LEARNING MATHEMATICS

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ABSTRACT. To record many interesting things I learnt in case I forgot. To keep myself busy ...

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1. June 2021

Something I would like to get done this month:

(1) Learn how much 2-dimensional gauge theory of finite group G tells us about representation of G. This is firstly motivated by Frobenius formula that Nam showed few months ago, interpreting number $\mathrm{Hom}(\pi_1(\sum), G)$ as sum over irreducible representations of G where $\sum$ is (closed, compact?) Riemann surface. I then found a "topological" proof using topological field theory `https://math.berkeley.edu/~qchu/TQFT.pdf` and `https://upennig.weebly.com/uploads/7/4/0/3/74037187/2d-tqft.pdf` and `https://www.math.ru.nl/~mueger/TQFT/FQ.pdf`. In topological field theory language, there is a functor Z from category of 2-cobordisms to category of vector spaces. And in our situation, it sends a manifold M to $\mathrm{Map}(\mathrm{Bun}_G(M), \mathbb{C})$ where $\mathrm{Bun}_G(M)$ is the groupoid of principal G-bundles over M, which can then be identified with $\mathrm{Hom}(\pi_1(M), G)/G$ quotient by conjugation. Roughly in the proof of Frobenius using TQFT, somehow one can cut and glue M to get the desired formula. The further question is how much $Z(M)$ tells us about representation of G, when one varies the manifold M.

More to read from: "Bartlett Categorical aspects of topological quantum field theories" (arvix); `https://arxiv.org/abs/1705.05734v1` and Kock "Frobenius algebra and 2D topological quantum field theory" `https://www.mat.uab.cat/~kock/TQFT/FS.pdf` this has book version; `https://golem.ph.utexas.edu/category/2008/06/teleman_on_topological_constru.html`.

One can also ask why we choose $\mathrm{Bun}_G(M)$ as target for our cobordism functor and expect it to tell something about representations of G. I think this is because one can interpret representations of G as bundles of some sort (see wikipedia of "induced representations").

(2) For my thesis, I am trying to understand certain self-adjoint operator in Langlands' computation of volume of fundamental domain $G(\mathbb{Z}) \setminus G(\mathbb{R})$. I don't understand this operator and all and how it is linked to Eisenstein series.

(3) Gauge theory in representation theory, geometric representation theory: `https://people.maths.ox.ac.uk/tillmann/ASPECTSbenzvi.pdf` and `https://web.ma.utexas.edu/users/benzvi/GRASP/lectures/NWTFT/nwtft.pdf`, also `https://ncatlab.org/nlab/files/BenZviGeometricK` `pdf`.

Daily learning

02/06/2021 Today I learnt roughly what is a "rigid symmetric monoidal category" and how category of n-dimensional cobordisms nCob is one, following `https://arxiv.org/abs/q-alg/9503002`. To explain roughly, "monoidal" means the category has a product $\otimes$ operation, "symmetric" means we have a map $a \otimes b \to b \otimes a$, "rigid" means every object x has a dual object x^*. The main point to take away is that one can visualise the relations/commutative diagrams in the category via cobordisms, which make it easier to remember. For example, relations between counit and unit maps is seen as straightening the curve S (see p. 5). For more examples of this, see p. 21 of `https://arxiv.org/pdf/math/0512103v1.pdf`.

Another note, on p. 4, it mentions that relations between morphisms in nCob can be understood using Morse theory, where we can stratify a bordism N (i.e. a n-manifold N) by giving a Morse function on N to pick up critical points (something relates to handle decomposition in Morse theory). *I see some familiar words like "stratification" and "Morse function" when reading about symplectic geometry, would like to learn more about this at some point*

06/06/2021 I am trying to understand first few sections of `https://ncatlab.org/nlab/files/BenZviGeometricFunct` `pdf` (with the hope of getting to know more about `https://ncatlab.org/nlab/show/geometric+infinity-function+theory`). Here is what I have so far:

Given two sets X, Y with a G-action on these two and a G-equivariant map $\phi : X \to Y$. We can pullback to give a map ϕ^* of G-equivariant complex-valued functions $Fun_G(Y)$ on Y to that $Fun_G(X)$ on X. Pushforward $\phi_* : Fun_G(X) \to Fun_G(Y)$ is a bit more tricky. Firstly, it is better to view X, Y as groupoids, then

$$\phi^* : f \in Fun_G(X) \mapsto \left(y \mapsto \sum_{x \in |\phi^{-1}(y)|} \frac{f(x)}{\#\mathrm{Aut}_{\phi^{-1}(y)}(x)} \right).$$

Here $|X|$ refers to isomorphic classes of objects in groupoid X. Note $\phi^{-1}(y)$ is also a groupoid with natural automorphisms. *How to come up with this pushforward? What condition should a good pushforward satisfy? Usually pushforward is very nontrivial to realise, unlike pullback. Is there a general rule to come up with something like this?*

The two are adjoint in following sense: One can define inner product on $Fun_G(X)$ by $(f, g) = \sum_{x \in X} \frac{f(x)\overline{g(x)}}{\#\mathrm{Aut}(x)}$ then ϕ^* and ϕ_* are adjoint with respect to this inner product. *What is the relation of this with adjointness as functors? Do we have something like Frobenius reciprocity in representation theory, where adjointness in inner product is the same as adjointness as adjoint functors due to semisimplicity of representations? What constitutes a good inner product?*

Perhaps this would help: `https://golem.ph.utexas.edu/category/2007/03/canonical_measures_on_configur_1.html` or `https://golem.ph.utexas.edu/category/2011/09/universal_measures.html` or `https://golem.ph.utexas.edu/category/2008/07/news_on_measures_on_groupoids.html`.

06/06/2021 Regarding item 1, I managed to figure out how to describe $Z_G(\bigcirc) : Z_G(pt) \to Z_G(S^1)$ where $N = \bigcirc$ is a half-phere with boundaries pt (a point) and S^1. To do this, start with the more geometric correspondence

$$\mathrm{Bun}_G(pt) \longleftarrow \mathrm{Bun}_G(N) \longrightarrow \mathrm{Bun}_G(S^1)$$

obtained by restricting G-bundles to corresponding boundaries (i.e. pull back). This gives us morphism of groupoids

$$pt \xleftarrow{p} pt \xrightarrow{q} G/G.$$

Indeed, N is just a disk so $\mathrm{Bun}_G(N) = \mathrm{Bun}_G(pt) = pt$, a point with a G action on it. We know $\pi_1(S^1) = \mathbb{Z}$ so $\mathrm{Bun}_G(S^1) = \mathrm{Hom}(\mathbb{Z}, G)/G = G/G$, groupoid with elements in G and automorphisms are conjugations by G. Then p is the identity map, q sends to the identity 1 in G (as we have group hom $\pi_1(S^1) \to 1 = \pi_1(N)$ inducing $\mathrm{Bun}_G(N) \to \mathrm{Bun}_G(S^1)$).

By definition, $Z_G(N) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Bun}_G(N), \mathbb{C})$ so this gives $Z_G(N) := q_* \circ p^* : Z_G(pt) \to Z_G(S^1)$. Note $Z_G(pt) = \mathbb{C}, Z_G(S^1) = \mathrm{Hom}(G/G, \mathbb{C}) = \mathbb{C}[G]^G$ so from pushforward described in 06/06/2021 for groupoids, we find $Z_G(N)$ sends $\lambda \in \mathbb{C}$ to $g \mapsto \lambda \delta_{g,1}/|G|$ in $\mathbb{C}[G]^G$.

A remark: Bun_G is a geometric object, but doing computation it seems to be easier to deal with $\mathrm{Hom}(\pi_1(\cdot), G)/G$.

16/06/2021 (Continued from 06/06/2021) It seems the idea of TQFT has been applied to study character varieties $\mathrm{Hom}(\pi_1(\sum), G)$ by Angel Gonzalez Prieto in `https://arxiv.org/abs/1812.11575` (his PhD thesis) or `https://arxiv.org/abs/1810.09714` (a relevant paper) , `http://www.mat.ucm.es/~joseag12/investigacion/documentos/SeminarioTesis.pdf` (PhD presentation), `http://www.mat.ucm.es/~joseag12/investigacion/documentos/TFMJAngelGonzalez.pdf` (his master thesis),... It seems there are many things unexplored here.

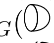
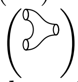
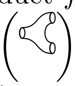
17/06/2021 (Continued from 06/06) Understand most of the computations in `https://math.berkeley.edu/~qchu/TQFT.pdf` and `https://upennig.weebly.com/uploads/7/4/0/3/74037187/2d-tqft.`

pdf. These two notes present a "topological" proof of

$$\frac{\#\operatorname{Hom}(\pi_1(\Gamma_g), G)}{|G|} = \frac{1}{|G|^{\chi(\Gamma_g)}} \sum_V (\dim V)^{\chi(\Gamma_g)}. \tag{1}$$

Here G is a finite group, Γ_g is a closed connected orientable surface of genus g, $\chi(\Gamma_g)$ is Euler characteristic of Γ_g, the sum is over all complex irreducible represention of G.

Here are the main computations:

- $Z_G($$) : Z_G(pt) \to Z_G(S^1)$. We did this on 06/06/2021. This map sends $\lambda \in \mathbb{C} = Z_G(pt)$ to $\lambda\delta_{g,1}/|G|$.
- $Z_G($$) : Z_G(S^1) \to Z_G(pt)$ sends $f \in Z_G(S^1) = \mathbb{C}[G]^G$ to $f(1)$.
- $Z_G\left(\right.$$\left.\right) : Z(S^1) \otimes Z(S^1) \to Z(S^1)$ sends $f \otimes g \in \mathbb{C}[G]^G \otimes \mathbb{C}[G]^G$ to the convolution product $f * g$ ...
- $Z_G\left(\right.$$\left.\right) : Z(S^1) \to Z(S^1) \otimes Z(S^1)$
- $Z_G(M_1 \sqcup_{M'} M_2) = Z(M_2) \circ Z(M_1)$ for surfaces M_1, M_2 with common boundary M'.
- As $Z_G(M) = \operatorname{Map}(\operatorname{Bun}_G(M), \mathbb{C})$ so if M is homotopic to M' then $Z_G(M) = Z_G(M')$.

From this, to compute $Z_G(\Gamma_g)$, it suffices to chop Γ_g into pieces and compute each piece, then compose everything together. On the other hand, $Z_G(\Gamma_g) = \operatorname{Hom}(\pi_1(\Gamma_g), G)/G$ so we can obtain (3).

We also have general formula when G is a Lie group (due to Witten 1991) or a quantum group (Rouchet-Szenes 2000) (I learnt this info from reading the slide https://www2.ist.ac.at/fileadmin/user_upload/group_pages/hausel/Aarhus07.pdf).

22/06/2021 (Things I would like to understand in a far way future) In this paper https://arxiv.org/pdf/1511.06271.pdf of Michael Groechenig claimed to give a generalisation to Weil's description of correspondence between groupoid of vector bundles on algebraic curve X (defined over algebraically closed field) and groupoid of the double quotient.

My impression of adeles is that it seems to be the right notion to study analysis on moduli space of bundles?

22/06/2021 (Things I would like to understand in a far way future) There seems to be some mysterious applications of p-adic integrations (and furthermore, motivic integrations) to various sorts of problems: equal Betti numbers of birational Calabi-Yau n folds https://www.math.uni-bonn.de/people/huybrech/Magni.pdf, Fundamental Lemma in Langlands program https://arxiv.org/abs/1810.06739, Topological Mirror Symmetry Conjecture by HauselThaddeus for smooth moduli spaces of Higgs bundles https://arxiv.org/abs/1707.06417v3.

22/06/2021 I attend lectures of Geordie Williamson about Spectra in representation theory and of Anna Romanov about Whittaker categories. I just want to write down what I understand (no matter how vague and imprecise it can be).

(a) In Geordie's talk: There are three ways to define cohomology $H^i(X, \mathbb{Z})$, one is via map $\triangle^n \to X$ ("maps in to X"), second is via Eilenberg-Maclane space $K(\mathbb{Z}, i)$ as $H^i(X, \mathbb{Z}) = [X, K(\mathbb{Z}, i)]$ (maps out of X), third is via constant sheaf $\underline{\mathbb{Z}}_X$, i.e. $H^i(X, \mathbb{Z}) = H^i(R\Gamma(X, \underline{\mathbb{Z}}_X))$ ("on X").
Geordie said that Grothendieck-Quillen's dream is that every generalised cohomology theory can be described as in the third way (i.e. "on X", without replying on $\triangle^n X$ or $K(\mathbb{Z}, i)$). He said Lurie has achieved this dream for K-theory. Then he goes to define spectra, which should be seen as an analogue of $\mathbb{Z}$. Then he mentioned that there is on going research trying to generalised Geometric Satake over spectra KU (instead of over

$\mathbb{Z}$) which gives the quantum group version on the RHS (instead of just representation of Langlands dual group).
What is spectra, KU rigorously? Do we have a visual(?) easier(?) explaination on why KU is an analogue of $\mathbb{Z}$? Why do we expect quantum groups on the RHS of geometric Satake, or is there some sort of deformation from $\mathbb{Z}$ to KU that explains this?
Some notes about spectra (other than Lurie ...): by Rok Gregoric `https://web.ma.utexas.edu/users/gregoric/Spectra%20Are%20Your%20Friends.pdf`, his summary of higher algebra `https://web.ma.utexas.edu/users/gregoric/Appendix.pdf`. I found a table giving the analogy between Lurie's theory and classical theory in `https://sites.duke.edu/scshgap/files/2018/05/Pandit-Imperial.pdf`.
Actually, there is an even elementary question I don't know how to answer, as I don't know anything about K-theory: Why K-theory is considered to be an upgraded version of cohomology theory? (or something along this question...)

(b) In Anna's talk: There is a lot of technical details going on so I didn't get much out of it. But here it is: A Whittaker module is the induced representation from the upper Borel $\mathfrak{n}$ to $\mathfrak{g}$ from some character of $\mathfrak{n}$. The motivation for Whittaker model is that one has an "explicit" (?) description of representation as functions on some space. The motivation for Whittaker module (I think explained by Masoud in the talk but I may need to hear it again) is that Whittaker module of $\mathrm{GL}_n(\mathbb{F}_q)$ gives almost all the irrep(?). *What is the motivation of Whittaker story here? In what sense it is a good generalisation of irreducible representations?*
Whittaker module contains all the finite-dimensional rep of $\mathfrak{g}$ then she studies category $\mathcal{N}$, roughly similar to category $\mathcal{O}$ but with simple objects being Whittaker modules.
Travis told me that the motivation of category $\mathcal{O}$ is that it is the nicest (?) category that contains finite-dimensional rep of $\mathfrak{g}$ and the Verma module. *Why Verma module? is it because all irrep of $\mathfrak{g}$ (even infinite one) can be described from Verma module ? What sort of results do people expect when studying this category? Something like characters of irreducible modules, Kazhdan Lusztig theory?.*

23/06/2021 I learnt about some geometric constructions of representation of a group G. Let's just say G is finite for simplicity. Let X be a space with an action of G.

To obtain a representation of G, one can linearise by considering a vector space $Fun(X)$ of complex-valued functions on X (one can also choose $Fun(X, V)$ of V-valued functions on X, where V is any vector space). Then G acts on $f \in Fun(X)$ by $(g \cdot f)(x) = f(g^{-1}x)$. We will explain a geometric analogue of this construction as follows. Again, we are given a space X with an action of G. The process of "linearising", i.e. associating X with $Fun(X)$, requires the language of vector bundles.

First, we recall notion of a vector bundle of rank n. It is a surjective map $\pi : V \to X$ satisfying local triviality condition: for every $x \in X$, there exists open neighborhood U of X containing x and a homeomorphism $\pi^{-1}(U) \to U \times \mathbb{C}^n$ compatible with projections to U, such that its restriction to x induces a vector space isomorphism $V_x := \pi^{-1}(x) \cong \{x\} \times \mathbb{C}^n$. Intuitively, a vector bundle associate each point $x \in X$ a vector space of dimension n. To see vector bundles as $Fun(X, \mathbb{C}^n)$, we consider its global section $\Gamma(X, V)$, which is a vector space because the fibers are vector spaces. Hence, one can think of vector bundle on X as vector-valued functions on X. Then line bundle (i.e. vector bundle of rank 1) is $\mathbb{C}$-valued functions on X.

Now, go back to the definition of G action on $Fun(X)$ and we want to interpret this in the language of vector bundles, i.e. how much information is needed to associate an action of G on sections $\Gamma(X, V)$ of a vector bundle $\pi : V \to X$? Note that if $s \in \Gamma(X, V)$ then $s(x) \in V_x$. Now we want to say something like "$(g \cdot s)(x) = s(g^{-1} \cdot x)$" but LHS is in V_x, while the RHS is in $V_{g,x}$. Hence, what we need is a linear isomorphism $g : V_x \to V_{g\cdot x}$ for every $g \in G$, so that we can write the action as $(g\cdot s)(x) = g\cdot s(g^{-1}x)$. This motivates the notion of G-equivariant vector bundles:

A G-equivariant vector bundle over X is a vector bundle $\pi : V \to X$ with an action of G on P such that π is G-equivariant and the induced map $g : V_x \to V_{gx}$ is a linear map (a priori this is just a bijection).

