^n e^{int}\right| + \left|cr_2^n e^{int} - f(r_2e^{it})\right|, \\
&\le \varepsilon r_1^n + c|r_1^n - r_2^n| + \varepsilon r_2^n, \\
&< (|c| - \varepsilon) r_1^n < |f(r_1e^{it})| = |\gamma_{r_1}(t)|.
\end{aligned}$$

The second last inequality can be claimed by taking r_1 sufficiently close to r_2 and $5\varepsilon < |c|$. Thus, $|\gamma_{r_1}(t) - \gamma_{r_2}(t)| < |\gamma_{r_1}(t)|$ for all $t \in [0, 1\pi]$ and for $|r_1 - r_2| < \delta$. Thus, γ_r is continuous as a function of r.

From (a), (b), (c), we conclude $n = 0$, a contradiction.

exer:rudin_chap8_28 28. For $0 \le r \le 1, 0 \le t \le 2\pi$, put $\gamma_r(t) = g(re^{it})$, and put $\psi(t) = e^{-it}\gamma_1(t)$ Hence, if $g(z) \ne -z$ for every $z \in T$ then $\psi(t) = e^{-it}\gamma_1(t) \ne -1 = e^{-it} \cdot -e^{it}$ for all $t \in [0, 2\pi]$. Note that $|\psi(t)| = 1$ and $\psi(t) \ne -1$ for all $t \in [0, 2\pi]$ so the range of $\psi(t)$ does not intersect the negative real axis. According to exercise 24 and exercise 26, this implies $\text{Ind}(\psi) = 0$.

We know that $|\psi(t)| = 1$ for all $t \in [0, 1]$ and that $\text{Ind}(\psi) = 0$ so according to exercise 26, there exists a trigonometric polynomial P such that $\text{Ind}(P) = \text{Ind}(\psi) = 0$ and that $|P(t) - \psi(t)| < 1/4$ which follows $|e^{it}P(t) - \gamma_1(t)| < 1/4$. Since $|\gamma_1(t)| = 1$ for all $t \in [0, 2\pi]$ so again from exercise 26, we find $\text{Ind}(\gamma_1) = \text{Ind}(e^{it}P(t)) = 1$.

On the other hand, $\gamma_0 = g(0)$ constant so $\text{Ind}(\gamma_0) = 0$. From exercise 27, we know $\text{Ind}(\gamma_r)$ is a continuous function of r so this gives the contradiction.

exer:rudin_chap8_29 29. (Two dimensional Brouwer's fixed point theorem) Assume $f(z) \ne z$ for every $z \in \overline{D}$. Associate to each $z \in \overline{D}$ the point $g(z) \in T$ which lies on the ray that starts at $f(z)$ and passes through z. Then g maps $\overline{D}$ into T, $g(z) = z$ if $z \in T$.

Next, we show that g is continuous. Indeed, $z - g(z)$ is the vector from $g(z)$ to z while $f(z) - z$ is the vector from z to $f(z)$. Since z lies between $f(z)$ and $g(z)$ so two vectors $f(z) - z$ and $z - g(z)$ are of the same direction so their ratio is equal to the ratio of their length, i.e. $z - g(z) = \frac{|g(z)-z|}{|f(z)-z|}[f(z) - z] = s(z)[f(z) - z]$. If we express $s(z) = \frac{|g(z)-z|}{|f(z)-z|}$ in terms of real and imaginary parts (which are continuous), we find that $s(z)$ is continuous. This implies $g(z) = z - s(z)[f(z) - z]$ is continuous.

From exercise 28, we find that there exists $z \in T$ such that $g(z) = -z$. This implies $z, -z, f(z)$ lie on the same line, which means $f(z)$ lies on the diameter of the unit circle (since $z \in T$). However, the ray starts at $f(z)$, passes through z and then meets $-z$ so $f(z) = z$, a contradiction.

