Advanced topics in algebra: Hochschild (co)homology, and the Hochschild–Kostant–Rosenberg decomposition

Pieter Belmans

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Introduction

What you are reading now are the lecture notes for a course on Hochschild (co)homology, taught at the University of Bonn, in the Sommersemester of 2017–2018. They are currently being written, and regularly updated. The table of contents is provisional.

The goal of the course is to give an introduction to Hochschild (co)homology, focussing on

1. its applications in *deformation theory* of algebras (and schemes)
2. and the role of the *Hochschild–Kostant–Rosenberg* decomposition in all this.

There are by many several texts on various aspects of Hochschild (co)homology. In particular the following books dedicate some chapters on Hochschild (co)homology:

1. chapter IX in Cartain–Eilenberg's *Homological algebra* [5],
2. the first chapters of Loday's *Cyclic homology* [16],
3. chapter 9 of Weibel's *An introduction to homological algebra* [20],
4. chapter 2 by Tsygan in Cuntz–Skandalis–Tsygan's *Cyclic homology in noncommutative geometry* [6],
5. chapter II by Schedler in the Bellamy–Rogalski–Schedler–Stafford–Wemyss' *Noncommutative algebraic geometry* [2].

There are also the following unpublished lecture notes:

1. Ginzburg's *Lectures on noncommutative geometry* [10]
2. Kaledin's Tokyo lectures [13] and Seoul lectures [14].

There is also Witherspoon's textbook-in-progress called *An introduction to Hochschild cohomology* [21], which is dedicated entirely to Hochschild cohomology and some its applications. So far this is the only textbook dedicated entirely to Hochschild (co)homology, and it is a good reference for things not covered in these notes.

Compared to the existing texts these notes aim to focus more on Hochschild (co)homology in algebraic geometry, using derived categories of smooth projective varieties. This point of view has been developed in several papers [3, 4, 15] and applied in many more dealing with semiorthogonal decompositions. But there is no comprehensive treatment, let alone starting from the basics of Hochschild (co)homology for algebras. These notes aim to fill this gap, where we start focussing on smooth projective varieties starting in the second half of chapter 2.

Now that we know what is supposed to go in this text, let us mention that the following will not be discussed: the relationship with algebraic K-theory via Chern characters, support varieties, deformation theory of abelian and dg categories, applications to Hopf algebras, topological versions of Hochschild (co)homology and related constructions, …

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Contents

Chapter 1

Algebras

Conventions Throughout these notes we will let k be a field. It is possible to develop much of the theory in the case for algebras which are flat over a commutative base ring without much extra effort, but we will not do so explicitly. The interested reader is invited to do so. There are also versions which are valid in a more general setting, but will refrain from discussing these.

At some points we will take k of characteristic zero, or algebraically closed. This will be mentioned explicitly.

If A is a k-algebra we will denote the *enveloping algebra* $A \otimes A^{\mathrm{opp}}$ of A by A^{e}, so that A-bimodules are the same as left A^{e}-modules.

1.1 Definition and first properties

1.1.1 Hochschild (co)chain complexes

We start with a seemingly ad hoc definition.

Definition 1. Let A be a k-algebra. The *bar complex* $\mathrm{C}^{\mathrm{bar}}_{\bullet}(A)$ of A is the cochain complex

$$(1.1)\quad \ldots \xrightarrow{\mathrm{d}_2} A \otimes_k A \otimes_k A \xrightarrow{\mathrm{d}_1} A \otimes_k A \to 0,$$

of A-bimodules, where we have $\mathrm{C}^{\mathrm{bar}}_n(A) := A^{\otimes n+2}$, hence $A \otimes_k A$ lives in degree 0, and the differentials $\mathrm{d}_n\colon \mathrm{C}^{\mathrm{bar}}_n(A) \to \mathrm{C}^{\mathrm{bar}}_{n-1}(A)$ are given by

$$(1.2)\quad \mathrm{d}_n(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}.$$

The A-bimodule structure (or equivalently left A^{e}-module structure) on $\mathrm{C}^{\mathrm{bar}}_n(A)$ is given by

$$(1.3)\quad (a \otimes b) \cdot (a_0 \otimes \ldots \otimes a_{n+1}) = aa_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}b.$$

We will also consider the morphism $\mathrm{d}_0\colon A \otimes_k A \to A$, which by the formula for d_n is nothing but the multiplication morphism $\mu\colon A \otimes_k A \to A$.

Remark 2. The terminology "bar complex" originates from the fact that an element $a_0 \otimes \ldots \otimes a_{n+1}$ is sometimes denoted $a_0[a_1|\ldots|a_n]a_{n+1}$.

Before we start studying the bar complex (for instance, at this point we haven't proven it is a complex), we introduce the following morphisms:

$$s_n\colon A^{\otimes n+2} \to A^{\otimes n+3} : a_0 \otimes \ldots \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_{n+1}. \tag{1.4}$$

Given that this is the first proof we will give details. We will see many similar proofs throughout the beginning of the notes, we will leave some of them as exercises.

Lemma 3. We have that

$$\mathrm{d}_{n+1} \circ s_n + s_{n-1} \circ \mathrm{d}_n = \mathrm{id}_{A^{\otimes n+2}}. \tag{1.5}$$

Proof. One computes that

$$\begin{aligned} &s_{n-1} \circ \mathrm{d}_n(a_0 \otimes \ldots \otimes a_n) \\ &\qquad = \sum_{i=0}^{n} (-1)^i 1 \otimes a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}, \\ &\mathrm{d}_{n+1} \circ s_n(a_0 \otimes \ldots \otimes a_n) \\ &\qquad = a_0 \otimes \ldots \otimes a_{n+1} + \sum_{i=1}^{n+1} (-1)^i 1 \otimes a_0 \otimes \ldots \otimes a_{i-1} a_i \otimes \ldots \otimes a_{n+1}, \end{aligned} \tag{1.6}$$

so everything but the identity cancels after reindexing. □

We can check that the d_i's indeed turn $\mathrm{C}^{\mathrm{bar}}_{\bullet}(A)$ into a chain complex.

Lemma 4. We have that $\mathrm{d}_{n-1} \circ \mathrm{d}_n = 0$.

Proof. Let us consider $n = 1$ first. Then $\mathrm{d}_0 \circ \mathrm{d}_1(a_0 \otimes a_1 \otimes a_2) = (a_0 a_1)a_2 - a_0(a_1 a_2)$, which is zero as A is associative.

For $n \geq 2$ we use induction, using (1.5). We have

$$\mathrm{d}_n \circ \mathrm{d}_{n+1} \circ s_n = \mathrm{d}_n - \mathrm{d}_n \circ s_{n-1} \circ \mathrm{d}_n = s_{n-2} \circ \mathrm{d}_{n-1} \circ \mathrm{d}_n = 0, \tag{1.7}$$

but as the image of s_n generates $A^{\otimes n+3}$ as a left A-module we get that $\mathrm{d}_n \circ \mathrm{d}_{n+1} = 0$. □

The bar complex didn't include A, but if we use the morphism $\mathrm{d}_0\colon A \otimes_k A \to A$ as defined above we get the following proposition.

Proposition 5. The bar complex of A is a free resolution of A as an A-bimodule, where the augmentation $\mathrm{d}_0\colon A \otimes_k A \to A$ is given by the multiplication.

Proof. By lemma 3 we see that the s_i's provides a contracting homotopy, hence the bar complex is exact, as a complex of A-bimodules.

We also check that the cokernel of d_1 is indeed the multiplication $A \otimes_k A \to A$. For this it suffices to observe that

$$\mathrm{d}_1(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2, \tag{1.8}$$

and that there exists a map $\operatorname{coker} \mathrm{d}_1 \to A$ mapping the class of $a_0 \otimes a_1$ to $a_0 a_1$. By the definition of d_1 it sends elements of $\operatorname{im} \mathrm{d}_1$ to zero, so it is well-defined. Its inverse is given by the morphism which sends a to $1 \otimes a$.

That $\mathrm{C}_n^{\mathrm{bar}}(A)$ is free as an A-bimodule follows from the isomorphisms of A-bimodules

(1.9) $$A^{\otimes n+2} \cong A^{\mathrm{e}} \otimes A^{\otimes n} \cong \bigoplus_{i\in I} A^{\mathrm{e}} \cdot 1 \otimes 1 \otimes a_i$$

where $\{a_i \mid i \in I\}$ is a vector space basis of $A^{\otimes n}$, and the first isomorphism is

(1.10) $$a_0 \otimes \ldots \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes a_1 \otimes \ldots \otimes a_n.$$

□

Definition 6. Let A be a k-algebra, and M an A-bimodule. The *Hochschild chain complex* $\mathrm{C}_\bullet(A, M)$ is $M \otimes_{A^{\mathrm{e}}} \mathrm{C}_\bullet^{\mathrm{bar}}(A)$, considered as a complex of k-modules, with differential $\mathrm{id}_M \otimes \mathrm{d}_n$.

Its homology is the *Hochschild homology of A with values in M*, and will be denoted $\mathrm{HH}_\bullet(A, M)$. If $M = A$, we'll write $\mathrm{HH}_n(A)$.

Dual to this we could instead of the tensor product use the Hom-functor, and obtain the dual notion of Hochschild cohomology.

Definition 7. Let A be a k-algebra, and M an A-bimodule. The *Hochschild cochain complex* $\mathrm{C}^\bullet(A, M)$ is $\mathrm{Hom}_{A^{\mathrm{e}}}(\mathrm{C}_\bullet^{\mathrm{bar}}(A), M)$, considered as a complex of k-modules, with differential $\mathrm{Hom}(\mathrm{d}_n, \mathrm{id}_M)$.

Its cohomology is the *Hochschild cohomology of A with values in M*, and will be denoted $\mathrm{HH}^\bullet(A, M)$. If $M = A$, we'll write $\mathrm{HH}^n(A)$.

Remark 8. Observe that one can recover the bar complex from the Hochschild complex:

(1.11) $$\mathrm{C}_\bullet^{\mathrm{bar}}(A) = \mathrm{C}_\bullet(A, A^{\mathrm{e}}).$$

Reinterpreting the Hochschild cochain complex The Hochschild (co)chain complexes were obtained by considering a specific free resolution of A as an A-bimodule, and constructing a (co)chain complex of vector spaces out of it. We can rephrase this complex of vector spaces a bit, where instead of $\mathrm{Hom}_{A^{\mathrm{e}}}(-,-)$ and $- \otimes_{A^{\mathrm{e}}} -$, we use $\mathrm{Hom}_k(-,-)$ and $- \otimes_k -$. This will be very useful for computations later on.

The proofs of the following two propositions follow from the fact that A^{e} only involves the first and last tensor factor of a bimodule in the bar complex. The explicit formula for the Hochschild differentials in (1.15) and (1.19) will be important for us in section 1.1.3.

Proposition 9. There exists an isomorphism of k-modules

(1.12) $$\varphi\colon\ \mathrm{C}^n(A, M) \xrightarrow{\cong} \mathrm{Hom}_k(A^{\otimes n}, M),$$

given by

(1.13) $$g \mapsto \left[a_1 \otimes \ldots \otimes a_n \mapsto g(1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1)\right],$$

whose inverse is given by

(1.14) $$f \mapsto \left[a_0 \otimes \ldots \otimes a_{n+1} \mapsto a_0 f(a_1 \otimes \ldots \otimes a_n) a_{n+1}\right].$$

The differential in $\mathrm{Hom}_k(A^\bullet, M)$ is then given by

(1.15) $$\begin{aligned} &\mathrm{d}_{\mathrm{Hoch}} f(a_1 \otimes \ldots \otimes a_{n+1}) \\ &\quad = a_1 f(a_2 \otimes \ldots \otimes a_{n+1}) \\ &\qquad + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\ &\qquad + (-1)^{n+1} f(a_1 \otimes \ldots \otimes a_n) a_{n+1} \end{aligned}$$

for $f \in \mathrm{Hom}_k(A^{\otimes n}, M)$.

Proposition 10. There exists an isomorphism of k-modules

$$(1.16)\quad \psi\colon\ \mathrm{C}_\bullet(A,M) \xrightarrow{\cong} M \otimes_k A^\bullet$$

given by

$$(1.17)\quad \psi(m \otimes_{A^e} a_0 \otimes \ldots \otimes a_{n+1}) = a_{n+1} m a_0 \otimes a_1 \otimes \ldots \otimes a_n,$$

whose inverse is given by

$$(1.18)\quad m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes_{A^e} 1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1.$$

The differential $\mathrm{d}_{\mathrm{Hoch}}\colon M \otimes_k A^{\otimes n} \to M \otimes_k A^{\otimes n-1}$ is then given by

$$(1.19)\quad \begin{aligned} &\mathrm{d}_{\mathrm{Hoch}}(m \otimes a_1 \otimes \ldots \otimes a_n) \\ &\quad = m a_1 \otimes \ldots \otimes a_n \\ &\quad\quad + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ &\quad\quad + (-1)^n a_n m \otimes a_1 \otimes \ldots \otimes a_{n-1} \end{aligned}$$

for $m \otimes a_1 \otimes \ldots \otimes a_n \in M \otimes_k A^{\otimes n}$.