Thus, we have obtained a map

$$\{G\text{-equivariant vector bundles over } X\} \longrightarrow \{G\text{-representations}\}\,.$$

More questions:

(a) *How to motivate notion of equivariant sheaf as generalisation of equivariant vector bundle? See Chriss Ginzburg or Achar book, wikipedia.*

(b) *What choice of space X gives you irreducible representations or all representations? For example, in Bott-Weil theory, X is a flag variety (so why flag varieties? Masoud told me that on the Lie algebra version, irreducible representations appear as quotients of Verma module, which is the induced representation from the Borel $\mathfrak{b}$; hence we expect irreducible representations to appear in bundle over $X = G/B$, which is the geometric way to describe induced representation) then considering certain line bundles over it (why line bundles but not general vector bundles? Is it because the regular representation of G on $Fun(G,\mathbb{C})$ contains all irreducible representations?) give you irreducible representations. What about Ginzburg construction of irreducible representations?*

(c) *Is there a map going backwards? Starting from a G-representation, how to get a G-equivariant vector bundle over some space X. Some ideas: Given G-module V, then $V \times G \to V$ is a principal G-bundle. Try to obtain a vector bundle out of this (for example, using associated bundle construction, or replace G with vector space $Fun(G)$, as we know action of G on V induces action of $Fun(G)$ on V).*

(d) *What representations will appear if we consider higher cohomology groups? Global sections is H^0.*

From this perspective, the construction of induced representation is quite natural. Consider a representation $\rho : H \to \mathrm{GL}(V)$ of subgroup H of G. Then we can construct a G-equivariant vector bundle as follows. Note $\pi : G \to G/H$ is a principal H-bundle, so to get a vector bundle, we use associated bundle construction: define $G\times_\rho V = G\times V/\sim$ where $(g(g')^{-1}, v) \sim (g, \rho(g')v)$ then $G\times_\rho V \to G/H$ is a vector bundle with fiber V. Note that G acts on $G\times_\rho V$ by $g(g', v) = (gg', v)$, making it into a G-equivariant vector bundle over G/H. In our analogy, this corresponds to $Fun(G/H, V)$. Taking global sections should give us back to the (analytic?) construction of induced representation.

Question:

(a) *I am still a bit confused about associated bundle construction. In particular, would the following construction be the same as associated bundle construction: Given group hom $f : G \to H$ then we have morphism $Bf : BG \to BH$ of classifying spaces, which induces a map from principal G-bundles to principal H-bundles. Now if we let $H = \mathrm{GL}(V)$ then is this the same as associated bundle construction?*

(b) *Check the details of the induced representation construction above. More is said here `https://mathoverflow.net/q/5772/89665` and here `https://math.stackexchange.com/q/1704622/58951`.*

We also have the correspondence

$$\{\text{principal G-bundles over } X\}/G \longleftrightarrow \{\text{group hom } \pi_1(X) \to G\}/G.$$

I know one direction from left to right: a principal G-bundle over X corresponds to homotopy $\phi : X \to BG$, taking fundamental group induces $\pi_1(X) \to \pi_1(BG) = G$.

Another way to describe this without using BG: consider principal G-bundle $\pi : P \to X$. Pick $x \in X, p \in P$ so $\pi(p) = x$. Consider a closed loop $\gamma : [0,1] \to X$ in X at x, i.e. $[\gamma] \in \pi_1(X, x)$. Then there exists a unique lift of γ to a curve $\tilde{\gamma}$ on P starting at p (but not necessarily closed curve). Then as $\tilde{\gamma}(1) = q$ and $\tilde{\gamma}(0) = p$ have the same fiber over x, and π is a principal G-bundle, there exists $g \in G$ so $q = p \cdot g$. We define $\pi_1(X) \to G$ by sending $[\gamma]$ to g.

Questions:

(a) *What is the reverse direction of this correspondence?*
(b) *If I choose $G = \mathrm{GL}(V)$, $X = BG$ then the RHS is representation of G. Compare with previous correspondence, can I associate principal $\mathrm{GL}(V)$-bundle over BG with G-equivariant vector bundles this way?*
(c) *This seems to be irrelevant, but what is the connection between principal GL_n-bundle and vector bundle? How to get one from the other?*

See more connection between vector bundles and representation theory at `http://www.numdam.org/item/PMIHES_1961__9__23_0.pdf`*..*

23/06/2021 For a Lie group G, I learnt how to describe a connection of a principal G-bundle $\pi : P \to X$ as $\mathfrak{g}$-vaulued 1-form on P.

First, we can define a vertical tangent bundle $T^vP = \{(p, v_p) : d\pi(v_p) = 0\}$. This gives us short exact sequence $0 \to T^vP \xrightarrow{f} TP \xrightarrow{g} \pi^*TX \to 0$. A connection on $P \to X$ is a choice of splitting of this short exact sequence. We have three equivantly ways to describe this:

(a) A map $s : TP \to T^vP$ s.t. $s \circ f = \mathrm{id}_{T^vP}$.
(b) A map $t : TX \to TP$ s.t. $g \circ t = \mathrm{id}_{TX}$.
(c) A direct sum decomposition $TP = T^vP \oplus H$.

If we use (a), then note that T^vP can be identified with $P \times \mathfrak{g}$, i.e. $(T^vP)_p$ can be identified with $\mathfrak{g}$ via the linear isomorphism $v_p \mapsto \frac{d}{dt}\big|_{t=0} p \cdot \exp(tv_p)$.

What is the geometric intuition of a connection for principal G-bundle?

2. July 2021

Some more things to learn

1. Tamagawa numbers:
   (a) Finish editing your thesis.
   (b) Learn how to define Tamagawa number properly for semisimple simply connected group.
   (c) Compute Tamagawa for other groups, read Weil's book: integration along fibers of differential forms, ...
   (d) Learn Langlands' proof of Tamagawa: Fourier inversion `https://people.reed.edu/~jerry/311/mats.html`, Mellin transform `https://people.reed.edu/~jerry/361/lectures/mats.html` `https://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf`, Riemann-Zeta functions, spectral theory `https://www.math.nagoya-u.ac.jp/~richard/teaching/s2019/Operators.pdf`, `https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/specthm.pdf`, `https://en.wikipedia.org/wiki/Spectral_theorem`.
2. Geometry:
   (a) Learn about classification of vector bundles (Hatcher `http://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html`), characteristic classes (e.g. Tu's book, `https://web.ma.utexas.edu/users/a.debray/lecture_notes/u17_characteristic_classes.pdf`), connections, equivariant cohomology (Tu's book, Geordie's note, see folder) -¿ Read Atiyah, Bott paper.

Daily learning:

2.1. **07/07/2021: Kemp-Ness theorem.**

From 01/07 till 07/07, we have a workshop on Heron island `https://sites.google.com/view/hiwgrt/home` about Kempf-Ness theorem, which gives a link between Geometric Invariant Theory and Symplectic Geometry.

Here are some unresolved thoughts I have during this workshop:

2.1.1. $\mathrm{GL}_n(\mathbb{C})$ *acts by conjugation on* $\mathfrak{gl}_n(\mathbb{C})$. One of the main example is letting $G = \mathrm{GL}_n(\mathbb{C})$ acting on $\mathfrak{gl}_n(\mathbb{C})$ by conjugation then stability of the action can be described as follows. Let $0 \neq V \in \mathfrak{gl}_n(\mathbb{C})$ then V can be written as $V = D + N$ where D is a diagonalisable matrix and N is nilpotent.

1. V is unstable, i.e. $0 \notin \overline{G \cdot V}$ iff $D = 0 \neq N$,
2. V is polystable, i.e. $\overline{G \cdot V}$ is closed, iff $N = 0 \neq D$.
3. V is semistable ($0 \notin \overline{G \cdot V}$) but is not polystable (note polystable implies semistable as $V \neq 0$) iff $D \neq 0 \neq N$.

Here the topology of $\mathfrak{gl}_n(\mathbb{C})$ is the classical topology of $\mathbb{C}^{n^2}$. I can show the following:

1. If V is nilpotent then $0 \in \overline{G \cdot V}$. Indeed, because $G = \mathrm{GL}_n(\mathbb{C})$ acts on V by conjugation, every matrix is conjugate to an upper triangular matrix, we can assume V is upper triangular with 0's on the diagonal, $V = (a_{ij})$ where $a_{ij} = 0$ for $i \geq j$. Let $f(t)$ be a diagonal matrix whose (i,i)th entry is t^i. Then $f(t)Vf(t)^{-1} = (t^{i-j}a_{ij})_{1 \leq i,j \leq n}$. Hence, as $|t| \to \infty$ in $\mathbb{C}^\times$, $f(t)Vf(t)^{-1} \to 0$. Hence, $0 \in \overline{G \cdot V}$.
2. If V is not nilpotent then V is semistable, i.e. $0 \notin \overline{G \cdot V}$. Indeed, V has nonzero eigenvalue $\lambda \in \mathbb{C}$ and hence gVg^{-1} also has nonzero eigenvalue λ for any $g \in \mathrm{GL}_n(\mathbb{C})$. Because all norm on a finite dimensional vector space is equivalent, we can choose the operator norm $\|.\|$ on $\mathfrak{gl}_n(\mathbb{C})$, giving $\|gVg^{-1}\| \geq |\lambda| > 0$ for all $g \in \mathrm{GL}_n(\mathbb{C})$ (operator norm of a matrix is at least the spectral radius, i.e. largest eigenvalue, of that matrix). Thus $0 \notin \overline{G \cdot V}$.

I don't know how to show that if V is diagonalisable (hence we can assume V is a diagonal matrix) with at least one nonzero eigenvalue, then $G \cdot V$ is closed (and vise versa).

I can do for example when $n = 2$, $V = diag(\lambda, 0)$ with $\lambda \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} V \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{\lambda}{ad - bc} \begin{pmatrix} ad & -ab \\ cd & -bc \end{pmatrix}.$$

Then $G \cdot V = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{gl}_2(\mathbb{C}) : x_{11} + x_{22} = \lambda, x_{11}x_{22} = x_{21}x_{12} \right\}$, which is closed in $\mathfrak{gl}_2(\mathbb{C})$.

What is the moment map here? Can I draw a picture for this example, showing that the moment map is some kind of critical point of some norm?

2.1.2. *Torus acting on a vector space.* Another example is letting $T = (\mathbb{C}^\times)^n$ acting on a (finite dimensional?) complex vector space V, equipped with a Hermitian inner product. Then V can be decomposed as direct sum of orthogonal vector spaces $\bigoplus_{\chi \in X^*(T)} V_\chi$ where $V_\chi = \{v \in V : t \cdot v = \chi(t)v\}$. Let $v \in V$ and $v = \sum_{i=1}^r v_i$ where $v_i \in V_{\chi_i}$, where $\chi_1, \ldots, \chi_r \in X^*(T)$ are distinct characters. Note that each character can be identified with an element in $\mathbb{Z}^n$ as $X^*(T) \cong \mathbb{Z}^n$ by sending $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ to $(t_1, \ldots, t_n) \in (\mathbb{C}^\times)^n \mapsto \prod_{i=1}^n t_i^{m_i}$. Then the stability of the action can be described as follows:

(1) $v \in V$ is unstable, i.e. $0 \in \overline{T \cdot v}$, iff 0 does not lie in the convex hull of $\chi_1, \ldots, \chi_r$.
(2) v is polystable, i.e. $T \cdot v$ is closed, iff 0 lies in the interior of the convex hull of $\chi_1, \ldots, \chi_r$.

I can prove (a). It is essentially lies in the proof of Hilbert-Mumford criterion `https://www.isibang.ac.in/~sury/hilbmumf.pdf` by B.Sury that I read in order to present to the workshop.

Let me try to decode the definition of convex hull to see why it is essentially in B. Sury's paper. One definition is that it is the set $\{\sum_{i=1}^r b_i\chi_i \in \mathbb{R}^n : 0 \leq b_i \leq 1, \sum b_i = 1\}$. Another way to think about this is that 0 does not lie in the convex hull iff there exists a hyperplane passing through 0, i.e. $\sum_i c_i x_i = 0$, such that χ_i's lie on the same half-plane separated by that hyperplane, i.e. $\langle \chi_i, (c_1, \ldots, c_n) \rangle > 0$. With this, we can choose $t_\ell = (\ell^{c_1}, \ldots, \ell^{c_n})$ then $\|t_\ell \cdot v\|^2 = \sum_{i=1}^r |\ell|^{2\langle \chi_i, (c_1, \ldots, c_n) \rangle} \|v_i\|^2$, which goes to 0 as $\ell \to 0$. Conversely, if $0 \in \overline{T \cdot v}$ then this is Sury's proof of Hilbert Mumford criterion for $\mathrm{GL}_n(\mathbb{C})$. *Need to write this down in case I forgot.*

How to prove (b)? Can I draw a picture of this example?

Maybe two subexamples we can draw are

(1) $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by multiplication via $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. The orbits are $O_\lambda = \{(x, y) \in \mathbb{C}^2 : xy = \lambda\}$ for $\lambda \in \mathbb{C}^\times$, $\{(x, 0) : x \neq 0\}$ and $\{(0, y) : y \neq 0\}$. One can see that the quotient topology of this action is not Hausdorff. Ramiro mentioned that the moment map is then somesort of shortest distance to each orbit. *Need to work this out.* The non-closed orbits are the x,y-axes without the origin. If we throw out these two orbits, we expect to get a nicer topology when taking quotient. *What does this mean? Can you draw it?*
(2) $\mathbb{C}^\times$ acts on $\mathbb{C}^2$ by multiplication $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$. Then the orbits are $\{(0, 0)\}$ and lines through the origin but minus the origin. When we take out the bad orbit $((0, 0))$ we get $\mathbb{P}^1$.

2.1.3. *Kempf-Ness fancy version.* I don't think I have seen the relation between Kempf-Ness theorem in Kempf-Ness paper and the fancy version of this (i.e. a homeomorphism between some GIT quotient and symplectic reduction). I would like to learn this someday. For example, I don't know much about the process of throwing away bad orbits to get better quotient topology. *Maybe work out the examples above or read this paper `https://arxiv.org/pdf/math/0512411.pdf` in more details.* Geordie mentions that this Kempf-Ness paper essentially embeds in Atiyah-Bott paper and I would like to understand more what did he mean by this.

2.1.4. *Complexification, real vs complex.* This is about complexification, real form, compact form, $\mathbb{R}$ vs $\mathbb{C}$ structure, reductive etc... of Lie groups. During the workshop, we have done this for torus, i.e. what is the most natural way to get a compact torus from a complex torus and vise versa?

There are two ways to define compact torus. A non-canonical way is that it is a real Lie group diffeomorphic to $U(1)^n$ for some n, where $U(1) = S^1 = \{z \in \mathbb{C} : z\overline{z} = 1\}$. A canonical definition is that it is a connected compact abelian real Lie group.

There are also two ways to define complex torus. A noncanonical way is that it is a complex Lie group diffeomorphic to $(\mathbb{C}^\times)^n$. A canonical definition is that it is a connected reductive abelian complex Lie group (see how compact is replaced by reductive when comparing with compact torus definition, i.e. one can think of reductive complex Lie group as complexification of compact real Lie group).

From a complex torus, you can take its maximal compact real Lie subgroup, which will be unique up to conjugation (every Lie group has a unique maximal compact subgroup up to conjugation?). From a compact torus T then the exponential map $\exp : \mathfrak{t} \to T$ is surjective, i.e. $T = \exp(\mathfrak{t})$. Indeed, $\exp(t)$ is a subgroup of T (as T is abelian) and contains a neighborhood of the identity. Hence, T is a disjoint union of cosets of $\exp(\mathfrak{t})$. But as T is connected so $T = \exp(t)$. Hence, if $\Gamma = \ker(\exp)$ then T is diffeomorphic and group isomorphic to $\mathfrak{t}/\Gamma$. The complexification of T is then $T_\mathbb{C} = (\mathfrak{t} \otimes_\mathbb{R} \mathbb{C})/\Gamma$. This way of defining complexification seems to work only for torus. In general, it is more complicated. *How to define complexification in general?* Ramiro mentioned there is a universal property about this. Masoud told us another way to define complexification, which is based on the idea that categories of representations of compact Lie group and of its complexification are the same (?).

Rohin told me that real/compact form is some sort of reverse process of complexification. From a compact Lie group, you can get a complex Lie group unique in some sense, but the reverse process is not unique, and a real form refers to a "real"isation of that ...

In the first line of Kemp-Ness proof for theorem in section 4: For $\mathrm{GL}_n(\mathbb{C})$, one can give it an algebraic structure over $\mathbb{R}$ such that its real locus is $U(n)$. Indeed, write down the equation $AA^* = I$ of $U(n)$ as polynomials over $\mathbb{R}$. The claim is that over $\mathbb{C}$, this gives $\mathrm{GL}_n(\mathbb{C})$. For example, when $n = 1$, then $U(1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. And we can identify $\left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} : x^2 + y^2 = 1 \right\}$ being isomorphic to $\mathbb{C}^\times$ over $\mathbb{C}$. *Need to write this down. I have done this before but I forgot.*

2.2. **19/07/2021: Line bundles on $\mathbb{P}^1$.** Past few days I have been trying to learn about line bundles on $\mathbb{P}^1$, how to classify topological/ holomorphic/ algebraic line bundles on $\mathbb{P}^1$, how to describe their (global) sections, to compute their transition functions, how to draw some of them.

2.2.1. *Transition functions of* $\mathcal{O}(-1)$. First is the (topological) canonical line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$ over $\mathbb{C}$. Since each point in $\mathbb{P}^1$ corresponds to a line in $\mathbb{C}^2$, as a set $\mathcal{O}(-1) = \{(\ell, x) \in \mathbb{P}^1 \times \mathbb{C}^2 : x \in \ell\}$ with the obvious projection p to $\mathbb{P}^1$.

Now, I want to compute the transition functions of this line bundle. $\mathbb{P}^1$ can be thought abstractly as open cover of $U_0 = \{[x, y] \in \mathbb{P}^1 : x \neq 0\}$ and $U_1 = \{[x, y] : y \neq 0\}$. Each U_i is isomorphic to affine space $\mathbb{C}$, for example, $U_0 \cong \mathbb{C}$ via $[x, y] \mapsto y/x$.

We have an isomorphism $\pi_0 : p^{-1}(U_0) \to U_0 \times \mathbb{C}$ sending $([1 : z_\ell], (x, y))$ to $([1 : z_\ell], x)$. The inverse π_0^{-1} sends $([1 : z_\ell], c)$ to $([1 : z_\ell], c(1, z_\ell))$. Notice that here we have picked a representative $(1, z_\ell) \in \mathbb{C}^2$ of $\ell = [1 : z_\ell]$ to define π_0. The reason for this is that we want our map to be a homeomorphism. Visually, imagine $\mathcal{O}(-1)$ as collection of lines on $\mathbb{C}^2$ passing through 0. Choosing representatives for elements in U_0 as above means the representatives lie on the line $x = 1$ in $\mathbb{C}^2$, which guarantee continuity (i.e. if $|z_\ell| < 1$ and $|c| < 1$ then the image $c(1, z_\ell)$ is open in $\mathbb{C}^2$, looking like a paper fan).

