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Tamagawa number for SL_2

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Contents

1. Introduction

1.1. **Adeles and Tamagawa measure.** In number theory, the Hasse principle is the idea that one can find integer solutions to an equation by combining together solutions modulo prime powers. This process is handled by considering the equation over all the completions of the rational numbers: the real numbers $\mathbb{R}$ and the p-adic numbers $\mathbb{Q}_p$. The adeles $\mathbb{A}$ of $\mathbb{Q}$ is a ring that combines all these completions together, with the purpose that instead of trying to do analysis over each completion separately, one should put them on an equal footing by simply working over the adeles. For a first concrete indication of this phenomenon, the adeles lies inside the product $\mathbb{R} \times \prod_p \mathbb{Q}_p$ of all completions of $\mathbb{Q}$. Many statements in number theory, such as class field theory, have more enlightning adelic formulations than their classical accounts.

For a linear algebraic group G over $\mathbb{Q}$ (for example, $\mathrm{SL}_n, \mathrm{GL}_n$), one can study analysis on the adelic points $G(\mathbb{A})$ of G. In particular, we have a *canonical* Haar measure on $G(\mathbb{A})$, called the *Tamagawa measure*.

As $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$, $G(\mathbb{Q})$ is also a discrete subgroup of $G(\mathbb{A})$ and this induces a $G(\mathbb{A})$-invariant measure on $G(\mathbb{Q}) \setminus G(\mathbb{A})$. In fact, for semisimple G (for example, $\mathrm{SL}_n, \mathrm{SO}_n$), the volume of $G(\mathbb{Q}) \setminus G(\mathbb{A})$ with respect to this measure, called the *Tamagawa number*, is finite, and contains interesting arithmetic informations. For example, the volume of $\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$ encodes the value of the Riemann-zeta function $\zeta(2)$. Knowing this volume for $G = O(n)$ implies the Smith-Minkowski-Siegel mass formula in the theory of quadratic forms, as we will later briefly describe.

The goal of our thesis is to study the following theorem

Theorem 1 (Weil's conjecture on Tamagawa numbers)**.** *Let G be a simply connected semisimple linear algebraic group over a number field or a function field k, then the Tamagawa number of G over k is* 1*.*

The theorem was firstly conjectured by Weil [Wei60] and was proven over number fields by Langlands and Kottwitz [Lan65, Kot88]. Over function fields, a completely different proof was given by Gaitsgory and Lurie [GL19].

For this mid-year review, we have completely defined the Tamagawa measures and explicitly computed the Tamagawa numbers for $\mathbb{G}_a$ and SL_2 over $\mathbb{Q}$.

1.2. **Motivation.** We will motivate the Tamagawa numbers via the arithmetic theory of quadratic forms, following [GL19].

A *quadratic space* (V, q) over $\mathbb{Q}$ is a finitely generated free $\mathbb{Q}$-module V equipped with a *quadratic form*, i.e. a map $q : V \to \mathbb{Q}$ satisfying the following conditions:

(1) The map $V \times V \to \mathbb{Q}$ given by $(v, w) \mapsto q(v + w) - q(v) - q(w)$ is $\mathbb{Q}$-bilinear.
(2) For every $\lambda \in \mathbb{Q}$ and every $v \in V$, we have $q(\lambda v) = \lambda^2 q(v)$.

A morphism between two quadratic spaces (V, q) and (V', q') is a linear map $f : V \to V'$ such that $q' \circ f = q$. The automorphism group of a quadratic space (V, q) over $\mathbb{Q}$ is denoted as $O_q(\mathbb{Q})$, the *orthogonal group* of (V, q).

If we fix a choice of basis $\{e_1, \dots, e_n\}$ for V, a quadratic form q on V then corresponds to a matrix B_q defined by $(B_q)_{ij} = \frac{1}{2}(q(e_i + e_j) - q(e_i) - q(e_j))$. One can show that two quadratic forms p, q on V are isomorphic if and only if $B_p = T^t B_q T$ for some invertible matrix $T \in \mathrm{GL}_n(\mathbb{Q})$.

One could then ask the question of classifying quadratic spaces over $\mathbb{Q}$ up to isomorphism; or equivalently, classifying $n \times n$ matrices over $\mathbb{Q}$ up to the equivalence relation $A \sim B \iff A = T^t B T$ for some $T \in \mathrm{GL}_n(\mathbb{Q})$. To achieve this, one first base changes the quadratic space (V, q) over $\mathbb{Q}$ to create a quadratic space $(V \otimes_{\mathbb{Q}} \mathbb{Q}_v, q_{\mathbb{Q}_v})$ over $\mathbb{Q}_v$ for each completion $\mathbb{Q}_v$ of $\mathbb{Q}$. The Hasse principle for quadratic forms then states that two quadratic forms are equivalent over $\mathbb{Q}$ if and only if they are equivalent over $\mathbb{Q}_v$. Over $\mathbb{Q}_v$'s, the classification of quadratic spaces is easier to describe, giving us the classification over $\mathbb{Q}$ (see [Ser73, Chapter IV]).

If we now restrict our attention to quadratic spaces over $\mathbb{Z}$ then a similar statement to the Hasse principle fails; i.e. even if s, q are two quadratic forms over $\mathbb{Z}$ such that they equivalent under extension of scalars to $\mathbb{Z}_p$ and $\mathbb{R}$ (we say s, q *have the same genus*), it does not necessarily follow that they are equivalent over $\mathbb{Z}$.

However, it is 'almost true' in the following sense: for a fixed positive-definite quadratic form q over $\mathbb{Z}$ [1], there are only finitely many quadratic spaces of the same genus to q (up to isomorphism). In fact, one obtains a bijection (or an equivalence of groupoids)

$$\{\text{genus of } q\} \longleftrightarrow O_q(\mathbb{Q}) \setminus O_q(\mathbb{A})/O_q(\widehat{\mathbb{Z}} \times \mathbb{R}).$$

Furthermore, we can also count the size of the isomorphism group of each quadratic form in the genus of g by the following formula (called the *mass* of the genus of q)

$$m(q) = \sum_{q'} \frac{1}{|O_{q'}(\mathbb{Z})|}$$

where the sum is over all quadratic forms of the same genus to q up to isomorphism. One can show that

$$m(q) = \sum_{q'} \frac{1}{|O_{q'}(\mathbb{Z})|} = 2^{k-1} \frac{\mu_{\mathrm{Tam}}(SO_q(\mathbb{Q}) \setminus SO_q(\mathbb{A}))}{\mu_{\mathrm{Tam}}(SO_q(\widehat{\mathbb{Z}} \times \mathbb{R}))}.$$

where μ_{Tam} is the Tamagawa measure on $SO_q(\mathbb{A})$, the group of automorphisms in $O_q(\mathbb{A})$ having determinant 1; k is the number of primes p for which $SO_q(\mathbb{Z}_p) = O_q(\mathbb{Z}_p)$. The numerator is the Tamagawa number for SO_q, and as one can also compute the denominator, this gives an explicit mass formula, called the Smith-Minkowski-Siegel mass formula.

1.3. **Tamagawa numbers for $\mathbb{G}_a$ and SL_2.** In this section, we will highlight the key ingredients used to compute the Tamagawa numbers for $\mathbb{G}_a$ and SL_2.

First, we will describe the adelic quotients $\mathbb{Q} \setminus \mathbb{A}$ or $\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$ via its local parts. We do this by proving strong approximation theorems for the two groups $\mathbb{G}_a$ and SL_2 (see section 5.2 and section 5.4), which then give homeomorphisms

$$\mathbb{Q} \setminus \mathbb{A} \cong (\mathbb{Z} \setminus \mathbb{R}) \times \widehat{\mathbb{Z}}$$

and

$$\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A}) \cong (\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) \times \prod_p \mathrm{SL}_2(\mathbb{Z}_p).$$

Hence, the Tamagawa number, i.e. the volume of the adelic quotient, can be computed by taking the product of volumes of the local parts in the above homeomorphisms:

$$\mu_{\mathrm{Tam}}(\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})) = \mu_\infty(\mathrm{SL}_2(\mathbb{Z} \setminus \mathrm{SL}_2(\mathbb{R})) \times \prod_p \mu_p(\mathrm{SL}_2(\mathbb{Z}_p))$$

where μ_v's are the corresponding measures on each local parts. For $\mathbb{G}_a$, the computation follows easily from our choice of Haar measure on each local field $\mathbb{Q}_v$ (see section 8.2).

For SL_2, to make sense of μ_v's, we first choose a left-invariant non-vanishing top differential form ω of SL_2 over $\mathbb{Q}$. As $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{R})$ are analytic manifolds, we can integrate over ω to obtain a measure $\mu_{\mathrm{SL}_2(\mathbb{Q}_v),\omega}$ on each of these spaces (see section 4). The Tamagawa number is in fact independent of this choice of ω.

[1] q being positive-definite means $q_{\mathbb{R}}$ is positive-definite, i.e. $q_{\mathbb{R}}(v) > 0$ for every nonzero vector v

The computation of
$$\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(\mathrm{SL}_2(\mathbb{Z}_p)) = \frac{|\mathrm{SL}_2(\mathbb{F}_p)|}{p^3} = 1 - p^{-2}$$
is achieved in section 9. The key idea is that we have a surjective map $p : \mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{F}_p)$. This allows us to write $\mathrm{SL}_2(\mathbb{Z}_p)$ as a disjoint union of cosets of $\ker p$, where computing the measure is easier.

The computation of
$$\mu_{\mathrm{SL}_2(\mathbb{R}),\omega}(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) = \zeta(2) = \frac{\pi^2}{6}$$
is achieved in section 10. We provide two methods to do this. In the first method, we give an explicit description for the fundamental domain of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$ and integrate over this domain to obtain $\pi^2/6$. For the second method, we apply the Poisson summation formula.

1.4. **Outline for the mid-year review.** In section 2, we discuss valuation theory, i.e. how to equip a field k with an absolute value and take completions of k with respect to this absolute value. Our main example is $k = \mathbb{Q}$ with its completions $\mathbb{Q}_p$ and $\mathbb{R}$, where p is a prime.

In section 3, we firstly review the theory of measures and integrations. We then focus on discussing Haar measures on locally compact topological groups and establish some results that will be used to do computations with Haar measures in later sections.

In section 4, we define the notion of a k-analytic manifold for any complete valued field k. When k is a local field (e.g. $\mathbb{R}$ or $\mathbb{Q}_p$), we show that there is a theory of integration on such manifolds, resembling the corresponding classical theory for smooth manifolds.

In section 5, we define the ring adeles $\mathbb{A}$ of $\mathbb{Q}$ and study its topology. We show that $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$ and that $\mathbb{Q} \setminus \mathbb{A}$ is compact. We also describe a functorial way to give a topology on $G(\mathbb{A})$ for any linear algebraic group G over $\mathbb{Q}$. After this, we then focus on the case where $G = \mathrm{SL}_2$ and prove the strong approximation theorem for SL_2.

In section 6, we discuss Fourier analysis on locally compact abelian groups. In particular, we describe the Pontryagin duals for $\mathbb{R}, \mathbb{Q}_p, \mathbb{A}$ together with their quotients $\mathbb{Z} \setminus \mathbb{R}, \mathbb{Z}_p \setminus \mathbb{Q}_p$ and $\mathbb{Q} \setminus \mathbb{A}$. We then prove the Poisson summation formula with the focus on these groups.

In section 7, we discuss the linear algebraic group SL_2 in the language of algebraic geometry. We determine its Lie algebra and its non-vanishing left-invariant global top forms, and then show how to relate these two notions.

In section 8, we define the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$ and the Tamagawa number $\tau(\mathrm{SL}_2)$. We show that $\tau(\mathrm{SL}_2)$ can be computed by knowing the volume of its local parts $\mu_\infty(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}))$ and $\mu_p(\mathrm{SL}_2(\mathbb{Z}_p))$.

In section 9, we compute $\mu_p(\mathrm{SL}_2(\mathbb{Z}_p))$.

In section 10, we give two ways to compute $\mu_\infty(\mathrm{SL}_2(\mathbb{Z})\setminus\mathrm{SL}_2(\mathbb{R}))$, as described in previous section.

2. Absolute values, completions of $\mathbb{Q}$

In this section, following [Mil, Sut19, Neu99], we will discuss the completion of a field that is equipped with an absolute value. We focus on describing the completions $\mathbb{Q}_p$'s and $\mathbb{R}$ of $\mathbb{Q}$.

2.1. Absolute values.

Definition 2. *An absolute value of a field k is a map $|\cdot| : k \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in k$:*

(1) $|x| = 0$ iff $x = 0$,
(2) $|xy| = |x||y|$,
(3) $|x + y| \leq |x| + |y|$.

The field k is then called a valued field. If the stronger condition

(4) $|x + y| \leq \max(|x|, |y|)$.

also holds, then the absolute value is nonarchimedean, otherwise it is archimedean.

The condition $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in k$ is equivalent to $|\ |$ being bounded on $\{n1 : n \in \mathbb{Z}\}$. In particular, this implies that if k is of positive characteristic then every absolute value on k is nonarchimedean.

For valued field k with nonarchimedean absolute value $|\ |$, the set $\mathcal{O}_k := \{x \in k : |x| \leq 1\}$ is a subring of k with groups of units $U := \{x \in k : |x| = 1\}$ and unique maximal ideal $\mathfrak{m} := \{x \in k : |x| < 1\}$.

Example 3. The map $|\ | : k \to \mathbb{R}_{\geq 0}$ defined by $|x| = 1$ if $x \neq 0$ and $|0| = 0$, is the *trivial absolute value* on k. It is nonarchimedean.

When k is equipped with an absolute value then k has a metric space topology. Two absolute values $|\ |_1$ and $|\ |_2$ on k are called equivalent if they define the same topology on k. This is the same as saying that there exists real number $s > 0$ such that $|x|_1^s = |x|_2$ for all $x \in k$. We call an equivalence class of absolute values on k a *place* of k.

For $\mathbb{Q}$, we have the usual absolute value $|\ |_\infty$, being an archimedean absolute value. For each prime p, we can define an archimedean absolute value $|\ |_p$ on $\mathbb{Q}$ as follows.

Example 4. Let p be a prime number. As every element in $\mathbb{Q}^\times$ can be written as $\pm \prod_q q^{e_q}$ where the product ranges over the primes of $\mathbb{Z}$ and the exponents $e_q \in \mathbb{Z}$ are uniquely determined. We have a map (called the p-adic valuation) $v_p : \mathbb{Q} \to \mathbb{Z}$, defined by

$$v_p\left(\pm \prod_q q^{e_q}\right) = e^p,$$

and $v_p(0) := \infty$. The *p-adic absolute value* on $\mathbb{Q}$ is defined by $|x_p| := p^{-v_p(x)}$, where $|0|_p = p^{-\infty}$ is understood to be 0.

Theorem 5 (Ostrowski)**.** *Every nontrivial absolute value on $\mathbb{Q}$ is equivalent to either $|\ |_\infty$ or $|\ |_p$ for some prime p.*

Sketch. For any $m, n \in \mathbb{Z}$, we can write $m = a_0 + a_1 n + \cdots + a_r n^r$ where $a_i \in \mathbb{Z}, 0 \leq a_i < n, n^r \leq m$. Letting $N := \max\{1, |n|\}$, we obtain a bound $|m| \leq N^{\log m / \log n}$.

If for all $n > 1$, we have $|n| > 1$, then $N = |n|$. From the previous inquality, we find $|m|^{1/\log m}$ is constant for all $m \in \mathbb{Z}_{>1}$. It follows $|n| = |n|_\infty^{\log c}$ for all integer $n > 1$, hence $|\ |$ is equivalent to $|\ |_\infty$.

If there is $n \in \mathbb{Z}$ such that $n > 1$ but $|n| \leq 1$, then $N = 1$ and hence $|m| \leq 1$ for all $m \in \mathbb{Z}$, meaning the absolute value is nonarchimedean. Let $\mathcal{O}$ be the corresponding local ring and $\mathfrak{m}$ be its maximal ideal. We find $\mathbb{Z} \subset \mathcal{O}$ and $\mathfrak{p} \cap \mathbb{Z}$ is a nonzero prime ideal, hence this ideal is (p) for some prime p. This implies $|m| = 1$ if m is not divisible by p, hence $|ap^r| = |p|^r$ if n is rational number whose numerator and denominator are not divisible by p. If $a \in \mathbb{R}$ such that $|p| = (1/p)^a$ then $|x| = |x|_p^a$ for all $x \in \mathbb{Q}$. $\square$

For a number field k, i.e. a finite extension of $\mathbb{Q}$, we can describe the places of k as follows

Theorem 6. *There exists exactly one place of k*

(1) for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$,
(2) for each real embedding of k (i.e. an injective field homomorphism $k \hookrightarrow \mathbb{R}$),
(3) for each conjugate pair of complex embeddings.

Example 7. When $k = \mathbb{Q}[x]/(x^2+1)$, we have one conjugate pair of complex embeddings $k \hookrightarrow \mathbb{C}$ sending $x \mapsto \pm i$. This corresponds to the completion $\mathbb{C}$ of k. On the other hand, ring of integers $\mathcal{O} = \mathbb{Z}[x]/(x^2+1)$ of k has prime ideals

(1) $(1+i) = (1-i)$,
(2) $(a+ib)$ where $a^2+b^2 = p$ is a prime with $p \equiv 1 \pmod 4$,
(3) (p) where $p \in \mathbb{Z}$ is a prime such that $p \equiv 3 \pmod 4$.

The absolute value of k corresponding to each prime ideal is defined analogously as in the case of p-adic absolute value for $\mathbb{Q}$.

2.1.1. *Nonarchimedean absolute values from discrete valuations.* The class of nonarchimedean valued fields that is of interest for us comes from discrete valuations.

Definition 8. *A* valuation *on a field k is a group homomorphism $k^\times \to \mathbb{R}$ such that for all $x, y \in k^\times$*

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

We may extend v to a map $k \to \mathbb{R} \cup \{\infty\}$ by defining $v(0) := \infty$. For any $0 < c < 1$, defining $|x|_v := c^{v(x)}$ yields the same nonarchimedean absolute value up to equivalence. We say v is a (normalised) discrete valuation *if $v(k^\times) = \mathbb{Z}$. We call $A := \{x \in k : v(x) \geq 0\}$ the* valuation ring *of k. A* discrete valuation ring *is an integral domain that is the valuation ring of its fraction field with respect to a discrete valuation.*

Example 9. For $k = \mathbb{Q}$, p-adic absolute value comes from discrete valuation v_p as in Example 4.

For a discrete valuation ring A, there holds $v(A) = \mathbb{Z}_{\geq 0}$, so there exists elements $\pi \in A$ such that $v(\pi) = 1$, which we call them *uniformisers* of A. If we fix a uniformiser π then every element $x \in k^\times$ can be written uniquely as $x = u\pi^n$, where $n = v(x)$ and $u = x/\pi^{v(x)} \in A^\times$. Every nonzero ideal of A is equal to $(\pi^n) = \{a \in A : v(a) \geq n\}$ for some integer $n \geq 0$. Hence, A has a unique maximal ideal $\mathfrak{m} = (\pi) = \{a \in A : v(a) \geq 0\}$.

Discrete valuation rings enjoys many properties which gives it many equivalent definitions. At the moment, we will direct the reader to [Sut19, Lecture 1], [Ser79], [Mil] for further discussions about this.

Example 10. The p-adic valuation v_p of $\mathbb{Q}$ as in Example 4 has valuation ring $\mathbb{Z}_{(p)}$, which is the localisation of $\mathbb{Z}$ at the multiplicative set $\mathbb{Z} \setminus (p)$. Concretely, it is a subring of $\mathbb{Q}$, with elements of the form $\frac{a}{b} \in \mathbb{Q}$ where $p \nmid b$. The residue field is $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$.

Example 11. For any field k, the valuation $v : k(\!(t)\!) \to \mathbb{Z} \cup \{\infty\}$ on the field of Laurent series over k defined by

$$v\left(\sum_{n \geq n_0} a_n t^n\right) := n_0,$$

where $a_{n_0} \neq 0$, has valuation ring $k[[t]]$, the ring of power series with coefficients in k. For $f \in k(\!(t)\!)^\times$, $v(f) \in \mathbb{Z}$ is the order of vanishing of f at 0.

2.2. **Completions of global fields.**

Definition 12. *Let k be a field with nontrivial absolute value. A sequence (a_n) of elements in k is called a* Cauchy sequence *if for every $\varepsilon > 0$, there is $N > 0$ such that $|a_n - a_m| < \varepsilon$ for all $n, m > N$. The field k is said to be* complete *if every Cauchy sequence has limit in k.*

Theorem 13. *Let k be a field with absolute value $|\ |$. There exists a complete value field $(\hat{k}, |\ |)$ and a homomorphism $k \to \hat{k}$ of topological fields, preserving the absolute value that is universal in the following sense: every homomorphism $k \to l$ from k to a complete value field $(l, |\ |)$ preserving the absolute value extends uniquely to a homomorphism $\hat{k} \to l$.*

Sketch. Construct $\hat{k}$ to be the set of equivalence classes of Cauchy sequences in $\hat{k}$, in the sense that two Cauchy sequences (a_n) and (b_n) are equivalent when $\lim_{n\to\infty} |a_n - b_n| = 0$. One can then define addition and multiplication in the obvious way and show that $\hat{k}$ is a field. An element $a \in k$ has image $(a, a, \dots)$ inside $\widehat{k}$. □

We are interested in completed valued fields that come from taking completions of a *global field*, i.e. a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_q((t))$. The resulting completed fields are called local fields, which have the following equivalent but simple description.

Definition 14. *A* local field *is a valued field k with nontrivial absolute value such that k is a locally compact.*

Note that if k is locally compact then k is complete [2]. All archimedean local fields are isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.

2.2.1. *Completions from discrete valuations.* This section is about completed valued field with discrete valuation, which, in particular, is where all nonarchimedean local fields come from.

Let $|\ |$ be a nonarchimedean absolute value on k obtained via a discrete valuation v. Let $A, \mathfrak{m}, \pi$ be the corresponding valuation ring, maximal ideal and uniformiser of k.

Proposition 15. *(a) For a completion $\widehat{k}$ of k with respect to $|\ |$ then $|\ |$ is also a discrete absolute value on $\widehat{k}$. Its maximal ideal $\widehat{\mathfrak{m}}$ is generated by π. The residue field of $\widehat{k}$ is $A/\mathfrak{m} \cong \widehat{A}/\widehat{\mathfrak{m}}$.*

(b) If $S \subset A$ is a set of representatives of $A/\mathfrak{m}$ then every element in $\widehat{k}$ has a unique representative of the form

$$a_{-n}\pi^{-n} + \cdots + a_0 + a_0\pi + \cdots + a_m\pi^m + \dots, \qquad a_i \in S.$$

(c) Furthermore, we have an isomorphism of topological rings

$$\widehat{A} \cong \varprojlim_{n\to\infty} \frac{A}{\pi^n A}.$$

Sketch. (a) Let $a \in \widehat{k}^\times$ then a corresponds to a sequence (a_n) in k converging to a. Then $|a_n| \to |a|$, so $|a|$ is a limit point of $|k^\times|$. But $|k^\times|$ is discrete of $\mathbb{R}$, hence closed,hence $|a| \in |k^\times|$. Thus $|\ |$ is a discrete absolute value on $\widehat{k}$ and we denote v to be also valuation on $\widehat{k}$ extending the one on k. This follows $\widehat{\mathfrak{m}}$ is generated by π.