Functoriality of Hochschild (co)homology Given an algebra morphism $f\colon A \to B$, or a bimodule morphism $g\colon M \to N$, we would like to understand how this interacts with taking Hochschild (co)homology. First of all: *Hochschild homology is covariantly functorial in both arguments.*

Proposition 11. Let $f\colon A \to B$ be an algebra morphism, and M a B-bimodule (which has an induced A-bimodule structure, denoted $f^*(M)$). Then

$$(1.20)\quad f_*\colon\ \mathrm{C}_\bullet(A, f^*(M)) \to \mathrm{C}_\bullet(B,M) : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes f(a_1) \otimes \ldots \otimes f(a_n)$$

gives a functor $\mathrm{HH}_\bullet(-, M)$.

Let $g\colon M \to N$ be an A-bimodule morphism. Then

$$(1.21)\quad g_*\colon\ \mathrm{C}_\bullet(A,M) \to \mathrm{C}_\bullet(A,N) : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto g(m) \otimes a_1 \otimes \ldots \otimes a_n$$

gives a functor $\mathrm{HH}_\bullet(A, -)$.

In particular, taking $M = A$ we can use the covariant functoriality in both arguments for Hochschild homology to get the following.

Corollary 12. Hochschild homology $\mathrm{HH}_\bullet(-)$ is a covariant functor from the category of associative k-algebras to the category of k-modules.

For Hochschild cohomology the situation is different: *Hochschild cohomology is contravariantly functorial in the first argument, and covariantly functorial in the second.*

Proposition 13. Let $f\colon A \to B$ be an algebra morphism, and M a B-bimodule (which has an induced A-bimodule structure). Then

$$(1.22)\quad f^*\colon\ \mathrm{C}^n(B,M) \to \mathrm{C}^n(A,M) : \varphi \mapsto \varphi \circ f^{\otimes n}$$

gives a (contravariant) functor $\mathrm{HH}^\bullet(-,M)$.

Let $g\colon M \to N$ be an A-bimodule morphism. Then

$$g_*\colon \mathrm{C}^n(A,M) \to \mathrm{C}^n(A,N) : \varphi \mapsto g \circ \varphi \tag{1.23}$$

gives a functor $\mathrm{HH}^\bullet(A,-)$.

Remark 14. So $\mathrm{HH}^\bullet(-)$ is *not* a functor (at least when we consider arbitrary morphisms between k-algebras), despite its appearance. We will come back to this in remark 20, and we will partially remedy this deficiency in section 2.3.

At this point it is also important that in some sources it is written that $\mathrm{HH}^\bullet(-)$ *is* a functor, see e.g. [17, §1.5.4]. But this is not the same functor, despite the similarity in notation! Indeed, in those situations one takes $M = A^\vee = \mathrm{Hom}_k(A,k)$ as the second argument. This makes the construction functorial (as the covariant functor in the second argument becomes contravariant), but one does not obtain the interpretation of Hochschild cohomology which will be used in this text. The construction in op. cit. has applications in studying cyclic cohomology and generalisations of the Chern character, which we will not go into here.

In section 2.3 we will greatly extend this functoriality for Hochschild homology, and discuss what can be done in the case of Hochschild cohomology. Remark that in the next section's corollary 16 we will obtain that Hochschild cohomology is a functor for Morita equivalences.

1.1.2 Hochschild (co)homology as Ext and Tor

In these notes we have *defined* Hochschild (co)homology as the (co)homology of an explicit (co)chain complex, which might seem ad hoc at first. But the bar complex of A being a free resolution of A as a bimodule over itself allows us to interpret Hochschild (co)homology in terms of more familiar constructions as explained in section 1.1.3.

Morover, the definition via the bar complex gives us an explicit description which will prove to be very useful in section 1.2 when we are discussing the extra structure on the Hochschild (co)chain complexes, which can conveniently be described by extra structure before taking cohomology. But it is of course an interesting question to find good intrinsic descriptions of the extra structure, and we will give further comments on this.

Theorem 15. There exist isomorphisms

$$\mathrm{HH}^i(A,M) \cong \mathrm{Ext}^i_{A^e}(A,M) \tag{1.24}$$

and

$$\mathrm{HH}_i(A,M) \cong \mathrm{Tor}_i^{A^e}(A,M). \tag{1.25}$$

Proof. By proposition 5 the bar complex is a free resolution of A as an A-bimodule. In particular it can serve as a flat (resp. projective) resolution when computing the derived functors of $A \otimes_{A^e} -$ (resp. $\mathrm{Hom}_{A^e}(A,-)$). ☐

In particular, we have that

$$\begin{aligned} \mathrm{HH}^0(A,M) &\cong \mathrm{Hom}_{A^e}(A,M), \\ \mathrm{HH}_0(A,M) &\cong M \otimes_{A^e} A. \end{aligned} \tag{1.26}$$

But these descriptions are not necessarily very illuminating at this point. In section 1.1.3 we will give more concrete interpretations.

An important observation using theorem 15 is that the Hochschild cohomology of the A-bimodule M only depends on the category of A-bimodules. In this generality it is due to Rickard [18].

Corollary 16. Hochschild (co)homology is Morita invariant.

Proof. Assume that A and B are Morita equivalent through the bimodules ${}_AP_B$ and ${}_BQ_A$. The equivalences of categories are given by $P \otimes_A -$ and $Q \otimes_B -$, and these functors preserve projective resolutions. We obtain isomorphisms

$$\begin{aligned}
&\mathrm{Ext}^n_A(P \otimes_B -, -) \cong \mathrm{Ext}^n_B(-, Q \otimes_A -)\\
&\mathrm{Ext}^n_A(-, P \otimes_B -) \cong \mathrm{Ext}^n_B(Q \otimes_A -, -)\\
&\mathrm{Tor}^A_n(P \otimes_B -, -) \cong \mathrm{Tor}^B_n(-, Q \otimes_A -)\\
&\mathrm{Tor}^A_n(-, P \otimes_B -) \cong \mathrm{Tor}^B_n(Q \otimes_A -, -)
\end{aligned} \tag{1.27}$$

where we are only using the left module structure, and we have similar expressions when using the right module structure.

Using theorem 15 and these isomorphisms we get for every A-bimodule M that

$$\begin{aligned}
\mathrm{HH}^n(A, M) &\cong \mathrm{Ext}^n_{A\otimes A^{\mathrm{opp}}}(A, M)\\
&\cong \mathrm{Ext}^n_{A\otimes A^{\mathrm{opp}}}(P \otimes_B Q, M)\\
&\cong \mathrm{Ext}^n_{B\otimes A^{\mathrm{opp}}}(Q, Q \otimes_A M)\\
&\cong \mathrm{Ext}^n_{B\otimes B^{\mathrm{opp}}}(B, Q \otimes_A M \otimes_A P\\
&\cong \mathrm{HH}^n(B, Q \otimes_A M \otimes_A P)
\end{aligned} \tag{1.28}$$

and likewise for Hochschild homology. ◻

In section 2.3 we will greatly extend this Morita invariance to derived Morita invariance.

1.1.3 Interpretation in low degrees

We will now give an interpretation for Hochschild (co)homology in low degrees, where we can explicitly manipulate the bar complex, or rather its reinterpretation as in propositions 9 and 10. For this we observe that the differential of the Hochschild chain complex in low degrees is given by

$$\begin{array}{ccccc}
M \otimes_k A \otimes_k A & \xrightarrow{\mathrm{d}} & M \otimes_k A & \xrightarrow{\mathrm{d}} & M\\
m \otimes a \otimes b & \longmapsto & ma \otimes b - m \otimes ab + bm \otimes a & & \\
 & & m \otimes a & \longmapsto & ma - am,
\end{array} \tag{1.29}$$

whilst for the Hochschild cochain complex $\mathrm{C}^\bullet(A, M)$

$$
(1.30)\quad \begin{aligned}
M &\xrightarrow{\mathrm{d}} \mathrm{Hom}_k(A, M) \xrightarrow{\quad\mathrm{d}\quad} \mathrm{Hom}_k(A \otimes_k A, M) \\
m &\longmapsto \mathrm{d}(m) \colon a \mapsto am - ma \\
& \qquad\qquad\quad f \longmapsto \mathrm{d}(f) \colon a \otimes b \mapsto af(b) - f(ab) + f(a)b
\end{aligned}
$$

and

$$
(1.31)\quad \begin{aligned}
\mathrm{Hom}_k(A \otimes_k A, M) &\xrightarrow{\quad\mathrm{d}\quad} \mathrm{Hom}_k(A \otimes_k A \otimes_k A, M) \\
g &\longmapsto \mathrm{d}(g) \colon a \otimes b \otimes c \mapsto ag(b \otimes c) - g(ab \otimes c) + g(a \otimes bc) - g(a \otimes b)c.
\end{aligned}
$$

Using these explicit descriptions in low degrees we can obtain the following.

Zeroth Hochschild homology

Proposition 17. We have that

$$(1.32)\quad \mathrm{HH}_0(A, M) \cong M/\langle am - ma \mid a \in A, m \in M\rangle$$

is the *module of coinvariants*. In particular, we have

$$(1.33)\quad \mathrm{HH}_0(A) \cong A/[A, A] = A_{\mathrm{ab}}.$$

Proof. This is immediate from the description of the morphism in (1.29). □

Remark 18. The vector space $[A, A]$ is usually not an ideal in A, so there is no obvious algebra structure on $\mathrm{HH}_0(A)$.

There is no one-size-fits-all description for Hochschild homology in higher degrees. But if A is commutative then a description in terms of differential forms is possible. We will come back to this in section 1.3.

Zeroth Hochschild cohomology

Proposition 19. We have that

$$(1.34)\quad \mathrm{HH}^0(A, M) \cong \{m \in M \mid \forall a \in A \colon am = ma\}$$

is the *submodule of invariants*. In particular, we have

$$(1.35)\quad \mathrm{HH}^0(A) \cong \mathrm{Z}(A).$$

Proof. This is immediate from the description of the morphism in (1.30). □

Remark 20. We can now give a new explanation of the non-functoriality of Hochschild cohomology using the interpretation of $\mathrm{HH}^0(A)$ as the center: taking the center of an algebra isn't a functor.

First Hochschild cohomology

Definition 21. A morphism $f\colon A \to M$ is a k-*derivation* if

$$f(ab) = af(b) + f(a)b. \tag{1.36}$$

We will denote the k-module of derivations by $\mathrm{Der}(A, M)$.

If $f = \mathrm{ad}_m$ for $m \in M$, where

$$\mathrm{ad}_m(a) = [a, m] = am - ma \tag{1.37}$$

then f is an *inner derivation*. We will denote the k-module of inner derivations by $\mathrm{InnDer}(A, M)$.

When $A = M$, we will use the notation $\mathrm{OutDer}(A)$ and $\mathrm{InnDer}(A)$. When A is commutative we will discuss derivations in more detail in section 1.3. For now, observe that in the commutative case there are no inner derivations.

Proposition 22. We have that

$$\mathrm{HH}^1(A, M) \cong \mathrm{OutDer}(A, M) \coloneqq \mathrm{Der}(A, M)/\,\mathrm{InnDer}(A, M) \tag{1.38}$$

are the *outer derivations*. In particular we have that

$$\mathrm{HH}^1(A) \cong \mathrm{OutDer}(A). \tag{1.39}$$

Proof. The description of the morphism in (1.30) tells us that Hochschild 1-cocycles are derivations, whilst Hochschild 1-coboundaries are inner derivations. □

At this point the first Hochschild cohomology $\mathrm{HH}^1(A)$ is just a vector space. But we can equip it with a Lie bracket. This is just a small piece of the extra structure that we will see in section 1.2.

Lemma 23. Let $D_1, D_2\colon A \to A$ be derivations. Then $[D_1, D_2] \coloneqq D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation. Moreover, if $D_2 = \mathrm{ad}_a$ is an inner derivation, for some $a \in A$, then $[D_1, \mathrm{ad}_a] = \mathrm{ad}_{D_1(a)}$.

Proof. From

$$\begin{aligned}
[D_1, D_2](ab) &= D_1(D_2(ab)) - D_2(D_1(ab)) \\
&= D_1\left(aD_2(b) + D_2(a)b\right) - D_2(aD_1(b) + D_1(a)b)) \\
&= aD_1(D_2(b)) + D_1(a)D_2(b) + D_2(a)D_1(b) + D_1(D_2(a))b \\
&\qquad - aD_2(D_1(b)) - D_2(a)D_1(b) - D_1(a)D_2(b) - D_1(D_2(a))b \\
&= aD_1(D_2(b)) - aD_2(D_1(b)) + D_1(D_2(a))b - D_2(D_1(a))b \\
&= a[D_1, D_2](b) + [D_1, D_2](a)b
\end{aligned} \tag{1.40}$$

we get that $[D_1, D_2]$ is indeed a derivation.

Similarly we compute

$$\begin{aligned}
[D_1, \mathrm{ad}_a](b) &= D_1(\mathrm{ad}_a(b)) - \mathrm{ad}_a(D_1(b)) \\
&= D_1(ab - ba) - (aD_1(b) - D_1(b)a) \\
&= aD_1(b) + D_1(a)b - bD_1(a) - D_1(b)a - aD_1(b) + D_1(b)a \\
&= D_1(a)b - bD_1(a) \\
&= \mathrm{ad}_{D_1(a)}(b).
\end{aligned} \tag{1.41}$$

□

Corollary 24. $\mathrm{HH}^1(A)$ has the structure of a Lie algebra.