Similarly, $\pi_1 : p^{-1}(U_1) \to U_1 \times \mathbb{C}$ sends $([z_\ell, 1], (x, y))$ to $([z_\ell, 1], y)$ and π_1^{-1} sends $([z_\ell, 1], c)$ to $([z_\ell, 1], c(z_\ell, 1))$.

The transition function $g_{01} : U_1 \cap U_0 \to \mathrm{GL}_1(\mathbb{C})$ is defined via

$$\begin{aligned}(U_1 \cap U_0) \times \mathbb{C} &\xrightarrow{\pi_0^{-1}} p^{-1}(U_1 \cap U_0) \xrightarrow{\pi_1} (U_1 \cap U_0) \times \mathbb{C} \\ (\ell = [1, z_\ell], c) &\mapsto ([1, z_\ell], c(1, z_\ell)) \mapsto ([1, z_\ell], cz_\ell)\end{aligned}$$

Hence, g_{01} sends $\ell = [1, z_\ell] \in U_1 \cap U_0$ to $(c \mapsto cz_\ell)$ in $\mathrm{GL}_1(\mathbb{C})$. Under identification of $U_1 \cap U_0$ with $\mathbb{C}^\times$ via indentification $U_0 \cong \mathbb{C}$, i.e. $\ell = [1, z_\ell] \mapsto z_\ell$ (we choose $U_0 \cong \mathbb{C}$ instead of $U_1 \cong \mathbb{C}$ because our map is $\pi_1 \circ \pi_0^{-1}$ so the domain $U_1 \cap U_0$ lies in U_0 originally) and $\mathrm{GL}_1(\mathbb{C})$ with $\mathbb{C}^\times$, g_{01} can be view simply as a map $\mathbb{C}^\times \to \mathbb{C}^\times$ sending z to z.

Similarly, one can check the other transition function $g_{10} : U_1 \cap U_0 \to \mathrm{GL}_1(\mathbb{C})$ defined by $\pi_0 \circ \pi_1^{-1}$ sends $z \mapsto z$, under identification of spaces $U_1 \cap U_0$ with $\mathbb{C}^\times$ via $U_1 \cong \mathbb{C}$. Let me spell this out. We have

$$\begin{aligned}(U_1 \cap U_0) \times \mathbb{C} &\xrightarrow{\pi_1^{-1}} p^{-1}(U_1 \cap U_0) \xrightarrow{\pi_0} (U_1 \cap U_0) \times \mathbb{C} \\ (\ell = [z_\ell, 1], c) &\mapsto ([z_\ell, 1], c(z_\ell, 1)) \mapsto ([z_\ell, 1], cz_\ell)\end{aligned}$$

Notice that it is true $g_{01}(\ell)g_{10}(\ell) = 1$ for all $\ell \in U_1 \cap U_0$, but upon correct identification of spaces, both these maps can be viewed as $z \mapsto z$ from $\mathbb{C}^\times$ to $\mathbb{C}^\times$.

This is the point that confuses me the most because it seems many other sources claiming that the transition map should send $z \mapsto z^{-1}$ (for example, this an this, p.30 while one reference agrees with my choice). The reason for this ambiguity seems to be because there are two ways to identify $U_1 \cap U_1$ with $\mathbb{C}^\times$, one via $U_0 \cong \mathbb{C}$ and the second via $U_1 \cong \mathbb{C}$ (see how I get g_{01} as $z \mapsto z$). To avoid ambiguity, the best way to phrase this is that g_{01} sends $[x_0, x_1] \to x_1/x_0$ and g_{10} sends $[x_0, x_1]$ to x_0/x_1 (later we will see that we can define line bundle $\mathcal{O}(k)$ with $k \in \mathbb{Z}$ with transition functions $g_{01}([x_0, x_1]) = (x_1/x_0)^k$).

One may then ask why name the bundle to be $\mathcal{O}(-1)$ instead of $\mathcal{O}(1)$? One reason, as we will later see that algebraic/holomorphic $\mathcal{O}(-1)$ has no nontrivial global section, while algebraic/holomorphic $\mathcal{O}(1)$ has global section being a $\mathbb{C}$-vector space of degree 1 homogeneous polynomials in two variables x_0, x_1. For more reasons, see `https://math.stackexchange.com/q/256482/58951`.

2.2.2. *$\mathcal{O}(-1)$ as a Mobius strip.* How do you visualise $\mathcal{O}(-1)$? I claim that you can draw this as an "infinite" Mobius strip. At least this is the picture over $\mathbb{R}$, but as a drawing, you can just pretend that any field k is just a line.

Before drawing it out, let me first explain why $\mathcal{O}(-1)$ over $\mathbb{R}$ is the infinite Mobius strip, i.e. $[0, 1] \times \mathbb{R}/\sim$ where $(0, t) \sim (1, -t)$. Over $\mathbb{R}$, every line in $\mathbb{R}^2$ intersects the circle S^1 at exactly two points, so we will define a map sending $x \in [0, 1]$ to $e^{\pi i x} \in S^1$ to indicate our line. Hence, $(x, t) \in [0, 1] \times \mathbb{R}$ is sent to a line ℓ in $\mathbb{R}^2$ passing through $e^{\pi i x}$ and the corresponding point $te^{\pi i x}$ on that line. To get a bijection, we need identification $(0, t) \sim (1, -t)$ because both represent the same point on the x-axis.

Now, let's try to draw this out for $1 < t < 2$. We would get fig. 1 where under $(0, t) \sim (1, -t)$ we need to glue the paper fan above along the arrows on the x-axis. As you let x goes from 0 to 1 to get back to the x-axis, by looking at the positive direction of each line, you can notice the twist.

Let me offer another explanation of $\mathcal{O}(-1)$ being the Mobius bundle. In the previous explanation, I take the definition of $\mathcal{O}(-1)$ as lines on $\mathbb{R}^2$, this time I want to view $\mathcal{O}(-1)$ abstractly via its transition functions. The goal would be the same, i.e. how to see the twist in $\mathcal{O}(-1)$, but I don't want to describe the homeomorphism explicitly (because it relies too much of the fact that we are in $\mathbb{R}$). If successful, I want an argument that works for any topological field.

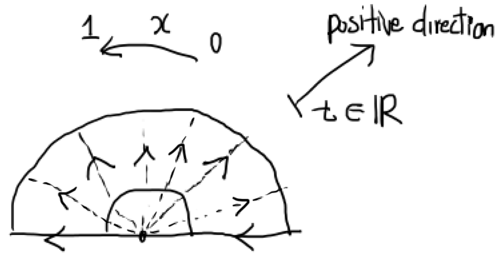


FIGURE 1. Visualise $\mathcal{O}(-1)$

Now, $\mathcal{O}(-1)$ is defined via its transition function $g_{01} : U_0 \cap U_1 \to \mathrm{GL}_1(\mathbb{R})$ sending $[x_0, x_1] \mapsto x_1/x_0$. This means that $\mathcal{O}(-1)$ is obtained by glueing $U_1 \times \mathbb{R}$ and $U_0 \times \mathbb{R}$ via $(U_1 \cap U_0) \times \mathbb{R} \to (U_1 \cap U_0) \times \mathbb{R}$ sending $([1, z], c) \mapsto ([1, z], zc)$. Now, I would draw $U_0 \cong \mathbb{R}$ as a circle with a point $[0, 1]$ removed and $U_1 \cong \mathbb{R}$ as a circle with a point $[1, 0]$ removed.

In U_0, on one side of $[1, 0]$ are points $[1, z]$ with $z < 0$ and on the other side are those $[1, z]$ with $z > 0$. On each fiber of $U_0 \times \mathbb{R} \to U_0$, we choose the positive directions as pointing outwards from the circle. This choice is possible because we are working over $\mathbb{R}$. Now we look at those fibers in the circle of U_1 via the glueing $([1, z], c) \mapsto ([1, z], zc)$, keeping in mind of the positive direction of each fiber (see fig. 2).

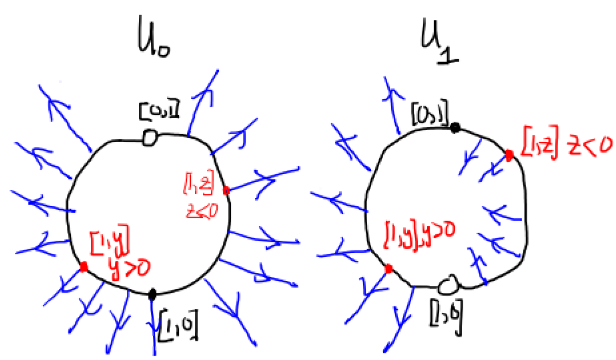


FIGURE 2. Visualise $\mathcal{O}(-1)$

With this, we observe that we have twisted one connected component of $(U_0 \cap U_1) \times \mathbb{R}$ by half.

Is there an easier way to see why $\mathcal{O}(-1)$ is not the trivial line bundle $\mathbb{P}^1 \times \mathbb{C}$? I would say that intuitively, because all lines/fibers of $\mathcal{O}(-1)$ "have a common point 0", while for $\mathbb{P}^1 \times \mathbb{C}$, the fibers do not intersect each other. *How to rigorously describe this phenomenon?*

2.2.3. *Dual line bundle* $\mathcal{O}(1)$. The line bundle $\mathcal{O}(1)$ is the dual line bundle to $\mathcal{O}(-1)$. What does this mean? One could guess that it means the fibers of $\mathcal{O}(1)$ are the dual spaces to the fibers of $\mathcal{O}(-1)$. But this is not a good enough description. To describe this vector bundle, we need to determine its transition functions.

In general, the dual vector bundle E^* of $E \to V$ is the vector bundle whose fibers are the dual spaces to the fibers of E. From this information, I will (heuristically) derive that the most natural vector bundle E^* that has this property is the one that has transition functions $g^*_{ij} = (g^T_{ij})^{-1}$. Because $\pi_j \circ \pi_i^{-1} : \{u\} \times \mathbb{C}^n \xrightarrow{\sim} E_u \xrightarrow{\sim} \{u\} \times \mathbb{C}^n$ is an isomorphism of vector spaces so $f_{ij} := \pi_j^{-1} \circ (g_{ij}u) \circ \pi_i \in \mathrm{GL}(E_u)$. We want the transition functions for E^*, i.e. how to cook up $\{u\} \times \mathbb{C}^n \xrightarrow{\sim} E^*_u \xrightarrow{\sim} \{u\} \times \mathbb{C}^n$? This is not possible as we don't know local trivialisations of E^*, but we can guess $f^*_{ij} : E^*_u \xrightarrow{\sim} \{u\} \times \mathbb{C}^n \xrightarrow{\sim} E^*_u$ in $\mathrm{GL}(E^*_u)$, i.e. the most natural one is $(\phi \in E^*_u) \mapsto ((v \in E_u) \mapsto \phi(f^{-1}_{ij} v))$. Notice I put f^{-1}_{ij} instead of f_{ij} because f^*_{ij} needs to satisfy the cocycle condition $f^*_{jk} f^*_{ij} = f^*_{ik}$. We will show that $f^*_{ij} = (f^{-1}_{ij})^T$ (note that for $\phi \in \mathrm{GL}_(V)$ then ϕ^T means $\phi^T \in \mathrm{GL}(V^*)$ sending $f \in V^*$ to $f \circ \phi$) and therefore implying $g^*_{ij} = (g^{-1}_{ij})^T$.

2.2.4. *Line bundles* $\mathcal{O}(k)$. We define $\mathcal{O}(k)$ to be $|k|$-times tensor product of $\mathcal{O}(1)$ if $k > 0$ or $\mathcal{O}(-1)$ if $k < 0$. We want to compute the transition functions of these bundles.

For vector bundles E_1, E_2 over V with transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_m(\mathbb{C}^m), f_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C}^n)$, the transition functions of $E_1 \otimes E_2$ are $g_{ij} \otimes f_{ij} : U_i \cap U_j \to \mathrm{GL}_{mn}(\mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^n)$.

We show $\mathcal{O}(-1) \otimes \mathcal{O}(1)$ is the trivial line bundle $\mathbb{P}^1 \otimes \mathbb{C}$. The transition function of $\mathcal{O}(-1) \otimes \mathcal{O}(1)$ is $g_{01} \otimes g^*_{01} : U_0 \cap U_1 \to \mathrm{GL}_1(\mathbb{C})$, sending $[x_0, x_1]$ to $\frac{x_1}{x_0} \otimes \frac{x_0}{x_1} = 1$ and similarly for the other transition function.

The transition function of $\mathcal{O}(2)$ is $g_{01} \otimes g_{01}$, sending $[x_0, x_1]$ to $(x_1/x_0)^2$. Now, if you try to draw out $\mathcal{O}(2)$ as I did for $\mathcal{O}(-1)$, you will notice that topologically, it is just the trivial line bundle $\mathbb{P}^1 \times \mathbb{R}$. Similarly, $\mathcal{O}(1)$ over $\mathbb{R}$ is also just a Mobius bundle. In fact, over $\mathbb{P}^1$ (or S^1), there are just two topological line bundles up to isomorphism!

Can I interpret $\mathcal{O}(k)$ where k means number of twists? Just like $\mathcal{O}(-1)$ is the Mobius strip over $\mathbb{R}$ (and I think $\mathcal{O}(1)$ over $\mathbb{R}$ is also a Mobius strip)? See `https://math.stackexchange.com/q/220203/58951` or `https://math.stackexchange.com/q/2219539/58951` for example. For example, can you draw $\mathcal{O}(2)$ or $\mathcal{O}(-2)$?.

2.2.5. *Only two topological line bundles over* $\mathbb{P}^1(\mathbb{R})$. Read `https://ayoucis.wordpress.com/2014/12/12/line-bundles-on-the-circle/`.

2.2.6. *What about topological line bundles over other fields?*

2.2.7. $\mathcal{O}(-1)$ *as an algebraic line bundle over* $\mathbb{C}$. So far, we have describe $\mathcal{O}(-1)$ as a topological line bundle, i.e. a continuous map $p : \mathcal{O}(-1) \to \mathbb{P}^1$ with local trivialisations $\pi_0 : p^{-1}(U_0) \xrightarrow{\sim} U_0 \times \mathbb{C}$ and $\pi_1 : p^{-1}(U_1) \xrightarrow{\sim} U_1 \times \mathbb{C}$ that commutes with projections to U_0, U_1, respectively. Furthermore, these local trivialisations must induce isomorphisms of vector spaces $p^{-1}(\ell) \cong \{\ell\} \times \mathbb{C}$ of each fiber of p.

How do we describe $\mathcal{O}(-1)$ as an algebraic/holomorphic/smooth/etc line bundle? `https://math.stackexchange.com/q/3481443/58951`.

2.2.8. $\mathbb{P}^1$ *as a scheme.* Before, we describe $\mathbb{P}^1$ abstractly as a topological space over $\mathbb{C}$, in fact the construction works over any field k, by glueing two k's, namely $\mathbb{A}_0 = k$ and $\mathbb{A}_1 = k$. This is glued via $\mathbb{A}_0 \setminus \{0\} = k^\times \to k^\times = \mathbb{A}_1 \setminus \{0\}$ sending $z \mapsto z^{-1}$. In particular, U is open in $\mathbb{P}^1$ if $U \cap \mathbb{A}_0$ and $U \cap \mathbb{A}_1$ are both open.

Now we wish to equip $\mathbb{P}^1$ with an extra structure, i.e. $\mathbb{P}^1$ as a scheme. I will explain how to construct $\mathbb{P}^1$ over field k as a sheaf of rings by glueing two affine pieces $\mathbb{A}_0 = \operatorname{Spec} k[t_0]$ and $\mathbb{A}_1 = \operatorname{Spec} k[t_1]$. By gluing, I mean we have two do two things: glue two topological spaces $\mathbb{A}_0$ and $\mathbb{A}_1$ and then glue their structure sheaves $\mathcal{O}_{\mathbb{A}_1}$ and $\mathcal{O}_{\mathbb{A}_0}$.

Compute global section of $\mathcal{O}_{\mathbb{P}^1}$.

2.2.9. *Classifying algebraic vector bundles on* $\mathbb{P}^1$ *via double cosets.* Algebraic vector bundles on $\mathbb{P}^1$ of degree n are classified via the transition functions $g_{ij} : \operatorname{Spec} k[t, t^{-1}] \to \operatorname{Spec} \mathcal{O}(\mathrm{GL}_n)$ which corresponds to a point in $\mathrm{GL}_n(k[t, t^{-1}])$ ...

2.2.10. $\mathcal{O}(2)$ *as tangent bundle of* $\mathbb{P}^1$.

2.2.11. *Global sections of algebraic bundles.*

2.3. **23/07/2021: Categorial measure theory.** Masoud mentioned that I only defined push-forward measures but not pullback measures in my thesis. When I googled, I found this explanation `https://mathoverflow.net/q/122704/89665` saying there is no pullback of general maps. One needs extra conditions, such as being able to integrate on fibers. There is also a way to develop measure theory categorically with pushforward and pullback available this way, from `https://mathoverflow.net/a/20820/89665`. But the language is quite forgein to me. *Need to read more at some point.*

2.4. **23/07/2021: Sheaf of solutions of ODE is a local system.** I want to claim that local existence and uniqueness of a differential equation is the same as saying that its sheaf of solutions is a constant sheaf.