(b) Let $\alpha \in \widehat{k}$ then $\alpha = \pi^n\alpha_0$ for α_0 unit in $\widehat{A}$. There exists $a_0 \in S$ such that $\alpha_0 - a_0 \in \widehat{\mathfrak{m}}$. Then $\frac{\alpha_0 - a_0}{\pi} \in \widehat{A}$ so there exists $a_1 \in S$ such that $\frac{\alpha_0 - a_0}{\pi} - a_1 \in \widehat{\mathfrak{m}}$. Keep going, we can write $\alpha_0 = a_0 + a_1\pi + \dots$ and $\alpha = \pi^n\alpha_0$.

We refer to [Sut19, Lecture 8] for the proof of part (c). □

[2] Let $(x_n)_{n=1}^\infty$ be a sequence in k that converges to $x \in \widehat{k}$. Let $U \subset k$ be a compact neighborhood of x_1 then $x_n x_1^{-1} U$ is a compact neighborhood of x_n. We should be able to find $x \in \bigcup_n x_n x_1^{-1} U \subset k$.

Example 16. Let $k = \mathbb{Q}$, v_p be the p-adic valuation of $\mathbb{Q}$ and $|x|_p := p^{-v_p(x)}$ be the corresponding p-adic absolute value. The completion of $\mathbb{Q}$ with respect to $|\ |_p$ is the field $\mathbb{Q}_p$ of p-adic numbers. For $x = a_m p^m + a_{m+1} p^{m+1} \cdots \in \mathbb{Q}_p$ where $m \in \mathbb{Z}, a_i \in \mathbb{F}_p, a_m \neq 0$ then $|x|_p = p^{-m}$. From Example 10, v_p over $\mathbb{Q}$ has valuation ring $\mathbb{Z}_{(p)}$, we have $\widehat{\mathbb{Z}_{(p)}} = \mathbb{Z}_p$, the p-adic integers. The basis of open sets of $0 \in \mathbb{Q}_p$ are $p^k \mathbb{Z}_p$ where $k \in \mathbb{Z}$.

Example 17. Let $k = k(t)$, v_t be t-adic valuation on $k(t)$, $|x|_t := q^{-v_t(x)}$ ($q > 1$ be any fixed real number) be the corresponding absolute value with $\pi = t$ being the uniformiser. The completion of $k(t)$ with respect to $|\ |_t$ is isomorphic to field $k((t))$ of Laurent series over k. The valuation ring of $k(t)$ with respect to v_t is $k[t]_{(t)}$, ring of rational functions whose denominators have nonzero constant term. With $\pi = t$ as out uniformiser, we find $\widehat{k[t]_{(t)}} = k[[t]]$, the power series over k.

Proposition 18. *k is locally compact if and only if it is complete and and has finite residue field $A/\mathfrak{m}$.*

Proof. If k is locally compact then k is complete. As $\pi^n A$ where $n \in \mathbb{Z}$ form a fundamental system of closed neighborhoods of 0, at least one of them is compact. Multiplying by π^{-n}, which is a homeomorphism, shows that A is compact. Let S be set of representatives for $A/\mathfrak{m}$ then compact subset A is a disjoint union of open sets $s + \mathfrak{m}$ for $s \in S$, implying S is finite.

Conversely, if $A/\mathfrak{m}$ is finite then $A/\pi^n A$ is finite, hence from previous proposition, $\widehat{A}$ is a projective limit of finite rings, hence is compact. If k is complete then $A = \widehat{A}$ is compact, meaning k is locally compact. □

Example 19. 1) The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to p-adic valuation v_p is locally compact, hence a nonarchimedean local field.

2) $\mathbb{F}_q(\!(t)\!)$ is locally compact as it is the completion of $\mathbb{F}_q(t)$ with respect to t-adic valuation and residue field $\mathbb{F}_q$.

3. Measures and integration

In this section we review the theory of measure spaces and integration on locally compact spaces, in particular Haar measure on locally compact groups. We refer to [Fol15, VR99, Kna02, BSU96] for the proofs of the results in this section.

Convention 20. From now on, all locally compact spaces are assumed to be Hausdorff.

3.1. **Measure.** Let X be a set, and let $\mathcal{M}$ be a collection of subsets of X.

Definition 21. *$\mathcal{M}$ is a σ-algebra if $\mathcal{M}$ is closed under taking complements in X and countable unions. Elements of $\mathcal{M}$ are called measurable sets.*

Example 22. Let X be a topological space. The collection of *Borel sets* is the σ-algebra $\mathcal{B}(X)$ generated by open subsets of X.

Definition 23. *A function $f : X \to Y$ is called measureable if the preimages of any measurable subset in Y is measureable in X.*

Remark 24. Let f be a complex-valued function on a σ-algebra, where the measurable sets in $\mathbb{C}$ are the Borel sets of $\mathbb{C}$. For f to be measurable, it suffices to check $f^{-1}(S)$ is measurable for open disks in $\mathbb{C}$. When f is real-valued, f is measurable iff $f^{-1}(S)$ is measurable for any $S = (a, \infty) \subset \mathbb{R}$ where $a \in \mathbb{R}$.

Definition 25. *A measure on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that $\mu\left(\bigcup A_i\right) = \sum \mu(A_i)$ for any countable (or finite) collection of disjoint measureable sets A_i. In the special case where $\mathcal{M} = \mathcal{B}$, a measure is called a Borel measure.*

A set $N \subset X$ is called a *null set* if N is contained in a measure-0 set. It is convenient to enlarge $\mathcal{M}$ so that all null sets are measurable. We call $f : X \to \mathbb{C}$ a *null function* if $\{x \in X : f(x) \neq 0\}$ is a null set.

Given a measure (X, μ) and a measurable map $f : X \to Y$ then the *pushforward* of μ is a measure on Y where $(f_*\mu)(B) := \mu(f^{-1}(B))$ for any measurable subset B of Y. We are not aware of any reference discussing pullback of measures in general. However, if X, Y are smooth manifolds and f is submersive, pullback of measures can be defined via fiber integrations.

3.2. **Integration.** We fix the notation $(X, \mathcal{M}, \mu)$ where X is a set with σ-algebra $\mathcal{M}$ and measure μ. We will briefly define integration with respect to this space. We refer to [BSU96] for a more details discussion of this construction.

Given $S \in \mathcal{M}$ with $\mu(S) < \infty$, let $1_S : X \to \{0, 1\}$ be the indicator function on S, i.e. it has value 1 on S and 0 outside of S, and define $\int_X 1_S d\mu := \mu(S)$. A *simple* function f is a function of the form $f = \sum_{i=1}^n a_i 1_{S_i}$ where $a_i \in \mathbb{R}$ and S_i's are pairwise disjoint sets in $\mathcal{M}$ of finite measure. For such a simple function $f = \sum a_i 1_{S_i}$, define $\int_X f d\mu := \sum_i a_i \mu(S_i)$. For any real-valued nonnegative measurable function f on X, we define

$$\int_X f(x) d\mu(x) := \sup_\phi \int_X \phi(x) d\mu(x),$$

where ϕ ranges over all real-valued simple functions on X with $0 \leq \phi \leq f$. We say that a measurable function $f : X \to \mathbb{C}$ is *integrable* if $\int_X |f(x)| dx < \infty$. If f is integrable, we can write $f = (u^+ - u^-) + i(v^+ - v^-)$ where $u^+(x) = \max\{\mathrm{Re}(f(x)), 0\}$, $u^-(x) = -\min\{\mathrm{Re} f(x), 0\}$ and similarly for v^+, v^-. We then define

$$\int_X f(x) d\mu(x) := \int_X u^+ d\mu - \int_X u^- d\mu + i \int_X v^+ d\mu - i \int_X v^- d\mu.$$

We define $L^1(X, \mu)$ to be the Banach space of measurable functions $f : X \to \mathbb{C}$ that have finite L^1-norm $\|f\|_1 := \int_X |f| d\mu$.

Proposition 26 (Change of variables formula)**.** *Given a measure (X,μ) and a measurable map $f : X \to Y$. Then, for a measurable function $g : Y \to \mathbb{C}$, $g \circ f$ is measurable and*

$$\int_Y g d(f_*\mu) = \int_X g \circ f d\mu.$$

3.3. **Measures and integrals on locally compact Hausdorff space.** Let X be a locally compact topological space.

Definition 27. *A function $f : X \to \mathbb{C}$ has compact support if the closure of $\{x \in X : f(x) \neq 0\}$ is compact. Define $C_c(X)$ to be space of continuous $f : X \to \mathbb{C}$ of compact support.*

Definition 28. *An outer Radon measure on X is a Borel measure $\mu : \mathcal{B} \to [0,\infty]$ that is*

- *locally finite: every $x \in X$ has an open neighborhood U such that $\mu(U) < \infty$*
- *outer regular: every $S \in \mathcal{B}$ satisfies $\mu(S) = \inf \mu(U)$ over all open $U \supset S$,*
- *inner regular on open sets: every open $U \subset X$ satisfies $\mu(U) = \sup \mu(K)$ over all compact $K \subset U$.*

A Radon integral on X is a $\mathbb{C}$-linear map $I : C_c(X) \to \mathbb{C}$ such that $I(f) \geq 0$ whenever $f \geq 0$.

For a Radon measure space (X,μ), $C_c(X)$ is a subspace of $L^1(X,\mu)$.

Theorem 29 (Riez representation theorem)**.** *Given an outer Radon measure μ, we define a linear functional*

$$\begin{aligned} I_\mu : C_c(X) &\to \mathbb{C} \\ f &\mapsto \int_X f d\mu. \end{aligned}$$

When X is locally compact Hausdorff, there is a bijection between outer Radon measures on X and Radon integrals on X, where one direction is given by $\mu \mapsto I_\mu$. The other direction is by sending $I : C_c(X) \to \mathbb{C}$ to the measure μ on X defined by $\mu(S) = I(1_S)$.

Example 30. Let $X = \mathbb{R}^n$, the map sending $f \in C_c(\mathbb{R}^n)$ to the Riemann integral $\int_{\mathbb{R}^n} f \in \mathbb{C}$ is a Radon integral. The Lebesgue measure μ_n is defined to be the corresponding outer Radon measure on $\mathbb{R}^n$. Note that we have $\mu_n(gA) = |\det(g)|\mu_n(A)$ for any $g \in \mathrm{GL}_n(\mathbb{R})$ and $A \in \mathcal{B}(\mathbb{R}^n)$.

3.4. **Haar measure.** Let G be a locally compact Hausdorff topological group. In this section, we will define Haar measures on G and studies some properties of this kind of measures.

Definition 31. *A Borel measure μ on G is left-invariant if $\mu(gS) = \mu(S)$ for all $g \in G$ and $S \in \mathcal{B}$. A left Haar measure on G, denoted d_lg, is a nonzero left-invariant outer Radon measure on G. Right Haar measure d_rg is defined similarly.*

Remark 32. In terms of Radon integrals, the condition $\mu(gS) = \mu(S)$ for any measurable S is equivalent to

$$\int_G f(x)d\mu(x) = \int_G f(g^{-1}x)d\mu(x)$$

for any $f \in C_c(G)$. Indeed, it suffices to check this for $f = 1_S$ where $S \subset G$ is measurable.

Convention 33. To be more precise, a left Haar measure μ is a map from Borel sets of G to $[0,\infty]$. However, for convenience, we will usually denote a left Haar measure of G to be d_lg and a right Haar measure by d_rg, where g is understood to be an element of G. For example, left-invariant property is short-handed as $d_l(hg) = d_l(g)$, where $d_l(hg)$ is understood to be the measure of G obtained by pushforward d_lg via left-multiplication by h^{-1}, i.e. $\Omega \mapsto \mu(h\Omega)$. Sometimes $d_l(hg)$ would cause ambiguity, where it could mean either pushing forward d_lg via left-multiplication by h^{-1}, i.e. $\Omega \mapsto d_l(h\Omega)$, or pushing forward d_lh via right-multiplication by g^{-1}, i.e. $\Omega \mapsto d_l(\Omega g)$, but we will try to be more precise when the situation arises. A better convention is $d(L_{h^{-1}}g)$ or $d(R_{g^{-1}}h)$.

Theorem 34 (Uniqueness of Haar measure)**.** *Let G be a locally compact topological group. There exists a left Haar measure μ on G and every other left Haar measure on G is $c\mu$ for some $c \in \mathbb{R}_{>0}$.*

Example 35. On $\mathbb{R}^n$, the Lebesgue measure is a Haar measure.

Remark 36. A left Haar measure need not be right-invariant. For example, consider

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\}$$

then G has a left Haar measure given by $\mu_L(S) = \int_S \frac{1}{a^2} dadb$ and a right Haar measure given by $\mu_R(S) = \int_S \frac{1}{|a|} dadb$.

Let μ be a left Haar measure on G, then G is compact if and only if $\mu(G) < \infty$. The normalised Haar measure on G is the unique Haar measure μ such that $\mu(G) = 1$.

Example 37. Let k be a nonarchimedean local field with valuation ring $\mathcal{O}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$ and π be a uniformiser of $\mathcal{O}$. There is a Haar measure μ on k satisfying $\mu(\mathcal{O}) = 1$.

For example, we will show $\mu(\mathfrak{m}) = (\#\mathcal{O}/\mathfrak{m})^{-1}$ (here $\#\mathcal{O}/\mathfrak{m}$ refers to number of elements of this finite field). Indeed, as $\mathcal{O}_k$ is a disjoint union of $a + \mathfrak{m}$'s where $a \in \mathcal{O}/\mathfrak{m}$ and that μ is left-invariant, we find

$$1 = \mu(\mathcal{O}) = \sum_{a \in \mathcal{O}/\mathfrak{m}} \mu(a + \mathfrak{m}) = (\#\mathcal{O}/\mathfrak{m})\mu(\mathfrak{m}).$$

Similarly, one can show that $\mu(\pi^n\mathcal{O}) = (\#\mathcal{O}/\mathfrak{m})^{-n}$ for $n \in \mathbb{Z}$.

For another example of a computation with μ, we will show $\mu(aA) = |a|_k\mu(A)$ for any open A of k and $a \in k^\times$. Indeed, let $a = u\pi^n$ where $u \in \mathcal{O}^\times, n \in \mathbb{Z}$ and if $A = \mathcal{O}$ then

$$\mu(aA) = \mu(\pi^n\mathcal{O}) = (\#\mathcal{O}/\mathfrak{m})^{-n} = |a|_k\mu(A).$$

As $\pi^n\mathcal{O}$'s form a basis of open neighborhoods of $0 \in k$ so from the above computation, we are done. In fact, $\mu(aA) = |a|_k\mu(A)$ holds for any choice of Haar measure on k.

3.4.1. *Modular quasicharacter.* Denote d_rg to be a right Haar measure of G. Then $d_r(hg)$ is also a right Haar measure. Therefore, by uniqueness of right Haar measure, there exists a positive real $\delta_G(h)$ so $d_r(hg) = \delta_G(h)d_rg$. We define the *modular quasicharacter* to be the corresponding group homomorphism $\delta_G : G \to \mathbb{R}_{>0}$ [3]. Note that δ_G does not depend on the choice of a left/right Haar measure on G.

Proposition 38. *Let d_rg, d_lg be right, left Haar measures of G, respectively. Then the following are equivalent ways to define the modular quasicharacter:*

(a) $d_r(hg) = \delta_G(h)d_rg$ *for all* $h \in G$,
(b) $d_l(gh) = \delta_G(h)^{-1}d_lg$ *for all* $h \in G$,
(c) $d_r(g^{-1}) = \delta_G(g)^{-1}d_rg$,
(d) $d_l(g^{-1}) = \delta_G(g)d_lg$.

Furthermore, if we given d_rg, we can choose d_lg to be such that $d_lg = d_r(g^{-1})$, or equivalently, $d_rg = \delta_G(g)d_lg$.

Finally, every left Haar measure is right Haar measure iff $\delta_G \equiv 1$ on G. If this is the case, we say G is unimodular.

[3] Some authors define modular quasicharacter to be the multiplicative inverse of δ_G, such as in [Fol15]. Our choice for the definition of δ_G is reflected in Proposition 113

Sketch. We will prove (a) implies (c). Note that $d_r(g^{-1})$ is a left Haar measure. Next, we show $\delta_G(g)^{-1}d_rg$ is also a left Haar measure. We have

$$\begin{aligned}\int_G f(hg)\delta_G(g)^{-1}d_rg &= \int_G f(g)\delta_G(h^{-1}g)^{-1}d_r(h^{-1}g),\\ &= \int_G f(g)\delta_G(h^{-1}g)^{-1}\delta_G(h^{-1})d_rg,\\ &= \int_G f(g)\delta_G(g)^{-1}d_rg.\end{aligned}$$

By uniqueness of left Haar measure, we find $d_r(g^{-1}) = c\delta_G(g)^{-1}d_rg$ for some constant c. Changing g to g^{-1} (i.e. pushforward two measures under taking inversion, which should give us the same equality), we find

$$d_r(g) = c\delta_G(g)d_r(g^{-1}) = c^2\delta_G(g)\delta_G(g^{-1})d_r(g^{-1}) = c^2d_r(g^{-1}),$$

hence $c = 1$.

To show (c) implies (b). As $d_r(g^{-1})$ is a left Haar measure so we have $d_l(g) = cd_r(g^{-1})$ for some $c \in \mathbb{R}_{>0}$. Then we have

$$d_l(gh) = cd_r(h^{-1}g^{-1}) = \delta_G(h^{-1})cd_r(g^{-1}) = \delta_G(h)^{-1}d_lg.$$

The other equivalences of (a), (b), (c), (d) can be done similarly.

Next, we show $d_rg = \delta_G(g)d_l(g)$ implies $d_lg = d_r(g^{-1})$. Indeed, by (c) and (d), we find $d_r(g^{-1}) = \delta_G(g)^{-1}d_r(g) = d_l(g)$.

Finally, we show if every left Haaf measure is right Haar measure then $\delta_G = 1$. Let $d_r = cd_l$, then from (a), as we fix h, we find

$$c\delta_G(h)d_lg = \delta_G(h)d_rg = d_r(hg) = c^{-1}d_l(hg) = c^{-1}d_lg.$$

This follows $\delta_G(h) = c^{-2}$, a constant. As δ_G is a group homomorphism, we find $\delta_G \equiv 1$. $\square$

3.4.2. *Haar measure on homogeneous space.* In this section, let G be a locally compact group with closed subgroup H. Then G acts on G/H by left-multiplication. We say a measure μ on G/H is *G-invariant* if $\mu(A) = \mu(xA)$ for any $x \in G$ and measurable $A \subset G/H$.

Theorem 39. *Let H be a closed subgroup of G with corresponding modular quasicharacters δ_H, δ_G. A necessary and sufficient condition for G/H to have nonzero G-invariant Borel measure $\mu_{G/H}$ is that the restriction to H of δ_G equals δ_H. In this case, such measure is unique up to positive scalar, and it can be normalised so that for any $f \in C_c(G)$, we have*

$$\int_{G/H} f^H d\mu_{G/H} = \int_G f d\mu_G$$

where $f^H \in C_c(G/H)$ is defined by

$$f^H(x) = \int_H f(xh)d\mu_H.$$

Sketch. We sketch the proof when G, H are unimodular. We denote the projection $p : G \to G/H$.

In fact, the map $C_c(G) \to C_c(G/H)$ sending $f \mapsto f^H$ is onto, which we will not prove here but refer to [Fol15, p.62]. To show $\mu_{G/H}$ can be defined as in the theorem, we need to show $f^H \mapsto \int_G f d\mu_G$ is a well-defined G-invariant positive linear functional on $C_c(G/H)$. By surjectivity of $C_c(G) \to C_c(G/H)$, it suffices to show that if $f \in C_c(G)$ and $f^H = 0$ then $\int_G f d\mu_G = 0$. Let

$\varphi \in C_c(G/H)$ such that $\varphi = 1$ on $p(\text{supp } f)$, then there exists $g \in C_c(G)$ so $g^H = \varphi$. Assuming $f^H = 0$, we find

$$\begin{aligned} 0 &= \int_G g(x) f^H(x) dx = \int_G \int_H g(x) f(xh) dh dx, \\ &= \int_H \int_G g(x) f(xh) dx dh = \int_H \int_G g(xh) f(x) dx dh, \\ &= \int_G \int_H g(xh) f(x) dh dx = \int_G f(x) g^H(x) dx, \\ &= \int_G f(x) dx. \end{aligned}$$

We are done. □

Remark 40. One can consider right coset $H \backslash G$ and modify the above theorem accordingly.

In general, one can define left G-invariant measure of a space under continuous transitive action of G as follows.

Definition 41. *Let S be a locally compact topological space then S is a* G-space *if there is a continuous left action of G on S, i.e. a continuous map from $G \times S$ to S such that $s \mapsto xs$ is a homeomorphism of S, and $x(ys) = (xy)s$ for all $x, y \in G, s \in S$. A G-space is called transitive if for every $s, t \in S$ there exists $x \in G$ such that $xs = t$.*

If S is a transitive G-space then for any $s_0 \in S$, the isotropy/stabiliser group $H = \{x \in G : xs_0 = s_0\}$ of s_0 is a closed subgroup of G and $\phi : G \to S$ by $x \mapsto xs_0$ is a continuous surjection of G onto S. This induces a continuous bijection $\Phi : G/H \to S$ such that $\Phi \circ p = \phi$ where $p : G \to G/H$ is the quotient map. Note that it is generally not the case that Φ has continuous inverse. For example, consider $G = \mathbb{R}$ with the discrete topology, acting by translation on $S = \mathbb{R}$ with the usual topology. We call S a *homogeneous space* if Φ is a homeomorphism. With this, we can identify S with G/H and G-invariant measure on G/H with G-invariant measure on S.

3.4.3. *Haar measure from fundamental domain.* When H is a discrete subgroup of G, one can determine $\mu_{G/H}$ by integrating with respect to μ_G over a fundamental domain F of G.