Thus, there exists $z \in \overline{D}$ such that $f(z) = z$.

exer:rudin_chap8_30 30. Apply Stirling's formula 7.6.0.7 for $\Gamma(x+1)$ and $\Gamma(x+c-1+1)$, we have

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^c\sqrt{2\pi x}} = \lim_{x\to\infty} \frac{\Gamma(x+c)}{\left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}} = 1.$$

Note that $\Gamma(x+1) = x\Gamma(x)$ so by dididing both sides

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} \cdot x^{c-1} \cdot \frac{(x/e)^x\sqrt{2\pi x}}{\left(\frac{x+c-1}{e}\right)^{x+c-1}\sqrt{2\pi(x+c-1)}} = 1.$$

Therefore, it suffices to show that

$$\lim_{x\to\infty} \sqrt{\frac{x+c-1}{x}} \cdot \frac{1}{e^{c-1}} \cdot \left(1 + \frac{c-1}{x}\right)^{x+c-1} = 1.$$

Note that $\lim_{x\to\infty}\left(1+\frac{c-1}{x}\right)^x = e^{c-1}$ according to exercise 4 (d) so the above limit is true.

exer:rudin_chap8_31 31. Firstly, since $(1-x^2)^n$ is an even function so

$$\int_{-1}^{1} (1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx.$$

Our goal is to use theorem 7.6.0.5, i.e. for $u, v > 0$ then

$$\int_0^1 t^{u-1}(1-t)^{v-1}dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

We want to appear $(1-x^2)$ in this integral so this suggests us to let $t = x^2$ do $dt = 2xdx$, which implies

$$\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = 2\int_0^1 x^{2u-1}(1-x^2)^{v-1}dx.$$

We want to choose u such that the integral on the right is just $(1-x^2)^{v-1}$. Hence, with $u = 1/2 > 0$ then we have

$$\frac{\Gamma(1/2)\Gamma(v)}{\Gamma(v+1/2)} = 2\int_0^1 (1-x^2)^{v-1}dx.$$

From example 7.6.0.6, we know $\Gamma(1/2) = \sqrt{\pi}$ and from previous exercise 30, we know

$$\lim_{v\to\infty} \frac{\sqrt{v}\Gamma(v)}{\Gamma(v+1/2)} = \lim_{v\to\infty} \frac{\sqrt{v-1}\Gamma(v)}{\Gamma(v+1/2)} = 1$$

so we find

$$\lim_{v\to\infty} 2\sqrt{v-1}\int_0^1 (1-x^2)^{v-1}dx = \sqrt{\pi}.$$

Chapter 8

Multivariable calculus

8.1 Functions on Euclidean Space

A function $f : A \to \mathbf{R}^m$ determines m component functions $f_1, \dots, f_m : A \to \mathbf{R}$ by $f(x) = (f_1(x), \dots, f_m(x))$.

theo:.7 **Theorem 8.1.0.1** A subset of $\mathbf{R}^n$ is compact if and only if it is closed and bounded.

theo:.8 **Theorem 8.1.0.2** If $A \subseteq \mathbf{R}^n$, a function $f : A \to \mathbf{R}^m$ is continuous if and only if for every open set $U \subset \mathbf{R}^m$ there is some open set $V \subset \mathbf{R}^n$ such hat $f^{-1}(U) = V \cap A$.

theo:.9 **Theorem 8.1.0.3** If $f : A \to \mathbf{R}^m$ is continuous, where $A \subseteq \mathbf{R}^n$, and A is compact, then $f(A) \subseteq \mathbf{R}^m$ is compact.

8.1.1 Exercises

anifold_spivak:1.23

1. If $\lim_{x\to a} f(x) = b$, which means for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - b| < \varepsilon$ for all $x \in A$ which satisfy $|0 < |x - a| < \delta$. We have $|f(x) - b| = \sqrt{\sum_{i=1}^{m}(f_i(x) - b_i)^2}$ so $|f_i(x) - b_i| \le |f(x) - f(a)| < \varepsilon$ for all $x \in A$ and $0 < |x - a| < \delta$. Therefore, $\lim_{x\to a} f_i(x) = b_i$ for all $i = 1, \dots, m$. The converse is similar.

anifold_spivak:1.24

2. Apply exercise 1.

anifold_spivak:1.25

3. Let $e_1, \dots, e_m$ be standard basis of $\mathbf{R}^m$ then note that $\|Te_i\|$ are constant. Let $h = \sum_{i=1}^{m} \alpha_i e_i$ then $\|h\|^2 = \sum_{i=1}^{m} |\alpha_i|^2$. We obtain

$$\|T(h)\| \le \sum_{i=1}^{m} |\alpha_i| \cdot \|Te_i\| \le M \sum_{i=1}^{m} |\alpha_i| \le M/\sqrt{m} \cdot \|h\|.$$

From this, the linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous, as $\|T(h) - T(a)\| \le N\|h - a\|$ for some constant N.

anifold_spivak:1.26

4. (a) Consider line $y = ax$ through $(0, 0)$. If $a > 0$ then with $B = \{(x, ax) : x \in (-\infty, a)\}$ we have $B \subseteq \mathbf{R}^2 \setminus A$. If $a \le 0$ then all points on the line are not in A. Consider the vertical line $x = 0$, which is also true.