Proof. By lemma 23 we have that $\mathrm{Der}(A)$ is a Lie algebra (bilinearity and alternativity are trivial, the Jacobi identity is an easy computation), whilst $\mathrm{InnDer}(A) \subseteq \mathrm{Der}(A)$ is a Lie ideal. So $\mathrm{OutDer}(A)$ has the structure of a Lie algebra, and so does $\mathrm{HH}^1(A)$ via proposition 22. □

Second Hochschild cohomology The following discussion is the first aspect of why we care about Hochschild cohomology in the context of these lecture notes: deformation theory.

Definition 25. Let A be a k-algebra, and M an A-bimodule. A *square-zero extension* of A by M is a surjection $f\colon E \twoheadrightarrow A$ of k-algebras, such that

1. $(\ker f)^2 = 0$ (which implies that it has an A-bimodule structure),
2. $\ker f \cong M$ as A-bimodules[1].

To see that $\ker f$ indeed has an A-bimodule structure, let e be a lift of $a \in A$. We will define $a \cdot m = em$ and $m \cdot a = me$ for $m \in \ker f$. If e' is another lift, then $e - e' \in \ker f$, so $(e-e')m \in (\ker f)^2 = 0$ means $em = em'$ and $me = me'$.

So we have a sequence

$$0 \to M \to E \to A \to 0. \tag{1.42}$$

We will impose an equivalence relation on square-zero extensions.

Definition 26. We say that $f\colon E \to A$ and $f'\colon E' \to A$ are *equivalent* if there exists an algebra morphism $\varphi\colon E \to E'$ (necessarily an isomorphism), such that

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & A \arrow[r, "f"] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & E' \arrow[r, "f'"] & A \arrow[r] & 0 \end{tikzcd} \tag{1.43}$$

commutes.

Under our standing assumption on k being a field the sequence (1.42) is split as a sequence of vector spaces. If we choose a splitting $s\colon A \to E$ we get an isomorphism $E \cong A \oplus M$ of vector spaces. Using this decomposition the multiplication law on E can be written as

$$(a, m) \cdot (b, n) = (ab, an + mb + g(a, b)) \tag{1.44}$$

for $g\colon A \otimes_k A \to M$. This morphism is called the *factor set.* The factor set is determined by the splitting s, which is not necessarily an algebra morphism, by $g(a, b) = s(ab) - s(a)s(b)$. One can check that the unit of E corresponds to $(1, -g(1, 1))$ in this description.

If we consider the multiplication $(a, 0) \cdot (b, 0) \cdot (c, 0)$ inside E, then the associativity of E is equivalent to

$$ag(b \otimes c) + g(a \otimes bc) - g(ab)c - g(ab \otimes c) = 0, \tag{1.45}$$

which corresponds to g being a Hochschild 2-cocycle, by (1.31).

[1] We will introduce an equivalence relation to deal with the choice of isomorphism here.

But there was a choice of splitting $s\colon A \to E$ involved in the definition of g. If $s'\colon A \to E$ is another splitting, then we obtain a different factor set g'. Comparing them gives

$$
\begin{aligned}
g'(a,b) - g(a,b) &= (s'(a)s'(b) - s'(ab)) - (s(a)s(b) - s(ab)) \\
&= s'(a)\,(s'(b) - s(b)) - (s'(ab) - s(ab)) + (s'(a) - s(a))\,s(b).
\end{aligned} \tag{1.46}
$$

But this is precisely the Hochschild differential applied to $s - s'$, which is a morphism $A \to M$ by construction, using the M-bimodule structure on M as discussed above. So the choice of a factor set gives a well-defined cohomology class.

If $g = 0$, then we call E the *trivial extension*.

Theorem 27. There exists a bijection

$$\mathrm{HH}^2(A,M) \cong \mathrm{AlgExt}(A,M) \tag{1.47}$$

such that $0 \in \mathrm{HH}^2(A,M)$ corresponds to the equivalence class of the trivial extension.

We will mostly be interested in the case where $M = A$. In this case we will call E an *square-zero deformation* or *first order deformation*. This is a particular case of an infinitesimal deformation, as will be discussed in section 1.5. When $M = A$, we are describing algebra structures on $A \oplus At$ such that $t^2 = 0$, so we can equivalently describe square-zero deformations of A as a $k[t]/(t^2)$-algebra E, such that $E \otimes_{k[t]/(t^2)} k \cong A$. The notion of equivalence becomes that of a $k[t]/(t^2)$-module automorphism which reduces to the identity when t is set to 0.

So far we haven't seen any examples of Hochschild cohomology, let alone an example where $\mathrm{HH}^2(A) \neq 0$. The following example gives an ad hoc description of a (non-trivial) infinitesimal deformation of the polynomial ring in 2 variables.

Example 28. Let $A = k[x,y]$. Then we can equip $k[x,y] \oplus tk[x,y]$ with a multiplication for which $y \cdot x = yx + t$, i.e. using the factor set $g(y,x) = 1$. This is an infinitesimal deformation of $k[x,y]$ in the direction of the Weyl plane. We will come back to this.

If $\mathrm{HH}^2(A) = 0$, then A does not have any square-zero deformations, and vice versa. Such algebras are called (infinitesimally, or absolutely) *rigid*[2].

Third Hochschild cohomology One can show that $\mathrm{HH}^3(A,M)$ classifies crossed bimodules, see [17, exercise E.1.5.1]. We will not discuss this here.

But we should at this point mention that the combination of $\mathrm{HH}^1(A)$, $\mathrm{HH}^2(A)$ and $\mathrm{HH}^3(A)$ will play an important role in the deformation theory of algebras, as discussed in section 1.5. The third Hochschild cohomology group will take on the role of obstruction space.

1.1.4 Examples

Explicit computations with the bar complex are often difficult, and only work in very elementary cases. We will collect a few of these examples, but we will also discuss some examples in which there exists a much smaller resolution that we can use, instead of the bar complex.

From now on we will focus on the case where $M = A$, occasionally we will mention what happens in the general case.

[2]Hochschild called such algebras *segregated* in his original paper.

Example 29 (The polynomial ring $k[t]$)**.** Instead of the bar complex we can use a very concrete resolution of $k[t]$ as a bimodule over itself. Observe that $k[t]^{\mathrm{e}} \cong k[x,y]$, and $k[t]$ as a $k[x,y]$-module has a free resolution

$$0 \to k[x,y] \overset{\cdot(x-y)}{\to} k[x,y] \to k[t] \to 0. \tag{1.48}$$

From this we immediately see that

$$\mathrm{HH}_i(k[t]) \cong \begin{cases} k[t] & i = 0,1 \\ 0 & i \geq 2. \end{cases} \tag{1.49}$$

and

$$\mathrm{HH}^i(k[t]) \cong \begin{cases} k[t] & i = 0,1 \\ 0 & i \geq 2. \end{cases} \tag{1.50}$$

This agreement between Hochschild homology and cohomology is no coincidence: $k[t]$ is a so called 1-Calabi–Yau algebra, so Poincaré–Van den Bergh duality applies, as in appendix B.2.

Example 30 (Finite-dimensional algebras)**.** If A is a finite-dimensional k-algebra, then it is possible to construct a small projective resolution of A as an A-bimodule. For details one is referred to [11, §1.5][3]

Applying this to $A = kQ$, where Q is a connected acyclic quiver, the resolution takes on the form

$$0 \to \bigoplus_{\alpha \in Q_1} A^{\mathrm{e}} e_{s(\alpha)} \otimes e_{t(\alpha)} \to \bigoplus_{v \in Q_0} A^{\mathrm{e}} e_v \otimes e_v \to A \to 0. \tag{1.51}$$

From the length of this resolution it is immediate that path algebras do not have deformations. Imposing relations on the quiver yields more complicated finite-dimensional algebras, and the explicit description of the resolution can be implemented in computer algebra, notably QPA[4].

Example 31 (Truncated polynomial algebras $k[t]/(t^n)$)**.** Again we want to use a small resolution of $A = k[t]/(t^n)$ as a bimodule over itself. We will use a 2-periodic resolution for this, which immediately tells us that the Hochschild (co)homology is itself 2-periodic, i.e.

$$\begin{aligned} \mathrm{HH}^i(A,M) &\cong \mathrm{HH}^{i+2}(A,M) \\ \mathrm{HH}_i(A,M) &\cong \mathrm{HH}_{i+2}(A,M) \end{aligned} \tag{1.52}$$

for any A-bimodule M, and $i \geq 1$. This would of course be impossible to read off from the definition using bar resolution.

This 2-periodic resolution is defined as follows: let $u = t \otimes 1 - 1 \otimes t$ and $v = \sum_{i=0}^{n-1} t^{n-1-i} \otimes t^i$. Then we will use

$$\ldots \xrightarrow{v\cdot} A^{\mathrm{e}} \xrightarrow{u\cdot} A^{\mathrm{e}} \xrightarrow{v\cdot} A^{\mathrm{e}} \xrightarrow{u\cdot} A^{\mathrm{e}} \xrightarrow{\mu} A \longrightarrow 0. \tag{1.53}$$

In exercise 32 a method of proving the exactness is suggested.

By applying $\mathrm{Hom}_{A^{\mathrm{e}}}(-,M)$ or $- \otimes_{A^{\mathrm{e}}} M$ to this sequence we get

$$0 \longrightarrow M \xrightarrow{0} M \xrightarrow{nt^{n-1}\cdot} M \xrightarrow{0} M \xrightarrow{nt^{n-1}\cdot} M \xrightarrow{0} \ldots \tag{1.54}$$

[3] I should probably give a self-contained discussion.

[4] `https://www.gap-system.org/Packages/qpa.html`

We always have that

$$\mathrm{HH}^0(A,M) \cong \mathrm{HH}_0(A,M) \cong M, \tag{1.55}$$

which we could also deduce from propositions 17 and 19.

For $i \geq 1$ the description depends on $\operatorname{char} k$. If $\gcd(n, \operatorname{char} k) = 1$ we obtain for i even

$$\mathrm{HH}^i(A,M) \cong \mathrm{HH}_i(A,M) \cong M/t^{n-1}M \tag{1.56}$$

and for i odd

$$\mathrm{HH}^i(A,M) \cong \mathrm{HH}_i(A,M) \cong tM. \tag{1.57}$$

On the other hand, if $\gcd(n, \operatorname{char} k) \neq 1$, then the morphism which is multiplication by nt^{n-1} is the zero morphism, so the sequence splits, and we obtain

$$\mathrm{HH}^i(A,M) \cong \mathrm{HH}_i(A,M) \cong M \tag{1.58}$$

for all $i \geq 1$.

1.1.5 Exercises

Exercise 32. Show that (1.53) is exact by showing that the maps s_i give a contracting homotopy, where for $i = -1$ we take $s_{-1}(1) = 1$, whilst for $m \geq 0$ we define

$$\begin{aligned} s_{2m}(1 \otimes t^j) &= -\sum_{l=1}^{j} t^{j-l} \otimes t^{l-1} \\ s_{2m+1}(1 \otimes x^j) &= \begin{cases} \delta_j^{n-1} \otimes 1 & j = n-1 \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{1.59}$$

Exercise 33. Let us denote $A = \mathrm{A}_1(k)$ the *first Weyl algebra*, defined as $k\langle x, y\rangle/(yx - xy - 1)$. It is the ring of differential operators on $\mathbb{A}^1_k = \operatorname{Spec} k[x]$, where y corresponds to $\partial/\partial x$.

Let V be a 2-dimensional vector space, and choose a basis $\{v, w\}$. Show that

$$0 \longrightarrow A^{\mathrm{e}} \otimes \textstyle\bigwedge^2 V \xrightarrow{f} A^{\mathrm{e}} \otimes V \xrightarrow{g} A^{\mathrm{e}} \longrightarrow 0 \tag{1.60}$$

where

$$f(1 \otimes 1 \otimes v \wedge w) = (1 \otimes x - x \otimes 1) \otimes w - (1 \otimes y - y \otimes 1) \otimes v \tag{1.61}$$

and

$$\begin{aligned} g(1 \otimes 1 \otimes v) &= 1 \otimes x - x \otimes 1 \\ g(1 \otimes 1 \otimes u) &= 1 \otimes y - y \otimes 1 \end{aligned} \tag{1.62}$$

gives a free resolution of A. Using this, show that

$$\begin{aligned} \mathrm{HH}^i(A) &= \begin{cases} k & i = 0 \\ 0 & i \neq 0 \end{cases}, \\ \mathrm{HH}_i(A) &= \begin{cases} k & i = 2 \\ 0 & i \neq 2 \end{cases}. \end{aligned} \tag{1.63}$$

This apparent duality between Hochschild homology and cohomology is not a coincidence in this case, see appendix B.2.

Exercise 34. We have seen that $\mathrm{HH}_\bullet(-)$ is a (covariant) functor. Show that

1. it sends products to direct sums, i.e.

$$\mathrm{HH}_\bullet(A \times B) \cong \mathrm{HH}_\bullet(A) \oplus \mathrm{HH}_\bullet(B), \tag{1.64}$$

2. it preserves sequential limits, i.e. if $A_i \to A_{i+1}$ for $i \in \mathbb{N}$ is a sequence of algebra morphisms, then

$$\mathrm{HH}_\bullet(\varinjlim A_i) \cong \varinjlim \mathrm{HH}_\bullet(A_i). \tag{1.65}$$

Now fixing A, show that $\mathrm{HH}_\bullet(A, -)$ sends a short exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{1.66}$$

of A-bimodules to a long exact sequence

$$\ldots \to \mathrm{HH}_n(A, M') \to \mathrm{HH}_n(A, M) \to \mathrm{HH}_n(A, M'') \to \ldots. \tag{1.67}$$

Exercise 35. Prove propositions 11 and 13.