Definition 42. *Given a locally compact topological group G and with a discrete subgroup H, a measurable set $F \subset G$ is a* strict fundamental domain for $H \setminus G$ *if the projection $\pi : F \to H \setminus G$ is a bijection. A measurable set $F \subset G$ is a* fundamendal domain *for $H \setminus G$ if F differs from a strict fundamental domain by a set of Haar measure 0.*

When we have such fundamental domain F, we can define a G-invariant measure on $H \setminus G$ by integrating over F:

Proposition 43. *Let G be a locally compact topological group with a left Haar measure $d\mu_G$ and H is a countable discrete subgroup of G, $F \subset G$ is a fundamental domain for $H \setminus G$. Then the quotient measure $H \setminus G$ can be given by*

$$\int_{H \backslash G} f(Hg) d\mu_{H \backslash G}(Hg) = \int_F f(g) d\mu_G(g).$$

Proof. By uniqueness of G-invariant measure on $H \setminus G$, it suffices to check that

$$\int_G f(g) dg = \int_F \sum_{h \in H} f(hg) dg$$

for all $f \in C_c(G)$. As $G = \bigcup_{h\in H} hF$, we find

$$\int_G f(g)dg = \sum_{h\in H} \int_{hF} f(g)dg = \sum_{h\in H} \int_F f(hg)dg = \int_F \sum_{h\in H} f(hg)dg$$

by left-invariance of G and the fact that we can exchange the sum and the integral by Fubini's theorem. ☐

3.4.4. *Haar measure from closed subgroups.* We have the following useful proposition that describe a Haar measure on $G = ST$ in terms of Haar measures on its closed subgroups S and T.

Proposition 44 (Theorem 8.32 of [Kna02]). *Suppose S and T are closed subgroups of G with compact intersection and the product map $S \times T \to G$ is open with image exhausting G except possibly for a set of Haar measure 0. Then one can normalise the left and right Haar measures on S and T, respectively, so that*

$$\int_G f(g)d_lg = \int_{S\times T} f(st)\frac{\delta_T(t)}{\delta_G(t)}d_lsd_rt = \int_{S\times T} \frac{f(st)}{\delta_G(t)}d_lsd_rt.$$

In particular, if G is unimodular, then

$$\int_G f(g)dg = \int_{S\times T} f(st)d_lsd_rt.$$

Proof. The group $S \times T$ acts continuously on $ST \subset G$ by $(s,t)w = swt^{-1}$, and the isotropy group at 1 is $K \times K$ where $K = S \cap T$. Thus, we have a bijective continuous map $\Phi : (S\times T)/(K\times K) \to ST$ sending $(s,t) \mapsto st^{-1}$. This map is a homeomorphism (i.e. has continuous inverse) since multiplication $S \times T \to G$ is an open map. Hence, a left Haar measure d_lg of G restricts to a Borel measure on ST, and hence obtaining a Borel measure $d\mu$ on $(S \times T)/(K \times K)$ via change of variables formula Proposition 26 for measures:

$$\int_{(S\times T)/(K\times K)} f(\Phi(s,t))d\mu = \int_{(S\times T)/(K\times K)} f(st^{-1})d\mu = \int_{ST} f(g)d_lg.$$

We denote $L_g, R_g : G \to G$ to be left/right translation maps. From Proposition 38, we have $d_l(L_{s_0}R_{t_0^{-1}}g) = \delta_G(t_0)d_lg$, which gives

$$\begin{aligned}
\int_{(S\times T)/(K\times K)} f(s,t)d\mu(L_{(s_0,t_0)}(s,t)) &= \int_{(S\times T)/(K\times K)} f(s_0^{-1}s, t_0^{-1}t)d\mu,\\
&= \int_{(S\times T)/(K\times K)} (f\circ\Phi^{-1}\circ L_{s_0}R_{t_0}^{-1}\circ\Phi)(s,t)d\mu,\\
&= \int_{ST} (f\circ\Phi^{-1})(L_{s_0^{-1}}R_{t_0}g)d_lg,\\
&= \int_{ST} (f\circ\Phi^{-1})(s_0^{-1}gt_0)d_lg,\\
&= \int_{ST} (f\circ\Phi^{-1})(g)d_l(s_0gt_0^{-1}),\\
&= \int_{ST} (f\circ\Phi^{-1})(g)\delta_G(t_0)d_l(g),\\
&= \delta_G(t_0)\int_{(S\times T)/(K\times K)} f(s,t)d\mu,
\end{aligned}$$

or in our convention,

$$d\mu(L_{(s_0,t_0)}x) = \delta_G(t_0)d\mu(x) \tag{1}$$

on $(S\times T)/(K\times K)$. We define measure $d\tilde{\mu}(s,t)$ on $S\times T$ by

$$\int_{S\times T} f(s,t)d\tilde{\mu}(s,t) = \int_{(S\times T)/(K\times K)} \left[\int_K f(sk,tk)dk\right] d\mu((s,t)K),$$

where dk is Haar measure on compact K normalised to have volume 1. From (1), we have $d\tilde{\mu}(s_0 s, t_0 t) = \delta_G(t_0)d\tilde{\mu}(s,t)$. Note that $\delta_G(t)d_l s d_r t$ also satisfies this condition. Therefore, $d\tilde{\mu}(s,t) = \delta_G(t)d_l s d_r t$ for suitable normalisation of $d_l s d_r t$ (to see this, mimic proof of left Haar measure is unique up to scalar, see [Kna02, Theorem 8.23]). Hence, we find

$$\int_{ST} f(g)d_l g = \int_{S\times T} f(st^{-1})\delta_G(t)d_l s d_r t$$

for all $f\in C_c(ST)$. Changing t by t^{-1} on the right hand side via Proposition 38 and replace ST by G on the left hand side, we are done. $\square$

3.4.5. *Haar measure on restricted product.* In this section, we will construct certain Haar measure on restricted products, which will be required later in defining Haar measure of adelic points of linear algebraic groups. We first define restricted products of a family of topological spaces.

Definition 45. *Let (X_i) be a family of topological spaces indexed by $i\in I$, and let (U_i) be a family of open sets $U_i\subset X_i$. The restricted product $\prod'_{i\in I} X_i$ with respect to U_i's is the topological space*

$$X = \prod_{i\in I}{}'(X_i,U_i) := \left\{(x_i)\in\prod X_i : x_i\in U \textit{ for almost all } i\in I\right\}.$$

with the basis of open sets

$$\left\{\prod V_i : V_i\subset X_i \textit{ is open for all } i\in I and V_i = U_i \textit{ for almost all } i\right\},$$

where almost all means all but finitely many.

Remark 46. We refer to [Sut19] for the proofs of the following remarks about restricted products:

(1) In general, the restricted product X is not the subspace topology from $\prod X_i$ as the former has more open sets [4].
(2) For a finite set $S\subset I$ then by letting

$$X_S := \prod_{i\in S} X_i \times \prod_{i\notin S} U_i$$

then X_S is an open set of X whose subspace topology is precisely the product topology of X_i's and U_i's. As $\prod' X_i = \bigcup_S X_S$ over all finite set $S\subset I$ so this gives another way to define restricted products as direct limit of X_S's.
(3) If X_i's are locally compact and almost all of U_i's are compact then the restricted product $\prod' X_i$ is locally compact.

Proposition 47 (p. 185 of [VR99])**.** *Let $G=\prod_{v\in J} G_v$ be the restricted direct product of locally compact groups G_v with respect to family of compact subgroups $H_v\subset G_v$ (except for some finite set of places J_∞). Let μ_v be a left Haar measure on G_v normalised so that $\prod_{v\notin J_\infty}\mu_v(H_v)$ converges. Then there is a unique Haar measure μ on G such that for each finite set of indices S containing J_∞, the restriction μ_S of μ to*

$$G_S = \prod_{v\in S} G_v \times \prod_{v\notin S} H_v$$

is the product measure.

[4] recall the product topology on $\prod_i V_i$ of topological spaces V_i is the coarsest topology for which all the projections are continuous

Proof. The finiteness of $\prod_{v \nmid \infty} \int_{H_v} dg_v$ guarantees that the product measure μ_S on G_S is a Haar measure, i.e. is finite on compact subsets $\prod_{v \nmid \infty} H_v$.

Next, we will show the existence of such a Haar measure on G. As G is locally compact, we can choose a left Haar measure μ on G such that for some fixed finite set of S of indices containing J_∞, the restriction of μ to G_S is the product measure μ_S. This measure μ is independent of the choice of S because if we consider another finite set S' of indices containing J_∞, again because of uniqueness of Haar measure on $G_{S \cup S'}$ whose restriction to G_S is μ_S, μ restricted to $G_{S \cup S'}$ must also be a product measure. Hence, μ restricted to $G_{S'}$ is also a product measure. $\square$

4. Analytic manifolds and integrations

Over a complete valued field k with respect to a nontrivial absolute value, one can develop a theory of k-analytic functions and k-analytic manifolds that closely resembles the classical setting of real analytic functions and real analytic manifolds. Furthermore, when k is a local field, one can also define integration of differential forms on k-analytic manifolds. In this section, we will describe this process, following [Igu00].

On a different note, unlike most references we find about differential geometry, we will discuss manifolds and its related objects in the language of sheaf theory. One reason is that this language is also used in algebraic geometry, so in our naive view, it seems to be a more universal language than describing manifolds via compatible charts. for example, such point of view is also taken in [Ram05], [Wed16].

4.1. **Analytic functions.** For every $a = (a_1, \dots, a_d) \in k^d$ and every $r \in \mathbb{R}_{>0}$, we denote the closed and open polydisks of radius r centered at a in k^d to be

$$D(a, r) = \{x \in k^d : |x_i - a_i| \leq r \ \forall i\},$$
$$D_0(a, r) = \{x \in k^d : |x_i - a_i| < r \ \forall i\}.$$

We consider power series $f(T) = \sum_{n \in \mathbb{Z}^d_{\geq 0}} f_n T^n \in k[[T_1, \dots, T_d]]$ of d variables with coefficients in k, where we denote $T^n := T_1^{n_1} \cdots T_d^{n_d}$ and $|n| = n_1 + \cdots + n_d$.

A power series $f = \sum_{n \in \mathbb{Z}^d_{\geq 0}} f_n T^n \in k[[T]]$ is said to be *convergent* if its radius of convergence, defined by $\rho(f) = \left(\limsup_{|n| \to \infty} |f_n|^{1/|n|}\right)^{-1}$ is positive. It follows, in particular, that for any $0 < r < \rho(f)$ then the series $\sum_n f_n T^n$ converges in k for all $T \in D_0(0, r)$.

Let U be an open subset of k^d. We say a function $f : U \to k$ is *k-analytic* in U if for each $a \in U$, there is a real number $r > 0$ and a convergent power series $f_a \in k[[T]]$ such that $f(x) = f_a(x - a)$ for all $x \in D(a, r) \subset U$. Every k-analytic function is continuous. If a k-analytic function on U does not vanish anywhere, then its inverse is k-analytic as well.

For a positive integer m, a function $f : U \to k^m$ defined by $u \mapsto (f_1(u), \dots, f_m(u))$ is k-analytic if each f_i is analytic for $1 \leq i \leq n$. Composition of k-analytic functions is k-analytic.

For a k-analytic function $f : U \to k$ on an open set U of k^d, one can define its partial derivatives at $a \in U$ to be

$$\frac{\partial f}{\partial x_i}(a) := \lim_{t \to \infty} \frac{f(a + t\varepsilon_i) - f(a)}{t},$$

for $i \in \{1, \dots, d\}$, where $\varepsilon_i = (0, \dots, 1, \dots, 0)$ which has 1 in the i-th place and 0 everywhere else. We also know that $\partial f / \partial x_i$'s are k-analytic. We define the Jacobian matrix of a k-analytic map $f : U \to k^d$ as $Df(a) = (\partial f_i / \partial x_j(a))$, where $f = (f_1, \dots, f_d)$. The determinant of $Df(a)$ defines an analytic map J_f on U, called the *Jacobian determinant* of f.

The inverse function theorem and implicit function theorem also work over any complete valued field k.

Theorem 48 (Inverse function theorem)**.** *Let $f : U \to k^d$ be a k-analytic function with open subset U of k^d. Let $a \in U$ such that the Jacobian matrix $Df(a)$ of f at a does not vanish. Then there exist an open neighborhood U_a of a such that $f(U_a)$ is an open neighborhood of $f(a)$ in K^d and a k-analytic function $g : f(U_a) \to U_a$ such that $g \circ f = id_{U_a}$ and $f \circ g = id_{f(U_a)}$.*

Theorem 49 (Implicit function theorem)**.** *Let $F = (F_1, \dots, F_m)$ where $F_1, \dots, F_m \in k[[x_1, \dots, x_n, y_1, \dots, y_m]]$ are k-analytic functions on a neighborhood of $(0, 0)$ such that $F_i(x, y) = 0$ for all $1 \leq i \leq m$. If $\det(\partial F_i / \partial y_j(0, 0)) \neq 0$ then there exists k-analytic functions $f_1, \dots, f_m \in k[[x_1, \dots, x_n]]$ on some open neighborhood U of $0 \in k^n$ such that for $f = (f_1, \dots, f_m)$, $f(0) = 0$ and $F(x, f(x)) = 0$ for all $x \in U$.*

For the proof of these two theorems, we refer to [Igu00, Section 2.1].

4.2. **Locally ringed space.** In this section, we would like to introduce the notion of locally ringed spaces, which we will later use to define k-analytic manifold.

Definition 50. *A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A k-ringed space is a ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of k-algebras. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ over Y.*

A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space $(X, \mathcal{O}_X)$ whose stalks are local rings. Given the stalk $\mathcal{O}_{X,x}$ at x with its unique maximal ideal $\mathfrak{m}_x$, the residue field of X at x is $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$. A morphism of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We say a ringed space $(X, \mathcal{O}_X)$ is locally isomorphic to $(Y, \mathcal{O}_Y)$ if for each $x \in X$, there exists an open neighborhood U of x and an isomorphism $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$ of sheaves where V is some open subset of Y.

Remark 51. For a locally ringed space $(X, \mathcal{O}_X)$, given $f \in \mathcal{O}_X(U)$, we can talk about the *value of f at $x \in U$* as the image of f in $\kappa(x)$. Hence, one would like to think of sections of $\mathcal{O}_X$ as functions on X.

Example 52. Let M be a real C^∞-manifold. Then we can define a structure sheaf $\mathcal{O}_M$ for M where $\mathcal{O}_M(U)$ is the ring of smooth functions $f : U \to \mathbb{R}$. $(M, \mathcal{O}_M)$ is then a locally $\mathbb{R}$-ringed space, as for $x \in M$, $\mathcal{O}_{M,x}$ is the ring of germs of smooth functions at x, which is a local ring with maximal ideal being functions that vanish at x. The value of $f \in \mathcal{O}_M(U)$ at $x \in U$, by definition above, is precisely $f(x)$. Furthermore, $(M, \mathcal{O}_M)$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ with its sheaf of C^∞-functions. Indeed, for any $x \in M$, we can choose a chart $(U, \varphi : U \to \mathbb{R}^n)$ of x, then $(U, \mathcal{O}_M|_U)$ is isomorphic to $(\varphi(U), \mathcal{O}_{\mathbb{R}^n}|_{\varphi(U)})$ by sending a smooth function $f : U \to \mathbb{R}$ on U to a smooth function $f \circ \varphi^{-1}$ on $\varphi(U) \subset \mathbb{R}^n$.

Combining with the previous example, the following proposition indicates that saying M is a real C^∞-manifold is the same as saying that M is a $\mathbb{R}$-ringed space that is locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ with its sheaf of C^∞-functions.

Theorem 53. *Let $(M, \mathcal{O}_M)$ be a $\mathbb{R}$-ringed space that is locally isomorphic to $(\mathbb{R}, \mathcal{O}_{\mathbb{R}^d})$ with its sheaf of C^∞-functions. Then M can be equipped with a structure of a real C^∞-manifold, with $\mathcal{O}_M$ being the sheaf of smooth functions on M.*

Proof. We can cover M by open sets U's such that for each U, there is an isomorphism $\varphi_U : (U, \mathcal{O}_M|_U) \xrightarrow{\sim} (V, \mathcal{O}_{\mathbb{R}^d}|_V)$, where V is open in $\mathbb{R}^d$. We say (U, φ_U) is a *chart* of M. The $\mathbb{R}$-algebra of $\mathbb{R}$-analytic functions on U is $\mathcal{O}_M(U)$. If we are given another chart $(U', \varphi_{U'})$ for M, we have an isomorphism of locally $\mathbb{R}$-ringed space

$$\varphi_{U'}|_{U\cap U'} \circ \varphi_U^{-1}|_{\varphi^{-1}(U\cap U')} : (\varphi_U(U \cap U'), \mathcal{O}_{\mathbb{R}^d}|_{\varphi_U(U\cap U')}) \to (\varphi_{U'}(U \cap U'), \mathcal{O}_{\mathbb{R}^d}|_{\varphi_{U'}(U\cap U')}).$$

The following lemma implies that the above morphism is precise the chart-compatibility condition in the classical definition of manifolds via charts and atlas.

Lemma 54. *Let $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ be the sheaf of C^∞-functions on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be open subsets with the induced structures of locally $\mathbb{R}$-ringed spaces $\mathcal{O}_U, \mathcal{O}_V$ from $\mathbb{R}^n, \mathbb{R}^m$, respectively. Then every morphism $f : (U, \mathcal{O}_U) \to (U, \mathcal{O}_V)$ of locally $\mathbb{R}$-ringed spaces is k-analytic. Furthermore, the morphism of sheaves is given by sending $g \in \mathcal{O}_V(V')$ to $g \circ f \in \mathcal{O}_U(f^{-1}(V'))$ for any open subset V' of V.*

Conversely, any $\mathbb{R}$-analytic map $f : U \to V$ induces a morphism of locally $\mathbb{R}$-ringed spaces via taking compositions.

Sketch of proof of lemma. Let V' be an open subset in V and $a \in f^{-1}(V')$. We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_V(V') \arrow[r] \arrow[d, "f(V')"] & \mathcal{O}_{V,f(a)} \arrow[r] \arrow[d, "f_a"] & \mathcal{O}_{V,f(a)}/\mathfrak{m}_{V,f(a)} \arrow[r, "\sim"] \arrow[d, "\overline{f_a}"] & \mathbb{R} \arrow[d, "\mathrm{id}"] \\
\mathcal{O}_U(f^{-1}(V')) \arrow[r] & \mathcal{O}_{U,a} \arrow[r] & \mathcal{O}_{U,a}/\mathfrak{m}_{U,a} \arrow[r, "\sim"] & \mathbb{R}
\end{tikzcd}$$

In this diagram, the first row corresponds to the evaluation of elements in $\mathcal{O}_V(V')$ at $f(a)$ and similarly for the second row. We have $f_a : \mathcal{O}_{V,f(a)} \to \mathcal{O}_{U,a}$ is a local ring map so it induces $\overline{f_a}$ which corresponds to the identity map on $\mathbb{R}$ because $f(V')$ is a morphism of $\mathbb{R}$-algebras. The commutativity of the diagram implies that $f(V')(g)(a) = g(f(a))$ for $g \in \mathcal{O}_V(V')$, as desired. □

We are done. □

Example 55. Let A be a commutative ring with unity. Let $X = \operatorname{Spec} A$ to be the set of all prime ideals of A. In this example, we show that X can be equipped with a structure of a ringed space. X is then called an affine scheme.

First, X is a topological space with closed sets being $V(S) = \{\mathfrak{p} \in \operatorname{Spec} A : S \subset \mathfrak{p}\}$ for all subsets S of A. One can also show that X has basis of open sets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\}$ where $f \in A$.

To define the structure sheaf $\mathcal{O}_X$ of X, it suffices to define this on the basis of open sets of X, i.e. we let $\mathcal{O}_X(D(f)) = A_f$, the localisation of A at the set $\{f, f^2, \ldots\}$.

In this case, for $f \in \mathcal{O}_X(X) = A$, the value of f at $\mathfrak{p} \in X$ is the image of f in $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, which is $f \pmod{\mathfrak{p}}$. A scheme is a ringed space that is locally isomorphic to affine schemes.

4.3. **Analytic manifolds.** Theorem 53 suggests us to define k-analytic manifolds as follows.

Definition 56. *A k-analytic manifold of dimension d is a k-ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to $(k^d, \mathcal{O}_{k^d})$ with its sheaf of k-analytic functions. This follows that $(M, \mathcal{O}_M)$ is a locally k-ringed space. A morphism $\phi : M \to N$ of two k-analytic manifolds is a morphism of locally k-ringed spaces.*

Remark 57. With the same argument as in Theorem 53, one can show that our definition of k-analytic manifolds is the same as the definition of k-analytic manifolds via charts and atlas.

Remark 58. In most situations, M is assumed to be paracompact, Hausdorff. For example, these conditions give existence of a continuous partition of unity on coverings of M (see [Cra11]), and we will later use this to define integration of top-forms on M.

Next, we will define (co)tangent bundles/vectors of a k-analytic manifolds as derivations. In fact, the following definitions work for any locally k-ringed space $(M, \mathcal{O}_M)$, but we will restrict our attention to M being a k-analytic manifold.

Definition 59 (Tangent bundle)**.** *Let $(M, \mathcal{O}_M)$ be a k-analytic manifold. A k-derivation of $\mathcal{O}_M$ is a k-linear homomorphism $D : \mathcal{O}_M \to \mathcal{O}_M$ of sheaves such that $D_U(fg) = fD_U(g) + gD_U(f)$ for all $U \subset M$ open, $f, g \in \mathcal{O}_M(U)$. Denote $\operatorname{Der}_k(\mathcal{O}_M)$ the k-vector space of k-derivations of $\mathcal{O}_M$. It is also an $\mathcal{O}_M(M)$-module via*

$$(g \cdot D)_U(f) := g_U D_U(f),\ D \in \operatorname{Der}_k(\mathcal{O}_M), g \in \mathcal{O}_M(M), f \in \mathcal{O}_M(U).$$

We define the tangent bundle TM to be the sheaf of $\mathcal{O}_M$-module via $TM(U) := \operatorname{Der}_k(\mathcal{O}_M|_U)$. A section of tangent bundle over U is called a vector field. The tangent space T_pM of M at p is the stalk of TM at p, which is a k-vector space of k-derivations $\operatorname{Der}_k(\mathcal{O}_{M,p})$. Equivalently, by composing with $\mathcal{O}_{M,p} \to \kappa(p) = \mathcal{O}_{M,p}/\mathfrak{m}_p \cong k$, we can describe T_pM as a k-vector space of k-derivations $\mathcal{O}_{M,p} \to k$ at p, i.e. $D \in T_pM$ then $D : \mathcal{O}_{M,p} \to k$ such that $D(fg) = f(a)D(g) + g(a)D(f)$.

Remark 60. Let (x, U) be a chart of a k-analytic manifold M with coordinate functions $x_1, \dots, x_d$ then $\frac{\partial}{\partial x_i} : \mathcal{O}_M|_U \to \mathcal{O}_M|_U$ sending $f \in \mathcal{O}_M(V)$ to $\frac{\partial f}{\partial x_i} \in \mathcal{O}_M(V)$ where $V \subset U$ is open. Here $\frac{\partial f}{\partial x_i} \in \mathcal{O}_M(V)$ is a k-valued function on V, sending $p \in V$ to $\frac{\partial f \circ x^{-1}}{\partial x_i}(x(p))$. And $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$ form a basis of the free $\mathcal{O}_M(U)$-module $\mathrm{Der}_k(\mathcal{O}_M|_U)$.