We have the following result for local existence and uniqueness of ODEs (cited from `https://www.math.utah.edu/~milicic/Eprints/de.pdf`, theorem 1.2): Let Ω be a simply connected region in $\mathbb{C}$, $z_0 \in \Omega$ and $A : \Omega \to \mathrm{GL}_n(\mathbb{C})$ a holomorphic map. For any $Y_0 \in \mathbb{C}^n$, there exists a unique holomorphic function $Y : \Omega \to \mathbb{C}^n$ such that $\frac{dY}{dz} = AY$ in Ω, and $Y(z_0) = Y_0$.

First, we can construct a sheaf $\mathcal{F}$ of solutions over Ω for this equation by letting $\mathcal{F}(U)$ to be set of all $Y : U \to \mathbb{C}$ satisfying the ODE. The claim is that $\mathcal{F}$ is a constant sheaf $\mathbb{C}^n$ (note that because Ω is simply connected so any continuous $\Omega \to \mathbb{C}^n$ with $\mathbb{C}^n$ having the discrete topology must be a constant function, meaning $\underline{\mathbb{C}^n}_{\Omega}(\Omega) = \mathbb{C}^n$). Indeed, $\mathcal{F}(\Omega) \to \mathbb{C}^n$ sending $Y \mapsto Y(z_0)$.

Now, if Ω is not simply connected then $\mathcal{F}$ is a $\mathbb{C}$-local system, i.e. Ω is union of its connected components and $\mathcal{F}$ restricted to each component is a constant sheaf, as shown below.

2.5. **23/07/2021: Shrawan Kumar's SMRI talk.** I attend Shrawan Kumar'S talk "Root components for tensor product of affine Kac-Moody Lie algebra modules" for the Sydney Mathematical Research Institute. Here are something news I learnt:

(1) Some history about tensor product decomposition problem for finite/affine/Kac-Moody Lie algebras.
(2) There is a problem that I think I can work out the proof: For integral dominant weight λ, μ, let $V(\lambda), V(\mu)$ be the corresponding highest weight reps. By complete irreducibility, we can decompose $V(\lambda) \otimes V(\mu)$ as direct sum of $V(\nu)$. Let $n^{\nu}_{\lambda,\mu}$ be the multiplicity of $V(\nu)$ in $n^{\nu}_{\lambda,\mu}$. The problem is that:

    Show, if $n^{\nu}_{\lambda,\mu} \neq 0$ then $n^{m\nu}_{m\lambda,m\mu} \neq 0$ for all positive integer m.

    Kumar claimed that there are two proofs of this, one use standard representation theoretic method and the second via Borel-Weil theorem, and I would like to know how to prove this for the above two methods.
(3) Do some google on "ample line bundle" as this terminology is mentioned in the talk. Roughly, a line bundle L on a proper [1] scheme X over field k is ample if $L^{\otimes n}$ has enough global sections to give a closed immersion (i.e. closed embedding) $X \to \mathbb{P}^N$ where $N = \dim H^0(X, L^{\otimes n}) - 1$. From what I know, this definition is important in order to classify algebraic varieties, i.e. describe a variety with certain properties as subvariety of certain projective space defined by equations of certain degrees. See `https://www.math.ucla.edu/~totaro/papers/public_html/algebraic.pdf` for example.

    Now, let me try to explain more about the part of "having enough global sections gives a morphism $X \to \mathbb{P}^N$". I read this from wikipedia `https://en.wikipedia.org/wiki/Ample_line_bundle`. Choose global sections $a_0, \ldots, a_{N-1} \in H^0(X, L)$ then we can defined $f : X \to \mathbb{P}^N$ sending $x \mapsto [a_0(x), \ldots, a_{N-1}(x)]$. Note that this is well-defined if over any $x \in X$, at least one of $a_i(x)$ is non-zero, i.e. intersection of zero sets of all global sections is empty. This is what is called "basedpoint-free" line bundle. "Semi-ample" line bundle

[1] finite dimensional of global section of line bundles on X?

L is when $L^{\otimes n}$ is basedpoint-free for some n. "Very-ample L is when $X \to \mathbb{P}^N$ is a closed immersion. "Ample" L is when $L^{\otimes r}$ is very-ample. So it's a bunch of definitions. *Workout the examples in wikipedia: $\mathcal{O}(d)$ on $\mathbb{P}^1$ is based-point free iff $d \geq 0$, and very ample iff $d \geq 1$.*

What is the role of being "ample" in representation theory, Kumar mentioned that $L(\lambda)$ as ample line bundle corresponding to a dominant weight λ, what does this mean?

2.6. **27/07/2021: Discriminant and different of field extension.** Let A be a Dedekind domain with field of fraction K, L/K is a finite separable extension and B is integral closure of A in L. Recall a prime $\mathfrak{q}|\mathfrak{p}$ of L is unramified if $e_{\mathfrak{q}} = 1$ and $B/\mathfrak{q}$ is separable extension of $A/\mathfrak{p}$. A prime $\mathfrak{p}$ of K is unramified if every prime $\mathfrak{q}|\mathfrak{p}$ lying above it are unramified.

In this notes, we will define *different* $\mathcal{D}_{B/A}$ and *discriminant* $D_{B/A}$ and show that these encodes information about ramification of L/K. In particular, we would like to explain the following:

The different is B-ideal that is divisible by the ramified primes $\mathfrak{q}$ of L, and the discriminant is A-ideal that is divisible by the ramified primes $\mathfrak{p}$ of K. The valuation $v_{\mathfrak{q}}(\mathcal{D}_{B/A})$ will give us information about the ramification index $e_{\mathfrak{q}}$ and its exact value when $\mathfrak{q}$ is tamely ramified.

This is created to summarised Lecture 12 in `https://math.mit.edu/classes/18.785/2019fa/lectures`.

2.6.1. *The different.* First, we need to define these two objects. We have trace pairing $L \times L \to K$ defined by $(x, y) \mapsto \mathrm{Trace}_{L/K}(xy)$. When L/K is separable, this is a perfect pairing (i.e. it induces K-module isomorphism L with $L^\vee = \mathrm{Hom}_K(L, K)$). B is a A-lattice in L (i.e. finitely generated A-module that spans L as K-vector space) and we have a corresponding *dual lattice* for B, defined as

$$B^* := \{x \in L : \mathrm{Trace}_{L/K}(xb) \in K \ \forall b \in B\}.$$

It is an A-lattice in L isomorphic to dual A-module $M^\vee := \mathrm{Hom}_A(M, A)$. One can show $B^* \in \mathcal{I}_B$ ($\mathcal{I}_B$ is the *ideal class group* of B, i.e. group of invertible fractional ideals of B, i.e. finitely generated B-submodules lying of L; here fractional ideal I being invertible means $IJ = B$ for some fractional ideal J). We define the *different* $\mathcal{D}_{L/K}$ of L/K (or the different $\mathcal{D}_{B/A}$ of B/A) to be the inverse of B^* in $\mathcal{I}_B$. Explicitly, we have

$$\mathcal{D}_{L/K} := \mathcal{D}_{B/A} := (B^*)^{-1} = \{x \in L : xB^* \subset B\}.$$

Note $B \subset B^*$ as $\mathrm{Trace}_{L/K}(ab) \in A$ for any $a, b \in B$, we find that the different is an ideal of B, not just fractional ideal.

Different respects localisation and completion:

- Let S be multiplicative subset of A. Then $S^{-1}\mathcal{D}_{B/A} = \mathcal{D}_{S^{-1}B/S^{-1}A}$. To prove this, it suffices to show inverses and duals commutes with localisation.
- Let $\mathfrak{q}|\mathfrak{p}$ be a prime of B. Then $\mathcal{D}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}} = \mathcal{D}_{B/A}\hat{B}_{\mathfrak{q}}$ as $\hat{B}_{\mathfrak{q}}$-ideals. Here $\hat{B}_{\mathfrak{q}}$ and $\hat{A}_{\mathfrak{p}}$ are completions of B and A at $\mathfrak{q}, \mathfrak{p}$, respectively.

2.6.2. *The discriminant.* Let $n := [L : K]$. For B an A-lattice in L, we can define the *discriminant of L/K (or of B/A)* to be the A-module $D_{L/K}$ (or $D_{B/A}$) generated by

$$\mathrm{disc}(x_1, \ldots, x_n) := \det[\mathrm{Trace}_{B/A}(x_i x_j)]_{ij} \in A$$

where $x_1, \ldots, x_n \in B$. This is infact a fractional ideal of A. When B is free A-lattice in L (such as when $A = \mathbb{Z}$) then $D_{B/A}$ is a principal fractional ideals, generated by $\mathrm{disc}(e_1, \ldots, e_n)$ where $e_1, \ldots, e_n$ is K-basis for L in B.

Depending on the situations, we have few ways to compute the discriminant:

- Let Ω/K be field extension for which there are discint $\sigma_1, \dots, \sigma_n \in \mathrm{Hom}_K(L, \Omega)$ then for any $e_1, \dots, e_n \in L$, we have
$$\mathrm{disc}(e_1, \dots, e_n) = \det[\sigma_i(e_j)]_{ij}^2,$$
- For polynomial $f(x) = \prod_i (x - \alpha_i)$ of degree n, then the discriminant of extension $A[x]/(f)$, where α is the image of x in $A[x]/(f)$, is generated by
$$\mathrm{disc}(1, \alpha, \dots, \alpha^{n-1}) = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$
People also define this to be the *discriminant of f*.

Discriminant also respects localisation and completion:

- For S multiplicative subset of A then $S^{-1} D_{B/A} = D_{S^{-1}B/S^{-1}A}$.
- For prime $\mathfrak{q}|\mathfrak{p}$ of B then *Need to learn more ... Check Serre after this.*

2.7. **28/07/2021: Representations of $\mathfrak{sl}_2$.** Recall on 23/07/2021, I found the following question on Kumar's talk that he mentioned as an "easy observation" (see `https://youtu.be/gph8XNkpdBM?t=368`):

Problem 1. *Show that if $V(\nu)$ appears in the direct sum decomposition of $V(\lambda) \otimes V(\mu)$ into irreducible representations, then $V(m\nu)$ appears in the direct sum decomposition of $V(m\lambda) \otimes V(m\mu)$ for any positive integer m.*

As a first step to answer a question, I just want to review the construction of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ via Verma modules.

The Lie algebra $\mathfrak{sl}_2$ has basis $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ with relations $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.

The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{GL}(\mathfrak{g})$ taking $(x \in \mathfrak{g}) \mapsto (\mathrm{ad}\ x : y \mapsto [x, y])$ gives us the root system Φ for $\mathfrak{g}$ and the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. For $\mathfrak{sl}_2$, we find $\mathfrak{sl}_2 = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f$ where $\alpha \in \Phi = \{\pm\alpha\} \subset \mathfrak{h}^*$ is defined by $\alpha(h) = 2$ (because $[h, e] = 2e$). We choose α to be a simple root, so $\Phi^+ = \{\alpha\}$.

Next, I want to determine the coroots, (co)weights. In order to do this, I first need an identification between $\mathfrak{h}$ and $\mathfrak{h}^*$. We have the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ sending $\kappa(a, b) = \mathrm{Trace}(\mathrm{ad}\ a \circ \mathrm{ad}\ b)$ which is nondegenerate when restricting to $\mathfrak{h} \times \mathfrak{h}$. This gives us an inner product on $\mathfrak{h}$. In our case of $\mathfrak{sl}_2$, we find $\kappa(h, h) = 8$.

Because of nondegeneracy of κ, we have an isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$ sending $\lambda \mapsto t_\lambda \in \mathfrak{h}$ defined by $\lambda(h) = \kappa(t_\lambda, h)$. Hence, we can turn κ into an inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*$ by $(\alpha, \beta) := \kappa(t_\alpha, t_\beta)$. In the literatures, there are two ways to define coroots. One is Bourbaki style, where the coroots lying inside $\mathfrak{h}$ while in Humphreys' style, the coroots lying in $\mathfrak{h}^*$. In Bourbaki's style, the coroot h_λ of $\lambda \in \Phi$ (in Bourbaki, this notation is $\lambda^\vee$, but we will save this for Humphreys style) is the unique element in $\mathfrak{h}$ such that $\lambda(h_\lambda) = 2$ (in particular, $h_\lambda = 2t_\lambda/(t_\lambda, t_\lambda)$). In Humphreys style, coroot of $\lambda \in \Phi$ is $\lambda^\vee := \frac{2\lambda}{(\lambda,\lambda)}$. One can show that $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} = \alpha(h_\beta) \in \mathbb{Z}$ for all $\beta \in \Phi$. In our case of $\mathfrak{sl}_2$, with simple root $\alpha \in \Phi$, we find $t_\alpha = \frac{h}{4}$ as $\alpha(h) = 2$. Hence $h_\alpha = h$ and $\alpha^\vee = \alpha$.

From now on, we will stick to Humphreys style. We can then define the (integral) weight lattice by
$$\Lambda := \{\alpha \in \mathfrak{h}^* | \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}\ \forall \beta \in \Phi\}.$$
In the case of $\mathfrak{sl}_2$, we find $\Lambda = \frac{1}{2}\mathbb{Z}\alpha$. Let $\omega = \alpha/2$ then ω is the fundamental weight of α, i.e. $\langle \omega, \alpha^\vee \rangle = 1$. Upon a choice of positive roots Φ^+, we can define the dominant (integral) weight lattice $\Lambda^+ := \{\alpha \in \mathfrak{h}^* | \langle \alpha, \beta^\vee \rangle \in \mathbb{Z}_{>0}\ \forall \beta \in \Phi^+\}$. In the case of $\mathfrak{sl}_2$, we find $\Lambda^+ = \mathbb{Z}_{>0}\omega$.

Next, we will construct the Verma module $M(\lambda)$ for every $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \mathfrak{n}$ to be the Borel subalgebra corresponding to the Cartan subalgebra $\mathfrak{h}$. In $\mathfrak{sl}_2$, $\mathfrak{b} = \mathbb{C}h \oplus \mathbb{C}e, \mathfrak{n} = \mathbb{C}e, \mathfrak{n}^- = \mathbb{C}f$. We define $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ where $\mathbb{C}_\lambda$ is a 1-dimensional representation of $\mathfrak{b}$ with a trivial action by $\mathfrak{n}$ and $\mathfrak{h}$ acts on $\mathbb{C}_\lambda$ by λ. Concrely, let $v^+ := 1 \otimes 1 \in M(\lambda)$ then $M(\lambda)$ has $\mathbb{C}$-basis $U(\mathfrak{n}^-)v^+$, with $\mathfrak{n} \cdot v^+ = 0$ and $h \cdot v^+ = \lambda(h)v^+$. For the case $\mathfrak{sl}_2$, $M(\lambda) = \operatorname{span}_{\mathbb{C}}\{f^i v^+\}$, $e \cdot v^+ = 0$ and $h \cdot v^+ = \lambda(h)v^+$. Letting $v_i := \frac{f^i v^+}{i!}$ for $i = 0, 1, \ldots$ then we can show that (here we have abused of notation to write λ for $\lambda(h)$):

$$\begin{aligned} h \cdot v_i &= (\lambda - 2i)v_i, \\ e \cdot v_i &= (\lambda - i + 1)v_{i-1}, \\ f \cdot v_i &= (i+1)v_{i+1}. \end{aligned}$$

$M(\lambda)$ always has a maximal proper submodule $L(\lambda)$ with quotient being a simple module $V(\lambda)$. $V(\lambda)$ is finite dimensional iff $\lambda \in \Lambda^+$. To see this for the case of $\mathfrak{sl}_2$, we observe:

(1) If $\lambda \notin \mathbb{Z}_{\geq 1}$ then $\lambda - i + 1 \neq 0$ for all $i = 1, \ldots,$ implying $M(\lambda)$ is irreducible. Indeed, starting with any $v \in M(\lambda)$, one can keep applying e to get v_0.
(2) If $\lambda \in \mathbb{Z}_{\geq 1}$ then $e \cdot v_{\lambda+1} = 0$. This means $U(\mathfrak{g}) \cdot v_{\lambda+1} = \operatorname{span}_{\mathbb{C}}(v_{\lambda+1}, v_{\lambda+2}, \ldots)$ is a maximal submodule of $M(\lambda)$ that is isomorphic to $M(-\lambda-2) \cong V(-\lambda-2)$. Its quotient $V(\lambda)$ has dimension $\lambda + 1$ and is irreducible. In $V(\lambda)$, we have $f^{\langle \lambda, \alpha^\vee \rangle + 1} v_0 = f^{\lambda+1} v_0 = 0$.

Now, coming back to Kumar's problem on 23/07/2021:

Problem 2. *Show that if $V(\nu)$ appears in the direct sum decomposition of $V(\lambda) \otimes V(\mu)$ into irreducible representations, then $V(m\nu)$ appears in the direct sum decomposition of $V(m\lambda) \otimes V(m\mu)$ for any positive integer m.*

I want to do an example for $\mathfrak{sl}_2$, based on the description of $V(\lambda)$ above.

Example 3. For $\mathfrak{sl}_2$, I will show that $V(1) \otimes V(2) = V(3) \oplus V(1)$. From the problem, I then should have $V(2)$ in $V(2) \otimes V(4)$.

Denote $V(m) := \operatorname{span}_{\mathbb{C}}\{v_{im} : i = 0, 1 \ldots\}$ where the v_{im}'s is defined above. Then $v_{01} \otimes v_{02}$ is a highest weight vector in $V(1) \otimes V(2)$ of weight 3, implying $V(3)$ appears in $V(1) \otimes V(2)$. Because $V(1) \otimes V(2)$ has dimension $2 \times 3 = 6$, $V(3)$ has dimension 4 so it can only be $V(1) \otimes V(2) = V(3) \oplus V(1)$. In fact, we can describe $V(1)$ explicitly as a submodule of $V(1) \otimes V(2)$. Note that $v_{01} \otimes v_{12}$ and $v_{11} \otimes v_{02}$ have weight 1. When applying e to both of these, we find a highest weight vector $v_{01} \otimes v_{12} - 2v_{11} \otimes v_{02}$ of weight 1.