1.2 Extra structure on Hochschild (co)homology

Hochschild homology and cohomology have a rich structure: they are more than just k-modules, which is how we defined them in the previous section. We will discuss the following structure in these notes.

1. Hochschild cohomology has both the structure of an associative algebra and a Lie algebra;
2. Hochschild homology is both a module and a representation over Hochschild cohomology;
3. if A is commutative, then Hochschild homology itself has an algebra structure.

We will take $A = M$ throughout here.

Observe that this is not an exhaustive list of the extra structure. We will not discuss the action of $\mathrm{HH}^\bullet(A)$ on $\mathrm{Ext}_A^\bullet(M,N)$ (see [21, §1.6]), the cut coproduct on Hochschild homology, generalisations of the structures discussed here when the A-bimodule has an algebra structure of its own, similar structures on the variations on cyclic homology, ...

1.2.1 Hochschild cohomology is a Gerstenhaber algebra

The first aspect that we deal with is the algebraic structure on Hochschild cohomology (and Hochschild cochains): it is both

- a graded-commutative algebra,
- a graded Lie (super-)algebra,

and these structures are compatible: we will call such a structure a Gerstenhaber algebra, see definition 52.

For Hochschild cochains the situation is somewhat more complicated, as some properties are only true *up to homotopy*. For now we will not go into many details regarding this, this might change later on in the notes.

Observe that we have already seen a small part of the algebra structure in proposition 19, and of the Lie algebra structure in corollary 24. We will now extend these structures to the entire Hochschild cohomology of A, and discuss their compatibility.

Originally the Lie bracket on Hochschild cochains was introduced by Gerstenhaber in [8] to prove that the multiplication on Hochschild cohomology is graded-commutative. But this Lie bracket is also very important for deformation theory, we will come back to this in section 1.5.

Associative algebra structure: cup product We will start with introducing the associative multiplication, both on $\mathrm{C}^\bullet(A)$, and by compatibility with the differential, on $\mathrm{HH}^\bullet(A)$. The graded-commutativity will have to wait for now.

Definition 36. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. The *cup product* of f and g is the element $f \cup g$ defined by

(1.68) $f \cup g(a_1 \otimes \ldots \otimes a_{m+n}) = f(a_1 \otimes \ldots \otimes a_m)g(a_{m+1} \otimes \ldots \otimes a_{m+n}).$

Lemma 37. The cup product makes $\mathrm{C}^\bullet(A)$ into a differential graded algebra, i.e. the cup product is associative and satisfies the *graded Leibniz rule*

(1.69) $\mathrm{d}_{m+n+1}(f \cup g) = \mathrm{d}_{m+1}(f) \cup g + (-1)^m f \cup \mathrm{d}_{n+1}(g).$

where $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$.

Proof. Associativity is immediate, as $(f \cup g) \cup h$ and $f \cup (g \cup h)$ involve multiplication inside A, which is associative.

The compatibility with the differential is the following computation, which follows immediately from the definitions. For the left-hand side we have

$$\begin{aligned}
&\mathrm{d}_{m+n+1}(f \cup g)(a_1 \otimes \ldots a_{m+n+1}) \\
&\quad = a_1(f \cup g)(a_2 \otimes \ldots \otimes a_{m+n+1}) \\
&\qquad + \sum_{i=1}^{m+n} (-1)^i (f \cup g)(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{m+n+1}) \\
&\qquad + (-1)^{m+n+1}(f \cup g)(a_1 \otimes \ldots \otimes a_{m+n}) a_{m+n+1}
\end{aligned} \tag{1.70}$$

whilst for the right-hand side we have

$$\begin{aligned}
&(\mathrm{d}_{m+1}(f) \cup g)(a_1 \otimes \ldots \otimes a_{m+n+1}) \\
&\quad = a_1 f(a_2 \otimes \ldots \otimes a_{m+1}) g(a_{m+2} \otimes \ldots \otimes a_{m+n+1}) \\
&\qquad + \sum_{i=1}^{m} f(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{m+1}) g(a_{m+2} \otimes \ldots \otimes a_{m+n+1}) \\
&\qquad + (-1)^{m+1} f(a_1 \otimes \ldots \otimes a_m) a_{m+1} g(a_{m+2} \otimes \ldots \otimes a_{m+n+1})
\end{aligned} \tag{1.71}$$

and

$$\begin{aligned}
&(-1)^m (f \cup \mathrm{d}_{n+1}(g))(a_1 \otimes \ldots \otimes a_{m+n+1}) \\
&\quad = (-1)^m f(a_1 \otimes \ldots \otimes a_m) a_{m+1} g(a_{m+2} \otimes \ldots \otimes a_{m+n+1}) \\
&\qquad + \sum_{i=1}^{n} (-1)^{m+i} f(a_1 \otimes \ldots \otimes a_m) g(a_{m+1} \otimes \ldots \otimes a_{m+i} a_{m+i+1} \otimes \ldots \otimes a_{m+n+1}) \\
&\qquad + (-1)^{m+n+1} f(a_1 \otimes \ldots \otimes a_m) g(a_{m+1} \otimes \ldots \otimes a_{m+n}) a_{m+n+1}.
\end{aligned} \tag{1.72}$$

It suffices to identify the last and first terms of (1.71) and (1.72), and reindex the summation in (1.72) to run from $m+1$ to $n+m$ to get the equality. ◻

By taking cohomology of the Hochschild cochain complex we get the following corollary.

Corollary 38. The Hochschild cohomology $\mathrm{HH}^\bullet(A)$ is a graded associative algebra.

This is only the first aspect of the algebraic structure of $\mathrm{C}^\bullet(A)$. Before we define the Lie bracket, we should mention that the cup product on the level of cohomology is actually commutative! This is one of the main results of [8].

Proposition 39. The Hochschild cohomology $\mathrm{HH}^\bullet(A)$ is a graded-commutative algebra, i.e. for $f \in \mathrm{HH}^m(A)$ and $g \in \mathrm{HH}^n(A)$ we have that

$$f \cup g = (-1)^{mn} g \cup f. \tag{1.73}$$

The proof of this result will require the Gerstenhaber bracket which will be defined shortly. We will show that the difference between $f \cup g$ and $g \cup f$ for two Hochschild cochains has a precise description as the differential of the circle product $f \circ g$, so that it vanishes in cohomology.

Observe that in proposition 19 we saw that $\mathrm{HH}^0(A) \cong \mathrm{Z}(A)$, so we at least already knew that the degree zero part was a commutative subalgebra. It turns out that in a precise sense Hochschild cohomology can be seen as a *derived center*.

Remark 40. Using theorem 15 we have another graded-commutative algebra structure on Hochschild cohomology, given by the Yoneda product on Ext-groups. One can show that the cup product and Yoneda product are actually identified under the isomorphism (1.24). We refer to [21] for details.

Lie algebra structure: Gerstenhaber bracket Next up is a Lie bracket on Hochschild cochains, which like the product is compatible with the Hochschild differential, hence descends to a Lie bracket on Hochschild cohomology.

Definition 41. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. Let us denote[5] the element $f \circ_i g$ of $\mathrm{C}^{m+n-1}(A)$, for $i = 1, \ldots, m$, defined by

$$(1.74)\quad f \circ_i g(a_1 \otimes \ldots \otimes a_{m+n-1}) = f\,(a_1 \otimes \ldots \otimes a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \ldots \otimes a_{m+n-1})\,.$$

The *circle product* of the Hochschild cochains f and g is the element $f \circ g \in \mathrm{C}^{m+n-1}(A)$ defined by

$$(1.75)\quad f \circ g \coloneqq \sum_{i=1}^{m} (-1)^{(i-1)(n+1)} f \circ_i g.$$

This circle product equips $\mathrm{C}^\bullet(A)$ with the structure of a so called *pre-Lie algebra*. In particular, it is not associative. We will not be interested in this structure on its own, as we are only interested in the structure induced by the following definition.

Definition 42. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. Then their *Gerstenhaber bracket* is the element $[f, g] \in \mathrm{C}^{m+n-1}(A)$ defined by

$$(1.76)\quad [f, g] \coloneqq f \circ g - (-1)^{(m-1)(n-1)} g \circ f.$$

The way Gerstenhaber proves essential properties of his bracket depends greatly on a detailed analysis of $- \circ_i -$ and $- \circ -$, and for details[6] one is referred to [8]. We will only summarise the intermediate steps in what follows.

Example 43. The Gerstenhaber product will play an important role when studying the deformation theory of algebras. If $m = n = 2$, and $f \in \mathrm{C}^2(A)$, then (at least when the characteristic is not 2)

$$(1.77)\quad [f, f] = 2\,(f(f(a \otimes b) \otimes c) - f(a \otimes f(b \otimes c)))\,.$$

Remark 44. Observe that the definition of the Gerstenhaber bracket did *not* use the algebra structure on A. But one can observe that the multiplication morphism $\mu\colon A \otimes_k A \to A$ is the coboundary of the identity, and

$$(1.78)\quad \mathrm{d}(f) = [f, -\mu]$$

makes the link between the algebra structure on A, the Hochschild differential and the Gerstenhaber bracket.

Even more is true: the cup product can also be expressed in terms of the $- \circ_i -$, as

$$(1.79)\quad f \cup g = (\mu \circ_0 f) \circ_{m-1} g$$

where $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$.

[5] The ambiguity with composition of functions is intentional: indeed, for $m = n = 1$ the circle product really *is* the composition of Hochschild 1-cochains.

[6] Gerstenhaber himself writes on page 86 of [9] that the Poisson identity relating the Gerstenhaber bracket to the cup product follows from "a (nasty) computation". I am not going to argue with this judgment.

The skew symmetry and Jacobi identity are discussed in [8, theorem 1]. These follow rather straightforwardly from the pre-Lie structure. Establishing that $\mathrm{C}^\bullet(A)$ has a pre-Lie structure is done by using that of a *pre-Lie system*, which takes all the $- \circ_i -$ into account. It is shown in [8, theorem 2] how such a pre-Lie system induces a pre-Lie algebra structure.

Proposition 45. Let $f \in \mathrm{C}^m(A,A)$, $g \in \mathrm{C}^n(A)$ and $h \in \mathrm{C}^p(A)$ be Hochschild cochains. Then

skew symmetry $[f,g] = -(-1)^{(m-1)(n-1)}[g,f]$

Jacobi identity $(-1)^{(m-1)(p-1)}[f,[g,h]] + (-1)^{(p-1)(n-1)}[h,[f,g]] + (-1)^{(n-1)(m-1)}[g,[h,f] = 0$

Remark 46. The skew symmetry means that we are considering graded Lie superalgebras, we will not consider graded Lie algebras that in the strict sense of the word.

Proof. The skew symmetry follows easily by replacing $[-,-]$ with its definition as the commutator of the circle product, and observing that the four terms appear with opposite signs.

For the proof of the Jacobi identity, one is referred to [8], as explained above. □

The next step is the compatibility with the Hochschild differential. In other words

Proposition 47. Let $f \in \mathrm{C}^m(A,A)$, $g \in \mathrm{C}^n(A)$ and $h \in \mathrm{C}^p(A)$ be Hochschild cochains. Then

$$\mathrm{d}([f,g]) = (-1)^{n-1}[\mathrm{d}(f),g] + [f,\mathrm{d}(g)]. \tag{1.80}$$

Proof. This follows from (1.78) and the Jacobi identity from proposition 45, applying (1.78) to $[f,g]$. □

From this we get the following corollary, which will be important for the deformation theory of algebras, see section 1.5. Recall that the axioms for a differential graded Lie algebra are precisely given by the results of proposition 45, except that there is a shift in the degree appearing.

Corollary 48. $\mathrm{C}^{\bullet+1}(A,A)$ is a differential graded Lie algebra.

Recall that in corollary 24 we saw that $\mathrm{HH}^1(A)$ has the structure of a Lie algebra. The following result tells us that it is a Lie subalgebra in degree 0 of a graded Lie algebra. It is clear from the definition of the Gerstenhaber bracket for elements in $\mathrm{C}^1(A)$ and the definition of the Lie algebra structure on $\mathrm{HH}^1(A)$ that they agree.

Proposition 49. $\mathrm{HH}^{\bullet+1}(A)$ is a graded Lie algebra.

Let us consider this graded Lie algebra structure in a special case.

Example 50. The Lie algebra $\mathrm{HH}^1(A)$ consisting of outer derivations acts on the Hochschild cohomology space $\mathrm{HH}^0(A)$, which we have shown to be the center $\mathrm{Z}(A)$ of A. If D is a derivation, and $z \in \mathrm{Z}(A)$ a central element, then

$$[D,z] = D \circ z - z \circ D = D \circ z = D(z) \tag{1.81}$$

commutes with every element $a \in A$, as one checks easily.

Commutativity of the cup product We can now prove the commutativity of the cup product on the level of cohomology. The main ingredient is given in proposition 51, which is a computation depending on the notion of a pre-Lie algebra that can be found in [8, theorem 3]. We will not reproduce it here[7].

[7] It is an interesting exercise to compute things in low degree, to get a feel for the formulas and the role of the Hochschild differential.