Definition 61. *The cotangent bundle Ω^1_M of an analytic manifold $(M, \mathcal{O}_M)$ is the sheaf of $\mathcal{O}_M$-module $\mathcal{H}om(TM, \mathcal{O}_M)$. Concretely, its section over U is a morphism of sheaves $f : TM|_U \to \mathcal{O}_M|_U$, called differential 1-form over U. Furthermore, we can define $\Omega^i_M = \bigwedge^i \Omega^1_M$, whose section over $U \subset M$ is called differential p-form over U.*

We can define $d : \mathcal{O}_M \to \Omega^1_M$ a morphism of sheaves of k-vector spaces as follows:

$$\begin{aligned} d : \mathcal{O}_M &\to \Omega^1_M, \\ f \in \mathcal{O}_M(U) &\mapsto (df : D \in TM_U = \mathrm{Der}_k(\mathcal{O}_M|_U) \mapsto D(f) \in \mathcal{O}_M|_U). \end{aligned}$$

In particular, we have $d(fg) = fdg + gdf$.

Remark 62. Let (x, U) be a chart of M with coordinate functions $x_1, \dots, x_d : U \to k$. Then $dx_i \in \Omega^1_M(U)$ and $(dx_1, \dots, dx_d)$ is a basis of $\Omega^1_M(U)$. This basis is dual to the basis $\left(\frac{\partial}{\partial x_i}\right)$ of $TM(U)$. For $f \in \mathcal{O}_M(U)$ then

$$df = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} dx_i.$$

For $r \geq 1$ then $\Omega^i_M|_U$ is a free $\mathcal{O}_M|_U$-module with basis

$$dx_{i_1} \wedge \cdots \wedge dx_{i_r},\ 1 \leq i_1 < \cdots < i_r \leq d.$$

Remark 63. The cotangent bundle satisfies the following universal property: it is a sheaf of $\mathcal{O}_M$-modules equipped with differential $d : \mathcal{O}_M \to \Omega^1_M$, i.e. a morphism of sheaves of k-vector spaces satisfying $d(fg) = fdg + gdf$ where $f, g \in \mathcal{O}_M(U)$, that is universal among sheaves of $\mathcal{O}_M$-modules X eqquiped with differential $d : \mathcal{O}_M \to X$. The universal condition implies that given a chart (U, x) M with coordinate functions $x_1, \dots, x_d$, $\Omega^1_M(U)$ is a free $\mathcal{O}_M(U)$-module with basis dx_i.

A morphism between k-analytic manifold $\phi : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ will induce a morphism $\phi^* : \Omega^1_N \to \Omega^1_M$ of $\mathcal{O}_M$-modules. Concretely, for $f \in \mathcal{O}_N(U)$ then $d_N f \in \Omega^1_N(U)$ is sent to $d_M(f \circ \phi)$ where $f \circ \phi \in \mathcal{O}_M(f^{-1}(U))$.

Remark 64. In terms of coordinates, let $f : M \to N$ be a morphism of k-analytic manifold, $(V, y), (U, x)$ be charts of M, N respectively with coordinate functions $x_1, \dots, x_d$ for x and $y_1, \dots, y_e$ for y. If $\omega \in \Omega^p_N(U)$ can be written as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq d} \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $I = (i_1, \dots, i_p)$ and ω_I's are k-analytic functions on U. The morphism $f : M \to N$ of k-analytic manifold will induce a differential p-form $f^*\omega \in \Omega^p_M(f^{-1}(U) \cap V)$ on $f^{-1}(U) \cap V \subset M$ defined by

$$\begin{aligned} f^*\omega &= \sum_I (\omega_I \circ \phi) d(x_i \circ \phi), \\ &= \sum_{1 \leq i_1 < \cdots < i_p \leq d} \sum_{1 \leq j_1 < \cdots < j_p \leq e} (\omega_I \circ \phi) \det \left(\frac{\partial x_{i_m} \circ \phi}{\partial y_{j_n}} \right)_{1 \leq m \leq d, 1 \leq n \leq e} dy_{j_1} \wedge \cdots \wedge dy_{j_p}. \end{aligned}$$

4.4. **Integration of differential forms.** In this section, we assume that k is a local field with a Haar measure μ. Let M be a k-analytic manifold of dimension d and let ω be a global differential d-form on M. We will define a measure on M by defining integration of d-form ω.

First, we consider the case when $M = k^d$. We then can form a product measure $d\mu$ on M from a choice of measure on local field k. Suppose that over open U of M, ω can be written as $h(x)dx_1 \wedge \cdots \wedge dx_d$ where h is k-analytic function on U. With this, we can define measure $|\omega|$ on U to be

$$\int_U \varphi|\omega| := \int_U \varphi(x)|h(x)|_k d\mu$$

for any complex-valued $\varphi \in C_c(M)$ with compact support in U. To see that happen to this measure under k-bianalytic map $f : V \to V$, we first need the change of variables formula for k^d:

Theorem 65. *Let U be open set in k^d and $f : U \to k^d$ be an injective k-analytic map whose Jacobian J_f does not vanish on U. Then for any measurable positive (resp. integrable) function $\varphi : f(U) \to \mathbb{R}$, we have*

$$\int_{f(U)} \varphi(y)d\mu(y) = \int_U \varphi(f(x))|J_f(x)|_k d\mu(x).$$

Proof. We postpone the proof of this theorem to our final thesis version. At the moment, refer to [Igu00, Theorem 7.4.1] for the proof when k is a nonarchimedean local field. □

Now, consider a k-bianalytic map $f : V \to U$, where $U, V \subset k^d$ are open with coordinates $x_1, \ldots, x_d$ on U and $y_1, \ldots, y_d$ on V. Then $J_f(x) = \det(\partial(x_i \circ f)/\partial y_j)$. As

$$f^*\omega = h(f(x))J_f(x)dy_1 \wedge \cdots \wedge dy_d,$$

we have, by the change of variable formula

$$\begin{aligned}
\int_V (\varphi \circ f)|f^*\omega| &= \int_V (\varphi \circ f)|h(f(x))|_k |J_f(x)|_k dy_1 \wedge \cdots \wedge dy_d, \\
&= \int_U \varphi|h(x)|_k dx_1 \wedge \cdots \wedge dx_d, \\
&= \int_U \varphi|\omega|.
\end{aligned}$$

Next, we consider the case when M is any k-analytic manifold of dimension d.

Proposition 66. *There exists a unique measure $|\omega|$ such that for every chart (U, f) of M and every measurable positive (resp. integrable) function φ supported in U,*

$$\int_M \varphi|\omega| = \int_{f(U)} (\varphi \circ f^{-1})|(f^{-1})^*\omega|.$$

Sketch. To construct ω, by Riez's representation theorem, it suffices to do this for φ with compact support. One can consider charts (U_i, f_i) of M covering support of φ and consider continuous partition of unity subordinated for these charts, i.e. a family (λ_i) of continuous real-valued functions on M such that supp $\lambda_i \subset U_i$ and $\sum \lambda_i = 1$ on supp φ. We then have

$$\int_M \varphi|\omega| := \sum_i \int_{f_i(U_i)} (\lambda_i \circ f_i^{-1})(\varphi \circ f_i^{-1})|(f_i^{-1})^*\omega|.$$

We show the independence of charts. Suppose we have another chart (U, g) with same $U \subset M$. Then $f \circ g^{-1} : g(U) \to f(U)$ is k-bianalytic map on k^d, so

$$\int_{f(U)} |(f^{-1})^*\omega| = \int_{g(U)} |(f \circ g^{-1})^*(f^{-1})^*\omega| = \int_{g(U)} |(g^{-1})^*\omega|.$$

It is not difficult to show that our definition does not depend on the choice of a partition of unity of M. $\square$

5. Adeles

In this section, we will define the adeles and study its topology. We then describe a functorial way to give a topology on $G(\mathbb{A})$ for any linear algebraic group G. We also show that $G(\mathbb{Q}_v)$ is a $\mathbb{Q}_v$-analytic manifold for smooth G. We then focus on the case where $G = \mathrm{SL}_2$ and prove the strong approximation theorem for SL_2.

5.1. Adeles of $\mathbb{Q}$.

We will review the construction of adeles $\mathbb{A}$ of $k = \mathbb{Q}$. Let S be always a finite nonempty set of places of $\mathbb{Q}$ including the infinite place. For convenience, we sometimes refer to $\mathbb{R}$ as $\mathbb{Q}_\infty$.

Definition 67. *The adeles $\mathbb{A}_\mathbb{Q}$ of $\mathbb{Q}$ is the restricted product of locally compact spaces $\mathbb{Q}_p$ with respect to compact open subspace $\mathbb{Z}_p$ of $\mathbb{Q}_p$. In other words, $\mathbb{A}$ is a topological space whose elements are*

$$\mathbb{A}_\mathbb{Q} = \prod_{p \leq \infty}{}' \mathbb{Q}_p := \left\{ (a_p)_p \in \prod_{p \leq \infty} \mathbb{Q}_p : a_p \in \mathbb{Z}_p \text{ for almost all } p \right\},$$

here "almost all" means "all but finitely many"; $\mathbb{A}$ has basis of open sets given by

$$U_S \times \prod_{v \notin S} \mathbb{Z}_v,$$

where S is a finite set of places of $\mathbb{Q}$, U_S is an open set of

$$\mathbb{Q}_S := \prod_{v \in S} \mathbb{Q}_v$$

under the product topology.

Let

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v, \widehat{\mathbb{Z}}^S = \prod_{v \notin S} \mathbb{Z}_v,$$

we find $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ is an open subring of $\mathbb{A}$ with the induced topology being the product topology [5]. Indeed, the open sets of $\mathbb{A}$ restricted to $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ is of the form $\prod_{v \in S} U_v \times \prod_{v \notin S} V_v$ where U_v is open in $\mathbb{Q}_v$, V_v is open in $\mathbb{Z}_v$ where $V_v = \mathbb{Z}_v$ for almost all $v \notin S$. This is precisely the open basis of the product topology of $\mathbb{A}_S := \mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$. Furthermore,

$$\mathbb{A} = \bigcup_S \mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$$

over all finite set S of places of $\mathbb{Q}$. Also note that for $S \subset T$, we have a restriction continuous map $\mathbb{Q}_T \times \widehat{\mathbb{Z}}^S$ to $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ of topological rings. In other words, we find $\mathbb{A}_K = \varinjlim_S \mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$. With this, $\mathbb{A}$ is a topological ring, with componentwise addition and multiplication.

Proposition 68. *The adeles ring $\mathbb{A}$ is a locally compact Hausdorff topological ring.*

Proof. We first show $\mathbb{A}$ is locally compact. Note that by Tychonoff's theorem, $\prod_{v \notin S} \mathbb{Z}_v$ is compact as each $\mathbb{Z}_v$ is compact. It follows that $\mathbb{Q}_S \times \widehat{\mathbb{Z}}^S$ is a finite product of locally compact spaces, hence locally compact. As each point $x \in \mathbb{A}$ lies in one of these spaces, we find $\mathbb{A}$ is locally compact.

Next, we show $\mathbb{A}$ is Hausdorff. Note that $\prod_v \mathbb{Q}_v$ is Hausdorff as $\mathbb{Q}_v$ is Hausdorff and the topology on $\mathbb{A}$ is finer (has less open sets) than the subspace topology of $\mathbb{A}$ with respect to $\prod_v \mathbb{Q}_v$, hence $\mathbb{A}$ is Hausdorff. ∎

[5] this notation of $\widehat{\mathbb{Z}}^S$ is motivated from the fact that it is the profinite completion of $\mathbb{Z}^S = \{x \in k | x \in \mathcal{O}_v \ \forall v \notin S\}$

For each place v of $\mathbb{Q}$, we have a continuous embedding

$$\mathbb{Q}_v \hookrightarrow \mathbb{A} : x_v \mapsto (0, \dots, 0, x_v, 0, \dots, 0).$$

Indeed, the preimage of basis open set $\prod_{v\in S} U_v \times \prod_{v\notin S} \mathbb{Z}_v$ of $\mathbb{A}$ is either $\varnothing$ if $0 \notin U_t$ for some place $t \in S, t \neq v$; or U_v if $0 \in U_t$ for all $t \in S \setminus \{v\}$ and $v \in S$; or $\mathbb{Z}_v$ if $0 \in U_t$ for all $t \in S$ and $v \notin S$.

We have a diagonal embedding

$$\mathbb{Q} \hookrightarrow \mathbb{A} : x \mapsto (x, x, \dots, x).$$

This map is well-defined as $x \in \mathbb{Z}_v$ for almost all place v where $x \in \mathbb{Q}$. The image of $\mathbb{Q}$ under this embedding is called the *principal adeles*, which we will also denote $\mathbb{Q}$ for convenience.

Proposition 69. *$\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$.*

Proof. It suffices to show that $0 \in \mathbb{Q}$ has an open neighborhood U in $\mathbb{A}$ that does not intersect $\mathbb{Q} \setminus \{0\}$. Let $U = \{(x_v) \in \mathbb{A} : |x_v|_v < 1 \text{ if } v|\infty \text{ and } |x_v|_v \leq 1 \text{ if } v \nmid \infty\}$ then U is open, $0 \in U$. By prime factorisation in $k = \mathbb{Q}$, we find $U \cap (\mathbb{Q} \setminus \{0\}) = \varnothing$, as desired. ☐

Let

$$\mathbb{A}^S = \prod_{v\notin S}{}' \mathbb{Q}_v$$

then we can identify $\mathbb{Q}_S \times \mathbb{A}^S$ with $\mathbb{A}$ via $\mathbb{Q}_S \times \mathbb{A}^S \hookrightarrow \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ where the later map is addition on $\mathbb{A}$. This follows $\mathbb{Q}_S \times \mathbb{A}^S$ is isomorphic to $\mathbb{A}$ as topological rings, with the product topology on $\mathbb{Q}_S$ and the restricted product topology on $\mathbb{A}^S$.

5.2. **Approximation theorem for adeles.**

Theorem 70. *For any finite nonempty set S of places of $\mathbb{Q}$, then*

(a) (Weak approximation property) $\mathbb{Q}$ is dense in $\mathbb{Q}_S$ via the diagonal embedding.

(b) (Strong approximation property) $\mathbb{Q}$ is dense in $\mathbb{A}^S$ via the diagonal embedding.

Proof. (a) Without loss of generality, we assume S contains the infinite place. We need to show that any open set in $\mathbb{Q}_S$ contains a nonzero element in $\mathbb{Q}$. Indeed, a basis of open sets of $\mathbb{Q}_S$ consists of open sets $U \times \prod_{p\in S, p<\infty}(a_p + p^{k_p}\mathbb{Z}_p)$ where U is open in $\mathbb{R}$ and $k_p \in \mathbb{Z}, a_p \in \mathbb{Q}$. We choose $x \in U \cap \mathbb{Q}$ then by the Chinese Remainder Theorem, there exists $y \in \mathbb{Q}, z \in \mathbb{Z} \setminus \{0, 1\}$ such that $y \equiv a_p - x \pmod{p^{k_p}}, z \equiv 1 \pmod{p^{k_p}}$ for all $p \in S \setminus \{\infty\}$. Hence, for sufficiently small $\ell < 0$, $x + yz^\ell$ is our desired element in $\mathbb{Q}$.

(b) We first consider the case where S does contain the infinite place. A basis of open sets of $\mathbb{A}^S$ consists of open sets $\prod_{p\in T}(a_p + p^{k_p}\mathbb{Z}_p) \times \prod_{p\notin S\cup T} \mathbb{Z}_p$ where T is a finite set of places of $\mathbb{Q}$, $T \cap S = \varnothing$, $a_p \in \mathbb{Q}$ for all $p \in T$. By the Chinese Remainder Theorem, there exists $x \in \mathbb{Q}$ such that $x \equiv a_p \pmod{p^{k_p}}$ where the denominator of x only has prime powers of primes $p \in T$. This follows x lies in the open set.

If S does not contain the infinite place, then there exists a prime $q \in S$. An open set of $\mathbb{A}^S$ consists of open sets $U \times \prod_{p\in T}(a_p + p^{k_p}\mathbb{Z}_p) \times \prod_{p\notin S\cup T} \mathbb{Z}_p$ where T is a finite set of places of $\mathbb{Q}$, $T \cap S = \varnothing$, $a_p \in \mathbb{Q}$ for all $p \in T$, U is open in $\mathbb{R}$.

There exists $\ell \in \mathbb{Z}_{>0}, x \in \mathbb{Z}$ such that $\frac{x}{q^\ell} \in U$. Indeed, pick any $y \in U$ and let ℓ be sufficiently large such that $(y - q^{-\ell}, y + q^{-\ell}) \subset U$. As $\left[q^\ell y - \frac{1}{2}, q^\ell y + \frac{1}{2}\right]$ has length 1, there exists an integer x lying inside that interval, giving $xq^{-\ell} \in U$.

By the Chinese Riemainder theorem, there exists $z \in \mathbb{Q}, t \in \mathbb{Z}_{>1}$ such that $z \equiv a_p - xq^{-\ell} \pmod{p^{k_p}}$ and $q^t \equiv 1 \pmod{p^{k_p}}$ for all $p \in T$, where the denominator of z only has prime powers of primes $p \in T$. This follows $xq^{-\ell} + zq^{tk} \in \mathbb{Q}$ lies in the desired open set. ☐

Corollary 71. *If $S = \infty$ is the set of all infinite places then*
(a) $|\mathbb{Q} \setminus \mathbb{A}^\infty / \widehat{\mathbb{Z}}| = 1$
(b) We have an isomorphism $\mathbb{Q} \setminus \mathbb{A}/\widehat{\mathbb{Z}} \cong \mathbb{Z} \setminus \mathbb{R}$ *of topological spaces.*

Proof. (a) It suffices to prove that $\mathbb{A}^\infty = \mathbb{Q} + \widehat{\mathbb{Z}}$. Consider $x \in \mathbb{A}^\infty$, then $x + \widehat{\mathbb{Z}}$ is an open neighborhood of x as $\widehat{\mathbb{Z}}$ is open subgroup in $\mathbb{A}^\infty$. From strong approximation theorem for adeles, we know $\mathbb{Q}$ is dense in $\mathbb{A}^\infty$. Hence, there exists $\ell \in \mathbb{Q}$ so that $\ell \in x + \widehat{\mathbb{Z}}$, implying $x \in \mathbb{Q} + \widehat{\mathbb{Z}}$.

(b) We identify $\mathbb{A}$ with $\mathbb{R} \times \mathbb{A}^\infty$. Consider the map

$$\begin{aligned} \phi : \mathbb{Z} \setminus \mathbb{R} &\to \mathbb{Q} \setminus \mathbb{A}/\widehat{\mathbb{Z}} \\ \mathbb{Z} + x &\mapsto [x, 0] \end{aligned}$$

where we denote $[x, y]$ for $(x, y) \in \mathbb{R} \times \mathbb{A}^\infty$ to be the double coset $\mathbb{Q} + (x, y) + \widehat{\mathbb{Z}}$. Note that $\widehat{\mathbb{Z}}$ and $\mathbb{Q}$ are embedded diagonally into $\mathbb{R}$ and $\mathbb{A}$, respectively.

We first show this ϕ is injective. If for $x \in \mathbb{R}$ so $[x, 0] = 0$ then $(x, 0) = (\ell, \ell + y)$ for $y \in \widehat{\mathbb{Z}}$ and $\ell \in \mathbb{Q}$. This follows $\ell = -y \in \widehat{\mathbb{Z}}$ so $\ell \in \mathbb{Q} \cap \widehat{\mathbb{Z}} = \mathbb{Z}$. Hence, $x = \ell \in \mathbb{Z}$, as desired.

To show ϕ is surjective. From (a), we find $\mathbb{A}^\infty = \mathbb{Q} + \widehat{\mathbb{Z}}$. Hence, $\mathbb{A} = \mathbb{Q} + \mathbb{R} + \widehat{\mathbb{Z}}$ and surjectivity follows.

To show ϕ is continuous. Consider $0 \in U \subset \mathbb{R}$ to be representatives of an open subset U in $\mathbb{Q} \setminus \mathbb{A}/\widehat{\mathbb{Z}}$. Then $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ is open in $\mathbb{A}$. As $\mathbb{A} = \mathbb{R} \times \mathbb{A}^\infty$, we can cover $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ by open subsets (X_i, Y_i) where X_i, Y_i open of $\mathbb{R}, \mathbb{A}^\infty$, respectively. As $0 \in U$ and $\widehat{\mathbb{Z}}$ is open in $\mathbb{A}^\infty$, we find $\mathbb{Q}+(X_i, \widehat{\mathbb{Z}})$ is also open in $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ and these subsets cover $\ell + (U, \widehat{\mathbb{Z}})$ for $\ell \in k, X_i$ open subsets of $\mathbb{R}$.

On the other hand, as $\mathbb{Z} = \mathbb{Q} \cap \widehat{\mathbb{Z}}$, $\mathbb{Q} + (U, \widehat{\mathbb{Z}})$ is disjoint union of $\ell + (\mathbb{Z} + U, \widehat{\mathbb{Z}})$ for $\ell \in \mathbb{Q}, \ell \notin \mathbb{Z}$. Combining with previous argument, we find $(\mathbb{Z} + U, \widehat{\mathbb{Z}})$ must be obtained from taking union of $(X_i, \widehat{\mathbb{Z}})$ where X_i open subsets of $\mathbb{R}$. This follows $U + \mathbb{Z}$ is open in $\mathbb{R}$, meaning inverse image of U under ϕ is open in $\mathbb{Z} \setminus \mathbb{R}$.

To show ϕ has homeomorphic inverse, it suffices to show ϕ is open map. Consider $U \in \mathbb{R}$ so $\mathbb{Z} + U$ is open in $\mathbb{R}$, we need to show $\mathbb{Q} + U + \widehat{\mathbb{Z}}$ is open in $\mathbb{A}$. This holds because $\mathbb{Q} + U + \widehat{\mathbb{Z}}$ is union of open sets $\ell + (\mathbb{Z} + U, \widehat{\mathbb{Z}})$ where $\ell \in \mathbb{Q}$. ☐

Corollary 72. *The quotient* $\mathbb{Q} \setminus \mathbb{A}$ *is compact.*

Proof. From previous corollary, we obtain an homeomorphism of topological spaces

$$\mathbb{Q} \setminus \mathbb{A} \cong \mathbb{Z} \setminus \mathbb{R} \times \widehat{\mathbb{Z}}.$$

Note that $\widehat{\mathbb{Z}}$ is compact. As $\mathbb{Z}$ is a lattice in $\mathbb{R}$, $\mathbb{Z} \setminus \mathbb{R}$ is compact. Thus, $\mathbb{Q} \setminus \mathbb{A}$ is compact. ☐

5.3. **Topology of adelic points of linear algebraic groups.** Let X be an affine k-scheme of finite type. For a k-algebra R which is also a topological ring, we can endow $X(R)$ with a canonical topology. When $k = \mathbb{Q}, R = \mathbb{A}$, $X(\mathbb{A})$ is homeomorphic to the restricted product of $X(\mathbb{Q}_v)$ over all places v of $\mathbb{Q}$. Furthermore, if k is a complete valued field and X is a smooth k-scheme of finite type, we can endow $X(k)$ with a canonical structure of a k-analytic manifold.