(b) We can see that g_h represents the indicator function defined on set A with the domain is all points in line $y = \frac{b}{a}x$ where $h = (a, b)$ (or line $x = 0$ if $h = (0, b)$). From (a), if $a \ne 0$, we find that if $t < b/a^2$ which means $at < b/a$, then $g_h(t) = f(t(a, b)) = f(at, bt) = 1$. Thus, g_h continuous at 0.

f is not continuous at 0 we can always choose $x < 0$ to make $(x, y) \notin A$ so $f(x, y) = 0$ for $\|(x, y)\| < \delta$ for any $\delta > 0$.

anifold_spivak:1.27

5.

anifold_spivak:1.28

6. If $A \subseteq \mathbf{R}^n$ is not closed then $\mathbf{R}^n \setminus A$ is not open, which means there exists $x \in \mathbf{R}^n \setminus A$ so x is not interior point of $\mathbf{R}^n \setminus A$. Therefore, for any ε, there exists $y \in A$ so $|y - x| < \varepsilon$. This implies $f : A \to \mathbf{R}^n$ so $f(y) = 1/|y - x|$ is unbounded.

anifold_spivak:1.29

7. Since A is compact and f is continuous so from theorem 8.1.0.3, $f(A)$ is compact. From theorem 8.1.0.1, we find $f(A)$ is closed and bounded. This follows $\sup\{x : x \in f(A)\} \in f(A)$ and $\inf\{x : x \in f(A)\} \in f(A)$, which means f takes on a maximum and a minimum value.

anifold_spivak:1.30

8. Since f is an increasing function so $m(a, f, \delta) = f(a - \delta)$ and $M(a, f, \delta) = f(a + \delta)$. This follows $o(f, x_i) = \lim_{\delta\to 0}[f(a + \delta) - f(a - \delta)]$. WLOG, say $x_1 < \cdots < x_n$. Then there exists δ_i so $x_i + \delta_i < x_{i+1} - \delta_{i+1}$ and $a < x_1 - \delta_1$ and $x_n + \delta_n < b$. Since f is increasing so we have

$o(f, x_i) < f(x_i + \delta_i) - f(x_i - \delta_i)$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} o(f, x_i) < \sum_{i=1}^{n} f(x_i + \delta_i) - f(x_i - \delta_i),& \\
<& f(x_1 - \delta_1) - f(a) + \sum_{i=1}^{n} [f(x_i + \delta_i) - f(x_i - \delta_i)] \\
&+ \sum_{i=1}^{n-1} [f(x_{i+1} - \delta_{i+1}) - f(x_i + \delta_i)] + f(b) - f(x_n + \delta_n), \\
=& f(b) - f(a).
\end{aligned}$$

8.2 Differentiation

Definition 8.2.0.1. For open set $A \subset \mathbf{R}^n$. A function $f : A \to \mathbf{R}^m$ is differentiable at $a \in A$ if there is a linear map $Df(a) : \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - (Df(a))(h)|}{|h|} = 0.$$

The linear map $Df(a)$ is called the derivative of f at a.

8.2.1 Exercises

manifold_spivak:2.1

1. Since f is differentiable at a so $\lim_{h\to 0}\frac{|f(a+h)-f(a)-(Df(a))(h)|}{|h|}=0$ which implies $\lim_{h\to 0}|f(a+h)-f(a)-(Df(a))(h)|=0$ since $\lim_{h\to 0}|h|=0$. This follows $\lim_{h\to 0}f(a+h)-f(a)-(Df(a))(h)=0$. From exercise 3, we know that $|(Df(a))(h)|\le M|h|$ for some $M>0$. Therefore, $\lim_{h\to 0}|(Df(a))(h)|=0$ so $\lim_{h\to 0}(Df(a))(h)=0$. With this, we obtain $\lim_{h\to 0}f(a+h)-f(a)=0$ or f continuous at a.

manifold_spivak:2.2

2. Define $g:\mathbf{R}\to\mathbf{R}$ as $g(x)=f(x,0)$ then since f is independent of the second variable, we have $g(x)=f(x,y)$. Proceed similarly to page 17 of [1], we obtain $f'(a,b)=(g'(a),0)$.