Proposition 51. Let $f \in \mathrm{C}^m(A)$ and $g \in \mathrm{C}^n(A)$ be Hochschild cochains. Then

$$f \cup g - (-1)^{mn} g \cup f = \mathrm{d}(g) \circ f + (-1)^m \mathrm{d}(g \circ f) + (-1)^{m-1} g \circ \mathrm{d}(f) \tag{1.82}$$

But this leads us immediately to the proof of the graded-commutativity of $\mathrm{HH}^\bullet(A)$.

Proof of proposition 39. In the notation of proposition 51, if f and g are Hochschild cocycles, then (1.82) becomes

$$f \cup g - (-1)^{mn} g \cup f = \mathrm{d}_{n+m+1}(f \circ g). \tag{1.83}$$

So the difference between the commutator of two cocycles is a coboundary, and it vanishes when taking cohomology. □

Gerstenhaber algebra structure The cup product and Gerstenhaber bracket on Hochschild cohomology define the structure of a super-commutative algebra and a graded Lie superalgebra. They are moreover compatible in the following sense. We assign a name to this structure, because as it turns out, this is *not* the only natural example of such a structure. We will discuss polyvector fields, and their connection to Hochschild cohomology, in section 1.3.

Definition 52. A graded vector space $A^\bullet$ is a *Gerstenhaber algebra* if

1. $A^\bullet$ has an (associative) super-commutative multiplication of degree 0;
2. $A^\bullet$ has a super-Lie bracket of degree -1;
3. these two structures are related via the *Poisson identity*

$$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|} b[a, c]. \tag{1.84}$$

Written out in full detail, we have that

$$\begin{aligned} |ab| &= |a| + |b| \\ ab &= (-1)^{|a||b|} ba \end{aligned} \tag{1.85}$$

for the multiplication, and

$$\begin{aligned} |[a, b]| &= |a| + |b| - 1 \\ [a, b] &= -(-1)^{(|a|-1)(|b|-1)} [b, a] \end{aligned} \tag{1.86}$$

for the Lie bracket.

The Poisson identity then tells us that $a \mapsto [a, -] \colon A^p \to A^{p-1}$ is a derivation of degree $p - 1$.

Proposition 53. Let A be an associative k-algebra. Then $\mathrm{HH}^\bullet(A)$ is a Gerstenhaber algebra.

Proof. In proposition 39 and proposition 49 we have discussed the algebra and Lie algebra structure. The missing ingredient is the compatibility between these to structures through the Poisson identity. The proof of this goes along the same lines as the commutativity of the Gerstenhaber product: one shows that on the level of Hochschild cochains the obstruction to the Poisson identity is a certain coboundary given in [8, theorem 5]. This is a quite technical computation, and we will not reproduce it here. □

Remark 54. The cup product and Gerstenhaber bracket on the level of Hochschild cochain complexes do *not* satisfy the Poisson identity, nor is the dg algebra structure graded-commutative, so they do not give an immediate dg translation of a Gerstenhaber algebra structure. But there are homotopical versions of this structure, such as that of a B_∞-and G_∞-algebra, which fixes this incompatibility by introducing higher homotopies.

At this point we should mention that these (and other) homotopical structures form part of the program on the Deligne conjecture[8]). We will not go further into this for the time being, but this operadic picture is an important modern incarnation of the extra structure that we have discussed up to now.

In proposition 79 we will see another example of a Gerstenhaber algebra. These two examples are very closely related, and their story forms one of the main topics of these notes.

1.2.2 Hochschild homology is a Gerstenhaber module for Hochschild cohomology

For arbitrary algebras A there is no internal structure[9] on $\mathrm{HH}_\bullet(A)$ or $\mathrm{HH}_\bullet(A, M)$. But there are interesting *actions* of $\mathrm{HH}^\bullet(A)$ on $\mathrm{HH}_\bullet(A)$, such that $\mathrm{HH}_\bullet(A)$ is

- a module under the graded-commutative multiplication,
- a representation for the Gerstenhaber bracket

which are compatible in a certain way. The combination of these structures will be called a Gerstenhaber module, and they constitute an important part of the so-called Gerstenhaber (pre)calculus on the pair $(\mathrm{C}^\bullet(A), \mathrm{C}_\bullet(A))$. As we will not discuss this again until the very end[10] we will content ourselves with giving the definitions.

Observe that there are no good written proofs of the compatibility of these operations with the Hochschild differentials. Feel free to take this up as a challenge.

The cap product First up, the action by multiplication, i.e. the module structure.

Definition 55. Let M be an A-bimodule. Let $f \in \mathrm{C}^n(A)$ and $m \otimes a_1 \otimes \ldots \otimes a_p \in \mathrm{C}_p(A, M)$. Then their *cap product* is

$$f \cap (m \otimes a_1 \otimes \ldots \otimes a_p) = \begin{cases} (-1)^n m f(a_1 \otimes \ldots \otimes a_n) \otimes a_{n+1} \otimes \ldots a_p & p \geq n \\ 0 & p < n \end{cases} \tag{1.87}$$

which is an element of $\mathrm{C}_{p-n}(A, M)$.

One can then prove the following result.

Proposition 56. $\mathrm{C}_i(A, M)$ is a differential graded module over $\mathrm{C}^\bullet(A)$.

From this we get the following.

Corollary 57. $\mathrm{HH}_\bullet(A, M)$ is a graded module for the graded-commutative algebra $\mathrm{HH}^\bullet(A)$.

Remark 58. In particular we have that $\mathrm{HH}_i(A, M)$ is a module over $\mathrm{HH}^0(A) \cong \mathrm{Z}(A)$.

[8] Stated in 1993 in a letter to Gerstenhaber–May–Stasheff, now a theorem with proofs due to Tamarkin, McClure–Smith, Kontsevich–Soibelman, ...

[9] If A is commutative we discuss the shuffle product in section 1.2.3.

[10] At least for now. The interested reader is invited to prove the following properties him- or herself.

The Lie derivative The next step is the action by the Lie bracket.

Definition 59. Let $f \in \mathrm{C}^{n+1}(A)$ and $a_0 \otimes a_1 \otimes \ldots \otimes a_p \in \mathrm{C}_p(A, M)$. Then the *Lie derivative* of $a_0 \otimes a_1 \otimes \ldots \otimes a_p$ with respect to f is

$$\begin{aligned} \mathrm{L}_f(a_0 \otimes a_1 \otimes \ldots \otimes a_p) &= \sum_{i=0}^{p-n} (-1)^{ni} a_0 \otimes \ldots \otimes a_{i-1} \otimes f(a_i \otimes \ldots \otimes a_{i+n}) \otimes a_{i+n+1} \ldots \otimes a_p \\ &+ \sum_{j=p-n}^{p-1} (-1)^{p(j+1)} f(a_{j+1} \otimes \ldots \otimes a_p \otimes a_0 \otimes \ldots \otimes a_{n-p+j}) \otimes a_{p-n+j+1} \otimes \ldots \otimes a_j \end{aligned} \tag{1.88}$$

One can then prove the following result.

Proposition 60. $\mathrm{C}_i(A)$ is a differential graded Lie representation over $\mathrm{C}^{\bullet+1}(A)$.

From this we get the following.

Corollary 61. $\mathrm{HH}_\bullet(A)$ is a representation of the graded Lie algebra $\mathrm{HH}^{\bullet+1}(A)$.

We can combine these into the notion of a Gerstenhaber module, and discuss the notion of a Gerstenhaber (pre)calculus. We will not do this for now.

1.2.3 The shuffle product on Hochschild homology

In general $\mathrm{HH}_\bullet(A)$ is only a graded $\mathrm{HH}^\bullet(A)$-module. But if A is commutative we can equip it with its own product. The algebra structure on $\mathrm{HH}_\bullet(A)$ for A commutative is actually induced using a pairing

$$\mathrm{C}_\bullet(A, M) \otimes_k \mathrm{C}_\bullet(B, N) \to \mathrm{C}_\bullet(A \otimes_k B, M \otimes_k N) \tag{1.89}$$

which is defined for arbitrary algebras A and B, and bimodules M and N (unlike in the rest of this section we will use M and N to make the formulas a bit more transparent, but we will have $M = A$ and $N = B$ in applications). This will be the shuffle product from the title of this section.

Definition 62. A *(p, q)-shuffle* is an element σ of Sym_{p+q} such that $\sigma(i) < \sigma(j)$ whenever

1. $1 \leq i < j \leq p$,
2. or $p + 1 \leq i < j \leq p + q$.

The subset of (p, q)-shuffles inside the symmetric group is denoted $\mathrm{Sh}_{p,q}$.

We can define an action of Sym_n on $\mathrm{C}_n(A, M)$, by setting

$$\sigma \cdot (m \otimes a_1 \otimes \ldots \otimes a_n) \coloneqq m \otimes a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(n)} \tag{1.90}$$

for $\sigma \in \mathrm{Sym}_n$ and $m \otimes a_1 \otimes \ldots a_n \in \mathrm{C}_n(A, M)$.

Definition 63. The *(p, q)-shuffle product* for A and B is the morphism

$$\mathrm{sh}_{p,q}(-,-) = - \times -\colon \mathrm{C}_p(A, M) \otimes_k \mathrm{C}_q(B, N) \to \mathrm{C}_{p+q}(A \otimes_k B, M \otimes_k N) \tag{1.91}$$

which sends $(m \otimes a_1 \otimes \ldots a_p) \otimes (n \otimes b_1 \otimes \ldots \otimes b_q)$ to

$$\sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)\sigma \cdot \Big((m \otimes n) \otimes (a_1 \otimes 1) \otimes \ldots \otimes (a_p \otimes 1) \otimes (1 \otimes b_1) \otimes \ldots \otimes (1 \otimes b_q)\Big) \tag{1.92}$$

The next lemma shows that the Hochschild homology differential is a graded derivation for the shuffle product. For a proof, see [17, proposition 4.2.2].

Lemma 64. Let $m \otimes a_1 \otimes \ldots \otimes a_p \in \mathrm{C}_p(A, M)$ and $n \otimes b_1 \otimes \ldots \otimes b_q \in \mathrm{C}_q(B, N)$ be Hochschild chains. Then

(1.93)
$$\begin{aligned} &\mathrm{d}\left((m \otimes a_1 \otimes \ldots \otimes a_p) \times (n \otimes b_1 \otimes \ldots \otimes b_q)\right) \\ &\quad = \mathrm{d}(m \otimes a_1 \otimes \ldots \otimes a_p) \times (n \otimes b_1 \otimes \ldots \otimes b_q) + (-1)^p (m \otimes a_1 \otimes \ldots \otimes a_p) \times \mathrm{d}(n \otimes b_1 \otimes \ldots \otimes b_q). \end{aligned}$$

Proof. Let us write the ith summand of the differential as in (1.19) by d_i, indexed by $i = 0, \ldots, n$. Let us moreover write

(1.94) $$(m \otimes a_1 \otimes \ldots \otimes a_p) \times (n \otimes b_1 \otimes \ldots \otimes b_q) = \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma)(m \otimes n) \otimes c_1 \otimes \ldots \otimes c_{p+q}$$

where c_i is in the set $\{a_1 \otimes 1, \ldots, a_p \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_q\}$. Now consider

(1.95) $$\mathrm{d}_i((m \otimes n) \otimes c_1 \otimes \ldots \otimes c_{p+q},$$

for $i = 0, \ldots, n$. We now explain what happens with (1.95) on a case-by-case analysis.

- If $i = 0$, then $c_1 = a_1 \otimes 1$ (resp. $c_1 = 1 \otimes b_1$), and (1.95) appears in the first summand (resp. second summand) of the right-hand side of (1.93).
- The case $i = n$ is similar.
- If $i = 1, \ldots, n-1$ then we distinguish two cases:
  1. If c_i and c_{i+1} are elements of the form $a \otimes 1$ (resp. $1 \otimes b$) then they appear in the first (resp. second summand) of the right-hand side of (1.93).
  2. Otherwise we can permute them, as they will still arise from the application of a different (p, q)-shuffle, in which case we can cancel them, as they appear with opposite signs in the shuffle product.

□

Using the shuffle product we can construct the Künneth formula for Hochschild homology: we will combine the (p, q)-shuffles in the following way

(1.96) $$\mathrm{sh}_n \coloneqq \sum_{p+q=n} \mathrm{sh}_{p,q} \colon (\mathrm{C}_\bullet(A) \otimes_k \mathrm{C}_\bullet(B))_n = \bigoplus_{p+q=n} \mathrm{C}_p(A) \otimes_k \mathrm{C}_q(B) \to \mathrm{C}_n(A \otimes_k B).$$

Proposition 65. The morphism $\mathrm{sh}_\bullet$ is a morphism of chain complexes.

Proof. By lemma 64 we can express $\mathrm{d} \circ \mathrm{sh}_{p,q}(-,-)$ in terms of $\mathrm{sh}_{p-1,q}(\mathrm{d}(-),-)$ and $\mathrm{sh}_{p,q-1}(-,\mathrm{d}(-))$, which with the appropriate signs gives the differential in the tensor product of chain complexes. □

But $\mathrm{sh}_\bullet$ is not just an morphism of chain complexes: it is actually a quasi-isomorphism. The proof of this result can be found [20, §9.4].

Theorem 66 (Künneth formula)**.** The shuffle product $\mathrm{sh}_\bullet$ induces an isomorphism

(1.97) $$\mathrm{HH}_\bullet(A) \otimes_k \mathrm{HH}_\bullet(B) \cong \mathrm{HH}_\bullet(A \otimes_k B).$$

Remark 67. Observe that a similar statement is not true for Hochschild cohomology, at least not without conditions on A and B. In exercise 71 a suggestion for a counterexample is given. In [20, §9.4] the condition that at least one of them is finite-dimensional is used. It is not clear to me whether this can be generalised.