Proposition 73. *Let R be a topological ring. There exists a unique way to topologise $X(R)$ for all affine schemes X of finite type over R such that*

(1) the topology is functorial in X; that is, if $X \to Y$ is a morphism of affine schemes of finite type of R, then the induced map on points $X(R) \to Y(R)$ is continuous;
(2) the topology is compatible with fiber products: this means if $X \to Y$ and $Y \to Z$ are morphisms of affine schemes of finite type over R, then the topology on $(X \times_Z Y)(R)$ is the fiber product topology;

(3) *closed immersion of affine schemes $X \hookrightarrow Y$ (i.e. the map of coordinate rings $\mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective) induces topological embeddings $X(R) \hookrightarrow Y(R)$ (i.e. a continuous map that is homeomorphic onto its image);*

(4) *if $X = \operatorname{Spec} R[t]$ then $X(R)$ is homeomorphic with R under natural identification $X(R) \cong R$.*

If R is Hausdorff or locally compact, then so is $X(R)$. Moreover, if R is Hausdorff then closed immersion $X \to Y$ induces closed embedding $X(R) \to Y(R)$.

Sketch of proof. We refer to [Con12] for the proofs of these two propositions. Essentially, the topology of $X(R)$ is constructed by choosing an R-algebra isomorphism $\mathcal{O}(X) \cong R[t_1, \ldots, t_n]/I$ for the coordinate ring of X, for any ideal I. $X(R)$ can then be identified with the sets of elements in R^n on which the elements in I (we view I as R-valued functions on R^n) all vanish. We have an injection $X(R) \hookrightarrow R^n$ which we equip $X(R)$ with the subspace topology of R^n. One then has to check that the defined topology does not depend on the choice of isomorphism $\mathcal{O}(X) \cong R[t_1, \ldots, t_n]/I$ and satisfies all the functorial properties as above. □

Proposition 74. *Let $R \to R'$ be a continuous map of topological rings and let X be an affine scheme of finite type over R. Then $X(R) \to X(R')$ is continuous. Moreover, if $R \to R'$ is a*

(1) *topological embedding*

(2) *open topological embedding*

(3) *closed topological embedding*

(4) *topological embedding onto a discrete subset*

then so is $X(R) \to X(R')$.

Proof. The proposition follows from

$$\begin{array}{ccc} X(R) & \xhookrightarrow{i} & R^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X(R') & \xhookrightarrow{i'} & R'^n \end{array}$$

where i, i' are topological embeddings. For example, we find $f^{-1}(U) = X(R) \cap g^{-1}(U)$ for any $U \subset X(R')$ so f is continuous. If g a topological embedding then so is $g \circ i$, hence f is also a topological embedding. If g is open/closed then so is f. □

Example 75 (Topology of adeles and ideles). When $k = \mathbb{Q}$ and $R = \mathbb{A}$ then $\mathbb{G}_a(R) = \mathbb{A}$. From section 5, we know that $\mathbb{A}$ has the topology consisting of basis of open sets

$$U_S \times \prod_{v \notin S} \mathbb{Z}_v$$

where S is a finite set of places of k, U_S is open in $\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v$.

Next, we consider the ideles $\mathbb{G}_m(\mathbb{A}) = \mathbb{A}^\times$. We have a closed immersion $\mathbb{G}_m \hookrightarrow \mathbb{G}_a \times \mathbb{G}_a$ sending $t \mapsto (t, t^{-1})$. Therefore, we have a topological embedding $\mathbb{A}^\times \hookrightarrow \mathbb{A} \times \mathbb{A}$, giving $\mathbb{G}_m(\mathbb{A}) = \mathbb{A}^\times$ the topology consisting of basis of open sets

$$U_S \times \prod_{v \notin S} \mathbb{Z}_v^\times$$

where S is a finite set of places of k, U_S is open in $\mathbb{Q}_S^\times = \prod_{v \in S} \mathbb{Q}_v^\times$ under the product topology [6]. Note that this is not the same topology as giving $\mathbb{A}^\times \subset \mathbb{A}$ the subspace topology. In particular, $\prod_{p<\infty} \mathbb{Z}_p^\times$ is an open set in $\mathbb{A}^\times$ but it is not open under the subspace topology from $\mathbb{A}$. Indeed,

[6] To see this, consider open set $U \times V$ in $\mathbb{A} \times \mathbb{A}$. For $a = (a_v) \in \mathbb{A}$, if $(a, a^{-1}) \in U \times V$ then $a_v^{\pm 1} \in \mathbb{Z}_v$ for almost all v, meaning $a_v \in \mathbb{Z}_v^\times$ for almost all v; finally, note that $\mathbb{Q}_v^\times$ has the subspace topology from $\mathbb{Q}_v$ for all places v

if $\prod_{p<\infty} \mathbb{Z}_p^\times$ is open under the subspace topology from $\mathbb{A}$, it is a union of $\mathbb{A}^\times \cap U$'s where $U = U_S \times \prod_{v \notin S} \mathbb{Z}_v$ is open in $\mathbb{A}$. One can then choose sufficiently large prime p such that $a = (a_v)_v$ satisfies $(a_v)_{v\in S} \in U_S \cap \mathbb{Q}_S^\times$, $a_p = p$ and $a_v = 1$ for $v \notin S \cup \{p\}$. This follows $a \notin \prod_{v<\infty} \mathbb{Z}_p^\times$ but $a \in \mathbb{A}^\times \cap U$, a contradiction.

Thus, we have a homeomorphism

$$\mathbb{G}_m(\mathbb{A}) \cong \prod_v{}' \mathbb{G}_m(\mathbb{Q}_v).$$

Example 76 (Topology of $\mathrm{GL}_2(\mathbb{A})$). The map $\mathrm{GL}_2 \hookrightarrow M_2 \times \mathbb{G}_a$ sending $x \mapsto (x, \det^{-1} x)$ is a closed immersion of affine schemes since the associated k-algebra map $k[x_{11}, x_{12}, x_{21}, x_{22}] \otimes_k k[t] \to k[x_{11}, x_{12}, x_{21}, x_{22}, \det^{-1}]$ sending t to $\det^{-1} := (x_{11}x_{22} - x_{12}x_{21})^{-1}$ is surjective. Hence, we have a topological embedding $\mathrm{GL}_2(R) \hookrightarrow M_2(R) \times \mathbb{G}_a(R)$.

With the above embedding, we will describe the topology of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. It suffices to describe the basis of open neighborhoods of the identity of $\mathrm{GL}_2(\mathbb{A})$.

We first describe the topology of $\mathrm{GL}_2(\mathbb{A}_T)$ where $\mathbb{A}_T := \mathbb{Q}_T \times \widehat{\mathbb{Z}}^T = \prod_{v\in T} \mathbb{Q}_v \times \prod_{v \notin T} \mathbb{Z}_v$ for some fixed finite set T of places of $\mathbb{Q}$ containing the infinite place. We know that $(I_2 + p^k M_n(\mathbb{Z}_p)) \times (1 + p^k\mathbb{Z}_p)$ for $k \in \mathbb{Z}_{\geq 1}$ forms a basis of open neighborhoods of $(I_2, 1) \in M_2(\mathbb{Q}_p) \times \mathbb{Q}_p$. Therefore, for any finite set S of places of $\mathbb{Q}$ containing the infinite place, and $k \in \mathbb{Z}_{\geq 1}$,

$$\prod_{p \in S\cap T} \Big((I_2 + p^k M_2(\mathbb{Z}_p)) \times (1 + p^k \mathbb{Z}_p)\Big) \times \prod_{p \notin S \cup T} (M_2(\mathbb{Z}_p) \times \mathbb{Z}_p)$$

form a basis of open neighborhood of $(I_2, 1)$ in $M_2(\mathbb{A}_T) \times \mathbb{A}_T$ (by definition of product topology). Intersecting these sets with the image of $\mathrm{GL}_2(\mathbb{A}_T)$ from the embedding, we obtain

$$\prod_{p\in S\cap T} (I_2 + p^k M_2(\mathbb{Z}_p)) \times \prod_{p \notin S\cup T} \mathrm{GL}_2(\mathbb{Z}_p)$$

as basis of open neighborhoods of I_2 in $\mathrm{GL}_2(\mathbb{A})$. Thus, we have a homeomorphism

$$\mathrm{GL}_2(\mathbb{A}_T) \cong \prod_{v\in T} \mathrm{GL}_2(\mathbb{Q}_v) \times \prod_{v \notin T} \mathrm{GL}_2(\mathbb{Z}_v).$$

Now, as $\mathbb{A}_T \hookrightarrow \mathbb{A}$ is an open embedding so $\mathrm{GL}_2(\mathbb{A}_T) \hookrightarrow \mathrm{GL}_2(\mathbb{A})$ is also an open topological embedding. Furthermore, as $\mathrm{GL}_2(\mathbb{A}) = \bigcup_S \mathrm{GL}_2(\mathbb{A}_S)$ over all finite set S of places of $\mathbb{Q}$ containing the infinite place, we conclude that

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathrm{GL}_2(\mathbb{Z}_v)$$

where S is a finite set of places of $\mathbb{Q}$ containing the infinite place, U_v is open in $\mathrm{GL}_2(\mathbb{Q}_v)$, form a basis of open sets of $\mathrm{GL}_2(\mathbb{A})$. Thus, we have a homeomorphism

$$\mathrm{GL}_2(\mathbb{A}) \cong \prod_v{}' \mathrm{GL}_2(\mathbb{Q}_v).$$

The argument in the previous example holds for general GL_n, giving a homeomorphism $\mathrm{GL}_n(\mathbb{A}) \cong \prod_v' \mathrm{GL}_n(\mathbb{Q}_v)$. Hence, we have the following result.

Proposition 77. *For a linear algebraic group G over $\mathbb{Q}$ and a faithful representation $G \to \mathrm{GL}_n$, one has an isomorphism of topological groups*

$$G(\mathbb{A}) \cong \prod_v{}' G(\mathbb{Q}_v)$$

where the restricted product on RHS is defined with respect to the compact open subgroup $G(\mathbb{Q}_v) \cap \mathrm{GL}_n(\mathbb{Z}_v)$ of $G(\mathbb{Q}_v)$ where $v \nmid \infty$.

This proposition holds more general for any schemes/algebraic spaces X (see [Con12]). It indicates that adeles is the right notion to study some space over all completions of $\mathbb{Q}$ at once. It also provides a concrete description of the adelic points of the space via its local parts.

Example 78 (Topology of $\mathrm{SL}_2(\mathbb{A})$). We have a closed immersion $\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$ of affine schemes as $\mathcal{O}(\mathrm{SL}_2) = \mathcal{O}(\mathrm{GL}_2)/(x_{11}x_{22} - x_{12}x_{11} - 1)$, giving a topological embedding $\mathrm{SL}_2(R) \hookrightarrow \mathrm{GL}_2(R)$ for any topological ring R.

Since $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}_{\mathbb{Q}}$ so $\mathrm{SL}_2(\mathbb{Q})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{A})$.

Continuing from Proposition 73, we restrict X to be a smooth affine scheme of finite type over k, where k is a completed value field. Here smoothness of X means the following (see [GW20, §6.8])

Definition 79. *Let X be an affine scheme of finite type over k. We say X is smooth of dimension d over k if X can be covered by affine open sets $\operatorname{Spec} k[t_1, \dots, t_n]/(f_1, \dots, f_{n-d})$ for suitable n and f_i, such that the Jacobian matrix $(\partial f_i/\partial t_j)$ has rank $n-d$ everywhere on $X(k)$. Equivalently, the ideal in $k[t_1, \dots, t_n]$ generated by the f_i's and all the $(n-d) \times (n-d)$ minors of the Jacobian $(\partial f_i/\partial t_j)$ is the whole ring $k[t_1, \dots, t_n]$.*

Example 80. $\mathrm{GL}_2 = \operatorname{Spec} k[x_{11}, x_{12}, x_{21}, x_{22}, t]/(t(x_{11}x_{22} - x_{21}x_{12}) - 1)$ is a smooth scheme over k of dimension 4.

Example 81. $\mathrm{SL}_2 = \operatorname{Spec} k[x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{22} - x_{21}x_{12} - 1)$ is a smooth scheme over k of dimension 3.

Proposition 82. *There is a canonical structure of a k-analytic manifold on $X(k)$, which is characterised as follows:*

(1) Functorial in X: morphism of smooth k-schemes induce morphism of k-analytic manifolds; open (resp. closed) immersions induce open (resp. closed) immersions of k-analytic manifolds.
(2) When $X = \operatorname{Spec} k[x_1, \dots, x_d]$ then the structure of k-analytic manifold on $X(k) \cong k^d$ is the natural one.
(3) Etale morphisms of smooth k-schemes induce k-analytic local isomorphisms.

Sketch of proof. To define a structure of a k-analytic manifold on $X(k)$ amounts to describe which continuous functions are k-analytic, such that the locally ringed space is locally isomorphic to k^n with its sheaf of k-analytic functions. Let U be an open set of $X(k)$. A continuous function $f : U \to k$ is k-analytic at $x \in U$ if there exists an immersion of k-schemes $i : V \to \operatorname{Spec} k[t_1, \dots, t_n]$ on a Zariski-open neighborhood V of x in X and a k-analytic function $g : W \to k$ on an open neighborhood of x such that $f = g \circ i$ on some open neighborhood of x in U. We say f is analytic if it is analytic at every point in U.

We refer to [CLNS18, Chapter 0, §1.6] for the check of the functorial conditions with the above analytic structure. □

Example 83. For $k = \mathbb{Q}_p$ or $k = \mathbb{R}$ then $\mathrm{GL}_2(k), \mathrm{SL}_2(k)$ are k-analytic manifolds.

5.4. **Approximation theorem for SL_2 over $\mathbb{Q}$.** For an affine algebraic group G over $\mathbb{Q}$. We say G satisfies strong approximation with respect to a finite set S of places of $\mathbb{Q}$ if $G(\mathbb{Q})$ is dense in $G(\mathbb{A}^S)$. We then have the following results, which we refer to [Rap13] for more discussions.

Theorem 84 (Strong approximation theorem)**.** *For semisimple and simply connected linear algebraic group G over $\mathbb{Q}$ then the strong approximation theorem holds. In other words, for any nonempty finite set S of places of $\mathbb{Q}$, $G(\mathbb{A})$ is dense in $G(\mathbb{A}^S)$.*

Corollary 85. *If affine algebraic group G over $\mathbb{Q}$ satisfies strong approximation with respect to a finite set $S = \{\infty\}$ of places of $\mathbb{Q}$ then*

(1) $|G(\mathbb{Q}) \setminus G(\mathbb{A}^\infty)/K^\infty| = 1$ *for any compact open subgroup* K^∞ *of* $G(\mathbb{A}^\infty)$.

(2) If $\Gamma = G(\mathbb{Q}) \cap K^\infty$ *then by embedding* $G(\mathbb{R})$ *to the infinite component of* $G(\mathbb{A})$*, we have a homeomorphism*

$$\Gamma \setminus G(\mathbb{R}) \to G(\mathbb{Q}) \setminus G(\mathbb{A})/K^\infty.$$

The proof of this corollary is similar to the proof of corollary 71 for $\mathbb{G}_a$.

We will give a proof of strong approximation theorem for SL_2 over $\mathbb{Q}$.

Proposition 86 (Strong approximation theorem for SL_2)**.** *For any non-empty finite set* S *of places of* $\mathbb{Q}$*,* $\mathrm{SL}_2(\mathbb{Q})$ *is dense in* $\mathrm{SL}_2(\mathbb{A}^S)$*.*

Proof. Let Z be the closure of $\mathrm{SL}_2(k)$ in $\mathrm{SL}_2(\mathbb{A}^S)$ then Z is a subsgroup of $\mathrm{SL}_2(\mathbb{A}^S)$. It suffices to prove that Z contains $\mathrm{SL}_2(\mathbb{Q}_v)$ for every $v \notin S$. Indeed, if such condition holds then subgroup Z will contain $\prod_{v\in S'} \mathrm{SL}_2(\mathbb{Q}_v) \times \prod_{v \notin S\cup S'} \mathrm{SL}_2(\mathbb{Z}_v)$ where S' is any finite set of places of $\mathbb{Q}$ disjoint from S. As this exhausts $\mathrm{SL}_2(\mathbb{A}^S)$, we find $Z = \mathrm{SL}_2(\mathbb{A}^S)$.

To show $\mathrm{SL}_2(\mathbb{Q}_v) \subset Z$, note that $\mathrm{SL}_2(\mathbb{Q}_v)$ is generated by $U^+(\mathbb{Q}_v) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$ and $U^-(\mathbb{Q}_v) = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}$ so it suffices to show Z contains $U^{\pm}(\mathbb{Q}_v)$. By definition, Z contains closure of $U^{\pm}(\mathbb{Q})$. As $U^+ \cong \mathbb{G}_a$ so by strong approximation theorem for $\mathbb{G}_a$, the closure of $U^+(\mathbb{Q})$ in $\mathrm{SL}_2(\mathbb{A}^S)$ is $U^+(\mathbb{A}^S)$, implying Z contains $U^+(\mathbb{Q}_v)$ for all $v \notin S$. $\square$

Corollary 87. *(a) We have* $\mathrm{SL}_2(\mathbb{A}^\infty) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{SL}_2(\widehat{\mathbb{Z}})$ *and*

$$\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q}) \left(\mathrm{SL}_2(\mathbb{R}) \times \prod_p \mathrm{SL}_2(\mathbb{Z}_p) \right).$$

Note that $\mathrm{SL}_2(\mathbb{Q})$ *embeds diagonally to* $\mathrm{SL}_2(\mathbb{A})$ *while* $\mathrm{SL}_2(\mathbb{R})$ *and* $\mathrm{SL}_2(\mathbb{Q}_p)$*, each is embedded to its* p*-component in* $\mathrm{SL}_2(\mathbb{A})$*.*

(b) We have

$$\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})/\,\mathrm{SL}_2(\widehat{\mathbb{Z}})$$

so

$$(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) \times \prod_p \mathrm{SL}_2(\mathbb{Z}_p) \cong \mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$$

as topological spaces.

One can repeat the proof of corollary 71 to prove this corollary.

6. Fourier analysis on locally compact abelian groups

In this section, we study Fourier analysis on locally compact abelian groups, in particular for the cases of $\mathbb{R}, \mathbb{Q}_p$ and $\mathbb{A}$. Our goal is to establish the Poisson summation formula. The references we use for this section are [Fol15, §4] and [Poo15].

Throughout the section, let G be always a locally compact Hausdorff abelian topological group. For example, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a locally compact abelian group.

6.1. Pontryagin dual.

In this subsection, we will define the Pontryagin dual $\widehat{G}$ of a locally compact abelian group G. The Pontryagin duality then claims that G is isomorphic to $\widehat{\widehat{G}}$ as topological groups. Furthermore, when G is a local field or G is the adeles, we also have an isomorphism $G \cong \widehat{G}$ of topological groups.

Definition 88. *A character of G is a continuous homomorphism $\chi : G \to \mathbb{C}^\times$. A unitary character of G is a continuous homomorphism $\chi : G \to S^1$. The Pontryagin dual $\widehat{G}$ of G is the group of unitary characters of G, with the group operation being pointwise multiplication. We can equip $\widehat{G}$ with the compact-open topology, i.e. the topology generated by $\{\chi \in \widehat{G} : \chi(K) \subset U\}$ for every compact $K \subset G$ and open $U \subset S^1$.*

In fact, $\widehat{G}$ is also a locally compact abelian group. Any continuous homomorphism $\phi : G \to H$ of locally compact abelian groups induces a continuous homomorphism $\widehat{H} \to \widehat{G}$ taking χ to $\chi \circ \phi$. In fact, taking the Protryagin dual is an contravariant and exact functor from the category of locally compact abelian groups to itself.

Example 89. If G is discrete then $\widehat{G}$ is compact. Indeed, the compact-open topology on $\widehat{G}$ is precisely the topology of pointwise convergence of all maps from G to S^1. With respect to the latter topology, $\widehat{G}$ is a closed subset of the space of all maps from G to S^1. The later space is compact as it is homeomorphic to $(S^1)^{|G|}$, therefore $\widehat{G}$ is also compact.

We will assume the following result (see [Fol15, p. 110] for the proof):

Theorem 90 (Pontryagin duality)**.** *We have a canonical isomorphism of topological groups*

$$\begin{aligned} G &\to \widehat{\widehat{G}}, \\ g &\mapsto (\chi \mapsto \chi(g)). \end{aligned}$$

In the next subsections, we will explain the following table:

G	$\widehat{G}$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}_p$	$\mathbb{Q}_p$
$\mathbb{A}$	$\mathbb{A}$
$\mathbb{Z}$	$\mathbb{R}/\mathbb{Z}$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Z}_p$
$\mathbb{Q}$	$\mathbb{Q} \setminus \mathbb{A}$

6.1.1. *Pontryagin duals of local fields.*

Let k be a local field.

Proposition 91. *For a local field k and a nontrivial unitary character ψ of $(k, +)$, we have an isomorphism $\Psi : k \to \widehat{k}$ of locally compact abelian groups, sending $a \mapsto \psi_a$, where $\psi_a(x) := \psi(ax)$.*

Proof. We check Ψ is injective. If $\psi_a = \psi_b$ for $a, b \in k$ then $\psi(ax) = \psi(bx)$ for all $x \in k$, or $\psi((a - b)x) = 1$ for all $x \in k$. As ψ is nontrivial, we find $a = b$.

We show that Ψ is a homeomorphism onto its image. From the topology of $\widehat{k}$, it suffices to show that k has $C(K,U) = \{a \in k : \psi_a(K) \subset U\} = \{a \in k : aK \subset \psi^{-1}(U)\}$ as basis of open neighborhoods of 0, where $K \subset k$ is compact and $1 \in U \subset S^1$ is open.

For any compact set K of k and open set U of S^1 containing 1, as $\psi^{-1}(U)$ contains an open disk around 0 and K is bounded, there exists $\delta > 0$ such that if $a \in k, |a| < \delta$ then $aK \subset \psi^{-1}(U)$. This shows $C(K,U)$ is open in k, as given $a_0 \in C(K,U)$, we know for all $a \in k$ such that $|a - a_0| < \delta$ then $(a - a_0)K \subset \psi^{-1}(U)$, implying $a \in C(K,U)$.