manifold_spivak:2.3

3. Similar to previous exercise 2. Constant function are independent of first variable and also of the second variable.

manifold_spivak:2.4

4. (a) We have $k(t+h)=|t+h|\cdot|x|g\left(\frac{x}{|x|}\right)$ for $t>0$ and $k(t+h)=-|t+h|\cdot|x|g\left(\frac{x}{|x|}\right)$ for $t<0$. Therefore, for $t>0$ then
$$\lim_{h\to 0}\frac{k(t+h)-k(t)}{h}=|x|g(x/|x|)\lim_{h\to 0}\frac{|t+h|-|t|}{h}=|x|g\left(\frac{x}{|x|}\right).$$
For $t<0$ then
$$\lim_{h\to 0}\frac{k(t+h)-k(t)}{h}=|x|g\left(\frac{x}{|x|}\right)\lim_{h\to 0}\frac{-|t+h|+|t|}{h}=|x|g\left(\frac{x}{|x|}\right).$$
Similarly, for $t=0$. Thus, h is differentiable.

   Imagine f is drawn in three-dimensional plane. If we take a line ℓ on the horizontal plane with slope defined by $x\in\mathbf{R}$. A vertical plane created by ℓ will cut graph of f to form a differentiable function $h:\mathbf{R}\to\mathbf{R}$.

   (b) If f is differentiable at 0 then $Df(0,0)$ must exist. This follows
$$\lim_{(h,k)\to 0}\frac{|f(h,k)-f(0,0)-(Df(0,0))(h,k)|}{|(h,k)|}=0.$$
With $k=0$ then
$$\lim_{(h,0)\to 0}\frac{||h|g((h,0)/|h|)-(Df(0,0))(h,0)|}{|h|}=0.$$
Since $g(1,0)=g(-1,0)=0$ so for any $h\neq 0$ then $g((h,0)/|h|)=0$. Therefore, this implies $\lim_{h\to 0}\frac{1}{|h|}(Df(0,0))(h,0)=0$ which follows $|(Df(0,0))(1,0)|=0$ since $(Df(0,0))(h,0)=h(Df(0,0))(1,0)$ as $Df(0,0):\mathbf{R}^2\to\mathbf{R}$ is a linear map. Similarly, by considering $h=0$, we obtain $Df(0,0)(0,1)=0$. Thus, $Df(0,0)=0$.

   From (a), for fixed $x\in\mathbf{R}$, we know that by choosing $(h,k)=tx\to 0$, we find $Df(0,0)(tx)=|x|g\left(\frac{x}{|x|}\right)$. Since $Df(0,0)$ is uniquely determined so $|x|g\left(\frac{x}{|x|}\right)=0$ or $g=0$. Thus, f is not differentiable at $(0,0)$ unless $g=0$.

manifold_spivak:2.5

5. For $x \in \mathbf{R}, x \neq 0$, Define $g : \mathbf{R} \to \mathbf{R}$ by $g(t) = f(t, xt)$. Consider when $t \to 0$ or $(t, xt) \to 0$ then
$$\lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{h|hx|}{|h|\sqrt{x^2+1}} = \frac{|x|}{\sqrt{x^2+1}}.$$
This follows $g'(0) = \frac{|x|}{\sqrt{x^2+1}}$, which means $Df(0,0)(t, xt) = \frac{|x|}{\sqrt{x^2+1}}$.
Now, consider $(x, 0) \to (0, 0)$ then we easily find $Df(0,0)(x, 0) = 0$. Since $Df(0,0)$ is uniquely determined so we find f not differentiable at $(0, 0)$.

manifold_spivak:2.6

6. Fixed a $x \in \mathbf{R}, x \neq 0$, consider path $(t, xt) \to 0$ or define $g : \mathbf{R} \to \mathbf{R}$ so $g(t) = f(t, xt)$. We find
$$\lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{|t|\sqrt{|x|}}{t}.$$
Note that the above limit is undefined, which means f is not differentiable at $(0, 0)$.

manifold_spivak:2.7

7. We have $|f(0)| = 0$ so $f(0) = 0$. Define linear map $Df(0) : \mathbf{R}^n \to \mathbf{R}$ so $Df(0) = 0$. Then we have
$$0 \leq \frac{|f(h) - f(0) - Df(0)(h)|}{|h|} = \frac{|f(h)|}{|h|} \leq |h|.$$
Taking the limit as $h \to 0$ will give the desired result.