If we now impose commutativity, then the multiplication gives us a morphism of algebras

$$\mu\colon A \otimes_k A \to A. \tag{1.98}$$

Using functoriality of the Hochschild chain complex, we obtain a morphism

$$\mathrm{C}_\bullet(A \otimes_k A) \to \mathrm{C}_\bullet(A). \tag{1.99}$$

One can then prove that this equips the Hochschild chain complex with the structure of a commutaitve differential graded algebra [20, proposition 9.4.2], and therefore we have the following.

Proposition 68. $\mathrm{HH}_\bullet(A)$ is a graded-commutative algebra.

1.2.4 Exercises

Exercise 69. Let $\mathfrak{g}$ be a Lie algebra. Equip $\bigwedge^\bullet \mathfrak{g}$ with the exterior product as multiplication, and the unique extension of the Lie bracket on $\bigwedge^1 \mathfrak{g}$ to all of $\bigwedge^\bullet \mathfrak{g}$. Show that this is a Gerstenhaber algebra.

Exercise 70. Use the definition of the circle product to check remark 44.

Exercise 71. Let K, L be fields of infinite transcendence degree over k. Then

$$\mathrm{HH}^\bullet(K \otimes_k L) \ncong \mathrm{HH}^\bullet(K) \otimes_k \mathrm{HH}^\bullet(L). \tag{1.100}$$

1.3 The Hochschild–Kostant–Rosenberg isomorphism

The goal of this section is to discuss the Hochschild–Kostant–Rosenberg isomorphism, which identifies the Hochschild (co)homology of a regular *commutative* k-algebra A with its polyvector fields and differential forms. It is given as [12, theorem 5.2], where the interpretation from theorem 15 is used to make the link with Hochschild (co)homology.

To understand where the isomorphism comes from, recall that we have identifications

$$\begin{cases} \mathrm{HH}^0(A) \cong A & \text{proposition 19} \\ \mathrm{HH}^1(A) \cong \mathrm{Der}(A) & \text{proposition 22} \end{cases} \tag{1.101}$$

and

$$\begin{cases} \mathrm{HH}_0(A) \cong A & \text{proposition 17} \\ \mathrm{HH}_1(A) \cong \Omega^1_A & \text{proposition 76} \end{cases} \tag{1.102}$$

where the identification for $\mathrm{HH}_1(A)$ stricto sensu is not yet known[11].

Then the Hochschild–Kostant–Rosenberg isomorphism (see theorem 94) tells us that we can get *all* of the Hochschild (co)homology by taking exterior powers of what we have in degree 1, and that this is an isomorphism of graded commutative algebras: by propositions 39 and 68 we have that $\mathrm{HH}^\bullet(A)$ and $\mathrm{HH}_\bullet(A)$ are graded commutative, and the exterior product is graded commutative by construction.

1.3.1 Polyvector fields and differential forms

Let us introduce the module Ω^1_A which already made an appearance in (1.102) without being defined.

Definition 72. The *module of Kähler differentials* Ω^1_A is the A-module which is generated by the symbols $\mathrm{d}a$ for $a \in A$, subject to the relations[12]

$$\mathrm{d}(\lambda a + \mu b) = \lambda \mathrm{d}a + \mu \mathrm{d}b \tag{1.103}$$

for all $\lambda, \mu \in k$ and $a, b \in A$, and

$$\mathrm{d}ab = a\mathrm{d}b + b\mathrm{d}a \tag{1.104}$$

for all $a, b \in A$.

The module of Kähler differentials appears in many ways in this context. First of all, it satisfies a well-known universal property: it co-represents the functor of derivations, via the *universal derivation*

$$\mathrm{d}\colon A \to \Omega^1_A : a \mapsto \mathrm{d}\,a. \tag{1.105}$$

Proposition 73. We have an isomorphism

$$\mathrm{Hom}_A(\Omega_A, M) \cong \mathrm{Der}(A, M) \tag{1.106}$$

sending $\alpha\colon \Omega_A \to M$ to $\mathrm{d} \circ \alpha : A \to M$, giving an isomorphism of functors $\mathrm{Hom}_A(\Omega_A, -) \cong \mathrm{Der}(A, -)$.

[11] The dilemma is whether to give a preliminary discussion of Ω^1_A in section 1.1.3, or postpone it until we have time to discuss it in detail. We have opted for the latter.

[12] To be precise, and consistent with the notation in the literature, we should denote this by $\Omega^1_{A/k}$, making the dependence on the base field explicit. But then we should also do this for $\mathrm{Der}(A)$, and likewise for $\mathrm{HH}^\bullet$ and $\mathrm{HH}_\bullet$, which we won't.

Recall from proposition 22 that $\mathrm{Der}(A, M) \cong \mathrm{HH}^1(A, M)$. So we have that $\mathrm{HH}^1(A) \cong \mathrm{Der}(A) \cong \mathrm{Hom}_A(\Omega_A, A)$. This is the first ingredient in the proof of the Hochschild–Kostant–Rosenberg isomorphism for Hochschild cohomology. In geometric notation, when $X = \operatorname{Spec} A$, we have $\mathrm{HH}^1(A) \cong \mathrm{T}_X$.

There is a second description of the Kähler differentials which is useful to us. For a proof of this standard fact one is referred to [19, tag 00RW].

Proposition 74. Let $I := \ker(\mu\colon A \otimes_k A \to A)$. Then the morphism

$$\Omega^1_A \to I/I^2 : a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1 \tag{1.107}$$

is an isomorphism of A-modules.

Remark 75. If A is noncommutative, then one denotes $\Omega^1_{\mathrm{nc}}(A) := I$ the bimodule of noncommutative differential forms on A. In that case (1.106) takes on the form

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_{A^e}(\Omega^1_{\mathrm{nc}}(A), M). \tag{1.108}$$

For more information, one is referred to [10, §10] or [21, §3.2].

Finally we can relate Ω^1_A to Hochschild homology, just like we have already done for Hochschild cohomology, which is the first step in understanding the Hochschild–Kostant–Rosenberg isomorphism for Hochschild homology.

Proposition 76. Let M be a symmetric A-bimodule. Then

$$\mathrm{HH}_1(A, M) \cong M \otimes_A \Omega^1_A. \tag{1.109}$$

In particular we have

$$\mathrm{HH}_1(A) \cong \Omega^1_A. \tag{1.110}$$

Proof. By assumption the morphism $M \otimes_k A \to M$ is the zero morphism, as this is the Hochschild differential as in (1.29) and M is symmetric, so $\mathrm{HH}_1(A, M)$ is the quotient of $M \otimes_k A$ by the subspace generated by $ma \otimes b - m \otimes ab + bm \otimes a$. So the morphism $\mathrm{HH}_1(A, M) \to M \otimes_A \Omega^1_A$ sending $m \otimes a$ to $m \otimes \mathrm{d}a$ is well-defined by (1.104).

In the other direction we consider the morphism $M \otimes_A \Omega^1_A \to \mathrm{C}_1(A, M)$ sending $m \otimes a\mathrm{d}b$ to $ma \otimes b$. This morphism lands in $\mathrm{Z}_1(A, M)$ by assumption, and one checks that the maps on cohomology are inverse. □

1.3.2 Gerstenhaber algebra structure on polyvector fields

Using $\mathrm{Der}(A)$ we can construct a new Gerstenhaber algebra, which will be closely related to the Gerstenhaber algebra structure on Hochschild cohomology. We will do this by considering $\bigwedge^\bullet \mathrm{Der}(A)$, the polyvector fields (or multiderivations) on A. On $\bigwedge^\bullet \mathrm{Der}(A)$ we can consider the exterior product of polyvector fields, which equips it with the structure of a graded commutative algebra.

The space of derivations is the algebraic version of the vector fields on a manifold. As such, it is equipped with a Lie bracket. We can extend this Lie bracket to all of $\bigwedge^\bullet \mathrm{Der}(A)$ in the following way.

Definition 77. Let $\alpha_1 \wedge \ldots \wedge \alpha_m \in \bigwedge^m \mathrm{Der}(A)$ and $\beta_1 \wedge \ldots \wedge \beta_n \in \bigwedge^m \mathrm{Der}(A)$ be polyvector fields. Their *Schouten–Nijenhuis bracket*[13] is given by

(1.111)

[13] Sometimes just Schouten bracket. I like to speculate that this is purely for pronunciation reasons.

$$[\alpha_1\wedge\ldots\wedge\alpha_m,\beta_1\wedge\ldots\wedge\beta_n] \coloneqq \sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j+m-1}[\alpha_i,\beta_j]\wedge\alpha_1\wedge\ldots\wedge\widehat{\alpha_i}\wedge\ldots\wedge\alpha_m\wedge\beta_1\wedge\ldots\wedge\widehat{\beta_j}\wedge\ldots\wedge\beta_n.$$

This bracket is the unique extension to a graded Lie algebra structure when one imposes that $[D,z]=D(z)$ for $D\in\mathrm{HH}^1(A)$ and $z\in\mathrm{HH}^0(A)\cong A$ as in example 50 and $[D_1,D_2]=D_1\circ D_2-D_2\circ D_1$ for $D_1,D_2\in\mathrm{HH}^1(A)$ as in corollary 24. The following lemma is proved by staring at the signs.

Lemma 78. The Schouten–Nijenhuis bracket equips $\mathrm{Der}(A)$ with the structure of a graded Lie algebra.

Because the Schouten–Nijenhuis bracket was defined in terms of the generators of the algebra, we obtain the following.

Proposition 79. The exterior product and Schouten–Nijenhuis bracket equip $\bigwedge^\bullet\mathrm{Der}(A)$ with the structure of a Gerstenhaber algebra.

Remark 80. We have not yet precisely defined what the dg version of a Gerstenhaber algebra is (as it requires to understand operations up to homotopy), so it's not clear what exactly the extra structure induced by the cup product and Gerstenhaber bracket on $\mathrm{C}^\bullet(A)$ is. But observe that we can equip $\bigwedge^\bullet\mathrm{Der}(A)$ with the zero differential, in which case it will be a (strict) dg Gerstenhaber algebra. Then the Hochschild–Kostant–Rosenberg isomorphism for Hochschild cohomology can be upgraded to Kontsevich formality: a quasi-isomorphism of "dg Gerstenhaber algebras" between $\bigwedge^\bullet\mathrm{Der}(A)$ and $\mathrm{C}^\bullet(A)$, i.e. Hochschild cochains are quasi-isomorphic to their cohomology, and we know exactly what this cohomology is. We might discuss formality results later on in these notes.

1.3.3 Gerstenhaber module structure on differential forms

This section will be expanded at some point.

1.3.4 The Hochschild–Kostant–Rosenberg isomorphism: Hochschild homology

We now come to the first proof of the Hochschild–Kostant–Rosenberg isomorphism, for smooth commutative algebras. We will do this in a rather classical fashion for now, based on [17, §1.3, §3.4]. The proof for smooth projective varieties in section 3.3 will use more advanced machinery. It should be remarked that it *is* actually possible to globalise the current proof without using the machinery of derived categories and Atiyah classes in an essential way, and maybe this will be discussed too at some point.

The statement of the Hochschild–Kostant–Rosenberg isomorphism in this setting is the following.

Theorem 81. Let A be a smooth k-algebra and M a symmetric A-bimodule. Then the antisymmetrisation morphism (1.113) induces an isomorphism

(1.112) $\Omega^n_{A/k}\otimes_A M\to\mathrm{HH}_n(A,M)$.

When $A=M$ this isomorphism is an isomorphism of graded k-algebras.

The proof given in this section naturally splits in two pieces:

1. constructing the antisymmetrisation morphism $\epsilon_n\colon M\otimes_A\Omega^n_A\to\mathrm{HH}_n(A,M)$ of A-modules (no smoothness is required here);
2. showing that it is an isomorphism by checking it at every maximal ideal, using the description of Hochschild homology as Tor and an explicit free resolution (the Koszul resolution) in the local setting (smoothness is required here).

The construction of the morphism is done in proposition 85, and checking that it locally is an isomorphism is done after we prove proposition 91.

Note that, if we would only be interested in $\mathrm{HH}_n(A)$ and not $\mathrm{HH}_n(A,M)$, the construction of the morphism in step 1 can be done via a universal property, based on the graded-commutative algebra structure from proposition 68. We will take this approach in the case of Hochschild cohomology in section 1.3.5. Observe that this was the generality of the original paper of Hochschild–Kostant–Rosenberg, i.e. they only considered $\mathrm{HH}_\bullet(A)$ and $\mathrm{HH}^\bullet(A)$.

The antisymmetrisation morphism In proposition 76 we saw that the first Hochschild homology is isomorphic to the Kähler differentials, with the morphism $\Omega^1_A \otimes_A M \to \mathrm{HH}_1(A,M)$ being of the form $m \otimes a\mathrm{d}b \mapsto ma \otimes b$. We can extend these morphisms to differential n-forms and $\mathrm{HH}_n(A,M)$ in the following way. First we introduce the *antisymmetrisation map*

$$\epsilon_n \colon M \otimes_k \bigwedge^n A \to \mathrm{C}_n(A,M) : m \otimes a_1 \wedge \ldots \wedge a_n \mapsto \sum_{\sigma \in \mathrm{Sym}_n} \mathrm{sgn}(\sigma)\sigma \cdot m \otimes a_1 \otimes \ldots \otimes a_n \tag{1.113}$$

where the action of σ is defined analogously to (1.90). Remark that from this point on we will have to be careful about whether $\otimes$ or $\bigwedge$ is taken over k or A.