For any $\delta > 0$, we show that there exists a compact K of k and an open set U of S^1 containing 1 such that the open disk $|a| < \delta$ contains $C(K,U)$. Indeed, we can choose $b \in k$ such that $\psi(b) \neq 1$ (ψ is nontrivial) and choose open $U \subset S^1$ containing 1 such that $\psi(b) \notin U$. Hence, $b \notin \psi^{-1}(U)$. We choose K to be a closed disk centered at 0 of radius at least $|b|/\delta$. Hence, $aK \subset \psi^{-1}(U)$ implies $b \notin aK$, meaning $|b| > |a| \cdot |b|/\delta$, so $|a| < \delta$.

Finally, we show Ψ is surjective. From the pairing $\langle , \rangle : k \times \widehat{k} \to S^1$, we have an order-reversing bijection between closed subgroups of $\widehat{k}$ and closed subgroups of k by taking orthogonal complements. Hence, to show $\Psi(k) = \widehat{k}$, it suffices to show $\Psi(k)^{\perp} = \{0\}$. If $x \in \Psi(k)^{\perp}$ then $\psi_a(x) = 1$ for all $a \in k$, implying $x = 0$. ☐

Remark 92. There is a standard nontrivial unitary character ψ for each local field k:

(1) If $k = \mathbb{R}$, we let $\psi(x) := e^{-2\pi i x}$.
(2) If $k = \mathbb{Q}_p$, ψ is defined by $\psi(\mathbb{Z}_p) = 1$ and $\psi(p^{-n}) = e^{2\pi i p^{-n}}$ for all $n \geq 1$.
(3) If $k = \mathbb{F}_p((t))$, define $\psi\left(\sum a_i t^i\right) := e^{2\pi i a_{-1}/p}$ (here we choose a lift of a_{-1} from $\mathbb{F}_p$ to $\mathbb{Z}$).
(4) If k_0 is either $\mathbb{R}, \mathbb{Q}_p$ or $\mathbb{F}_p((t))$ with the corresponding character ψ_0 as above, and k is a finite separable extension of k_0 then let $\psi : k \to S^1$ defined by the composition $k \xrightarrow{\mathrm{Tr}_{k/k_0}} k_0 \xrightarrow{\psi_0} S^1$.

Corollary 93. *We have $\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}$ and $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \cong \mathbb{Z}_p$ for prime p.*

Proof. It suffices to show that the image of the map $\widehat{\mathbb{R}/\mathbb{Z}} \to \widehat{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}$ is $\mathbb{Z}$. A nontrivial unitary character $f : \mathbb{R}/\mathbb{Z} \to S^1$ induces a nontrivial unitary character $f' : \mathbb{R} \to S^1$ of $\mathbb{R}$ whose kernel contains $\mathbb{Z}$. From previous proposition, it must be of the form $f'(x) = e^{2\pi i a x}$ for some $a \in \mathbb{R}$. Because $f'|_{\mathbb{Z}} = 1$ so $a \in \mathbb{Z}$. Hence, we can define a bijection $\widehat{\mathbb{R}/\mathbb{Z}} \to \mathbb{Z}$ sending f to a.

Similarly, the image of $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} \to \widehat{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{Q}_p$ is $\mathbb{Z}_p$ because for the standard character ψ of $\mathbb{Q}_p$ defined in Remark 92, $\psi(ax) = 1$ for all $x \in \mathbb{Z}_p$ iff $a \in \mathbb{Z}_p$. ☐

6.1.2. *Pontryagin dual of adeles.* Recall that the adeles $\mathbb{A}$ of $\mathbb{Q}$ is a locally compact abelian group under addition. From previous propositions, we have the following result

Proposition 94. *We have an isomorphism of topological groups*

$$\begin{aligned} \widehat{\mathbb{A}} &\to \prod_v{}'(\widehat{\mathbb{Q}_v}, \widehat{\mathbb{Q}_v/\mathbb{Z}_v}) \cong \mathbb{A}, \\ \psi &\mapsto (\psi|_{\mathbb{Q}_v}), \\ \prod_v \psi_v &\leftarrow\!\shortmid (\psi_v). \end{aligned}$$

In other words, to give a unitary character ψ of $\mathbb{A}$, it suffices to give a collection (ψ_v) of unitary characters of $\mathbb{Q}_v$ so that $\psi_v|_{\mathbb{Z}_v} = 1$ for almost all places v of $\mathbb{Q}$.

Furthermore, we can construct a nontrivial unitary character ψ on $\mathbb{A}$ by letting $\psi|_{\mathbb{Q}_v}$ to be the standard characters on $\mathbb{Q}_v$ as in Remark 92. Then

$$\begin{aligned} \Psi : \mathbb{A} &\to \widehat{\mathbb{A}} \\ a &\mapsto (\psi_a : x \mapsto \psi(ax)). \end{aligned}$$

is an isomorphism of topological groups.

Sketch. The first isomorphism holds for any restricted product, i.e. if G_v are locally compact abelian groups and H_v is open compact subgroup of G_v then

$$\widehat{\prod{}'(G_v, H_v)} \cong \prod{}'\left(\widehat{G_v}, \widehat{G_v/H_v}\right)$$

with the similar map as defined in the proposition. One can easily show that this map is a bijective group homomorphism. To show it is a homeomorphism requires more work.

For the second isomorphism, the standard character ψ_v on $\mathbb{Q}_v$ induces an isomorphism $\Psi_v : \mathbb{Q}_v \to \widehat{\mathbb{Q}_v}$ that sends $\mathbb{Z}_v$ to $\widehat{\mathbb{Q}_v/\mathbb{Z}_v}$, as shown in corollary 93. Hence, Ψ is precisely the map

$$\mathbb{A} = \prod_v{}'(\mathbb{Q}_v, \mathbb{Z}_v) \xrightarrow{\prod \Psi_v} \prod_v{}'(\widehat{\mathbb{Q}_v}, \widehat{\mathbb{Q}_v/\mathbb{Z}_v}) \xrightarrow{\sim} \widehat{\mathbb{A}}.$$

$\square$

Corollary 95. *Let ψ be the standard character on $\mathbb{A}$ as in the previous proposition. Then ψ is trivial on $\mathbb{Q}$ and the isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$ defined via ψ gives rise to an isomorphism of topological groups $\mathbb{Q} \cong \widehat{\mathbb{Q} \setminus \mathbb{A}}$.*

Proof. We first recall the definition of $\psi = \prod_v \psi_v$. Here $\psi_\infty : \mathbb{R} \to S^1$ is defined as $\psi_\infty(x) = e^{-2\pi i x}$ and for prime p, if $x = up^n \in \mathbb{Q}_p$ where $u \in \mathbb{Z}_p^\times$, then $\psi_p(x) = 0$ if $n \geq 0$ and $\psi_p(x) = e^{2\pi i p^n (xp^{-n} \mod p^{-n})}$ if $n < 0$. Hence, to show $\psi(x) = 1$ for $x \in \mathbb{Q}$, it suffices to show that if $x = p_1^{k_1} \cdots p_\ell^{k_\ell} \in \mathbb{Q}$ then $x - \sum_{p_i \text{ s.t. } k_i < 0} p_i^{k_i}(xp^{-k_i} \mod p^{-k_i}) \in \mathbb{Z}$, which is true. Thus, ψ is trivial on $\mathbb{Q}$.

Next, we will show $\mathbb{Q} \cong \widehat{\mathbb{Q} \setminus \mathbb{A}}$. Since $\mathbb{Q} \setminus \mathbb{A}$ is compact so $\widehat{\mathbb{Q} \setminus \mathbb{A}}$ is discrete. Under the identification $\psi : \widehat{\mathbb{A}} \cong \mathbb{A}$, $\mathbb{Q} \setminus \Psi(\widehat{\mathbb{Q} \setminus \mathbb{A}})$ is a discrete subgroup of the compact group $\mathbb{Q} \setminus \mathbb{A}$, implying $\mathbb{Q} \setminus \Psi(\widehat{\mathbb{Q} \setminus \mathbb{A}})$ is finite [7].

On the other hand, $\Psi(\widehat{\mathbb{Q} \setminus \mathbb{A}})$ is a $\mathbb{Q}$-subspace of $\mathbb{A}$, as if $\psi_a|_{\mathbb{Q}} = 1$ then $\psi_{qa}|_{\mathbb{Q}} = 1$ for all $q \in \mathbb{Q}$. This means $\mathbb{Q} \setminus \Psi(\widehat{\mathbb{Q} \setminus \mathbb{A}})$ is a finite $\mathbb{Q}$-vector space. As $\mathbb{Q}$ is infinite so $\mathbb{Q} = \Psi(\widehat{\mathbb{Q} \setminus \mathbb{A}})$, as desired. $\square$

6.2. **Fourier transform.** In this subsection, we will discuss Fourier transform on G, in particular when G is $\mathbb{Q}_p, \mathbb{R}$ or $\mathbb{A}_{\mathbb{Q}}$.

If $f \in L^1(G)$, we can define the *Fourier transform* $\widehat{f} : \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\chi) := \int_G f(g)\chi(g)dg.$$

One can show $\widehat{f} : \widehat{G} \to \mathbb{C}$ is always continuous. [8]

Under the condition that the function on G is nice enough, we have the following Fourier inversion formula

Theorem 96 (Fourier inversion formula)**.** *Let G be a locally compact abelian group. Let dg be a Haar measure on G. Then there exists a unique Haar measure $d\chi$ on $\widehat{G}$, called the* Plancherel measure*, such that if $f \in L^1(G)$ is such that $\widehat{f} \in L^1(\widehat{G})$ then*

$$f(g) := \int_{\widehat{G}} \widehat{f}(\chi)\overline{\chi(g)}d\chi \tag{2}$$

[7] We show that a discrete subgroup H of a compact group G has to be finite. Indeed, as H is discrete, there exists an open neighborhood U of 1 so $H \cap U = \{1\}$. This follows $aU \cap H$ is either empty if $a \notin H$ or $\{a\}$ if $a \in H$. Because G is compact, G is a finite union of aU's for $a \in G$, imlying H is finite.

[8] Some authors, such as [Fol15], define $\widehat{f}$ by taking complex conjugate of $\chi(g)$.

for almost everywhere g, i.e. there exists a null-set $N \subset G$ such that the above formula holds for all $g \in G \setminus N$.

We refer to [Fol15, p. 111] for the proof of this theorem. Note that under Pontryagin duality, the Fourier inversion formula can be written as $\hat{\hat{f}}(x) = f(-x)$.

Example 97. If G is discrete with the counting measure, the Plancherel measure on the compact group $\widehat{G}$ is the normalised Haar measure so that $\widehat{G}$ has volume 1. Indeed, consider $f \in L^1(G)$ defined as $f(x) = 1$ if $x = 1$ in G and 0 everywhere else. Hence, we find

$$\widehat{f}(\chi) = \sum_{g \in G} f(g)\chi(g) = \chi(1) = 1.$$

By the Fourier inversion formula,

$$1 = f(1) = \int_{\widehat{G}} \widehat{f}(\chi)\overline{\chi(1)}d\chi = d\chi(\widehat{G}).$$

In Theorem 96, the Fourier inversion formula depends on the condition that $\widehat{f} \in L^1(\widehat{G})$. One can define the space of Schwatz-Bruhat functions on a locally compact abelian group so that Fourier transform is an isomorphism on these spaces. In the next subsections, we will focus on defining such functions when G is a local field or G is the adeles. Furthermore, in such cases of G, there is a natural Haar measure on G such that its pushforward via $G \cong \widehat{G}$ (as discussed in the previous section) is the Plancherel measure on $\widehat{G}$ in the Fourier inversion formula.

6.2.1. *Fourier transform for local fields.*

Definition 98. *For a local field k, a function $f : k \to \mathbb{C}$ is called a Schwartz-Bruhat function if*

- *When $k = \mathbb{R}^n$, f is a C^∞-function whose derivatives are rapidly decreasing* [9]*, i.e. for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $D^\beta f := \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} f$, we have*

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right| < \infty.$$

- *When $k = \mathbb{C}^n$, f is viewed as a function on $\mathbb{R}^{2n}$ with rapidly decreasing derivatives.*
- *When k is a nonarchimedean local field, f is a locally contant function of compact support.*

We denote $S(k)$ to be the complex vector space of Schwartz-Bruhat functions on k.

Example 99. If $f(x) \in \mathbb{R}[x_1, \ldots, x_n]$ then $f(x)e^{-a|x|^2} \in S(\mathbb{R}^n)$. All compactly supported real C^∞-functions are Schwartz functions.

Example 100. Every Schwartz-Bruhat function $f \in S(\mathbb{Q}_p)$ can be written as $f = \sum_{i=1}^n c_i 1_{a_i + p^{k_i}\mathbb{Z}_p}$ where $a_i \in \mathbb{Q}_p, k_i \in \mathbb{Z}$ and $c_i \in \mathbb{C}$. Indeed, because every open set in $\mathbb{Q}_p$ is a disjoint union of open balls $a + p^k\mathbb{Z}_p$ (for some $a \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$) and that f is compactly supported, the support of f is a finite disjoint union of such open balls. As f is also locally constant, we are done.

Upon identifying k with $\widehat{k}$ via a choice of a standard unitary character as in Remark 92, we can rewrite the Fourier inversion formula as follows

Theorem 101. *Let k be a local field and ψ be the standard unitary character on k as in Remark 92. Under the identification $k \cong \widehat{k}$ via ψ, the Fourier transform*

$$\widehat{f}(y) := \int_k f(x)\psi(xy)dx$$

[9] in the case where $k = \mathbb{R}^n$, such f is also called Schwartz function

defines an automorphism of vector spaces on $S(k)$.

There is a unique Haar measure dx on k such that its pushforward via $k \cong \widehat{k}$ is the Plancherel measure on $\widehat{k}$. We call dx the self-dual Haar measure on k. Under such choice of measure, we have the Fourier inversion formula

$$f(x) = \int_k \widehat{f}(y)\overline{\psi(xy)}dy.$$

In particular, the self-dual Haar measure on k can be described explicitly as follows:

(1) If $k = \mathbb{R}$ then dx is the Lebesgue measure.
(2) If $k = \mathbb{C}$ then dx is twice the Lebesgue measure.
(3) If k is nonarchimedean then dx is the Haar measure for which its ring of integers $\mathcal{O}$ has measure $(\#\mathcal{O}/\mathcal{D})^{-1/2}$, where $\mathcal{D}$ is the different of the field extension $k/\mathbb{Q}_p$ or $k/\mathbb{F}_p((t))$.

Proof. We will defer the proof of this proposition for the final version of our thesis. At the moment, we will refer to [VR99, p. 300] for further discusisons. □

6.2.2. *Fourier transform for adeles.* In this section, we work with the adeles of $\mathbb{Q}$, but the statement will hold for the adeles of any global field k.

Definition 102. *On the adeles $\mathbb{A}$ of $\mathbb{Q}$, a Schwartz-Bruhat function $f : \mathbb{A} \to \mathbb{C}$ is a finite $\mathbb{C}$-linear combination of $\prod_v f_v : \mathbb{A} \to \mathbb{C}$, where $f_v \in S(\mathbb{Q}_v)$ and $f_v|_{\mathbb{Z}_v} = 1$ for almost all places v of k. We denote $S(\mathbb{A})$ to the space of Schwartz-Bruhat functions on $\mathbb{A}$.*

We can describe the Fourier transform on $\mathbb{A}$ in the same way as how we have done for local fields.

Theorem 103. *We fix a standard unitary character for $\mathbb{A}$ as given in Proposition 94 and let dx to be the self-dual measure with respect to ψ. For $f \in S(\mathbb{A})$, the Fourier transform*

$$\widehat{f}(y) := \int_{\mathbb{A}} f(x)\psi(xy)dx$$

defines an isomorphism on $S(\mathbb{A})$. We also have the Fourier inversion formula $\hat{\hat{f}}(x) = f(-x)$ for all $x \in \mathbb{A}$.

We will defer the proof of this proposition for the final version of our thesis.

6.3. **Poisson summation formula.** Consider an exact sequence of locally compact Hausdorff abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

where A, B, C are equipped with Haar measures $d\mu_A, d\mu_B, d\mu_C$ that make the following equation holds:

$$\int_B f(b)d\mu_B(b) = \int_A \int_C f(c+a)d\mu_A(a)d\mu_C(c)$$

for all $f \in C_c(B)$.

The Poisson summation formula is essentially the special case of the following result:

Theorem 104. *For any Schwartz-Bruhat function $f : B \to \mathbb{C}$, we have*

$$\int_A f(a)d\mu_A(a) = \int_{\widehat{C}} \widehat{f}(\widehat{c})d\mu_{\widehat{C}}$$

where $\widehat{f} : \widehat{B} \to \mathbb{C}$ is the Fourier dual of f, $d\mu_{\widehat{C}}$ is the dual Haar measure on $\widehat{C}$.

Sketch of proof. Define $F(x) = \int_A f(x+a)d\mu_A$ as a function on C. By Fourier inversion formula, we have

$$\begin{aligned}
\widehat{F}(\chi_C) &= \int_C F(c)\overline{\chi_C(c)}d\mu_C(c), \\
&= \int_C \int_A f(c+a)\overline{\chi_C(c)}d\mu_C(c)d\mu_A(a), \\
&= \int_B f(b)\overline{\chi_C(b)}d\mu_B(b), \\
&= \widehat{f}(\chi_C).
\end{aligned}$$

Again, by Fourier inversion formula, we find

$$F(c) = \int_{\widehat{C}} \widehat{F}(\chi_C)\overline{\chi_C(c)}d\mu_{\widehat{C}}(\chi_C)$$

which may be written as

$$\int_A f(c+a)d\mu_A(a) = \int_{\widehat{C}} \widehat{f}(\chi_C)\overline{\chi_C(c)}d\mu_{\overline{C}}(\chi_C).$$

By letting $c = 0$, we get the desired identity. $\square$

In the special case where L is a lattice in B (i.e. L is discrete and B/L is compact) then the dual space $L^\perp = \widehat{B/L}$ is a lattice inside $\widehat{B}$. From Example 97, the counting measure on $\widehat{B/L}$ is dual to the normalised Haar measure on B/L, giving

$$\sum_{x\in L} f(x) = \frac{1}{\mu_B(B/L)} \sum_{y\in L^\perp} \widehat{f}(y)$$

The measure on B is often chosen so that $\mu(B/L) = 1$.

Example 105. From the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{Z}\setminus\mathbb{R} \to 0$ and that $\widehat{\mathbb{Z}\setminus\mathbb{R}} \cong \mathbb{Z}$, we find

$$\sum_{x\in\mathbb{Z}} f(x) = \frac{1}{\mu_{\mathbb{R}}(\mathbb{Z}\setminus\mathbb{R})} \sum_{x\in\mathbb{Z}} \widehat{f}(x).$$

Applying the above equality to $\widehat{f}$ yields

$$\sum_{x\in\mathbb{Z}} \widehat{f}(x) = \frac{1}{\mu_{\mathbb{R}}(\mathbb{Z}\setminus\mathbb{R})} \sum_{x\in\mathbb{Z}} f(x).$$

Combining these two identities, we find $\mu_{\mathbb{R}}(\mathbb{Z}\setminus\mathbb{R}) = 1$.

The same exact argument for $0 \to \mathbb{Q} \to \mathbb{A} \to \mathbb{Q}\setminus\mathbb{A} \to 0$ shows $\mu_{\mathbb{A}}(\mathbb{Q}\setminus\mathbb{A}) = 1$.

7. SL_2

In this section, we will describe SL_2 as a linear algebraic group over a field k. We will then define and derive a nonvanishing, left-invariant global top form for SL_2.

7.1. Affine algebraic group SL_2.

The affine algebraic group SL_2 over a field k is the morphism of affine scheme $\operatorname{Spec} \mathcal{O}(\mathrm{SL}_2) \to \operatorname{Spec} k$ where

$$\mathcal{O}(\mathrm{SL}_2) = k[x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{22} - x_{21}x_{12} - 1).$$

The k-algebra $\mathcal{O}(\mathrm{SL}_2)$ is generated by x_{11}, x_{12} as for any $f \in \mathcal{O}(\mathrm{SL}_2)$ then $f = f(x_{11}x_{22} - x_{21}x_{12}) \in (x_{11}, x_{12})$. Therefore, $\operatorname{Spec} \mathcal{O}(\mathrm{SL}_2) = D(x_{11}) \cup D(x_{12})$ where for $f \in \mathcal{O}(\mathrm{SL}_2), D(f) := \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2) : f \notin \mathfrak{p}\}$ is the distinguished open set of $\operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)$.

For a k-algebra R, the R-points of SL_2, denoted $\mathrm{SL}_2(R)$ is the group $\operatorname{Hom}_{k-\mathrm{alg}}(\mathcal{O}(\mathrm{SL}_2), R)$, which can be identified with $\left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} : x_{ij} \in R, x_{11}x_{22} - x_{12}x_{21} = 1 \right\}$ with the usual group structure.

For a ring R, we denote $\mathrm{SL}_{2,R}$ to be SL_2 over R.

7.2. Lie algebra.

In this section, we define the Lie algebra of SL_2. We have a projection $k[\epsilon]/(\epsilon^2) \to k$ sending $a + \epsilon b$ to a. For k-algebra R, we define the *Lie algebra* of SL_2 over R to be

$$\operatorname{Lie}(\mathrm{SL}_2)(R) := \ker(\mathrm{SL}_2(R[\epsilon]/(\epsilon^2)) \to \mathrm{SL}_2(R)) \tag{3}$$

In particular, one can describe elements in $\operatorname{Lie}(\mathrm{SL}_2)(k)$ as 2-by-2 matrices of determinant 1, with entries over $k[\epsilon]/(\epsilon^2)$, such that by letting $\epsilon \mapsto 0$, we get the identity matrix. Concretely, elements of $\operatorname{Lie}(\mathrm{SL}_2)$ are of the form $\begin{pmatrix} 1 + \epsilon a_{11} & \epsilon a_{12} \\ \epsilon a_{21} & 1 + \epsilon a_{22} \end{pmatrix} = I_2 + \epsilon \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that $a_{11} + a_{22} = 0$, which we can identify with $\mathfrak{sl}_2(k)$, a k-vector space of 2-by-2 matrices having trace 0.

In general, for affine algebraic group G, it is more subtle to see the Lie algebra structure from definition (3) of tangent space. However, we can embed G into GL_n and the Lie algebra tructure of $\operatorname{Lie}(G)$ is induced from this embedding.

We can find that for a k-algebra R then $\operatorname{Lie}(\mathrm{SL}_{2,R}) \cong R \otimes_k \operatorname{Lie}(\mathrm{SL}_2)$ as R-modules.

The dual $\operatorname{Lie}(\mathrm{SL}_2)^*$ is a k-module generated by dx_{ij}, for $1 \leq i, j \leq 2$, modulo the relation $dx_{11} + dx_{22} = 0$.