manifold_spivak:2.8

8. If f_1, f_2 are differentiable at a then let $Df(a)(h) = (h(f_1)'(a), h(f_2)'(a))$. We have
$$\begin{aligned}
&\lim_{h \to 0} \frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|}, \\
=&\lim_{h \to 0} \frac{1}{|h|} |(f_1(a+h), f_2(a+h)) - (f_1(a), f_2(a)) - \big(h(f_1)'(a), h(f_2)'(a)\big)|, \\
=&\lim_{h \to 0} \frac{1}{|h|} \sqrt{[f_1(a+h) - (f_1(a) - h(f_1)'(a)]^2 + [f_2(a+h) - (f_2(a) - h(f_2)'(a)]^2}, \\
=&0.
\end{aligned}$$
Conversely, if f differentiable at a, say $Df(a)(h) = (h\alpha, h\beta)$, then we do the reverse of the above.

manifold_spivak:2.9

9. (a) With $a_0 = f(a), a_1 = f'(a)$, we have
$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - g(a+h)}{h} &= \lim_{h \to 0} \frac{f(a+h) - a_1 h - a_0}{h}, \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a) - hf'(a)}{h} = 0.
\end{aligned}$$
(b) Induction + L'Hospital's rule.

8.3 Fubini's theorem

Example 8.3.0.1 (When Fubini's theorem does not work)

See MSE.

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Chapter 9

Intuition

9.1 Summary

1. Purpose of definitions/theorems.

2. Intuitive explanations for definitions/theorems. The main point.

Differentiability and continuity describe the local change at a point. Integration describe the global change.

Differentiability implies continuity.

Continuity implies integrability when the function is closed and bounded.

> **Question 9.1.0.1.** Global vs Local.

> **Question 9.1.0.2.** Intuition for uniformly continuous function on $\mathbf{R}$?

The answer is from [6, Math Stack Exchange]:

> This is slightly tricky because your intuitive picture for "continuous" probably more closely matches "uniformly continuous." That is, if you can *actually* draw a graph without lifting your pencil then it's uniformly continuous. In order to get something continuous but not uniformly continuous you have to do something that you can't actually draw like going off to infinity on an open interval or oscillating really wildly.

> **Question 9.1.0.3.** What is the intuition behind a function being differentiable if its partial derivatives are continuous?

See this answer on Quora by Alon Amit.

Continuity at a point p describes the behaviour of the functions at points close to p. It implies that

> **Question 9.1.0.4.** What is the intuition behind differentiability?

The goal of differentiability of a function at p is to give a local linear approximation to the function at that point.

> **Question 9.1.0.5.** What is the intuition behind integration?

The integral is *defined* to be the (net signed) area under the curve.

> **Question 9.1.0.6.** Intuition behind the Fundamental Theorem of Calculus?

If you look at the goals of differentiation and of integration, you would think that they don't seem to have any connection. This is where the Fundamental Theorem of Caculus comes in. It turns out (perhaps somewhat magically) that the area can be computed using an antiderivative. See here for a bit of history of how this theorem is discovered.

The main point for the theorem is that: The total change is equal to sum of the small changes. $f'(x)dx$ is a tiny bit change in the value of f. You add up all the tiny changes to get the total change. See here for the explanation.

> **Question 9.1.0.7.** Intuition behind the Chain Rule?

See here for the answer. See here for more elaboration.

> **Question 9.1.0.8.** Why does integrability not imply continuity?

Essentially, you are asking a function satisfying global property (i.e. integrability) to satisfy local property (continuity).

(From MSE) The conditions of continuity and integrability are very different in flavour. Continuity is something that is extremely sensitive to local and small changes. It's enough to change the value of a continuous function at just one point and it is no longer continuous. Integrability on the other hand is a very robust property. If you make finitely many changes to a function that was integrable, then the new function is still integrable and has the same integral. That is why it is very easy to construct integrable functions that are not continuous.

> **Question 9.1.0.9.** Why $\arctan x = \int \frac{dx}{\sqrt{1+x^2}}$?

See here for the geometry intepretation.

> **Question 9.1.0.10.** Directional derivative? Gradient descent geometric explanation? Orthogonal?

> **Question 9.1.0.11.** An intuitive way to think about the difference between pointwise convergence and uniformly convergence of sequence of functions?

Source from [5, MathStackExchange] gives an nice perspective thinking about the difference between pointwisely convergence and uniformly convergence for sequence of functions

> Pointwise convergence means at every point the sequence of functions has its own speed of convergence (that can be very fast at some points and very very very very slow at others).
>
> Uniform convergence means there is an overall speed of convergence.

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