We want to turn this into a morphism $M \otimes_A \Omega^n_A \to \mathrm{HH}_n(A,M)$, so we need to show that

1. ϵ_n is compatible with the Hochschild differential;
2. it factors through $M \otimes_A \Omega^n_A$.

To do the first, we will use a technical trick, inspired by Chevalley–Eilenberg (co)homology for Lie algebras. If $\mathfrak{g}$ is a Lie algebra, and M a Lie module over it, then the *Chevalley–Eilenberg differential* is

$$\begin{aligned} \mathrm{d}_{\mathrm{CE}} \colon M \otimes_k \bigwedge^n \mathfrak{g} &\to M \otimes_k \bigwedge^{n-1} \mathfrak{g} \\ m \otimes g_1 \wedge \ldots \wedge g_n &\mapsto \sum_{i=1}^n (-1)^{i-1}[m, g_i] \otimes g_1 \wedge \ldots \wedge \widehat{g_i} \wedge \ldots g_n \\ &\qquad + \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} m \otimes [g_i, g_j] \wedge \ldots \wedge \widehat{g_i} \wedge \ldots \wedge \widehat{g_j} \wedge \ldots \wedge g_n. \end{aligned} \tag{1.114}$$

The role of this differential in Chevalley–Eilenberg cohomology, which is the cohomology theory for Lie algebras parallel to Hochschild cohomology for associative algebras, will eventually be explained in appendix A.2.

We will apply it to $\mathfrak{g} = A$, where A is considered as a Lie algebra via the commutator bracket. In particular, when A is commutative this is an abelian Lie algebra. But the following proposition holds without commutativity assumption.

Proposition 82. The diagram

$$\begin{tikzcd} M \otimes_k \bigwedge^n A \arrow[r, "\epsilon_n"] \arrow[d, "\mathrm{d}_{\mathrm{CE}}"] & \mathrm{C}_n(A,M) \arrow[d, "\mathrm{d}"] \\ M \otimes_k \bigwedge^{n-1} A \arrow[r, "\epsilon_{n-1}"] & \mathrm{C}_{n-1}(A,M) \end{tikzcd} \tag{1.115}$$

commutes for all $n \geq 0$.

The proof goes via induction. We will need the following technical (but easy) lemma, where

$$\begin{aligned} \mathrm{ad}_n(a)\colon\ &\mathrm{C}_n(A,M) \to \mathrm{C}_n(A,M) \\ (1.116)\qquad &m \otimes a_1 \otimes \ldots \otimes a_n \mapsto \sum_{i=0}^{n} m \otimes a_1 \otimes \ldots \otimes [a,a_i] \otimes \ldots \otimes a_n \end{aligned}$$

is an extension of the notion of inner derivation to $\mathrm{C}_n(A,M)$, and

$$\begin{aligned} h_n(a)\colon\ &\mathrm{C}_n(A,M) \to \mathrm{C}_{n+1}(A,M) \\ (1.117)\qquad &m \otimes a_1 \otimes \ldots \otimes a_n \mapsto \sum_{i=0}^{n} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i \otimes a \otimes a_{i+1} \otimes \ldots \otimes a_n \end{aligned}$$

will provide a null-homotopy for our newly defined $\mathrm{ad}_n(a)$, and an inductive way to describe ϵ_n as in (1.120).

Lemma 83. We have that

$$(1.118)\quad -\mathrm{ad}_n(a) = \mathrm{d} \circ h_n(a) + h_{n-1}(a) \circ \mathrm{d}\,.$$

In particular $\mathrm{ad}_n(a)\colon\ \mathrm{HH}_n(A,M) \to \mathrm{HH}_n(A,M)$ is zero, as for $n = 0$ in proposition 22.

Proof. The term $\mathrm{d} \circ h_n(a)$ gives $[a,a_i]$ by considering the Hochschild differential for the summands containing $a \wedge a_i$ and $a_i \wedge a$. The term $h_{n-1}(a) \circ \mathrm{d}$ cancels all the other summands. □

We can now give the proof of proposition 82.

Proof of proposition 82. The statement for $n = 0$ is vacuous as the lower line is zero. For $n = 1$ we have that $\epsilon_0 = \mathrm{id}_M$ and $\epsilon_1 = \mathrm{id}_{M \otimes_k A}$. As

$$(1.119)\quad \mathrm{d}_{\mathrm{CE}}(m \otimes a) = [m,a] = ma - am = \mathrm{d}(m \otimes a)$$

the diagram commutes.

Let us assume that $\mathrm{d} \circ \epsilon_n = \epsilon_{n-1} \circ \mathrm{d}_{\mathrm{CE}}$. By construction we have that

$$(1.120)\quad \epsilon_{n+1}(m \otimes a_1 \wedge \ldots \wedge a_n \wedge a_{n+1}) = (-1)^n h_n(a_{n+1})\epsilon_n(m \otimes a_1 \wedge \ldots \wedge a_n),$$

so

$$(1.121)\quad \begin{aligned} &\mathrm{d} \circ \epsilon_{n+1}(m \otimes a_1 \wedge \ldots \wedge a_{n+1}) \\ &\quad = (-1)^n\, \mathrm{d} \circ h_n(a_{n+1}) \circ \epsilon_n(m \otimes a_1 \wedge \ldots \wedge a_n) && (1.120) \\ &\quad = (-1)^n(-\mathrm{ad}_n(a_{n+1}) - h_n(a_{n+1}) \circ \mathrm{d}) \circ \epsilon_n(m \otimes a_1 \wedge \ldots \wedge a_n) && \text{lemma 83} \\ &\quad = (-1)^{n+1}\, \mathrm{ad}_n(a_{n+1}) \circ \epsilon_n(m \otimes a_1 \wedge \ldots \wedge a_n) \\ &\qquad\quad + (-1)^{n-1} h_{n-1}(a_{n+1}) \circ \epsilon_{n-1} \circ \mathrm{d}_{\mathrm{CE}}(m \otimes a_1 \wedge \ldots \wedge a_n) \\ &\quad = (-1)^{n+1}\, \mathrm{ad}_n(a_{n+1}) \circ \epsilon_n(m \otimes a_1 \wedge \ldots \wedge a_n) \\ &\qquad\quad + \epsilon_n\,(\mathrm{d}_{\mathrm{CE}}(m \otimes a_1 \wedge \ldots \wedge a_n) \wedge a_{n+1}) && (1.120) \\ &\quad = \epsilon_n \circ \mathrm{d}_{\mathrm{CE}}(m \otimes a_1 \wedge \ldots \wedge a_n) \wedge a_{n+1}). \end{aligned}$$

□

Corollary 84. If A is commutative and M symmetric, then $\mathrm{im}(\epsilon_n) \subseteq \mathrm{Z}_n(A)$. In particular, there exists a morphism

$$(1.122)\quad \epsilon_n \colon M \otimes_k \bigwedge^n A \to \mathrm{HH}_n(A, M).$$

Proof. The Chevalley–Eilenberg differential is identically zero in this case. □

Now we can check that the antisymmetrisation indeed defines a morphism of the desired form.

Proposition 85. Let A be commutative, and M a symmetric A-bimodule. Then the morphism (1.122) factors as

$$(1.123)\quad \begin{tikzcd} M \otimes_k \bigwedge^n A \arrow[r, "\epsilon_n"] \arrow[d] & \mathrm{HH}_n(A, M) \\ M \otimes_A \Omega_A^n \arrow[ur, "\epsilon_n"] & \end{tikzcd}$$

where we will recycle the symbol ϵ_n for the morphism that we are interested in.

Proof. Recall that Ω_A^1 is generated by the symbols $\mathrm{d}a$, and hence Ω_A^n by the symbols $\mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n$.

We need to check that ϵ_n is compatible with the relations imposed on Ω_A^1 and that we can go from a tensor product over k to a tensor product over A. By the definition of ϵ_n we can assume that the product ab is the first position. We need to show that

$$(1.124)\quad \epsilon_n(m \otimes ab \wedge a_2 \wedge \ldots \wedge a_n) - \epsilon_n(ma \otimes b \wedge a_2 \wedge \ldots \wedge a_n) - \epsilon_n(mb \otimes a \wedge a_2 \wedge \ldots \wedge a_n)$$

is actually zero in homology, as this expresses the relation $\mathrm{d}ab = a\mathrm{d}b + b\mathrm{d}a$, together with the change from $- \otimes_k -$ to $- \otimes_A -$.

If $n = 0$ then there is nothing to check. If $n = 1$ we have that (1.124) is $\mathrm{d}(m \otimes a \otimes b)$. More generally one can check that

$$(1.125)\quad (1.124) = -\mathrm{d}\left(\sum_{\sigma \in S} \mathrm{sgn}(\sigma)\sigma \cdot (m \otimes a \otimes b \otimes a_2 \otimes \ldots \otimes a_n \right)$$

where $S = \{\sigma \in \mathrm{Sym}_{n+1} \mid \sigma(1) < \sigma(2)\}$. □

So for now we have only used commutativity of A. We will continue the proof of the Hochschild–Kostant–Rosenberg isomorphism after a short digression.

The projection morphism Before we continue with the Hochschild–Kostant–Rosenberg decomposition for smooth algebras we can prove something for arbitrary commutative algebras over fields of characteristic 0, by constructing a morphism in the opposite direction.

$$(1.126)\quad \pi_n \colon \mathrm{C}_n(A, M) \to M \otimes_A \Omega_A^n : m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n.$$

This morphism is again compatible with the Hochschild differential.

Lemma 86. We have that $\pi_n \circ \mathrm{d} = 0$ for all $n \geq 0$.

Proof. Using the relation $\mathrm{d}ab = a\mathrm{d}b + b\mathrm{d}a$ after applying π_{n-1} to the expression (1.19) allows one to pair off terms with opposite signs. □

Corollary 87. There exists a morphism

$$\pi_n\colon \mathrm{HH}_n(A,M) \to M \otimes_A \Omega_A^n. \tag{1.127}$$

Proposition 88. The composition $\pi_n \circ \epsilon_n$ is multiplication by $n!$.

Proof. We have the equality

$$m \otimes \mathrm{d}a_{\sigma^{-1}(1)} \wedge \ldots \wedge \mathrm{d}a_{\sigma^{-1}(n)} = \mathrm{sgn}(\sigma) m \otimes \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n \tag{1.128}$$

so this term appears $n!$ times. □

In characteristic zero we therefore obtain the following corollary.

Corollary 89. If $\operatorname{char} k = 0$, then $M \otimes_A \Omega_A^n$ is a direct summand of $\mathrm{HH}_n(A,M)$.

This leads to the *λ-decomposition* or *Hodge decomposition* of Hochschild homology, but we will not develop this further for now. The interested reader is referred to [17, §4.5]. Just be warned that what is called the Hochschild–Kostant–Rosenberg decomposition in section 3.3 is sometimes referred to as the Hodge decomposition, especially in earlier papers. We should stress that

1. in the affine setting the Hodge decomposition is only interesting in the presence of singularities, and in the smooth case it reduces to the Hochschild–Kostant–Rosenberg isomorphism;
2. in the smooth and projective setting the Hochschild–Kostant–Rosenberg decomposition was originally proved only for Hochschild cohomology, whence the name Hodge decomposition was used, but as the Hochschild–Kostant–Rosenberg decomposition for Hochschild homology is a transpose (see section 3.3) of the Hodge decomposition arising in Hodge theory, this leads to an unfortunate clash of terminology, which is avoided in these notes.

Computing Tor via the Koszul resolution We have seen in theorem 15 that Hochschild homology can be described using Tor, as the bar complex provided a free resolution of A as a bimodule. We will need another explicit free resolution in the computation of Tor for the proof of the Hochschild–Kostant–Rosenberg isomorphism, when A is a smooth local k-algebra. This will be provided by the Koszul complex, which is a standard object in algebra and algebraic geometry. For more information one is referred to [7, §17], we will only recall some notation and facts.

Definition 90. Let A be a commutative ring. Let $f\colon M = A^{\oplus n} \to A$ be a morphism of A-modules. Then the *Koszul complex* associated to f is

$$0 \to \bigwedge^n M \to \ldots \to \bigwedge^1 M \to A \to 0 \tag{1.129}$$

where

$$\mathrm{d}\colon \bigwedge^j M \to \bigwedge^{j-1} M : m_1 \wedge \ldots \wedge m_j \mapsto \sum_{i=1}^{j} (-1)^{i+1} f(m_i) m_1 \wedge \ldots \wedge \widehat{m}_{i-1} \wedge \ldots \wedge m_j. \tag{1.130}$$

One can check that this is indeed a complex, but more importantly, when the morphism f corresponds to a regular sequence for an ideal I, then it is actually a free resolution of A/I.

Recall that $f = (a_1, \ldots, a_n)$ is a regular sequence if a_{i+1} is not a zero-divisor in $A/(a_1, \ldots, a_i)$. In definition 92 we relate this to smoothness of a k-algebra.

We prove the following general result, which will be applied to the local rings we encounter after applying the local-to-global principle.