7.3. Differential form.

To define cotangent sheaf of SL_2 over k, we first need to define the *module of relative differentials* $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$. It is a $\mathcal{O}(\mathrm{SL}_2)$-module equipped with a k-derivation $d : \mathcal{O}(\mathrm{SL}_2) \to \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ that is universal as initial object among $\mathcal{O}(\mathrm{SL}_2)$-modules M equipped with k-derivation $d : \mathcal{O}(\mathrm{SL}_2) \to M$. Concretely, $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is a $\mathcal{O}(\mathrm{SL}_2)$-module genereated by dx_{ij} for $x_{ij} \in \mathcal{O}(\mathrm{SL}_2), 1 \leq i, j \leq 2$ quotient out by the relation $x_{11}dx_{22} + x_{22}dx_{11} - x_{12}dx_{21} - x_{21}dx_{12}$. The map $d : \mathcal{O}(\mathrm{SL}_2) \to \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is the obvious one, as suggested by the notation.

We define the *cotangent sheaf* $\Omega_{\mathrm{SL}_2/k}$ [10] to be the sheaf of $\mathcal{O}_{\mathrm{SL}_2}$-modules associated to $\mathcal{O}(\mathrm{SL}_2)$-module $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$. Concretely, for $f \in \mathcal{O}(\mathrm{SL}_2)$, its section over distinguished open $D(f) = \{\mathfrak{p} \in \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2) : f \notin \mathfrak{p}\}$ in $\operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)$ is the localisation of $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ at f. Note also that the global section of $\Omega_{\mathrm{SL}_2/k}$ is precisely $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$.

Proposition 106. *(a) The cotangent sheaf $\Omega_{\mathrm{SL}_2,k}$ is a vector bundle of rank 3, hence a cotangent bundle.*

(b) The fiber of $\Omega_{\mathrm{SL}_2,k}$ at a point $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)$ is $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \otimes_{\mathcal{O}(\mathrm{SL}_2)} \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p}) = \mathcal{O}(\mathrm{SL}_2)_{\mathfrak{p}}/\mathfrak{m} \cong k$ and $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}(\mathrm{SL}_2)_{\mathfrak{p}}$. The fiber of $\Omega_{\mathrm{SL}_2/k}$ at $\mathfrak{p}$ is isomorphic as k-vector space to the cotangent space $\mathfrak{m}/\mathfrak{m}^2$.

[10] notice that there is a subtlety in our choice of notations, where $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is different from $\Omega_{\mathrm{SL}_2/k}$

A *differential 1-form over open U* in SL_2 is a section of $\Omega_{\mathrm{SL}_2/k}$ over U. For example, in $\Gamma(\Omega_{\mathrm{SL}_2/k}, D(x_{11})) = \left(\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}\right)_{x_{11}}$, we have

$$dx_{22} = \frac{x_{11}dx_{22}}{x_{11}} = \frac{1}{x_{11}}(x_{12}dx_{21} + x_{21}dx_{12} - x_{22}dx_{11}). \tag{4}$$

Therefore, any differential 1-form over $D(x_{11})$ can be written as

$$\frac{1}{x_{11}^k}(f_{12}dx_{12} + f_{21}dx_{21} + f_{11}dx_{11}),$$

where $f_{ij} \in \mathcal{O}(\mathrm{SL}_2)$.

We define the *canonical sheaf* ω_{SL_2} to be $\omega_{\mathrm{SL}_2} = \bigwedge^3 \Omega_{\mathrm{SL}_2/k}$. Its sections over open U of $\operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)$ are called *top (dimensional) forms of SL_2 over U*. For example, its sections over $D(x_{11})$ form a $\mathcal{O}(\mathrm{SL}_2)_{x_{11}}$-module generated by $dx_{11} \wedge dx_{12} \wedge dx_{21}$. If ω is a top form of SL_2 over U then we say ω is *nowhere vanishing* if $\omega_x \in (\omega_{\mathrm{SL}_2})_x$ is nonzero for all $x \in U$.

7.4. **Left-invariant differential form.** We consider a k-algebra isomorphism

$$\begin{aligned} L_a : \mathcal{O}(\mathrm{SL}_2) &\to \mathcal{O}(\mathrm{SL}_2), \\ x_{ij} &\mapsto a_{i1}x_{1j} + a_{i2}x_{2j},\ 1 \le i,j \le 2. \end{aligned}$$

corresponding to left-multiplication by $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{SL}_2(k)$. This induces an isomorphism of $\mathcal{O}(\mathrm{SL}_2)$-modules

$$\begin{aligned} L_a : \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} &\to \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \\ f dx_{ij} &\mapsto L_a(f)(a_{i1}dx_{1j} + a_{i2}dx_{2j}), \end{aligned}$$

for any $f \in \mathcal{O}(\mathrm{SL}_2)$, hence, an isomorphism of sheaves of $\mathcal{O}_{\mathrm{SL}_2}$-modules $L_a : \omega_{\mathrm{SL}_2} \to \omega_{\mathrm{SL}_2}$.

A top form ω over open set U is called *left-invariant* if $L_a\omega = \omega$ for any $a \in \mathrm{SL}_2(k)$.

Proposition 107. *There is a unique, nowhere vanishing, left-invariant, global top form for SL_2 up to scalar over $k^\times$.*

Proof. We first determine all left-invariant top forms ω_{11} over $D(x_{11})$. From previous section, we can write ω_{11} over $D(x_{11})$ as $f dx_{11} \wedge dx_{12} \wedge dx_{21}$ where $f \in \mathcal{O}(\mathrm{SL}_2)_{x_{11}}$. It follows that over $D(x_{11})$, we have

$$\begin{aligned} L_a\omega_{11} &= L_a(f)d(a_{11}x_{11} + a_{12}x_{21}) \wedge d(a_{11}x_{12} + a_{12}x_{22}) \wedge d(a_{21}x_{11} + a_{22}x_{21}), \\ &= \frac{L_a(f)}{x_{11}}(a_{11}x_{11} + a_{12}x_{21})dx_{11} \wedge dx_{12} \wedge dx_{21}. \end{aligned}$$

Therfore, for ω_{11} to be left-invariant, we must have $x_{11}f = (a_{11}x_{11} + a_{12}x_{21})L_a(f)$ for any $f \in \mathcal{O}(\mathrm{SL}_2)_{x_{11}}$ and any $a_{ij} \in k$ such that $a_{11}a_{22} - a_{12}a_{21} = 1$. This implies $f = Cx_{11}^{-1}$ for $C \in k$. Thus, $\omega_{11} = Cx_{11}^{-1}dx_{11} \wedge dx_{12} \wedge dx_{21}$ for some $C \in k^\times$.

Similarly, we also can find a left-invariant top form ω_{12} over $D(x_{12})$ to be $\omega_{12} = C'x_{12}^{-1}dx_{11} \wedge dx_{12} \wedge dx_{22}$ for some $C' \in k^\times$.

As $\operatorname{Spec} \mathcal{O}(\mathrm{SL}_2) = D(x_{11}) \cup D(x_{12})$, a global left-invariant top form $\omega \in \Gamma(\Omega_{\mathrm{SL}_2/k}, \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2))$, if exists, must correspond to ω_{11} and ω_{12} when restricting to $D(x_{11})$ and $D(x_{12})$, respectively. Hence, to find such global top form, it suffices to find $C, C' \in k^\times$ such that $\omega_{11} = \omega_{12}$ on $D(x_{11}) \cap D(x_{12}) = D(x_{11}x_{12})$. Indeed, on $D(x_{11}x_{12})$, dx_{22} can be written as in (4), hence $\omega_{12} = \frac{C'x_{12}}{x_{11}x_{12}}dx_{11} \wedge dx_{12} \wedge dx_{21}$. Therefore, $\omega_{11} = \omega_{12}$ gives $C = C'$. Thus, SL_2 has a unique , nowhere-vanishing, left-invariant global top form up to scalar over $k^\times$. $\square$

Next, we will identify left-invariant global top forms over k with $\bigwedge^3 \mathrm{Lie}(\mathrm{SL}_2)(k)^*$, following [BLR90, §4.2]. Indeed, the unit element in the group structure of SL_2 corresponds to the k-algebra morphism $\varepsilon : \mathcal{O}(\mathrm{SL}_2) \to k$ sending x_{ij} to 1 if $1 \leq i = j \leq 2$ and 0 everywhere else. This then corresponds to a morphism $\varepsilon : \operatorname{Spec} k \to \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2)$ of affine scheme. Therefore, one can pullback sheaf $\Omega_{\mathrm{SL}_2/k}$ of $\mathcal{O}_{\mathrm{SL}_2}$-module via ε to get a sheaf $\varepsilon^*\Omega_{\mathrm{SL}_2/k}$ of $\mathcal{O}_{\operatorname{Spec} k}$-modules, which is just a k-module $k \otimes_{\mathcal{O}(\mathrm{SL}_2)} \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$. We note that this k-module is isomorphic to $\mathrm{Lie}(\mathrm{SL}_2)(k)^*$.

On the other hand, via the structural morphism $p : \operatorname{Spec} \mathcal{O}(\mathrm{SL}_2) \to \operatorname{Spec} k$, we have a canonical isomorphism $p^*\varepsilon^*\Omega_{\mathrm{SL}_2/k} \xrightarrow{\sim} \Omega_{\mathrm{SL}_2/k}$ that is obtained by extending sections in $\varepsilon^*\Omega_{\mathrm{SL}_2/k} = \mathrm{Lie}(\mathrm{SL}_2)(k)^*$ to left-invariant sections in $\Omega_{\mathrm{SL}_2/k}$ (see [BLR90, page 102]). Thus, the k-module $\bigwedge^3 \mathrm{Lie}(\mathrm{SL}_2)(k)^*$ is identified with the k-module of left-invariant global top forms.

7.5. **Adjoint map.** Given a k-algebra R, an affine algebraic group G and its Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$, we define the *adjoint representation* $\mathrm{Ad} : G(R) \to \mathrm{Aut}(\mathfrak{g}(R))$ to be $\mathrm{Ad}(g)x = gxg^{-1}$ where $x \in \mathrm{Lie}(G)(R) \subset G(R[\varepsilon]/(\varepsilon^2))$.

In particular, if $\omega \in \bigwedge^3 \mathfrak{g}(R)^*$ then

$$\mathrm{Ad}(g)\omega = \det\left(\mathrm{Ad}(g) : \mathfrak{g}(R) \to \mathfrak{g}(R)\right)\omega.$$

8. Tamagawa measure of $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$

8.1. **Haar measure on local fields.** Let k be a local field. By uniqueness of Haar measure as in Theorem 34, there exists a Haar measure μ on locally compact group $(k,+)$ that is unique up to scalar over k.

If k is isomorphic to $\mathbb{R}$ then a Haar measure on k is the standard Lebesgue measure. If k is isomorphic to $\mathbb{C}$ then we use twice the standard Haar measure on $\mathbb{C}$, i.e. if $z = x+iy$ with $x, y \in \mathbb{R}$ then this Haar measure is $|dz \wedge d\overline{z}| = 2x \wedge dy$ where dx, dy are Lebesgue measure on $\mathbb{R}$.

If k is nonarchimedean with valuation ring $\mathcal{O}_k$, then there exists unique Haar measure on k such that $\mu(\mathcal{O}_k) = 1$.

Remark 108. Modular quasicharacter of a local field k gives a canonical way to define absolute value on k, i.e. we can define $|a| := \delta_G(a)$, then

- If $k = \mathbb{R}$ then this is the ordinary absolute value,
- If $k = \mathbb{C}$ then $|a|$ is the square of ordinary absolute value on $\mathbb{C}$,
- If k is nonarchimedean then $|a| = (\#A/aA)^{-1}$ where A is the valuation ring of k.

8.2. **Tamagawa measure on $\mathbb{A}$.** To define a Haar measure on $\mathbb{A}_{\mathbb{Q}}$, we first choose the normalised Haar measure $\mu_{\mathbb{Q}_v}$ on $\mathbb{Q}_v$ as in previous section. From section 3.4.5, this gives a Haar measure $\mu_{\mathbb{G}_{a,\mathbb{Q}}}$ on $\mathbb{A}_{\mathbb{Q}}$ such that restriction of $\mu_{\mathbb{G}_a,\mathbb{Q}}$ to open set $\prod_{v\in S}\mathbb{Q}_v \times \prod_{v\notin S}\mathbb{Z}_v$ is the product measure of $\mu_{\mathbb{Q}_v}$'s, where S is a finite set of places of $\mathbb{Q}$. We call this the *Tamagawa measure* of $\mathbb{A}_{\mathbb{Q}}$.

As $\mathbb{Q}$ is a unimodular closed subgroup of $\mathbb{A}_{\mathbb{Q}}$, from Theorem 39, the Tamagawa measure on $\mathbb{A}_{\mathbb{Q}}$ gives a $\mathbb{Q}$-invariant measure on $\mathbb{Q} \setminus \mathbb{A}_{\mathbb{Q}}$. Furthermore, as $\mathbb{Q} \setminus \mathbb{A}_{\mathbb{Q}}$ is compact (see corollary 72), it must have finite volume with this measure. We denote $\tau(\mathbb{G}_{a,\mathbb{Q}})$ to be volume of $\mathbb{Q} \setminus \mathbb{A}$ and call it the *Tamagawa number* of $\mathbb{G}_{a,\mathbb{Q}}$.

Proposition 109. *$\tau(\mathbb{G}_{a,\mathbb{Q}}) = 1$.*

Proof. From corollary 72, we know $\mathbb{Q} \setminus \mathbb{A} \cong \mathbb{Z} \setminus \mathbb{R} \times \widehat{\mathbb{Z}}$ so a fundamental domain of $\mathbb{Q} \setminus \mathbb{A}$ is

$$[0,1) \times \prod_p \mathbb{Z}_p.$$

The volume of $\mathbb{Q} \setminus \mathbb{A}$ is then the volume of this fundamental domain. From our construction of Tamagawa measure, this is precisely $\mu_{\mathbb{R}}([0,1)) \times \prod_p \mu_{\mathbb{Q}_p}(\mathbb{Z}_p) = 1$. $\square$

Remark 110. With this, we have a second proof of $\tau(\mathbb{G}_{a,\mathbb{Q}}) = 1$. The first proof using Poisson summation formula is given in section 6.3.

Remark 111. There are some modifications needed to define Tamagawa measure over base field other than $\mathbb{Q}$. At the moment, we will restrict to our definition of Tamagawa measure.

8.3. **Haar measure on $\mathrm{SL}_2(\mathbb{Q}_v)$.** Let G be an affine algebraic group over local field k and ω be a left-invariant global top form of G over k. We will define a Haar measure $d|\omega|_v$ on $G(k_v)$ in this subsection.

We can cover $\operatorname{Spec}\mathcal{O}(G)$ by affine schemes U_i over k such that

(1) when taking k_v-points, $G(k_v)$ is cover by charts $(U_i(k_v), (x_j)_j)$;
(2) for each U_i, the x_j's are elements in $k_v \otimes_k \mathcal{O}(U_i)$, where we denote $\mathcal{O}(U_i)$ to be the coordinate ring of U_i;
(3) on U_i, ω over k_v can be written as $f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ for some $f \in k_v \otimes_k \mathcal{O}(U_i)$. Note that f is a k-analytic function on $U_i(k_v)$.

Remark 112. For SL_2, as we did in the proof of Proposition 107, we can choose $U_{11} = \operatorname{Spec}\mathcal{O}(\mathrm{SL}_2)_{x_{11}}$ and $U_{12} = \operatorname{Spec}\mathcal{O}(\mathrm{SL}_2)_{x_{12}}$ to cover $\operatorname{Spec}\mathcal{O}(\mathrm{SL}_2)$. Hence, ω over $U_{11}(\mathbb{Q}_v) = \{(x_{11}, x_{12}, x_{21}) \in \mathbb{Q}_v^3 : x_{11} \neq 0\}$ can be written as $\frac{1}{x_{11}} dx_{11} \wedge dx_{12} \wedge dx_{21}$, and indeed x_{11}^{-1} is $\mathbb{Q}_v$-analytic on $U_{11}(\mathbb{Q}_v)$.

With this setup, we can define a measure $|f|_v dx_1 dx_2 \cdots dx_n$ on each $U_i(k_v)$. This measure is independent of choice of coordinates x_i because of the same argument as we did in section 4.4.

To define a measure on $G(k_v)$, we argue using partition of unity, analogously as in section 4.4. The resulting measure $d|\omega|_v$ on $G(k_v)$ also does not depend on the choice of partition of unity. It is a left Haar measure because ω is left-invariant.

Proposition 113. *The modular quasicharacter of $G(k_v)$ is*

$$\delta_{G(k_v)}(g) = \left|\det\left(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v)\right)\right|_v .$$

Proof. From section 7.5, we know $\mathrm{Ad}(g)\omega = \det\left(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v)\right)\omega$. Therefore, by Change of variables formula in Theorem 65, we find that $\mathrm{Ad}(g) : G(k_v) \to G(k_v)$ induces a new left Haar measure $d|\omega|_v(ghg^{-1})$ on $G(k_v)$ so that

$$d|\omega|_v(ghg^{-1}) = \left|\det\left(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v)\right)\right|_v d|\omega|_v.$$

As $d|\omega|_v$ is left Haar measure so from Proposition 38, we find

$$d|\omega|_v(ghg^{-1}) = d|\omega|_v(hg^{-1}) = \delta_{G(k_v)}(g) d|\omega|_v.$$

This gives $\delta_{G(k_v)}(g) = \left|\det\left(\mathrm{Ad}(g) : \mathfrak{g}(k_v) \to \mathfrak{g}(k_v)\right)\right|_v$, as desired. $\square$

8.4. **Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$.** From a choice of left-invariant global top form ω of SL_2 over $\mathbb{Q}$, we can construct left Haar measure $\mu_v = d|\omega|_v$ for $\mathrm{SL}_2(\mathbb{Q}_v)$. As $\mathrm{SL}_2(\mathbb{A})$ is the restricted product of $\mathrm{SL}_2(\mathbb{Q}_v)$ over compact open sets $\mathrm{SL}_2(\mathbb{Z}_v)$ for almost all places v (see Proposition 77) and as we will later show $\prod_{p<\infty} \mu_p(\mathrm{SL}_2(\mathbb{Z}_p))$ is finite, from section 3.4.5, we can construct a left Haar measure μ_{Tam} on $\mathrm{SL}_2(\mathbb{A})$ from left Haar measures μ_v's on $\mathrm{SL}_2(\mathbb{Q}_v)$. We call this the *Tamagawa measure* of SL_2 over $\mathbb{Q}$.

Proposition 114. *The definition of the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$ does not depend on the choice of a nowhere vanishing, left-invariant global top form on SL_2 over $\mathbb{Q}$.*

Proof. From Proposition 107, any left-invariant, nowhere vanishing global top form ω of SL_2 over $\mathbb{Q}$ is $c\omega$ for some $c \in \mathbb{Q}^\times$.

Therefore, if $\omega' = c\omega$ is another choice of a top form on SL_2 over $\mathbb{Q}$, from section 8.3, the corresponding Haar measure on $\mathrm{SL}_2(\mathbb{Q}_v)$ is $|c|_v\mu_v$ for each place v of $\mathbb{Q}$, where μ_v the Haar measure on $\mathrm{SL}_2(\mathbb{Q}_v)$ corresponding to ω. By similar construction, we denote μ'_{Tam} to be the restricted product measure on $\mathrm{SL}_2(\mathbb{A})$ corresponding to ω.

Consider an open subset $\mathrm{SL}_2(\mathbb{A}_\infty) = \mathrm{SL}_2(\mathbb{R}) \times \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p)$ of $\mathrm{SL}_2(\mathbb{A})$. By construction, the restriction of μ_{Tam} and μ'_{Tam} to $\mathrm{SL}_2(\mathbb{A}_\infty)$ is the product measure. On the other hand, by the product formula $\prod_v |c|_v = 1$ so $\mu_{\mathrm{Tam}} = \mu'_{\mathrm{Tam}}$ on $\mathrm{SL}_2(\mathbb{A}_\infty)$. By the uniqueness of Haar measure, $\mu_{\mathrm{Tam}} = \mu'_{\mathrm{Tam}}$ on $\mathrm{SL}_2(\mathbb{A})$. $\square$

8.5. **Tamagawa number for SL_2 over $\mathbb{Q}$.** Because $\mathrm{SL}_2(\mathbb{A})$ is unimodular and $\mathrm{SL}_2(\mathbb{Q})$ is a discrete closed subgroup of $\mathrm{SL}_2(\mathbb{A})$, from section 3.4.2, μ_{Tam} induces a $\mathrm{SL}_2(\mathbb{A})$-invariant measure to the quotient $\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$. The volume of $\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{A})$ is then called the *Tamagawa number of SL_2 over $\mathbb{Q}$*, denoted as $\tau(\mathrm{SL}_{2,\mathbb{Q}})$.

By the construction of Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$ and by corollary 87, we obtain

$$\tau(\mathrm{SL}_{2,\mathbb{Q}}) = \mu_{\mathrm{SL}_2(\mathbb{R}),\omega}(\mathrm{SL}_2(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{R})) \times \prod_{\mathrm{SL}_2(\mathbb{Q}_p),\omega} \mu_p(\mathrm{SL}_2(\mathbb{Z}_p))$$

where ω is a choice of a volume form over $\mathbb{Q}$ of SL_2; $\mu_{\mathrm{SL}_2(\mathbb{Q}_v),\omega}$ is the corresponding measure on $\mathrm{SL}_2(\mathbb{Q}_v)$ defined via ω, as described in previous section.

