






Last time: - Defined pseudo-diff operators $\text{Op}(a)$ for some "differential operator" $a(\xi)$

where ξ viewed as "differential symbol", x as variable for the coef of the operator

$$(\text{Op}(a)f)(x) = \int a(x, \xi) \hat{f}(\xi) e^{-ix\xi} d\xi.$$

→ e.g.: $a = -i\xi$ differentiate
 $a \hat{f} = (\frac{d}{dx} f)(\xi) \Rightarrow \text{Op}(a)f = \frac{df}{dx}$

- Describe conditions for $\text{Op}(a)$ to approximate an operator that localise both $f \in L^2(\mathbb{R}^2)$ and \hat{f} .

(can only be approximation because f and \hat{f} cannot be both localised) Last time $\text{Op}(a)f$ is localised but

$\widehat{\text{Op}(a)f}$ is near 0 far away from some interval

- a smooth approx of $\chi_{I \times J}$ where I, J closed intervals in \mathbb{R}
- Length $\ell(I)\ell(J) \gg 1$ for $\text{Op}(a)$ to be a projection.

Today: - Discuss decomposition of $L^2(\mathbb{R})$ using pseudo diff operators.

§ 3.4, 3.5 - Make sense of what is a precisely Venkatesh notes

Now, $\mathbb{R}^2 = \bigsqcup_{i \in \mathbb{N}} \Omega_i$ where $\Omega_i = \text{very large rectangle } I_i \times J_i$.

$$\text{then } \text{Id}_{\mathcal{B}(\mathbb{R})} = \text{Op}(1) = \text{Op}\left(\sum_{i \in \mathbb{N}} X_{\Theta_i}\right) \quad (1)$$

$\approx \sum_{i \in \mathbb{N}} \underbrace{\text{Op}(\alpha_i)}_{\hookrightarrow \approx \text{projection operator.}} \text{ where } \alpha_i \text{ smooth approx of } X_{\Theta_i}$

- Let $V_i := \bigcap_{m \in \mathbb{Z}(\mathbb{R})} \text{Op}(a_i)$ (Viewed approximately as functions
 \downarrow
 f supported on J_i^+ and \hat{f}
 \downarrow
 $\text{supported on } J_i^-$)

Then from (1), $E(\mu) = \sum_{i \in N} v_i$

In fact, $L^2(\mathbb{R}) = \bigoplus_{i \in N} V_i$ because $O_P(a_i)$ are projection operators.

Indeed, since $\text{Op}(a_i) : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ projection, we

$$\text{have } L^2(\mathbb{R}) = V_i \oplus \underbrace{(1 - O_p(a_i))}_{\text{range}} L^2(\mathbb{R})$$

$$\sum_{j \neq i} O_p(\alpha_j)$$

So if we have $f \in V_i \cap V_j$ then $f = 0 + f = f + 0$,
 a contradiction to above decomposition.

- Thus, we have a decomposition of $L^2(\mathbb{R})$ that is very geometric, i.e. it is based from partitioning \mathbb{R}^2 into

rectangles !!

- In fact, the $\dim V_i$ is related to trace of $\text{Op}(a_i)$. :

Claim: $\text{Tr}(\text{Op}(a_i)) = \int_{\Omega_i} a(x, \xi) dx d\xi \approx \text{Area}(\Omega_i)$.

Proof: Assume that f is Schwartz $\subset L^2(\mathbb{R})$:

$$\begin{aligned} (\text{Op}(a_i) f)(x) &= \int a(x, \xi) \hat{f}(\xi) e^{-ix\xi} d\xi \\ &= \int a(x, \xi) \left(\int e^{iy\xi} f(y) dy \right) e^{-ix\xi} d\xi \\ &= \iint a(x, \xi) e^{i(y-x)\xi} d\xi f(y) dy \end{aligned}$$

$K(x, y)$. kernel

If $\text{Op}(a)$ is of trace class (i.e. $L^2(\mathbb{R})$ Hilbert \rightarrow orthonormal

So trace of an integral kernel basis $\{e^{iy\xi}\} \rightarrow$ trace)

$$\text{is } \text{Tr}(\text{Op}(a)) = \int K(y, x) dy = \int a(y, \xi) dy d\xi \approx \text{Area}(\Omega_i).$$

because $a_p \approx \chi_{\Omega_i}$

□

What symbols a are allowed to make sense of ?

$$\text{Op}(a) \text{Op}(b) = \text{Op}(ab + i\partial_\xi a \partial_x b + \dots)$$

Answer: $a \in S_m$ = "diff operators of order $\leq m$ with
banded coeff".

$$\hookrightarrow \sup_{\gamma, \xi} \frac{|\partial_x^i \partial_\xi^j a(\gamma, \xi)|}{(1 + |\xi|)^{m-j}} \leq \text{const}(i, j) \quad \forall i, j \geq 0$$

Fact: 1) If $a \in S_m$, $b \in S_m$,

then $\exists c \in S_{m+m}$ so $\text{Op}(a) \text{Op}(b) = \text{Op}(c)$.

2) $c \approx \sum_{N=0}^w \frac{i^N}{N!} \partial_\xi^N a \partial_x^N b$ for large w .

Remark: The theory is not symmetric in the sense that it doesn't work if \mathbb{R}^2 is partitioned into rotated rectangles, for example.

§4 of Venkatesh's lectures

 Next task: Using orbit method, $L^2(\mathbb{R})$ should

be a rep of some group H so that the corresponding co-adjoint orbit of H is \mathbb{R}^2 .

- And the Kirillov character formula should correspond to

$$\text{Tr}(\text{Op}(a)) = \int_{\mathbb{R}^2} a(\gamma, \xi) d\gamma d\xi$$

- The decomposition of $L^2(\mathbb{R})$ and of \mathbb{R}^2 reflect our speculation.

Q: What is H then?

Suggests that localised functions (i.e. those in V_i) should be roughly eigenvectors under the action of H .

• \hat{f} being localised is equivalent to $f \approx e^{isx}$?

Why is this true?

$$\Rightarrow H \text{ should contain translation operator} \quad (\mathcal{T}_y f)(t) = e^{iyt} f(t)$$

Take the Fourier transform of the above to get that
 H should also contain multiplication operator ?

$$(\mathcal{M}_y f)(t) = e^{iyt} f(t).$$

$$\widehat{\mathcal{T}_y f}(s) = \widehat{f}(s-y) = e^{-isy} \widehat{f}(s) \quad f = e^{ist}$$

$$\int_{-\infty}^{\infty} e^{-2\pi i (y+t-s)z} dz$$