We can play the same game to see how $V(2)$ appears in $V(2) \otimes V(4)$. In particular, $v_{i2} \otimes v_{j4}$ for $i + j = 2, 0 \leq i \leq 2, 0 \leq j \leq 4$ are weight vectors of weight 2 in $V(2) \otimes V(4)$. By applying e to these vectors:

(1) $e(v_{02} \otimes v_{24}) = (4 - 2 + 1)v_{02} \otimes v_{14}$,
(2) $e(v_{12} \otimes v_{14}) = (2 - 1 + 1)v_{02} \otimes v_{14} + (4 - 1 + 1)v_{12} \otimes v_{04}$,
(3) $e(v_{22} \otimes v_{04}) = (2 - 2 + 1)v_{12} \otimes v_{04}$.

Hence, $2v_{02} \otimes v_{24} - 3v_{12} \otimes v_{14} + 12v_{22} \otimes v_{04}$ is a highest weight vector of weight 2.

However, I don't know how to see $V(2)$ lying inside $V(2) \otimes V(4)$ from the fact that $V(1)$ lies inside $V(1) \otimes V(2)$.

Maybe it will help if I go a bit more general, suppose $V(c)$ appears in $V(a) \otimes V(b)$ for $\mathfrak{sl}_2$. This means there exists a highest weight vector of weight c in $V(a) \otimes V(b)$. Such vector has the form

$\sum_{i+j=(a+b-c)/2} c_{ij} v_{ia} \otimes v_{jb}$. For this to be a highest weight vector, we need

$$\begin{aligned} 0 &= e \cdot \left(\sum_{i+j=(a+b-c)/2} c_{ij} v_{ia} \otimes v_{jb} \right), \\ &= \sum_{i+j=\ell} c_{ij} \left((a-i+1) v_{i-1,a} \otimes v_{j,b} + (b-j+1) v_{i,a} \otimes v_{j-1,b} \right), \ \ell = (a+b-c)/2 \\ &= \sum_{i+j=\ell-1} \left(c_{i+1,j}(a-i) + c_{i,j+1}(b-j) \right) v_{i,a} \otimes v_{j,b}. \end{aligned}$$

To show $V(mc)$ appears in $V(ma) \otimes V(mb)$ for some positive integer m, I want to find a weight vector $\sum_{i+j=m\ell} d_{ij} v_{i,ma} \otimes v_{j,mb}$ of weight mc in $V(ma) \otimes V(mb)$, where $d_{ij} \in \mathbb{C}, \ell = m(a+b-c)/2$, such that it is of highest weight, i.e. for all $i+j = m\ell - 1$ then $d_{i+1,j}(ma-i) + d_{i,j+1}(mb-j) = 0$.

Now this is where I got stuck for the $\mathfrak{sl}_2$ case ...

2.8. **More unresolved questions.**

2.8.1. *How to identify function as a Riemann surfaces.*

2.8.2. *Functional equation of Riemann zeta function. What is the proof of the functional equation of the Riemann zeta function using Poisson summation formula.* I read the proof from Terence Tao's blog `https://terrytao.wordpress.com/2008/07/27/tates-proof-of-the-functional-equation/`, `https://math.bu.edu/people/jsweinst/Teaching/MA843/TatesThesis.pdf` or `https://people.reed.edu/~jerry/361/lectures/mats.html` (the continuation and functional equation sections).

Need to explain Mellin transform is Fourier transform on $(\mathbb{R}_{>0}, \cdot)$ *by transfering the usual Fourier transform via* $\exp : \mathbb{R} \to \mathbb{R}_{>0}$. To learn more, read `https://people.reed.edu/~jerry/361/lectures/mats.html` (the Mellin transform section), `https://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf`, `https://mathoverflow.net/q/79868/89665`.

Relate this with Tate's thesis, read Fourier analysis on Number fields, `https://math.stackexchange.com/q/25090/58951`.

2.8.3. *Grothendieck's proof of classification of line bundles on* $\mathbb{P}^1$. See Marielle Ong's `https://drive.google.com/file/d/1yfe9lTjF48a0UiZJqEqb8PvRY5yUC2Os/view` or Sabin Cautis notes Vector bundles on Riemann surfaces.

See more things about $\mathbb{P}^1$ at `https://math.berkeley.edu/~qchu/Notes/256B.pdf` or `https://math.stanford.edu/~vakil/725/class21.pdf`.

2.8.4. *Fourier transform.* I want to first explain that Poisson transform is some sort of change of basis formula, i.e. given $f(x)$, we want to write it with respect to some basis $e^{2\pi i x}$ (this choice of basis is invariant under translation in some sense) and this is what the Fourier transform indicates .... *Where can I read more something along this line?* See Jacob Lurie `https://www.youtube.com/watch?v=w3f8KEcv4RE&t=2497s`.

2.8.5. *Algebraic groups.* I just want to verify the following facts: SO_n is connected semisimple but is not simply connected.

3.1. 04/08/2021: Tamagawa number for GL_1 over $\mathbb{Q}$. I will define Tamagawa measure for GL_1 over $\mathbb{Q}$ and then compute its Tamagawa number.

The ideles $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^\times$ is the restricted product of $\mathbb{Q}_v^\times$'s with respect to its compact open $\mathbb{Z}_v^\times$. We have a norm map

$$|\cdot| : \mathbb{A}^\times \to \mathbb{R}_{>0}$$

defined by sending $a = (a_v) \in \mathbb{A}^\times$ to $\prod_v |a_v|_v$. The product is well-defined because $a_v \in \mathbb{Z}_v^\times$ for almost all places v of $\mathbb{Q}$. Its kernel is denoted $\mathbb{A}^1$, which is closed in $\mathbb{A}^\times$. In fact, we have a homeomorphism

$$\begin{aligned} \phi : \mathbb{A}^\times &\to \mathbb{A}^1 \times \mathbb{R}_{>0}, \\ a = (a_v) &\mapsto ((a_\infty/|a|, a_2, a_3, \dots), |a|), \\ ra &\leftarrow\!\shortmid (a, r). \end{aligned}$$

Proof that ϕ is a homeomorphism. The map $(a, r) \mapsto ra$ from $\mathbb{A}^1 \times \mathbb{R} > 0 \to \mathbb{A}^\times$ is continuous because $\mathbb{A}^\times$ is a topological ring with multiplication map being continuous, and that the two maps $\mathbb{A}^1 \hookrightarrow \mathbb{A}^\times$ and $\mathbb{R}_{>0} \hookrightarrow \mathbb{A}^\times$ are continuous.

We just need to show its inverse ϕ is also continuous. $\mathbb{A}^1$ has basis of open sets

$$U_S := \left\{ a = (a_v) \in U_\infty \times \prod_{p \in S \setminus \{\infty\}} b_p(1 + p^{k_p}\mathbb{Z}_p) \times \prod_{v \notin S} \mathbb{Z}_v^\times : \prod_{v \in S} |a_v|_v = 1 \right\}.$$

where S is a finite set of places of $\mathbb{Q}$ containing the infinite place. Let U be an open subset in $\mathbb{R}_{>0}$. Then the preimage $\phi^{-1}(U_S \times U)$ is

$$U_\infty U \times \prod_{p \in S \setminus \{\infty\}} b_p(1 + p^{k_p}\mathbb{Z}_p) \times \prod_{v \notin S} \mathbb{Z}_v^\times$$

which is open. $\square$

To define the Tamagawa measure $\mu_{\mathrm{GL}_1, \mathbb{Q}}$ on $\mathbb{A}^\times$, we choose a left-invariant differential form $\omega = x^{-1}dx$ on $\mathrm{GL}_1(\mathbb{Q})$. It is left-invariant because left-multiplication by $a \in \mathbb{Q}$ gives $L_a(x^{-1}dx) = (ax)^{-1}d(ax) = x^{-1}dx$. Over each completion of $\mathbb{Q}$, this induces a left-invariant Haar measure $\mu_{\mathrm{GL}_1(\mathbb{Q}_v), \omega} = d|\omega|_v$ on $\mathbb{Q}_v^\times$ by integrating over ω. It is left-invariant because we have the change of variables formula, even over $\mathbb{Q}_p$, i.e.

$$\int_{\mathbb{Q}_v^\times} f(x)|x|_v^{-1} d|x|_v = \int_{\mathbb{Q}_v^\times} f(x) d|\omega|_v = \int_{\mathbb{Q}_v^\times} f(ax) d|L_a\omega|_v = \int_{\mathbb{Q}_v^\times} f(ax)|x|_v^{-1} d|x|_v,$$

where $f(x)$ is a complex-valued continuous function with compact support on $\mathbb{Q}_v^\times$, $d|x|_v$ is the Haar measure on $\mathbb{Q}_v$. Here $\mathbb{Q}_\infty^\times$ means $\mathbb{R}^\times$. For example, we can compute

$$\mu_{\mathbb{Q}_p^\times, \omega}(\mathbb{Z}_p^\times) = \int_{\mathbb{Z}_p^\times} |x|_p^{-1} d|x|_p = \int_{\mathbb{Z}_p^\times} d|x|_p = \sum_{i=1}^{p-1} \int_{i + p\mathbb{Z}_p} d|x|_p = (p-1) \int_{p\mathbb{Z}_p} d|x|_p = \frac{p-1}{p}.$$

Let the Tamagawa measure on $\mathbb{A}^\times$ over $\mathbb{Q}$ to be essentially the product measure

$$\mu_{\mathbb{R}^\times, \omega} \times \prod_p \left(1 - \frac{1}{p}\right)^{-1} \mu_{\mathbb{Q}_p^\times, \omega}$$

In other words, there is a unique Haar measure on $\mathbb{A}^\times$ such that over the open set $\prod_{v \in S} U_v \times \prod_{v \notin S} \mathbb{Z}_v^\times$ where S is a finite set of places of $\mathbb{Q}$ containing the infinite place, the measure is by taking the product of measures of each local part.

Because of the homeomorphism $\mathbb{A}^\times \cong \mathbb{A}^1 \times \mathbb{R}_{>0}$, if we give $\mathbb{R}_{>0}$ the natural Haar measure obtained by integrating $x^{-1}dx$, this determines a Haar measure da_1 on $\mathbb{A}^1$ satisfying

$$\int_{\mathbb{A}^\times} f(x)\mu_{\mathrm{GL}_1,\mathbb{Q}}(x) = \int_{\mathbb{R}_{>0}} \int_{\mathbb{A}^1} f(a_1 t)t^{-1}da_1 dt.$$

As $\mathbb{Q}^\times$ is discrete in $\mathbb{A}^1$, this induces a measure $\mu'_{\mathrm{GL}_1,\mathbb{Q}}$ on $\mathbb{Q}^\times \setminus \mathbb{A}^1$. We define the Tamagawa number for GL_1 over $\mathbb{Q}$ to be

$$\tau(\mathrm{GL}_{1,\mathbb{Q}}) := \int_{\mathbb{Q}^\times \setminus \mathbb{A}^1} \mu'_{\mathrm{GL}_1,\mathbb{Q}}.$$

To compute $\tau(\mathrm{GL}_{1,\mathbb{Q}})$, we will determine a fundamental domain for $\mathbb{Q}^\times \setminus \mathbb{A}^1$, based on the following proposition:

Proposition 4. *We have*

(a) *$\mathbb{Q}^\times$ is dense in $\mathrm{GL}_1(\mathbb{A}^\infty)$, implying $\mathbb{Q}^\times \setminus (\mathbb{A}^\times)^\infty / \widehat{\mathbb{Z}}^\times = \{1\}$.*
(b) *We have a homeomorphism $\mathbb{Q}^\times \setminus \mathbb{A}^\times \cong (\{\pm 1\} \setminus \mathbb{R}^\times) \times \widehat{\mathbb{Z}}^\times$.*

Proof. (a) We need to show every basis of open sets of $\mathrm{GL}_1(\mathbb{A}^\times)$ contains an element in $\mathbb{Q}^\times$. A basis of open sets of $\mathrm{GL}_1(\mathbb{A}^\times)$ consists of $\prod_{p\in S\setminus\{\infty\}} a_p(1+p^{k_p}\mathbb{Z}_p) \times \prod_{p\notin S\cup\{\infty\}} \mathbb{Z}_p^\times$, where S is a finite set of places of $\mathbb{R}$ containing the infinite place, $k_p \in \mathbb{Z}_{\geq 1}$ (as then $1+p^{k_p}\mathbb{Z}_p$ is open neighborhood of 1 in $\mathbb{Q}_p^\times$), $a_p \in \mathbb{Q}^\times$ (for any $a_p \in \mathbb{Q}_p^\times$, you can always find $a'_p \in \mathbb{Q}^\times$ so $a_p - a'_p \in p^{k_p}\mathbb{Z}_p$). By Chinese Riemander Theorem, there exists $q \in \mathbb{Q}^\times$ such that $q \equiv a_p \pmod{p^{k_p}}$ for all $p \in S \setminus \{\infty\}$ and q only has primes $p \in S \setminus \{\infty\}$ in its prime factorisation. This implies q lies in the desired open set.

To see $(\mathbb{A})^\infty = \mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times$, as $\widehat{\mathbb{Z}}^\times$ is compact open in $(\mathbb{A})^\times$, any $a \in (\mathbb{A})^\infty$ then the open set $a\widehat{\mathbb{Z}}^\times$ must contains an element in $\mathbb{Q}^\times$, as desired.

(b) We define the map

$$\begin{aligned}\phi : (\{\pm 1\} \setminus \mathbb{R}^\times) \times \widehat{\mathbb{Z}}^\times &\to \mathbb{Q}^\times \setminus \mathbb{A}^\times, \\ (r + \{\pm 1\}, z) &\mapsto rz.\end{aligned}$$

Note that $\{\pm 1\} = \widehat{\mathbb{Z}}^\times \cap \mathbb{Q}^\times$, one can then show that ϕ is indeed a homeomorphism. Be careful that $\widehat{\mathbb{Z}}^\times$ and $\mathbb{Q}^\times$ are embedded differently into $\mathbb{A}^\times$. ☐

From this proposition, as $\mathbb{A}^1 \cong \mathbb{A}^\times/\mathbb{R}_{>0}$ so $\mathbb{Q}^\times \setminus \mathbb{A}^1 \cong \mathbb{Q}^\times \setminus \mathbb{A}/\mathbb{R}_{>0} \cong \widehat{\mathbb{Z}}^\times$. Thus, we have

$$\begin{aligned}\tau(\mathrm{GL}_{1,\mathbb{Q}}) &= \int_{\widehat{\mathbb{Z}}^\times} \mu_{\mathrm{GL}_1,\mathbb{Q}}, \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{-1} \int_{\mathbb{Z}_p^\times} \mu_{\mathbb{Q}_p^\times,\omega}, \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{-1} \frac{p-1}{p}, \\ &= 1.\end{aligned}$$

Now, something I would like to learn next:

(1) How to write the above description but for GL_n.
(2) What happen over other number fields/function fields?

3.2. **07/08/2021: Global left-invariant top form of SL_2.** GL_2 has a global left-invariant top form $\det(x_{ij})^{-2}dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22}$. I want to use this and the map $\mathrm{SL}_2 \times \mathbb{G}_m \to \mathrm{GL}_2$ to find a global left-invariant top form of SL_2.

3.2.1. *Sheaf of differentials.* We want to minic the following construction in differential geometry to algebraic geometry language: Given a smooth map $\phi : X \to Y$ of smooth manifolds, this induces map $\phi^* : T^*Y \to T^*X$ of cotangent bundles.

I read this from Neron models book, chapter 2, p. 33.

First, we will describe the sheaf of differentials (i.e. the cotagent bundles in differential geometry) $\Omega_{X/k}$ for an affine scheme $X = \operatorname{Spec} A$. To do this, I will need to introduce the *module of differentials* $\Omega_{A/k}$. It is an A-module equipped with a k-derivation $d : A \to \Omega_{A/k}$ (i.e. a k-linear map such that $d(fg) = fd(g) + gd(f)$ for $g, f \in A$) such that it is universal among those A-modules M with A-derivation $d_M : A \to M$. The *sheaf of differentials* $\Omega_{X/k}$ is then the sheaf of $\mathcal{O}_X$-modules corresponding to the module of differential $\Omega_{A/k}$. In particular, over open set $D(f)$ of X where $f \in A$, its sections are $\Gamma(D(f), \Omega_{X/k}) = (\Omega_{A/k})_f$.

We start with $\phi : X = \operatorname{Spec} A \to Y = \operatorname{Spec} B$ a morphism of affine schemes over k. This gives us a ring map $B \to A$ and a morphism $\mathcal{O}_Y \to \phi_*\mathcal{O}_X$ of sheaves of rings over Y [2].

I claim that we have a morphism

$$\Phi : \phi_*\Omega_{Y/k} \to \Omega_{X/k}$$

of sheaves of $\mathcal{O}_X$-modules that resembles $T^*Y \to T^*X$ in differential geometry.

I will describe $\phi_*\Omega_{Y/k}$ first. It is a sheaf of $\mathcal{O}_X$-modules obtained by pulling back $\Omega_{Y/k}$ along $\phi : X \to Y$. Its sections over open U of X forms a $\mathcal{O}_X(U)$-module $\Omega_{Y/k}(\phi^{-1}(U)) \otimes_{\mathcal{O}_Y(\phi^{-1}(U))} \mathcal{O}_X(U)$. In other words, it is a sheaf of $\mathcal{O}_X$-modules corresponding to the A-module $\Omega_{B/k} \otimes_B A$.

Thus, to describe Φ, we just need to know a map $\Omega_{B/k} \otimes_B A \to \Omega_{A/k}$ of A-modules.

By composing the ring map $B \to A$ with d_A, we get a k-derivation map corresponding to the B-module $\Omega_{A/k}$, hence by universal property of $\Omega_{B/k}$, this induces a map of B-modules

$$\begin{aligned} \Omega_{B/k} &\to \Omega_{A/k} \\ f d_B(g) &\mapsto \phi(f) d_A(\phi(g)),\ f, g \in B. \end{aligned}$$

We then also have a morphism of A-modules

$$\Omega_{B/k} \otimes_B A \to \Omega_{A/k}.$$

3.2.2. *Global left-invariant top form of* GL_2. I want to determine a left-invariant global differential form of top degree for GL_2.