In the next two sections, we will show that

$$\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(\mathrm{SL}_2(\mathbb{Z}_p)) = \frac{|\,\mathrm{SL}_2(\mathbb{F}_p)|}{p^3} = 1 - p^{-2}$$

and

$$\mu_{\mathrm{SL}_2(\mathbb{R}),\omega}(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) = \zeta(2) = \frac{\pi^2}{6},$$

obtaining the following theorem

Theorem 115. $\tau(\mathrm{SL}_{2,\mathbb{Q}}) = 1.$

9. Volume of $\mathrm{SL}_2(\mathbb{Z}_p)$

In this section, we will use the Haar measure $\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}$ obtained via the top form $\omega = \frac{1}{x} dx \wedge dy \wedge dz$ (as defined in section 7.4) to compute the volume of $\mathrm{SL}_2(\mathbb{Z}_p)$. Indeed, we have a surjective map $p : \mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{F}_p)$ with kernel

$$N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, c \in 1 + p\mathbb{Z}_p; b, d \in p\mathbb{Z}_p \right\}.$$

The surjectivity of p is shown in the following lemma:

Lemma 116. *Let $N \in \mathbb{Z}_{>0}$. The group homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.*

Proof. Indeed, we want to show that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})$ such that $ad - bc - Nm = 1$ for some $m \in \mathbb{Z}$ then there exists $B \in \mathrm{SL}_2(\mathbb{Z})$ such that $B \equiv A \pmod N$. From $ad - bc - Nm = 1$, we know $\gcd(d, c, N) = 1$ so there exists $n \in \mathbb{Z}$ such that $\gcd(c, d + Nn) = 1$ (for example, by Chinese Remainder Theorem, we can choose n such that $d + Nn \equiv 1 \pmod p$ for $p | c, p \nmid N$ and $d + Nn \equiv d \pmod p$ for $p \mid c, p \mid N$, i.e. $p \nmid d$). By replacing d with $d + Nn$, we can assume that $\gcd(d, c) = 1$. We want to find $B = \begin{pmatrix} a + Ne & b + Nf \\ c & d \end{pmatrix}$ such that $ad - bc + N(de - cf) = 1$, or $m = de - cf$. As $\gcd(c, d) = 1$, there exists $e, f \in \mathbb{Z}$ such that $m = de - cf$, as desired. $\square$

Because $|\mathrm{SL}_2(\mathbb{F}_p)| = p(p^2 - 1)$ so by the left-invariance of the measure, we find

$$\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(\mathrm{SL}_2(\mathbb{Z}_p)) = |\mathrm{SL}_2(\mathbb{F}_p)| \mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(N) = p(p^2 - 1) \mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(N).$$

We have

$$\begin{aligned} \mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(N) &= \int_N |a^{-1}|_p dadbdc = \int_N dadbdc, \\ &= \int_{a,c \in 1+p\mathbb{Z}_p, b \in p\mathbb{Z}_p} dadbdc, \\ &= \int_{p\mathbb{Z}_p} \int_{p\mathbb{Z}_p} \int_{p\mathbb{Z}_p} dadbdc, \\ &= (\mu_p(p\mathbb{Z}_p))^3 = p^{-3}. \end{aligned}$$

Thus, $\mu_{\mathrm{SL}_2(\mathbb{Q}_p),\omega}(\mathrm{SL}_2(\mathbb{Z}_p)) = (1 - p^{-2})$.

10. Volume of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$

We present two ways of computing the volume of quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$: one by computing volume of fundamental domain and the second by using Poisson summation formula.

10.1. **First method via fundamental domain.** In this section, we compute the volume of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$ by determine its fundamental domain.

10.1.1. *Volume form of* $\mathrm{SL}_2(\mathbb{R})$. From section 7.4, we know that SL_2 over $\mathbb{Q}$ has a unique left-invariant volume form ω up to scalar over $\mathbb{Q}^\times$. In particular, over open set $\left\{\begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : x \neq 0\right\}$ of $\mathrm{SL}_2(\mathbb{R})$ then $\omega = x^{-1} dx \wedge dy \wedge dz$.

Over $\mathbb{R}$, every element in $\mathrm{SL}_2(\mathbb{R})$ is uniquely expressed as product

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

where $\varphi \in [0, 2\pi), \alpha > 0, u \in \mathbb{R}$. Hence, under change of coordinates $x = \alpha\cos\varphi, y = \alpha u \cos\varphi - \alpha^{-1}\sin\varphi$ and $z = \alpha \sin\varphi$, we find that $\omega_{\mathbb{R}}$ can be globally expressed as $\omega_{\mathbb{R}} = \alpha d\varphi \wedge d\alpha \wedge du$

10.1.2. *Fundamental domain of* $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$. First, we denote the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), z \in \mathbb{C}$, we define $gz := \frac{az+b}{cz+d}$.

Proposition 117. *We have a smooth action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathcal{H}$ *via*

$$\begin{aligned} \Phi : \mathrm{SL}_2(\mathbb{R}) \times \mathcal{H} &\to \mathcal{H} \\ (g, z) &\mapsto gz. \end{aligned}$$

This action is transitive with the special orthogonal group

$$\mathrm{SO}_2(\mathbb{R}) = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}\right\}$$

is the stabiliser of i, *inducing a homeomorphism*

$$\phi : \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R}) \to \mathcal{H}$$

sending $g \mapsto gi$. *Furthermore,* $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ *acts faithfully on* $\mathcal{H}$.

Proof. As $\Im(gz) = \frac{\Im(z)}{|cz+d|^2} > 0$ so $gz \in \mathcal{H}$, meaning each $g \in \mathrm{SL}_2(\mathbb{R})$ induces a smooth map from $\mathcal{H}$ to $\mathcal{H}$ (called *linear fractional transformation*) with inverse g^{-1}. Furthermore, one can also check $g(g'z) = (gg')z$ so we have an action of $\mathrm{SL}_2(\mathbb{R})$ onto $\mathcal{H}$.

Next, we show this action ϕ is smooth. We first choose a chart for $\mathrm{SL}_2(\mathbb{R})$. WLOG, let $U_a = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a \neq 0\right\}$ be a open subset of $\mathrm{SL}_2(\mathbb{R})$ with chart $\phi : U_a \to V$ where $V = \{(a,b,c) \in \mathbb{R}^3 : a \neq 0\}$ sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to (a,b,c). Under this chart, the map ϕ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times z \mapsto \frac{az+b}{cz + \frac{bc+1}{a}},$$

which is smooth on $U_a \times \mathcal{H}$ because it is composition of smooth maps.

This action is transitive as for any $z = x + iy$ then $y^{-1/2}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ maps i to z. One can also check that

$$\mathrm{SO}_2(\mathbb{R}) = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}\right\}.$$

being the stabiliser of i. Overall, we have a smooth and transitive action of Lie groups $\mathrm{SL}_2(\mathbb{R})$ onto smooth manifold $\mathcal{H}$, we obtain a diffeomorphism

$$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) = G/\mathrm{Stab}(i) \to \mathcal{H},$$

sending $g \mapsto gi$. $\square$

Under action of $\mathrm{PSL}_2(\mathbb{Z})$, $\mathcal{H}$ has fundamental domain:

$$D = \{z \in \mathcal{H} : |z| > 1, \mathrm{Re}(z) < 1/2\}.$$

Action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathcal{H}$ commutes with the left action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. We then find

Proposition 118. *(a) The fundamental domain for action of* $\mathrm{PSL}_2(\mathbb{Z})$ *on* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ *is*

$$\phi^{-1}(D) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : |u| < 1/2, 0 < \alpha < \frac{1}{\sqrt{1-u^2}} \right\}$$

(b) The fundamental domain for left-action of $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathrm{SL}_2(\mathbb{R})$ *is* $\phi^{-1}(D)K$ *where*

$$K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : \varphi \in [0, 2\pi), \alpha > 0 \right\} \cong \mathrm{SO}_2(\mathbb{R})/\{\pm 1\}.$$

Proof. (a) Indeed, as ϕ is homeomorphism, $\phi^{-1}(D)$ is open connected. As no two points in D belong to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit, no two points in $\phi^{-1}(D)$ belongs to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit. We also have $\phi^{-1}(\overline{D}) = \overline{\phi^{-1}(D)}$, hence, knowing $\mathcal{H} = \bigcup_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma \overline{D}$ implies $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) = \bigcup_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)}$. We are done.

(b) From (a), we have

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) &= \bigcup_{\gamma \in \mathrm{PSL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)}\, \mathrm{SO}_2(\mathbb{R}), \\ &= \bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)}\, \mathrm{SO}_2(\mathbb{R})/\{\pm 1\}, \\ &= \bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \gamma \overline{\phi^{-1}(D)} K. \end{aligned}$$

Also from (a), no two points in $\phi^{-1}(D)K$ belong to the same $\mathrm{SL}_2(\mathbb{Z})$-orbit, else we can find two points in $\phi^{-1}(D)$ belong to the same $\mathrm{PSL}_2(\mathbb{Z})$-orbit. $\square$

Thus, from this proposition, we find

$$\begin{aligned} \mu_{\mathrm{SL}_2(\mathbb{R}),\omega}(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})) &= \int_{\phi^{-1}(D)K} \alpha d\alpha du d\varphi, \\ &= \int_{u=-1/2}^{1/2} \int_{\alpha=0}^{(1-u^2)^{-1/2}} \int_{\varphi=0}^{\pi} \alpha d\varphi d\alpha du, \\ &= \pi^2/6. \end{aligned}$$

10.2. **Second method via Poisson summation.** Let

$$\begin{aligned}
G &= \mathrm{SL}_2(\mathbb{R}),\\
K &= \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} : \varphi \in [0, 2\pi) \right\},\\
A &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha > 0 \right\},\\
N &= \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{R} \right\},\\
P &= AN = NA = \left\{ \begin{pmatrix} \alpha & b \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha > 0, b \in \mathbb{R} \right\}.
\end{aligned}$$

10.2.1. *Haar measure on* $\mathrm{SL}_2(\mathbb{R})$. In this section, we identify a Haar measure on $\mathrm{SL}_2(\mathbb{R})$ via the Iwasawa decomposition. We denote da, dn, dk to be Haar measure on A, N, K, respectively. Note A is group isomorphic and homeomorphic to $(\mathbb{R}_{>0}, \cdot)$ so we can identify da with Haar measure on $(\mathbb{R}_{>0}, \cdot)$. Similarly, N is group isomorphic and homeomorphic to $(\mathbb{R}, +)$ so we can identify dn with Haar measure on $(\mathbb{R}, +)$.

Proposition 119. *Given decomposition $G = ANK$ into unimodular closed subgroups A, N, K such that A normalises N, one can normalise the Haar measure of A, N, K so that a Haar measure for $G = \mathrm{SL}_2(\mathbb{R})$ can be chosen to be* [11]

$$\int_G f(g)dg = \int_{A\times N\times K} f(ank)dadndk = \int_{A\times N\times K} f(nak)\alpha^{-2}dadndk.$$

Proof. We prove the first equality. Iwasawa decomposition gives us a homeomorphism $P \times K \to \mathrm{SL}_2(\mathbb{R})$ sending $(p, k) \to pk$, so by Proposition 44, as K and $\mathrm{SL}_2(\mathbb{R})$ are unimodular, we can normalise left Haar measure on P and Haar measure on K such that

$$\int_G f(g)dg = \int_{P\times K} f(pk)d_lpdk.$$

Symbolically, we can write $dg = dkd_lp$. Furthermore, we also have homeomorphism $A \times N \cong P$ sending $(a, n) \mapsto an$ where A, N are closed unimodular subgroups and A normalises N. We claim that $d_lp := dadn$ is a left Haar measure on P, i.e. we have

$$\int_P f(p)d_lp = \int_{A\times N} f(an)dadn. \tag{5}$$

Indeed, $dadn$ is left invariant under A. Let $n_1 \in N$, we have

$$\int_{A\times N} f(n_1an)dadn = \int_{A\times N} f(aa^{-1}n_1an)dadn.$$

As $a^{-1}n_1a \in N$ so by left invariance of N, we find $dadn$ is also left invariant under N. Thus, $dadn$ is left P-invariant, as desired. Thus, we obtain

$$\int_G f(g)dg = \int_{A\times N\times K} f(ank)dadndk.$$

To prove the second equality, we consider automorphism $n \mapsto ana^{-1}$ of N. This gives rise to a Haar measure $d(L_aR_{a^{-1}}n) = d(ana^{-1})$ of N. Indeed, as A normalises N so

$$d(L_aR_{a^{-1}}n'n) = d(an'a^{-1}ana^{-1}) = d(ana^{-1}) = d(L_aR_{a^{-1}}n).$$

[11] Notice that, decompositions AN and NA of P give different homeomorphisms $A \times N \to P$, hence induce different measures on $A \times N$ from the same left Haar measure of P.

This means there exists group homomorphism $\beta : A \to \mathbb{R}_{>0}$ such that

$$\int_N f(n)d(ana^{-1}) = \int_N f(a^{-1}na)dn = \beta(a)\int_N f(n)dn. \tag{6}$$

Replacing a with a^{-1} and then n with na for (6), we find

$$\int_N f(an)dnda = \beta(a)^{-1}\int_N f(na)dnda.$$

This follows from (5) that

$$\int_P f(p)d_lp = \beta(a)^{-1}\int_{A\times N} f(na)dadn,$$

hence

$$\int_G f(g)dg = \int_{A\times N\times K} f(nak)\beta(a)^{-1}dadndk.$$

To find $\beta(a)$ satisfying (6). Note that

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}^{-1}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} = \begin{pmatrix}1 & \alpha^{-2}u\\ 0 & 1\end{pmatrix},$$

so by identifying $N \cong \mathbb{R}$, (6) becomes

$$\int_{\mathbb{R}} f(\alpha^{-2}u)du = \beta(\alpha)\int_{\mathbb{R}} f(u)du.$$

With this, we find $\beta(\alpha) = \alpha^2$, as desired. $\square$

Remark 120. In fact, the modular quasicharacter of P is $\delta_P(an) = \beta(a)^{-1} = \alpha^{-2}$. Indeed, as $dadn$ is left Haar measure of P so denoting $\Phi : A \times N \to P$ to be the homeomorphism sending $(a, n) \mapsto an$, by pushing forward $dadn$ via right multiplication by $a_1^{-1} \in A$, we find

$$\begin{aligned}\int_{A\times N} f(\Phi(a,n)a_1)dadn &= \int_{A\times N} f(ana_1)dadn,\\ &= \int_{A\times N} f(\Phi(aa_1, a_1^{-1}na_1))dadn,\\ &= \int_{A\times N} f(\Phi(a,n))\beta(a_1)dadn.\end{aligned}$$

Therefore, from Proposition 38(b), we find $\delta_P(a_1) = \beta(a_1)^{-1}$, as desired.

10.2.2. *Poisson summation.* Let f be a Schwartz function on $\mathbb{R}^n$ and let

$$\widehat{f}(y) = \int_{\mathbb{R}^2} f(x)e^{-2\pi i\langle x,y\rangle}dx$$

be its Fourier transform. The Poisson summation formula says

$$\sum_{x\in\mathbb{Z}^2} f(x) = \sum_{y\in\mathbb{Z}^2} \widehat{f}(y).$$

On the other hand, we consider $G = \mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{R}^2$ on the right. Let $f_g(x) = f(xg)$ for $g \in \mathrm{SL}_2(\mathbb{R})$ then

$$\begin{aligned}
\widehat{f_g}(y) &= \int_{\mathbb{R}^2} f(xg)e^{-2\pi\langle x,y\rangle}dx, \\
&= \int_{\mathbb{R}^2} f(x)e^{-2\pi i\langle xg^{-1},y\rangle}dx, \\
&= \int_{\mathbb{R}^2} f(x)e^{-2\pi i\langle x,y(g^{-1})^T\rangle}dx, \\
&= \widehat{f}(y(g^{-1})^T).
\end{aligned}$$

Apply Poisson summation to f_g gives

$$\sum_{x\in\mathbb{Z}^2} f(xg) = \sum_{y\in\mathbb{Z}^2} \widehat{f}(y(g^{-1})^T). \tag{7}$$

10.2.3. *Computation of volume of quotient.* Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We follow [Gar14]'s notes of Siegel's computation of volume of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$. We choose to normalise the Haar measure on G such that

$$\int_G f(g)dg = \int_P\int_K f(pk)dkdp$$

where the Haar measure on K having total measure 2π, and the left Haar measure dp on P is normalised as $dp = d(na) = \frac{dnda}{\alpha^2}$ from previous section. Here we choose the Haar measure on $N \cong \mathbb{R}$ to be $dn = du$ and the Haar measure on $A \cong \mathbb{R}_{>0}$ to be $da = \alpha^{-1}d\alpha$. Note that this normalisation agrees with our normalisation in the first method.

Consider an auxilary Schwartz function f on $\mathbb{R}^2$ and define a left Γ-invariant function F on G by

$$F(g) = \sum_{v\in\mathbb{Z}^2} f(vg).$$

We will determine the volume of $\Gamma \setminus G$ by evaluating

$$\int_{\Gamma\setminus G} F(g)dg$$

in two different ways.

Lemma 121. *The action of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2 - \{0\}$ by right multiplication is transitive, with orbits $\{(c,d) : \gcd(c,d) = \ell\}$ over $\ell \in \mathbb{Z}_{>0}$. The stabiliser of $(0,1)$ in Γ is $N_{\mathbb{Z}} = N \cap \Gamma = \left\{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} : x \in \mathbb{Z}\right\}$, giving a bijection*

$$\begin{aligned}
\mathbb{Z}_{>0} \times N_{\mathbb{Z}} \setminus \Gamma &\to \mathbb{Z}^2 - \{0\}, \\
\ell \times N_{\mathbb{Z}}\gamma &\mapsto \ell \cdot (0,1)\gamma.
\end{aligned}$$

Proof. Note that if $(c,d) \in \mathbb{Z}^2 - \{0\}$ so $\gcd(c,d) = \ell$ then $(c,d)\cdot \mathrm{SL}_2(\mathbb{Z}) \subset \{(x,y) : \gcd(x,y) = \ell\}$. Therefore, to show $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{Z}^2 - \{0\}$, it suffices to show that for any $(x,y),(z,t) \in \mathbb{Z}^2 - \{0\}$ such that $\gcd(x,y) = \gcd(z,t) = 1$ then there exists $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $(x,y) = (z,t)\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. As $\gcd(x,y) = 1$, there exists $x', y' \in \mathbb{Z}$ so $\begin{pmatrix} x' & y' \\ x & y\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, implying

$(x, y) = (0,1)\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}$. Similarly, $(z, t) = (0,1)\begin{pmatrix} z' & t' \\ z & t \end{pmatrix}$. Therefore, $(x, y) = (z, t)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equivalent to

$$(0,1) = (0,1)\begin{pmatrix} z' & t' \\ z & t \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}^{-1}.$$

One can find a, b, c, d by letting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} z' & t' \\ z & t \end{pmatrix}^{-1}\begin{pmatrix} x' & y' \\ x & y \end{pmatrix}.$$

From above, we also find that $\{(c, d) : \gcd(c, d) = \ell\}$ is in bijection with $\{\ell \cdot (0,1) \cdot N_{\mathbb{Z}} \setminus \Gamma\}$ where $N_{\mathbb{Z}}$ is stabiliser of $(0,1)$ in Γ. Thus, we obtain bijection $\mathbb{Z}_{>0} \times N_{\mathbb{Z}} \setminus \Gamma \to \mathbb{Z}^2 - \{0\}$, as desired. $\square$

We can rewrite the integral as

$$\begin{aligned}
\int_{\Gamma\setminus G} F(g)dg &= \int_{\Gamma\setminus G} \sum_{v\in\mathbb{Z}^2} f(vg)dg, \\
&= \int_{\Gamma\setminus G} f(0)dg + \int_{\Gamma\setminus G} \sum_{v\in\mathbb{Z}^2, v\neq 0} f(vg)dg, \\
&= f(0)\mathrm{vol}(\Gamma \setminus G) + \int_{\Gamma\setminus G} \sum_{\ell=1}^{\infty} \sum_{\gamma\in N_{\mathbb{Z}}\setminus\Gamma} f(\ell \cdot (0,1) \cdot \gamma g)dg, \\
&= f(0)\mathrm{vol}(\Gamma \setminus G) + \sum_{\ell=1}^{\infty} \int_{N_{\mathbb{Z}}\setminus G} f(\ell \cdot (0,1) \cdot g)dg.
\end{aligned}$$

As $N_{\mathbb{Z}} = P \cap \Gamma$ so $N_{\mathbb{Z}} \setminus G \cong N_{\mathbb{Z}} \setminus P \times K$. Because of our normalisation for Haar measure on G, the integral becomes

$$f(0)\mathrm{vol}(\Gamma \setminus G) + \sum_{\ell=1}^{\infty} \int_{N_{\mathbb{Z}}\setminus P} \int_K f(\ell \cdot (0,1) \cdot pk)dpdk$$

One can choose f on $\mathbb{R}^2$ such that it is right K-invariant, giving

$$f(0)\mathrm{vol}(\Gamma \setminus G) + 2\pi \sum_{\ell=1}^{\infty} \int_{N_{\mathbb{Z}}\setminus P} f(\ell \cdot (0,1) \cdot p)dp$$

since the total measure of K is 2π. Because of our choice of measure of P, the integral becomes

$$f(0)\mathrm{vol}(\Gamma \setminus G) + 2\pi \sum_{\ell=1}^{\infty} \int_{N_{\mathbb{Z}}\setminus N} \int_A f(\ell(0,1)na)\alpha^{-2}dnda.$$

Note that N stabilises $(0,1)$ and the Haar measure on $N \cong \mathbb{R}$ gives $\int_{N_{\mathbb{Z}}\setminus N} 1dx = \int_{\mathbb{R}/\mathbb{Z}} 1dx = 1$. Therefore, the whole integral is

$$\begin{aligned}
\int_{\Gamma\setminus G} F(g)dg &= f(0)\mathrm{vol}(\Gamma \setminus G) + 2\pi \sum_{\ell} \int_A f(\ell(0,1)a)\alpha^{-2}da, \\
&= f(0)\mathrm{vol}(\Gamma \setminus G) + 2\pi \sum_{\ell} \int_0^{\infty} f(\ell(0,1)\alpha^{-1})\alpha^{-2}\frac{d\alpha}{\alpha}, \\
&= f(0)\mathrm{vol}(\Gamma \setminus G) + 2\pi \sum_{\ell} \int_0^{\infty} f(0, \ell\alpha)\alpha d\alpha,
\end{aligned}$$

where we substitute α by α^{-1} on the last line. Substituting α by $\ell\alpha$, we find

$$\begin{aligned}\int_{\Gamma\backslash G} F(g)dg &= f(0)\mathrm{vol}(\Gamma\setminus G) + 2\pi\sum_{\ell}\ell^{-2}\int_0^\infty f(0,\alpha)\alpha d\alpha,\\ &= f(0)\mathrm{vol}(\Gamma\setminus G) + 2\pi\zeta(2)\cdot\int_0^\infty f(0,\alpha)\alpha d\alpha.\end{aligned}$$

As f is right K-invariant, we find

$$\int_0^\infty f(0,\alpha)\alpha d\alpha = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)dx = \frac{1}{2\pi}\widehat{f}(0).$$

Hence, we find

$$\int_{\Gamma\backslash G} F(g)dg = f(0)\cdot\mathrm{vol}(\Gamma\setminus G) + \zeta(2)\widehat{f}(0).$$

As Γ is stable under taking transpose and inverse, we can do a completely analogous computation with roles of f and $\widehat{f}$ reversed. Combining with (7), we obtain

$$f(0)\cdot\mathrm{vol}(\Gamma\setminus G) + \zeta(2)\widehat{f}(0) = \int_{\Gamma\backslash G} F(g)dg = \widehat{f}(0)\cdot\mathrm{vol}(\Gamma\setminus G) + \zeta(2)f(0).$$

We find

$$(f(0)-\widehat{f}(0))\cdot(\mathrm{vol}(\Gamma\setminus G)-\zeta(2)) = 0.$$

We can choose f such that $f(0)\neq\widehat{f}(0)$ to obtain $\mathrm{vol}(\Gamma\setminus G)=\zeta(2)$.

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