Firstly, the module of differential $\Omega_{\mathcal{O}(\mathrm{GL}_2)/k}$ is the $\mathcal{O}(\mathrm{GL}_2)$-module generated by dx_{ij}, dt for $1 \le i, j \le 2$, modulo the relation

$$0 = d(t(x_{11}x_{22} - x_{21}x_{12}) - 1) = td(x_{11}x_{22} - x_{21}x_{12}) + (x_{11}x_{22} - x_{12}x_{21})dt.$$

As in $\mathcal{O}(\mathrm{GL}_2)$, $(x_{11}x_{22} - x_{12}x_{21})t = 1$ so we can write

$$dt = -t^2 d(x_{11}x_{22} - x_{21}x_{12}). \tag{2}$$

Consider left-multiplication by $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}_2(k)$ via

$$\begin{aligned} L_a : \mathcal{O}(\mathrm{GL}_2) &\to \mathcal{O}(\mathrm{GL}_2), \\ x_{ij} &\mapsto a_{i1}x_{1j} + a_{i2}x_{2j}, \\ t &\mapsto (a_{11}a_{22} - a_{12}a_{21})^{-1}t. \end{aligned}$$

[2]At some point, I was confused on whether the arrow should be $\mathcal{O}_Y \to \phi_*\mathcal{O}_X$ or the other way around. To convince myself on this, just view $\mathcal{O}_Y$ as functions on Y and $\mathcal{O}_X$ as functions on X, and we know a function on Y induces a function on X by precomposing with ϕ.

This induces a morphism of $\mathcal{O}(\mathrm{GL}_2)$-modules

$$L_a : \bigwedge^2 \Omega_{\mathcal{O}(\mathrm{GL}_2)/k} \to \bigwedge_{i,j=1}^2 \Omega_{\mathcal{O}(\mathrm{GL}_2)/k},$$

$$f \bigwedge_{i,j=1}^2 dx_{ij} \mapsto L_a(f) \bigwedge_{i,j=1}^2 (a_{i1}dx_{1j} + a_{i2}dx_{2j}).$$

Note that we only need to specify what $\bigwedge_{ij} dx_{ij}$ is sent to because dt is determined from (2). I want to find $f \in \mathcal{O}(\mathrm{GL}_2)$ so $L_a(f \bigwedge_{i,j} dx_{ij}) = f \bigwedge_{ij} dx_{ij}$, or

$$L_a(f)\det(a_{ij})^2 dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} = f dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22}.$$

Thus, we want $f \in \mathcal{O}(\mathrm{GL}_2)$ such that $L_a(f)\det(a_{ij})^2 = f$ for all $a \in \mathrm{GL}_2(k)$. Note that $f = t^2$ satisfies this.

3.2.3. *Global left-invariant top form of* SL_2. Now, I want to focus on an example $\phi : \mathrm{SL}_2 \times \mathbb{G}_m \to \mathrm{GL}_2$, defined by

$$\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, t\right) \mapsto \begin{pmatrix} tx_{11} & tx_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

From this map and the corresponding left-invariant top form of GL_2, I want to obtain a left-invariant global top differential form of SL_2.

We first have an isomorphism $\phi' : \mathcal{O}(\mathrm{GL}_2) \to \mathcal{O}(\mathrm{SL}_2) \otimes \mathcal{O}(\mathbb{G}_m)$ of k-algebras given by

$$\begin{aligned} \phi' : k[x_{ij}, t]/(t(x_{11}x_{22} - x_{21}x_{22}) - 1) &\to k[x_{ij}]/(x_{11}x_{22} - x_{21}x_{12} - 1) \otimes_k k[t, t^{-1}], \\ x_{11} &\mapsto tx_{11}, \\ x_{12} &\mapsto tx_{12}, \\ x_{21} &\mapsto x_{21}, \\ x_{22} &\mapsto x_{22}, \\ t &\mapsto t^{-1}. \end{aligned}$$

The module of differentials $\Omega_{\mathcal{O}(\mathrm{SL}_2)/k}$ is the $\mathcal{O}(\mathrm{SL}_2)$-module generated by dx_{ij} for $1 \le i, j \le 2$, modulo the relation

$$0 = d(x_{11}x_{22} - x_{12}x_{21} - 1) = x_{11}dx_{22} + x_{22}dx_{11} - x_{12}dx_{21} - x_{21}dx_{12}.$$

And $\Omega_{\mathcal{O}(\mathbb{G}_m)/k}$ is the $\mathcal{O}(\mathrm{GL}_1)$-module generated by dt with the k-derivation $dt^{-1} := -t^{-2}dt$.

The module of differentials $\Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k}$ is the $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)$-module generated by dx_{ij} for $1 \le i, j \le 2$ and dt modulo the relation $d(x_{11}x_{22} - x_{21}x_{12}) = 0$, with the differential $dt^{-1} := -t^{-2}dt$. In particular, one can show that

$$\Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k} \cong \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \otimes_{\mathcal{O}(\mathrm{SL}_2)} \mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m) \oplus \Omega_{\mathcal{O}(\mathbb{G}_m)/k} \otimes_{\mathcal{O}(\mathbb{G}_m)} \mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m),$$

which we will write $\Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k} \cong \Omega_{\mathcal{O}(\mathrm{SL}_2)/k} \oplus \Omega_{\mathcal{O}(\mathbb{G}_m)/k}$ for convenience. This induces an isomorphism of $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)$-modules

$$\bigwedge^4 \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k} \cong \Omega_{\mathcal{O}(\mathbb{G}_m)/k} \wedge \bigwedge^3 \Omega_{\mathcal{O}(\mathrm{SL}_2)/k}.$$

From ϕ', we have a morphism of $\mathcal{O}(\mathrm{GL}_2)$-module

$$\Omega_{\mathcal{O}(\mathrm{GL}_2)/k} \to \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k},$$

inducing a morphism of $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)$-module

$$\Omega_{\mathcal{O}(\mathrm{GL}_2)/k} \otimes_{\mathcal{O}(\mathrm{GL}_2)/k} (\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)) \to \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k}.$$

This is the map obtained by taking global sections of

$$\Phi : \phi_* \Omega_{\mathrm{GL}_2/k} \to \Omega_{\mathrm{SL}_2 \times \mathbb{G}_m/k}.$$

With this, we have a morphism of top forms $\bigwedge^4 \phi_* \Omega_{\mathrm{GL}_2/k} \to \bigwedge^4 \Omega_{\mathrm{SL}_2 \times \mathbb{G}_m/k}$, whose global sections sends

$$\begin{aligned}\bigwedge_{i,j=1}^{2} dx_{ij} &\mapsto d(tx_{11}) \wedge d(tx_{12}) \wedge dx_{21} \wedge dx_{22}, \\ &= (tdx_{11} + x_{11}dt) \wedge (tdx_{12} + x_{12}dt) \wedge dx_{21} \wedge dx_{22}, \\ &= t^2 dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} \\ &\quad + tx_{11}dt \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} \\ &\quad - tx_{12}dt \wedge dx_{11} \wedge dx_{21} \wedge dx_{22}.\end{aligned}$$

Note that $x_{11}x_{22} - x_{21}x_{12} = 1$ in the $\mathcal{O}(\mathrm{SL}_2) \otimes_k \mathcal{O}(\mathbb{G}_m)$-module $\bigwedge^4 \Omega_{\mathcal{O}(\mathrm{SL}_2 \times \mathbb{G}_m)/k}$ and that $d(x_{11}x_{22} - x_{21}x_{12}) = 0$ so

$$dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} = (x_{11}x_{22} - x_{21}x_{12}) dx_{11} \wedge dx_{12} \wedge dx_{21} \wedge dx_{22} = 0.$$

In the end, we find

$$t^2 \bigwedge_{i,j=1}^{2} dx_{ij} \mapsto t^{-1}dt \wedge (x_{11}dx_{12} \wedge dx_{21} \wedge dx_{22} - x_{12}dx_{11} \wedge dx_{21} \wedge dx_{22}).$$

From previous section, $t^2 \bigwedge_{i,j} dx_{ij}$ is a global left-invariant top form of GL_2. We also know $t^{-1}dt$ is the global left-invariant top form of $\mathbb{G}_m$. This should imply that

$$\omega = x_{11}dx_{12} \wedge dx_{21} \wedge dx_{22} - x_{12}dx_{11} \wedge dx_{21} \wedge dx_{22}$$

is a global left-invariant top form of SL_2. .... Is this correct? If yes, what is the quickest way to check this?

3.3. **11/08/2021: $V(m\nu)$ in $V(m\lambda) \otimes V(m\mu)$.** I want to continue to solve the problem I had on 28/07/2021. I learnt this proof from Travis.

Lemma 5. *There is a unique copy $V(m\lambda)$ in $V(\lambda)^{\otimes m}$ that contains $v^{\otimes m}$ for every $v \in V(\lambda)$.*

Proof. We denote v_λ to be the highest weight vector of weight λ in $V(\lambda)$. Then $v_\lambda^{\otimes m}$ is a highest weight vector of weight $m\lambda$ in $V(\lambda)^{\otimes m}$ so $V(\lambda)^{\otimes m}$ contains a copy of $V(m\lambda)$.

There is only one copy of $V(m\lambda)$ in $V(\lambda)^{\otimes m}$ because $v_\lambda^{\otimes m}$ is the only vector of highest weight λ in $V(\lambda)^{\otimes m}$ (up to linear independence). Indeed, suppose $\sum_{i=1}^n c_i v_{i1} \otimes v_{i2} \otimes \cdots \otimes v_{im}$ is a highest weight vector of weight $m\lambda$ where $0 \neq c_i \in \mathbb{C}$ and v_{ik} is a weight vector of weight λ_{ik}, then from $h \cdot \sum_{i=1}^n c_i v_{i1} \otimes v_{i2} \otimes \cdots \otimes v_{im} = (m\lambda)(h) \sum_{i=1}^n c_i v_{i1} \otimes v_{i2} \otimes \cdots \otimes v_{im}$, we find $m\lambda = \sum_{k=1}^m \lambda_{ik}$ for every $1 \leq i \leq n$. Because $\lambda_{ik} \leq \lambda$ for every i, k so we find $\lambda_{ik} = \lambda$ for every i, k, as desired.

$V(m\lambda)$ in $V(\lambda)^{\otimes m}$ contains $v^{\otimes m}$ for every $v \in V(\lambda)$ because $U(\mathfrak{g}) \cdot v^{\otimes m}$ contains $v_\lambda^{\otimes m}$. Indeed, let $v = \sum_{i=1}^{\ell} a_i v_i$ where v_i is a weight vector of weight $\mu_i \leq \lambda$ ($\mu_i \neq \mu_j$ for $i \neq j$) and $a_i \in \mathbb{C} \setminus \{0\}$. Let $\lambda - \mu_j = \sum_{i=1}^k c_{ij}\alpha_i$ where α_i's are the simple roots, $c_{ij} \in \mathbb{Z}_{\geq 1}$.

We can choose a μ_1 among μ_j's such that $c_{11} = \max_j\{c_{1j}\}$; c_{21} is maximal among those c_{2j}'s of μ_j's satisfying $c_{1j} = c_{11}$; c_{31} is maximal among those c_{3j}'s of μ_j's satisfying $c_{1j} = c_{11}, c_{21} = c_{2j}$; ...

With this, by letting $e_i \in \mathfrak{n}$ that corresponds to the simple root α_i, we find $e_1^{c_{11}} \cdots e_k^{c_{k1}} v_j = 0$ for $j \neq 1$ and $0 \neq e_1^{c_{11}} \cdots e_k^{c_{k1}} v_1 \in \mathbb{C}v_\lambda$. Thus, when $g = e_1^{c_{11}} \cdots e_k^{c_{k1}}$ then $gv \in \mathbb{C}v_\lambda$, while $g^2 v = 0$. This follows $g^m v^{\otimes m} = m!(gv)^{\otimes m}$, as desired. $\square$

Lemma 6. *For $\mathfrak{g}$-modules V, W then $V \otimes W \cong W \otimes V$ by sending $v \otimes w \mapsto w \otimes v$.*

Proof. It suffices to check that the map $f : V \otimes W \to W \otimes V$ is a $\mathfrak{g}$-module homomorphism. Indeed, $f(g(v\otimes w)) = f((gv)\otimes w + v\otimes(gw)) = w\otimes(gv)+(gw)\otimes v = g(w\otimes v) = g\cdot f(v\otimes w)$. This argument is the same as saying that $U(\mathfrak{g})$ is cocommutative, i.e. comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, defined by $\Delta(g) = g \otimes 1 + 1 \otimes g$ for $g \in \mathfrak{g}$, commutes with $\tau : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$, where τ is the permutation map. □

Back to the problem, we want to show that $V(m\lambda) \otimes V(m\mu)$ contains a copy of $V(m\nu)$. There exists a highest weight vector of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. By the construction of tensor product, we can write it as $v \otimes w$ where $v \in V(\lambda), w \in V(\mu)$.

We also know that $(V(\lambda) \otimes V(\mu))^{\otimes m} \cong V(\lambda)^{\otimes m} \otimes V(\mu)^{\otimes m}$ sends $(v \otimes w)^{\otimes m}$ to $v^{\otimes m} \otimes w^{\otimes m}$. We have $(v \otimes w)^{\otimes m}$ is of highest weight $m\nu$. From the previous lemma, we know $v^{\otimes m} \in V(m\lambda)$ and $w^{\otimes m} \in V(m\mu)$ so $U(\mathfrak{g}) \cdot v^{\otimes m} \otimes w^{\otimes m}$ is a submodule of $V(m\lambda) \otimes V(m\mu)$ in $V(\lambda)^{\otimes m} \otimes V(\mu)^{\otimes m}$. With this, we conclude $V(m\lambda) \otimes V(m\mu)$ contains a copy of $V(m\nu)$.

3.4. **12/08/2021: Tate vector spaces.** I read something interesting called Tate vector spaces. It refers to an infinite dimensional vector space V equipped with a set of lattices in V such that V is isomorphic to the inverse limit of V/L where L runs through lattices in V.

This notion generalises other spaces such as the Grassmannians, adeles, $k(\!(t)\!)$.

There are dimension theory (i.e. assign each lattice in V a number, called dimension) and determinant theory (i.e. assign each each lattice in V a line) for Tate vector space V generalise certain constructions in the above mentioned examples. For the determinant theory, one can use it to construct central extensions of $\mathrm{GL}(V)$ and somehow people want to do this ...

Need to read more at `http://page.mi.fu-berlin.de/groemich/chicago.pdf` by Michael Groechenig and `https://people.math.harvard.edu/~gaitsgde/grad_2009/SeminarNotes/Nov3-10(CentExt).pdf` by Dustin Clausen.

3.5. **13/08/2021: Inner product and Hom.** Madeline pointed out to the category theory reading group that in Etingof's book on representation theory (p. 189 of `http://www-math.mit.edu/~etingof/reprbook.pdf`), there is a myterious dictionary between a category and a vector space V equipped with a nondegenerate inner product. In particular, an inner product (x, y) in V is analogous to $\mathrm{Hom}(X, Y)$ in a category. I wonder if we can make this analogy more formal. I suspect it is some sort of (de)categorification, although I am not too sure about this.

Let me try to describe this analogy in the case our category is $\mathrm{Rep}(G)$, i.e. the category of finite-dimensional complex representations of a finite group G. Then the vector space is vector space V of class functions, i.e. complex-valued functions on G that is invariant under conjugation action of G. This the space where all characters of representations live. One can equipp V with a Hermitian inner product. For example, if χ_V, χ_W are characters of representations V, W of G, then $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}(V, W)$. In some sense, $\mathrm{Rep}(G)$ is the categorification of V, where orthonormal basis of V corresponds to irreducible representations.

This reminds me of a question I had on 06/06/2021 about the relation between adjoint operators and adjoint functors.

Nasos told us that John Baez have done something related to this dictionary in this paper of his: Higher-Dimensional Algebra II: 2-Hilbert Spaces `https://arxiv.org/pdf/q-alg/9609018.pdf`.

3.6. **14/08/2021: Principal G-bundles.** I want to digest the definition of principal G-bundles, as it seems to me that there are many ways to phrase this notion and different sources define principal bundles differently, depending on how nice the space of consideration is.

In this discussion, every topological space is assumed to be Hausdorff.

We start off with one definition of principal G-bundles, taken from Cohen's lecture notes on The Topology of Fiber Bundles or Stephen A. Mitchell's notes `https://sites.math.washington.edu/~mitchell/Notes/prin.pdf`.

Definition 7. *For a topological group G, a principal G-bundle P over X is a (surjective) continuous map $\pi : P \to X$ satisfying the following conditions:*

(1) G acts on the right on P;

(2) Local trivialisation: There is an open cover $\{U\}$ of X such that for each such U, we have a homeomorphism $\pi_U : \pi^{-1}(U) \to U \times G$ that is G-equivariant satisfying

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim", "\pi_U"'] \arrow[dr, "\pi"] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

Here g acts on $U \times G$ by $(u, g)g' = (u, gg')$.

Firstly, π_U being G-equivariant tells us that G acts on fiber P_x. Indeed, for a local trivialisation (U, π_U) at x, if $p \in P_x$ so $\pi_U(p) = (x, g)$, we find $\pi_U(pg') = \pi_U(p)g' = (x, gg')$. Hence, $pg' \in P_x$. To say G acts on fibers is the same as saying $\pi : P \to X$ is G-equivariant with trivial G-action on X.

Furthermore, π_U being bijective implies G acts simply transitively on P_x. Indeed, choose a local trivialisation (U, π_U) of x, for any $y, z \in P_x$, let $\pi_U(y) = (x, g_y)$ and $\pi_U(z) = (x, g_z)$ then there exists a unique $g \in G$ so $g_y g = g_z$, implying $\pi_U(z) = \pi_U(y)g = \pi_U(yg)$. Hence, there exists unique $g \in G$ so $z = yg$.

On the other hand, π_U being a G-equivariant bijection implies that G acts freely (i.e. trivial stabiliser) on P. Indeed, if we have $p \in P$ and $g \in G$ so $pg = p$ then by choosing a local trivialisation (U, π_U) of $\pi(p)$, we find $(\pi(p), g') = \pi_U(p) = \pi_U(pg) = \pi_U(p)g = (\pi(p), g'g)$. Hence, $g' = g'g$ so $g = 1$.

Because G acts simply transitively on P_x, we know that for any $y \in P_x$, the map $G \to P_x$ defined by $g \mapsto yg$ is a continuous bijection [3]. However, we cannot say anything more than this. In particular, this does not imply that G is homeomorphic to P_x, i.e. P_x is a G-torsor, which is what we want when we define principal G-bundles [4]. Hence, we would like to add the following condition to the definition

Definition 8. *We add the following condition to our definition of principal G-bundle $\pi : P \to X$.*

(1) For every $y \in P_x$, the morphism $G \to P_x$ defined by $g \mapsto yg$ is a homeomorphism.

In the case where our spaces are smooth manifolds and G is a Lie group, this condition automatically holds, see p.5 `https://www.mathi.uni-heidelberg.de/~lee/MenelaosSS16.pdf`. The reason roughly is that the map $G \to P_x$ is of constant rank and bijective, hence a diffeomorphism.

An equivalent way to phrase this condition is that the map $\phi : P \times G \to P \times P$, defined by $(x, g) \mapsto (x, xg)$, is a homeomorphism. Indeed, for closed subset U in G then $\phi(\{x\} \times U)$ is closed.

---

[3] One can show $f_y : G \to P_x$ is continuous when assuming P_x (or P) is Hausdorff. Indeed, since G acts continuously on P_x via $\phi : P_x \times G \to P_x$, for any closed subset U of P_x, we know $\phi^{-1}(U) \cap (\{y\} \times G)$ is closed, implying $\phi_y : G \to P_x$, defined by $\phi_y(g) = yg$, is continuous

[4] For example, consider $G = \mathbb{R}$ with discrete topology acts on $P = \mathbb{R}$ with the usual topology, then G is not homeomorphic to $P_x = \mathbb{R}$.

Hence, $\phi_x : g \mapsto xg$ is a closed map. We know ϕ_x is a continuous bijection so this implies ϕ_x is a homeomorphism.

An equivalent way to say $\pi : P \to X$ has local trivialisations is to say it has local sections.

If $\pi : P \to X$ admits local sections, i.e. for every x there is a continuous section $s : U \to P$ on some open neighborhood U of x, then this will imply local trivialisation condition. Indeed, we can define $\pi_U^{-1}(u, g) = s(u) \cdot g$. The map is bijective since G acts simply transitively on fibers. Indeed, if $s(u')g = s(u)g'$ then $u = u'$ and for any $p \in \pi^{-1}(U)$, there exists a unique $g \in G$ so $s(\pi(p)) \cdot g = p$. The map is continuous as it is the composition of $U \times G \xrightarrow{s \times \mathrm{id}_G} P \times G \to P$. *I cannot seem to show that this map is a homeomorphism, i.e. it is open/closed with the current assumption on X, P?* It seems for this to be true, s needs to be homeomorphic onto its image, then π_U^{-1} is an open map. For this to work, I can assume extra condition that X is locally compact Hausdorff, which means we can assume U is compact (or restrict U to a compact neighborhood). Then we have $s : U \to s(U)$ is a bijective continuous map from compact U to Hausdorff $s(U)$, implying s is a homeomorphism onto its image. Thus, π_U is a local trivialisation.

Conversely, if we are given a local trivialisation (U, π_U) of a principal G-bundle $\pi : P \to X$, we can define a local section $s : U \to P$ by $s(u) := \pi_U^{-1}(u, 1)$. To check s is a continuous map, given a closed set $V \subset \pi^{-1}(U)$ of $p \in \pi^{-1}(U)$, $V \cap P_{\pi(p)}$ is also a closed set of p, we have $s^{-1}(V \cap P_{\pi(p)})$ is either $\{\pi(p)\}$ or $\varnothing$, hence is closed.

We know if $\pi : P \to X$ is a principal G-bundle then G acts freely on P. In the other direction, if we are given a space P with a free G-action on the right of P, it is not enough to say $\pi : P \to P/G$ is a principal G-bundle. For example, let $\mathbb{R}$ acts on the torus $\mathbb{R}^2/\mathbb{Z}^2$ by translating $(1/2, a)$ where a is an irrational number. Even though the action is free, the orbits are dense and the quotient space is not even Hausdorff. One needs to add extra condition, for example, π having local sections. Some references would call P a (right) *free* G-space if $\pi : P \to P/G$ is a principal G-bundle.

Given a principal G-bundle $\pi : P \to X$ then by the universal property of quotient spaces, this induces a continuous bijective map $\phi : P/G \to X$, defined by $[p] \in P/G \mapsto \pi(p)$. In our setup, I don't think we can show ϕ is a homeomorphism. However, if we assume our spaces are smooth manifolds, G is a Lie group, then ϕ is a diffeomorphism (see for example `https://www.mathi.uni-heidelberg.de/~lee/MenelaosSS16.pdf`).

Some more references `https://web.ma.utexas.edu/users/dafr/M392C-2017/Notes/lecture13.pdf`, `https://ncatlab.org/nlab/show/principal+bundle`

Principal G-bundles in topological setting is discussed at Tammo tom Dieck's book Algebraic Topology

3.7. 14/08/2021: Gufang Zhao's first lecture: Bundles as double quotient space.

Just want to take some notes on Gufang Zhao lectures. The summary of his lecture series is as follows: One can construct certain bundles on Riemann surfaces (bundles of conformal blocks on the configuration space of points of the Riemann surface), equipped it with a flat connection, called Knizhnik-Zamolodchikov (KZ) connection. The solution of of the KZ equations can be obtained by counting algebraic curves.

His first lecture is about constructing such bundles.

3.7.1. *Representations from sections of equivariant vector bundles.*

Given a space X with a free (right) G-action (i.e. so that $\pi : X \to X/G$ is a principal G-bundle) and a representation $\rho : G \to \mathrm{GL}(V)$ of G, we can construct a vector bundle $X \times_G V \to X/G$ by letting $X \times_G V := (X \times V)/\sim$ where the equivalence relation is $(xg, v) \sim (x, \rho(g^{-1})v)$ [5]. It has a natural projection $p : X \times_G V \to X/G$, making it a vector bundle over X/G with fiber isomorphic to V. We can describe its global

[5] Some references define this equivalence relation as $(xg, v) \sim (x, \rho(g), v)$. I don't think it matters which one we choose, but we need to modify this discussion accordingly

sections $H^0(X/G, X \times_G V)$ as the space of G-equivariant functions $s : X \to V$, i.e. $s(xg) = g \cdot s(x)$ for all $g \in G$. Indeed, given such s, we can define $\overline{s} : X/G \to X \times_G V$ by $\overline{s}(\overline{x}) = (x, s(x))$ where $\pi(x) = \overline{x}$. This is well-defined because $(xg^{-1}, s(xg^{-1})) \sim (x, g \cdot s(xg^{-1})) \sim (x, s(x))$.

If we also have an action of a group H on X that commutes with the G-action, we can turn $X \times_G V \to X/G$ into a H-equivariant vector bundle by letting H act on $X \times_G V$ by $h(x, v) := (hx, v)$ and on X/G by $h(xG) = hxG$. This is well-defined because the action of H on X commutes with action of G on X. This induces a representation of H on the space of global sections $H^0(X/G, X \times_G V)$. Indeed, let $\overline{s} : X/G \to X \times_G V$ be in $H^0(X/G, X \times_G V)$ then $(h \cdot \overline{s})(x) := h \cdot s(h^{-1}x)$. As $\overline{s}$ corresponds to a G-equivariant $s : X \to V$, we have $(h \cdot s)(x) = s(h^{-1}x)$.

Example 9. Consider $X = \mathbb{C}^2 \setminus \{0\}$ and $G = \mathbb{C}^\times$ acts on X by $t(x, y) = (tx, ty)$, then X/G is $\mathbb{P}^1(\mathbb{C})$.

For every $d \in \mathbb{Z}$, we can construct a representation $\mathbb{C}(d)$ of $G = \mathbb{C}^\times$ on $V = \mathbb{C}$ by $t \cdot x = t^d x$.

Thus, this gives us a vector bundle $\mathcal{O}(d) := \mathbb{C}^2 \setminus \{0\} \times_{\mathbb{C}^\times} \mathbb{C}(d)$ over $\mathbb{P}^1$. Its global sections can be described as

$$H^0(\mathbb{P}^1, \mathcal{O}(d)) = \{f : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C} | f(tx) = t^d f(x) \ \forall t \in \mathbb{C}^\times\}.$$

When we only care about holomorphic/algebraic sections then

$$H^0(\mathbb{P}^1, \mathcal{O}(d)) = \{f \in \mathbb{C}[x, y] : f(tx, ty) = t^d f(x, y) \ \forall t \in \mathbb{C}^\times\}.$$

This is the space of homogeneous polynomials of degree d over two variables x, y.

Let $H = \mathrm{SL}_2(\mathbb{C})$ acts on $X = \mathbb{C}^\times \setminus \{0\}$ by left-mutiplication. Then $\mathrm{SL}_2(\mathbb{C})$ acts on $H^0(\mathbb{P}^1, \mathcal{O}(d))$ by $(a \cdot f)(x, y) = f(a^{-1}(x, y))$.

3.8. **15/08/2021: Tamagawa number for SL_n over $\mathbb{Q}$.** I would like to sketch the computation that the Tamagawa number of SL_n over $\mathbb{Q}$ is 1, i.e. $\mu_{\mathrm{SL}_n,\mathbb{Q}}(\mathrm{SL}_n(\mathbb{Q}) \setminus \mathrm{SL}_n(\mathbb{A})) = 1$. I learnt this from Garrett's notes `https://www-users.cse.umn.edu/~garrett/m/v/volumes.pdf` and Andre Weil's book Adeles and Algebraic Groups (p. 47, §3.4). From these sources, I know that Siegel came up with this proof for $\mathrm{SL}_n(\mathbb{Z}) \setminus \mathrm{SL}_n(\mathbb{R})$, which was then adapted by Weil to prove it for $\mathrm{SL}_n(\mathbb{Q}) \setminus \mathrm{SL}_n(\mathbb{A})$.

At the moment, to focus on describing the main idea of the proof, I will avoid analytic issues such as convergence of integrals, measure-preserving homeomorphisms and normalisations of measures. This is also because I haven't managed to figure out all these technical details.

For $G_n = \mathrm{SL}_n$, we are able to find a copy of SL_{n-1} in SL_n via the action of SL_n on k^n. This then allows us to use induction on n. More precisely, let $G_n(k)$ acts on k^n by left-multiplication. Let $G'_n(k)$ be the stabiliser of $e_1 = (1, 0, 0, \ldots, 0)^t \in k^n$ in $G(k)$. We then have the following identification of spaces

(1) $G'_n(k) \setminus G_n(k)$ is continuously bijective to $G_n(k)e_1$ for $k = \mathbb{Q}_v$ or $\mathbb{A}$. When k is a division algebra, $G_n(k)e_1 = k^n \setminus \{0\}$ as one can write out explicitly $M \in G_n(k)$ such that $Me_1 = y$ for any $y \in k^n \setminus \{0\}$. Note that $\mathbb{A}$ is not a division algebra, but if we only care about integrating over this quotient space, then it is good enough to know that the set $\mathbb{A}^n - G_n(\mathbb{A})e_1$ has measure 0.
(2) G'_n is the semidirect product of G_{n-1} with $\mathbb{G}_a^{n-1}$. In particular, elements in $G'_n(k)$ can be described as
$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$
where $u \in k^{n-1}$, $x \in G_{n-1}(k)$.

We start with the following formula that describe integration over $G'_n(\mathbb{Q}) \setminus G_n(\mathbb{A})$ in two ways

$$(3) \qquad \int_{x \in G_n(\mathbb{Q}) \setminus G_n(\mathbb{A})} \int_{y \in G'_n(\mathbb{Q}) \setminus G_n(\mathbb{Q})} f(xy) dx dy = \int_{z \in G'_n(\mathbb{A}) \setminus G_n(\mathbb{A})} \int_{t \in G'_n(\mathbb{Q}) \setminus G'_n(\mathbb{A})} f(zt) dz dt.$$

As $G'_n = \mathbb{G}_a^{n-1} \ltimes G_{n-1}$, by inductive hypothesis on G_n, we find the Tamagawa number of G'_n (i.e. the volume of $G'_n(\mathbb{Q}) \setminus G'_n(\mathbb{A})$) is

$$\tau(G'_n) = \tau(\mathbb{G}_a^{n-1})\tau(G_{n-1}) = 1.$$

Therefore, if we choose f to be a function on $G'_n(\mathbb{A}) \setminus G_n(\mathbb{A})$, i.e. trivial on $G'_n(\mathbb{A})$, and from the identification of $G'_n(k) \setminus G_n(k)$, we can rewrite (3) as

$$\int_{x \in G_n(\mathbb{Q}) \setminus G_n(\mathbb{A})} \sum_{y \in \mathbb{Q}^n \setminus \{0\}} f(xy) dx dy = \int_{\mathbb{A}^n} f(z) dz. \tag{4}$$

The Fourier transform of $f : \mathbb{A}^n \to \mathbb{C}$ is

$$\widehat{f}(y) = \int_{\mathbb{A}^n} f(x) \chi_{\mathbb{A}}(y^t \cdot x) dx,$$

where $\chi_{\mathbb{A}}$ is the standard unitary character on $\mathbb{A}$. Applying (4) for $\widehat{f}$, we find

$$\int_{x \in G_n(\mathbb{Q}) \setminus G_n(\mathbb{A})} \sum_{y \in \mathbb{Q}^n \setminus \{0\}} \widehat{f}(xy) dx = \int_{\mathbb{A}^n} f(z) dz.$$

On the other hand, noting that for $x \in G_n(\mathbb{A})$, as $\det x = 1$, we find $f(xy) = \widehat{f}((x^t)^{-1}y)$ for $y \in \mathbb{A}^n$. Thus, combining the above two equations, we find

$$\int_{\mathbb{A}^n} (f(x) - \widehat{f}(x)) dx = \int_{G_n(\mathbb{Q}) \setminus G_n(\mathbb{A})} (\widehat{f}(0) - f(0)) dx.$$

By Fourier inversion formula, the left-hand side is $\widehat{f}(0) - f(0)$. One can choose f such that $f(0) \neq \widehat{f}(0)$, giving $G_n(\mathbb{Q}) \setminus G_n(\mathbb{A})$ volume 1, as desired.

Remark 10. What I found surprising about this proof is the following:

(1) It works (almost) verbatim if we work with $\mathrm{SL}_n(\mathbb{Z}) \backslash \mathrm{SL}_n(\mathbb{R})$ instead. An induction argument can be used to show the answer is $\zeta(1)\zeta(2)\cdots\zeta(n)$.
(2) Comparing this proof with the one given by describing a fundamental domain for $\mathrm{SL}_n(\mathbb{Q}) \setminus \mathrm{SL}_n(\mathbb{A})$, for the later one, we need to compute the volume of $\mathrm{SL}_n(\mathbb{Z}_p)$, but this is nowhere to be seen for this proof. In some sense, the appearance of $\mathbb{Q}_p$ is completely suppressed.
(3) It seems to me that because this is an inductive argument, one does not have to make an explicit choice of a left-invariant top form for G. This proof also works for any global field I believe.
(4) The proof works mainly because for $G_n = \mathrm{SL}_n$, one can find a nice subgroup $G'_n = \mathbb{G}_a^{n-1} \ltimes G_{n-1}$. I wonder if this phenomenon holds for other groups. For example, it is mentioned in the references that the inductive strategy of this proof works for Sp_{2n}, (any more?)

3.9. **22/08/2021: Tamagawa number of** Sp_{2n}**.** Let $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ be a $2n$-by-$2n$ matrix. The symplectic group Sp_{2n} is defined as

$$\mathrm{Sp}_{2n}(k) = \{M \in M_{2n \times 2n}(k) : M^t J_n M = J_n\}.$$

In this section, we show that the Tamagawa number of the symplectic group Sp_{2n} is 1. We prove this by induction on n. For $n = 1$ then $\mathrm{Sp}_2 = \mathrm{SL}_2$ so $\tau(\mathrm{Sp}_2) = \tau(\mathrm{SL}_2) = 1$.

Let $G_n = \mathrm{Sp}_{2n}$ acts on k^{2n} by left-multiplication. Let $G'_n(k)$ be the stabiliser of $e = (1, 0, \ldots, 0)^t \in k^{2n}$. Elements in $G'_n(k)$ can be written as

$$\begin{pmatrix} 1 & * & * & * \\ 0 & a & * & b \\ 0 & 0 & 1 & 0 \\ 0 & c & * & d \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{n-1}(k)$ and other entries are suitably chosen.

Determine elements in G'_n. To determine other entries, let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D \in M_{n\times n}$, then such matrix lies in Sp_{2n} iff $A^tC - C^tA = B^tD - D^tB = 0$ and $A^tD - C^tB = I_n$. For X to stabilises e means $A = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ and $C = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$. Hence, we find $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x & x_2 \\ 0 & a & y_1 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & y_2 & d \end{pmatrix}$ where $(x_1, x_2) = (y_1, y_2)^t \begin{pmatrix} 0 & I_{n-1} \\ -I_{n-1} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2n-2}$. $\square$

This follows that G'_n is a semidirect product of G_{n-1} and $\mathbb{G}_a \ltimes (\mathbb{G}_a)^{2n-2}$ (corresponding to a choice of $x \in k$ and $(x_1, x_2) \in k^{2n-2}$). Thus, by inductive hypothesis, the Tamagawa number of G'_n is $\tau(G'_n) = \tau(G_{n-1})\tau(\mathbb{G}_a)^{2n-1} = 1$.

We can identify $G'_n(k) \setminus G_n(k)$ with $G'_n(k)e$. When k is a division algebra, $G'_n(k)e = k^{2n} \setminus \{0\}$. When $k = \mathbb{A}$ then $\mathbb{A}^{2n} \setminus G'_n(\mathbb{A})e$ has measure 0. Proceeding exactly as in SL_n case, we obtain $\tau(\mathrm{Sp}_{2n}) = 1$.

3.10. **22/08/2021: Tamagawa measure and restriction of scalars.** I will discuss on how to define a measure on the adelic points of a variety. After this, we will focus on the case for linear algebraic groups and define the Tamagawa measure on the adelic points of such groups.

The references we follow are: Weil's Adeles and Algebraic Groups, first chapter of Gaitsgory and Lurie's book Weils Conjecture for Function Fields I.

3.10.1. *Tamagawa measure on smooth schemes.* Let X be a separated [6] smooth (affine) scheme of finite type over k. A *volume form* on X is a nowhere-vanishing algebraic differential form of top degree on X. Suppose X has a volume form ω (in other words, the canonical line bundle on X has a nonzero global section).

Over a complete valued field k, $X(k)$ has a canonical structure of a k-analytic manifold. When k is a local field equipped with a Haar measure, from the volume form of X, one can define a measure on $X(k)$.

Let k be now a global field. Let k_v to be the completion of k with respect to a place v of k. Let $\mathcal{O}_{k_v}$ to be the ring of integers of k_v. Let μ_{k_v} to be standard Haar measure on k_v (i.e. when k_v is a nonarchimedean local field, we normalise μ_{k_v} so $\mu_{k_v}(\mathcal{O}_v) = 1$; when $k_v = \mathbb{R}$ then it is the Lebesgue measure; when $k_v = \mathbb{C}$ then μ_{k_v} is twice the Lebesgue measure).

From a volume form ω on X, we can define a measure ω_v on $X(k_v)$, where v is a place of k and k_v is the completion of k with respect to v. Denote $\mathcal{O}_{k_v}$ to be the ring of integers of k_v.

Let S be a nonempty finite set of places of k that contains every archimedean places of k and let $\mathcal{O}_S := \{x \in k : x \in \mathcal{O}_{k_v}\ \forall v \notin S\}$. Suppose there exists a smooth scheme $\overline{X}$ over $\mathcal{O}_S$ such that

[6] I don't know that well of this separatedness condition for schemes but it will guarantee that over complete valued field k, $X(k)$ is Hausdorff

$\overline{X}_k = X$ (here $\overline{X}_k$ is an affine scheme over k such that its coordinate ring is $\mathcal{O}(\overline{X}_k) = \mathcal{O}(\overline{X}) \otimes_{\mathcal{O}_S} k$) [7], and suppose that $\overline{X}$ has a volume form $\overline{\omega}$. Then this induces a volume form $\overline{\omega}|_X$ for X. As $\mathcal{O}_S \hookrightarrow \mathcal{O}_v$ or Spec $\mathcal{O}_v \to$ Spec $\mathcal{O}_S$ [8], we can make sense of $\mu_{\omega,v}(\overline{X}(\mathcal{O}_v))$ where $\mu_{\omega,v}$ is the induced measure from ω on $X(k_v) \supset \overline{X}(\mathcal{O}_v)$. We can also regard

$$\prod_{v \notin S} \overline{X}(\mathcal{O}_v) \times \prod_{v \in S} X(k_v)$$

as an open subgroup of $X(\mathbb{A}_k)$.

If the product

$$\prod_{v \notin S} \mu_{\omega,v} \overline{X}(\mathcal{O}_v)$$

converges absolutely to a nonzero real number, we say X *admits a Tamagawa measure.*

If X admits a Tamagawa measure, a Tamagawa measure μ_ω on $X(\mathbb{A}_k)$ with respect to ω is such that over open sets $\prod_{x \notin S} \overline{X}(\mathcal{O}_v) \times \prod_{v \in S} X(k_v)$ of $X(\mathbb{A}_k)$, μ_ω is the product measure

$$\tau(\mathbb{G}_a)^{-\dim X} \prod_v \mu_{\omega,v}.$$

Here $\tau(\mathbb{G}_a)$ is the Tamagawa number for $\mathbb{A}$ (i.e. we define the measure $\mu_{\mathbb{A}}$ on $\mathbb{A}$ to be the one such that over open sets $\prod_{v \notin S} \mathcal{O}_v \times \prod_{v \in S} k_v$, it is the product measure. We then let $\tau(\mathbb{G}_a) = \int_{\mathbb{Q} \setminus \mathbb{A}} \mu_{\mathbb{A}}$, which is well-defined because $\mathbb{Q} \setminus \mathbb{A}$ is compact).

This definition of μ_ω does not depend on the choice of a set S of places of k or the choice of an integral model $\overline{X}$ for X (because two choice of integral models become isomorphic after enlarging S).

The reason why we add a factor $\tau(\mathbb{G}_a)^{-\dim X}$ is because of the following result [9]

Theorem 11. *Let K/k be a finite and separable extension. Let V be a separated smooth scheme of finite type over K. Let W to be the Weil's restriction of scalars of V with respect to K/k, i.e. W is a scheme over k such that $W(R) := V(R \otimes_k K)$ where R is a k-algebra. Under the restriction of scalars, there is a canonical isomorphism $W(\mathbb{A}_k) \cong V(\mathbb{A}_K)$. We then have*

(1) W admits a Tamagawa measure iff V admits a Tamagawa measure.
(2) If the previous condition holds, the canonical isomorphism $W(\mathbb{A}_k) \cong V(\mathbb{A}_K)$ is a measure preserving map.

What to do next:

(1) Do some more examples over number fields and function fields.
(2) Examples with restriction of scalars.

3.10.2. *Tamagawa measure for semisimple algebraic groups.* Let G be a connected semisimple linear algebraic group over a global field k. The above discussion of Tamagawa measure applies for ω being a left-invariant volume form for G. In particular, G does admit a Tamagawa measure (in this case, Tamagawa measure for G is a Haar measure) and furthermore, such measure does not depend on the choice of ω.

[7] One says $\overline{X}$ is a *model* for X over $\mathcal{O}_S$. So the point of a model is as follows: for a linear algebraic group G over $\mathbb{Q}$. We would like to describe $G(\mathbb{Z})$ without having to embedd G to GL_n and write $G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$, which is less functorial

[8] For example, when $k = \mathbb{Q}$, $S = \{\infty, 2, 3\}$ then $\mathcal{O}_S = \mathbb{Z}[1/2, 1/3]$ and for $v = 5 \notin S$, as $1/2$ and $1/3$ are invertible in $\mathbb{Z}_5$, we have a map $\mathbb{Z}[1/2, 1/3] \to \mathbb{Z}_5$.

[9] Another way for people to define Tamagawa measure is to choose Haar measures on local fields k_v such that $\tau(\mathbb{G}_a) = 1$. A natural choice of Haar measures on k_v would be the one that is self-dual with respect to its Pontryagin dual $\widehat{k_v}$.

3.11. **27/08/2021: Affine Grassmannian.** Today I attend a WiSe talk about affine Grassmannian by Alex Weeks `https://sites.google.com/view/mathwise` and I just want to write down something I learnt from this talk.

We start by defining the ring of formal power series $\mathcal{O} = \mathbb{C}[[z]] = \{a_0 + a_1 z + a_2 z^2 + \dots | a_i \in \mathbb{C}\}$ and its fraction field, the field of formal Laurent series $K = \mathbb{C}((z)) = \{a_n z^n + a_{n+1} z^{n+1} + \dots | n \in \mathbb{Z}, a_i \in \mathbb{C}\}$.

An $\mathcal{O}$-lattice $L \subset K^n$ is a finitely-generated $\mathcal{O}$-module [10] such that $K \otimes_{\mathcal{O}} L \cong K^n$ As a set, the affine Grassmannian of GL_n is defined to be

$$\mathrm{Gr}_{\mathrm{GL}_n} = \{L \subset K^n : L \text{ is an } \mathcal{O} - \text{lattice}\}.$$

We will later see that $\mathrm{Gr}_{\mathrm{GL}_n}$ has more structure.

For example, let $n = 1$, then $\mathrm{Gr}_{\mathrm{GL}_1} \cong \mathbb{Z}$, where $n \in \mathbb{Z} \mapsto z^n \mathcal{O}$. The set $\{z^n \mathcal{O} : n \in \mathbb{Z}\}$ are all the $\mathcal{O}$-lattices in K because if L is an $\mathcal{O}$-lattice, there exists an element of lowest degree $a_n z^n + a_{n+1} + \dots$ in L, where $a_n \neq 0$. This follows $z^n \in L$ and hence $L = z^n \mathcal{O}$.

We can contruct $\mathcal{O}$-lattice L_λ, where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, in K^n, by $L_\lambda = z^{\lambda_1} \mathcal{O} e_1 \oplus \dots \oplus z^{\lambda_n} \mathcal{O} e_n$. Here $e_i = (0, \dots, 1, 0 \dots, 0)$.

The ind-scheme structure of $\mathrm{Gr}_{\mathrm{GL}_n}$ comes from the following observation: For any $L_1, L_2 \in \mathrm{Gr}_{\mathrm{GL}_n}$, we can find $a, b \in \mathbb{Z}_{\geq 0}$ such that $z^a L_1 \subset L_2 \subset z^{-b} L_1$. To see this, we can choose $\mathcal{O}$-basis $\{v_1, \dots, v_n\}$ for L_2 and $\mathcal{O}$-basis $\{w_1, \dots, w_n\}$ for L_1. We then can write $w_i = \sum a_{ij} w_j$ with $a_{ij} \in K$. We can choose sufficiently large $n \in \mathbb{Z}$ such that $z^n w_i \in L_2$ for every $1 \leq i \leq n$. This implies $z^n L_1 \subset L_2$. For the case $n = 1$, the following observation just says that for any integer n, there exists $a, b \in \mathbb{Z}$ so $a \leq n \leq b$ (which trivially holds ...).

From this observation, given $z^a \mathcal{O}^n \subset L \subset z^{-b} \mathcal{O}^n$, we find $L / z^a \mathcal{O}^n \subset z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$. Furthermore, $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$ is a finite dimensional complex vector space of dimension $(a + b)n$. Indeed, it has basis $z^{-b} e_i, z^{-b+1} e_i, \dots, z^{a-1} e_i$ over all i. If we write $d = \dim_{\mathbb{C}} (L / z^a \mathcal{O}^n)$, we find $L / z^a \mathcal{O}^n \in \mathrm{Gr}(d, z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n)$, our usual Grassmannian of d-dimensional subspaces in $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$.

Now, we wonder if a d-dimensional subspace $U \in \mathrm{Gr}(d, z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n)$ can be used to build an $\mathcal{O}$-lattice in K^n?

To see this, we first can define an operator T on $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$ by multiplying by z. This operator T is nilpotent with Jordan type $(a+b, \dots, a+b)$. Indeed, $T^{a+b}(z^j e_i) = 0$ where $1 \leq i \leq n, -b \leq j \leq a$ and $T(z^j e_i) = z^{j+1} e_i$ for $-b \leq j \leq a - 2$.

If L is an $\mathcal{O}$-lattice where $z^a \mathcal{O}^n \subset L \subset z^{-b} \mathcal{O}^n$ then $L / z^a \mathcal{O}^n$ is a T-invariant subspace of $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$. Back to our question: The converse also holds, i.e. $U \in \mathrm{Gr}(d, z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n)$ defines an $\mathcal{O}$-lattice in K^n if it is T-invariant.

Indeed, let L to be the kernel of

$$L := \ker \left(z^b \mathcal{O}^n \to z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n \to \left(z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n \right) / U \right)$$

We claim that L is a $\mathcal{O}$-lattice in K^n. Firstly, L is an $\mathcal{O}$-module because U is T-invariant. It is finitely generated because it is an $\mathcal{O}$-submodule of finitely generated module $z^b \mathcal{O}^n$ ($\mathcal{O} = \mathbb{C}[[z]]$ is Noetherian as $\mathbb{C}$ is, and any submodule of finitely genenerated module over a Noetherian ring is also finitely generated). L contains $z^a \mathcal{O}^n$ so $L \otimes_{\mathcal{O}} K = K^n$, as desired.

In summary, we have a bijection from d-dimensional $\mathcal{O}$-lattices $z^a \mathcal{O}^n \subset L \subset z^{-b} \mathcal{O}^n$ to d-dimensional subspaces of $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$.

Thus, the task of finding an $\mathcal{O}$-lattice L (of dimension d) between $z^a \mathcal{O}^n$ and $z^{-b} \mathcal{O}^n$ boils down to find a d-dimensional subspace of $z^{-b} \mathcal{O}^n / z^a \mathcal{O}^n$.

[10] Zhu's note on affine Grassmannian put finitely generated *projective* $\mathcal{O}$-module, which is a free module as any finitely generated projective module over a PID (e.g. $\mathcal{O}$) is free. This then means we can choose an $\mathcal{O}$-basis $e_1, \dots, e_n$ for L, i.e. $L = \mathcal{O} e_1 \oplus \dots \oplus \mathcal{O} e_n$.

Example 12. When $n = 2, a = 2, b = 0$ then $T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ by identifying basis ze_1, e_1, ze_2, e_2 of $z^2\mathcal{O}^2/\mathcal{O}^2$ with (e_1, e_2, e_3, e_4) of $\mathbb{C}^4$. To find $\mathcal{O}$-lattices $z^2\mathcal{O}^2 \subset L \subset \mathcal{O}^2$ of dimension 2, we want to find T-invariant 2-dimensional subspaces of $\mathbb{C}^4$. Note that $\ker T$ is 2-dimensional, so it is one candidate. If T-invariant subspace U is not $\ker T$, then there exists $v \in U$ so $T(v) \neq 0$. Then $U = \mathrm{span}_{\mathbb{C}}(v, Tv)$ because T has only eigenvalue 0 with eigenvector Tv, but $v \notin \ker T$ so v and Tv are linearly independent.

These are all the 2-dimensional subspaces of $\mathbb{C}^4$ that are T-invariant.

Our problem of finding 2-dimensional T-invariant subspaces is related to the nilpotent cone. Let Nil_2 to be the set of all 2×2 nilpotent matrices over $\mathbb{C}$. Equivalently, we can write

$$\mathrm{Nil}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) : a^2 + bc = 0 \right\}.$$

It is called a "cone" because under change of variables $b = x - y, c = x + y$ then the equation becomes $a^2 + x^2 = y^2$, which is a cone over the real points.

We can define a map

$$\begin{aligned} \mathrm{Nil}_2 &\to \mathrm{Gr}_{\mathrm{GL}_2} \\ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} &\mapsto L := \mathcal{O}(z + a, c) \oplus \mathcal{O}(b, -a + z). \end{aligned}$$

We notice $z^2\mathcal{O}^2 \subset L \subset \mathcal{O}^2$. L is a free $\mathcal{O}$-module because $a^2 + bc = 0$. To see $\dim_{\mathbb{C}}(L/z^2\mathcal{O}^2) = 2$, by identifying $z^2\mathcal{O}^2/\mathcal{O}^2 \cong \mathbb{C}^4$ via $(ze_1, e_1, ze_2, e_2) \mapsto (e_1, e_2, e_3, e_4)$, we can write

$$L/z^2\mathcal{O}^2 = \mathbb{C}(1, a, 0, c) \oplus \mathbb{C}(0, b, 1, -a).$$

This vector space is T-invariant because $a^2 + bc = 0$.

3.12. **Some unanswered questions.**

(1) Masoud asked what happen if we do not choose the standard character in identifying k with $\widehat{k}$ where k is a local field, what would be the dual Haar measure on k and $\widehat{k}$ in the Fourier inversion formula.

(2) What happen to Haar measure on local field under field extension. For example $\mathbb{C}/\mathbb{R}$?

4. More things to learn

(1) Everthing about $\mathrm{SL}_2, \mathrm{GL}_2$: Representations of $\mathrm{SL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{Q}_p), U(\mathfrak{sl}_2)$, quantum affine $\mathfrak{sl}_2$, ... Some links:
   (a) `https://jenseberhardt.com/teaching/S21Seminardata/plan.pdf`, `https://jenseberhardt.com/teaching/W2021Seminardata/plan.pdf`, `https://www.maths.usyd.edu.au/u/romanova/Talks/TwistSheavessl2.pdf`
   (b) Rep of $\mathrm{SL}_2(\mathbb{F}_q)$, book by Cedric Bonnafe (see rep lie group of finite type folder).
   (c) automorphic representations of $\mathrm{SL}_2, \mathrm{GL}_2$ over adeles `https://virtualmath1.stanford.edu/~conrad/conversesem/refs/NgoGL2.pdf`, trace formula `https://www.math.stonybrook.edu/~aknapp/pdf-files/355-405.pdf`,
   (d) More general refs: `https://virtualmath1.stanford.edu/~conrad/conversesem/`,
   (e) Langlands fundamental lemma for SL_2 (Bill Casselman Essays on the Fundamental Lemma).

(2) Perverse sheaves, Kazhdan-Lusztig conjectures. See Archar's book, Chriss Ginzburg, Goerdie and Anna's notes, Humphreys book, `https://chenhi.github.io/math7390-s21/`, Gelfand and Manin books, any many more ...

(3) Algebraic geometry, complex geometry
   (a) do Vakil exercises. Learn sheaf cohomology in more details.
   (b) Moduli space of vector bundles over Riemann surfaces, over $\mathbb{P}^1$ (see Sabin Cautis notes).

(4) Something is myterious to me about "hypergeometric", as it seems appear in many areas. Some references: Zoladek The Monodromy Group (see Geometry folder), Kapranov Hypergeometric functions on reductive groups (Hypergeometric folder), Macdonald hypergeometric functions, Hypergeometric functions over finite fields by Jenny Fuselier, Ling Long, Ravi Ramakrishna, Holly Swisher, Fang-Ting Tu.

   I would like to at least understand all the application in terms of te $._2F_1$ hypergeometric function.