Proposition 91. Let B be a commutative local ring, and I an ideal of B generated by a regular sequence $\mathrm{g} = (g_1, \ldots, g_n)$. Then the isomorphism

$$\epsilon_1 \colon I/I^2 \xrightarrow{\cong} \mathrm{Tor}_1^B(B/I, B/I) \tag{1.131}$$

induces an isomorphism

$$\epsilon_\bullet \colon \bigwedge\nolimits_{B/I}^{\bullet} I/I^2 \xrightarrow{\cong} \mathrm{Tor}_\bullet^B(B/I, B/I) \tag{1.132}$$

of graded-commutative algebras.

Proof. The Koszul complex provides a free resolution

$$0 \to \bigwedge\nolimits_B^n B^{\oplus n} \to \ldots \to \bigwedge\nolimits_B^2 B^{\oplus n} \to B^{\oplus n} \to B \to B/I \to 0 \tag{1.133}$$

of B/I as a B-module which we can use to compute Tor:

$$\begin{aligned} \mathrm{Tor}_\bullet^B(B/I, B/I) &\cong \mathrm{H}_\bullet\left((\bigwedge\nolimits_B^\bullet B^{\oplus n}) \otimes_B B/I, \mathrm{d}_{\mathrm{g}} \otimes_B \mathrm{id}_{B/I}\right) \\ &\cong \mathrm{H}_\bullet\left((\bigwedge\nolimits_B^\bullet B^{\oplus n}) \otimes_B B/I, 0\right) \\ &\cong \bigwedge\nolimits_B^\bullet (B/I)^{\oplus n} \\ &\cong \bigwedge\nolimits_B^\bullet I/I^2 \end{aligned} \tag{1.134}$$

The second isomorphism follows from the observation that d_{g} has coefficients landing in $I \subseteq B$, as $\mathrm{d}_{\mathrm{g}} \colon \bigwedge^{k+1} B^{\oplus n} \to \bigwedge^k B^{\oplus n}$ has the form

$$\mathrm{d}_{\mathrm{g}}(v_0 \wedge \ldots \wedge v_k) = \sum_{i=0}^{k} (-1)^i \mathrm{g}(v_i) v_0 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k. \tag{1.135}$$

As I is generated by a regular sequence, we have that I/I^2 is a free B/I-module of rank n, generated by the classes of the elements in the sequence.

Finally, to check that (1.132) is an isomorphism of graded *algebras*, observe that the algebra structure on the right is described by an external product (much like the shuffle product), which can be computed via the exterior product of Koszul complexes. □

As throughout the entirety of these notes we will let k be a field. We have the following equivalent definitions for smoothness.

Definition 92. Let A be a flat k-algebra, locally of finite type. We say that A is *smooth* (over k) if one of the following equivalent conditions holds:

1. for all $\mathfrak{m}$ a maximal ideal of A the kernel of $\mu_{\mathfrak{m}} \colon (A \otimes_k A)_{\mu^{-1}(\mathfrak{m})} \to A_{\mathfrak{m}}$ is generated by a regular sequence;
2. the kernel of $\mu \colon A \otimes_k A \to A$ is a locally complete intersection;
3. for all $\mathfrak{p}$ a prime ideal of A we have that $\dim_{k(\mathfrak{p})} \Omega^1_{A/k} \otimes_A k(\mathfrak{p}) = \dim_{\mathfrak{p}} \operatorname{Spec} A$.

Let us remark that in characteristic 0 smoothness at a point $\mathfrak{p} \in \operatorname{Spec} A$ is equivalent to $\Omega^1_{A/k,\mathfrak{p}}$ being free of finite rank, and the ring $A_{\mathfrak{p}}$ being regular[14] (which is an absolute notion).

Having introduced smoothness, we can put it to good use in proving the main theorem of this section.

[14]In positive characteristic it only implies regularity.

Proof of theorem 81. We have constructed a morphism

$$(1.136)\quad \epsilon_n\colon \Omega^n_A \otimes_A M \to \mathrm{HH}_n(A, M)$$

of A-modules. We can check whether it is an isomorphism by checking it after localising at every maximal ideal $\mathfrak{m}$ of A, i.e. $\epsilon_n \otimes_A A_{\mathfrak{m}}$ needs to be an isomorphism for every $\mathfrak{m}$. For the left-hand side we have the following compatibility with localisation

$$(1.137)\quad (\Omega^n_A \otimes_A M) \otimes_A A_{\mathfrak{m}} \cong \Omega^n_{A_{\mathfrak{m}}/k} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}}.$$

For the right-hand side we need an isomorphism

$$(1.138)\quad \mathrm{HH}_n(A, M) \otimes_A A_{\mathfrak{m}} \cong \mathrm{HH}_n(A_{\mathfrak{m}}, M_{\mathfrak{m}}),$$

so in the construction of $\epsilon_\bullet$ we can assume that $(A, \mathfrak{m})$ is a local ring. To do this, let us denote $I := \ker(\mu\colon A \otimes_k A \to A)$. As $\operatorname{Spec} A \to \operatorname{Spec} A \otimes_k A$ is a closed morphism we have that $\mathfrak{n} := \mu^{-1}(\mathfrak{m})$ is a maximal ideal of $A \otimes_k A \cong A^{\mathrm{e}}$. There exists an isomorphism

$$(1.139)\quad \mathrm{Tor}^{A^{\mathrm{e}}}_n(A, M) \otimes_A A_{\mathfrak{m}} \cong \mathrm{Tor}^{(A^{\mathrm{e}})_{\mathfrak{n}}}_n(A_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \mathrm{Tor}^{A_{\mathfrak{m}} \otimes_k A_{\mathfrak{m}}}_n(A_{\mathfrak{m}}, M_{\mathfrak{m}})$$

by flat base change for Tor.

By the definition of smoothness we have that $I_{\mathfrak{n}}$ is generated by a regular sequence of length $\dim A$. In the notation of proposition 91 we take $B := A \otimes_k A$, and I the ideal that cuts out A. □

So we get that the Hochschild homology $\mathrm{HH}_\bullet(A)$ for a smooth algebra is concentrated in finitely many degrees (where it consists of projective modules of finite rank). There is actually a converse to this, characterising smoothness in terms of the vanishing of Hochschild homology, see [1].

1.3.5 The Hochschild–Kostant–Rosenberg isomorphism: Hochschild cohomology

One could follow a similar approach to proving the Hochschild–Kostant–Rosenberg isomorphism for Hochschild cohomology. But in the special case of $\mathrm{HH}^\bullet(A)$ one can take a shortcut, avoiding checking explicitly that things are compatible with the differential, etc.

Indeed, as $\mathrm{HH}^\bullet(A)$ is a graded commutative algebra, the identification $\mathrm{HH}^1(A) \cong \mathrm{Der}(A)$ extends via the universal property of the exterior product to a morphism

$$(1.140)\quad \bigwedge\nolimits^\bullet \mathrm{Der}(A) \to \mathrm{HH}^\bullet(A).$$

To check that it locally is an isomorphism if A is a smooth k-algebra we will use the following result, which says that Ext commutes with localisation at a prime ideal. It is a special case of [20, proposition 3.3.10].

Lemma 93. Let A be a noetherian ring, and M, N be A-modules where M is moreover finitely generated. Let $\mathfrak{p}$ be a prime ideal of A, then

$$(1.141)\quad \mathrm{Ext}^n_A(M, N)_{\mathfrak{p}} \cong \mathrm{Ext}^n_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

for all $n \geq 0$.

Then one can recycle the argument for Hochschild homology verbatim to obtain the following result.

Theorem 94. Let A be a smooth k-algebra. Then there exist an isomorphism of graded-commutative algebras

$$\mathrm{HH}^{\bullet}(A) \cong \bigwedge^{\bullet} \mathrm{Der}(A). \tag{1.142}$$

Remark 95. Instead of Hochschild (co)homology we can also consider Hochschild (co)chains on one hand, and the exterior powers of derivations (resp. differential forms) as a complex with zero differential on the other. Then we have constructed a quasi-isomorphism between these complexes. But e.g. on the level of Hochschild cohomology it is not a quasi-isomorphism of differential graded algebras. Fixing this is part of the theory of Kontsevich's formality, which we might get back to in appendix B.1.

Remark 96. If we write $X = \operatorname{Spec} A$, then theorem 94 can be rewritten as

$$\mathrm{HH}^{\bullet}(A) \cong \Gamma(X, \bigwedge^{\bullet} \mathrm{T}_X) \tag{1.143}$$

and

$$\mathrm{HH}_{\bullet}(A) \cong \Gamma(X, \Omega^{\bullet}_{X/k}) \tag{1.144}$$

In section 3.3 we will generalise this result to the non-affine setting. In this situation the Hochschild–Kostant–Rosenberg isomorphism becomes a Hochschild–Kostant–Rosenberg decomposition, as in section 3.3: the higher cohomology of polyvector fields and differential forms starts playing a role.

1.3.6 Gerstenhaber calculus

This section will be extended at some point, but it seems that there is no operad-free proof of Hochschild cohomology being isomorphic to polyvector fields as Gerstenhaber algebras. That is unfortunate, as we want to avoid operads in this chapter.

1.4 Variations on Hochschild (co)homology

This will be skipped during the course, unless there is time and interest to revisit the noncommutative calculus of Hochschild (co)homology and cyclic homology at the end of the course.

1.5 Deformation theory of algebras

We have seen in theorem 27 that the second Hochschild cohomology group $\mathrm{HH}^2(A, M)$ parametrises extensions of A by M. As explained there, we will usually take $M = A$, so that we are effectively describing algebra structures on $A \oplus At$ with t^2 that reduce to the original multiplication on A when t is set to 0. In which case we call this a *(first order) deformation* of A, as is customary in algebraic geometry.

In this section we will discuss the higher-order deformation theory of an associative algebra A, not just up to first-order. It turns out that this is also controlled by the Hochschild cohomology, where we will also use

1. the Gerstenhaber bracket;
2. the third Hochschild cohomology $\mathrm{HH}^3(A)$.

We will also discuss the general formalism of differential graded Lie algebras governing deformation problems, using the Maurer–Cartan equation.

Summarising the results (at least on the infinitesimal level) we can draw the following picture, which gives the interpretation of the first, second and third Hochschild cohomology group in the deformation theory of algebras, together with the role of the Gerstenhaber bracket. Recall that it has degree -1, so we are landing in the appropriate spaces.

infinitesimal automorphisms

(1.145)
$$\begin{tikzcd} \mathrm{HH}^2(A) \arrow[loop above, "{[\mathrm{HH}^1(A),-]}"] \arrow[rr, "{\mu\mapsto[\mu,\mu]}"] & & \mathrm{HH}^3(A) \end{tikzcd}$$

deformations obstructions

To streamline the discussion we will in this section assume that k is not of characteristic 2. Whenever necessary we will even restrict ourselves to characteristic zero, but this will be mentioned explicitly.

1.5.1 Obstructions and the third Hochschild cohomology

We have seen in theorem 27 that $\mathrm{HH}^2(A)$ corresponds to first-order deformations of A. Given a first-order deformation, the first step to take is to (try to) extend it to a second-order deformation. Sometimes this is possible, sometimes it fails. When it fails it is because of an obstruction. In this section we discuss how to analyse the failure of extending, using $\mathrm{HH}^3(A)$. Once the first-order deformation is extended to a second-order deformation, we can try to extend it further. Again the third Hochschild cohomology and the Gerstenhaber bracket control this behaviour. If the extension is possible at each step we end up with a formal deformation, which is discussed in section 1.5.2.

From first-order to second-order Let $\mu_1\colon A \otimes_k A \to A$ be a Hochschild 2-cocycle, for which $\mu_0 + \mu_1 t$ is a first-order deformation, as in the discussion following theorem 27. We wish to extend this using a $\mu_2\colon A \otimes_k A \to A$, such that

(1.146) $a * b = \mu_0(a \otimes b) + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2$

gives an associative product on $A[t]/(t^3)$ (considered as a module, not as an algebra). The associativity relation

$$(1.147)\quad (a * b) * c = a * (b * c)$$

breaks up into two[15] conditions: one expression an equality for the coefficients of t and another for t^2. The first condition we have already discussed, and says precisely that μ_1 needs to be a Hochschild 2-cocycle. The associativity condition at t^2 can be written as

$$(1.148)\quad \mu_1(\mu_1(a \otimes b) \otimes c) - \mu_1(a \otimes \mu_1(b \otimes c)) = a\mu_2(b \otimes c) - \mu_2(ab \otimes c) + \mu_2(a \otimes bc) - \mu_2(a \otimes b)c.$$

The right-hand side of this equality is given by $\mathrm{d}(\mu_2)(a \otimes b \otimes c)$, so the left-hand side, considered as an element of $\mathrm{Hom}_k(A^{\otimes 3}, A)$ must be a coboundary.

But the left-hand side (which only depends on μ_1, which is a cocycle by the first associativity condition) is always a cocycle, because it is equal to $\mu_1 \circ \mu_1 = \frac{1}{2}[\mu_1, \mu_1]$, and the Gerstenhaber bracket of the cocycle μ_1 with itself is again a cocycle. So the equality (1.148) says that the cocycle is actually a coboundary, so it defines $0 \in \mathrm{HH}^3(A)$. In general, when we are only given μ_1 and we are looking for a compatible μ_2, we can define the following.

Definition 97. The class in $\mathrm{HH}^3(A)$ defined by the left-hand side of (1.148) is the *obstruction* to extending μ_1. We call $\mathrm{HH}^3(A)$ the *obstruction space* of A.

If this obstruction class vanishes there exists a $\mu_2 \in \mathrm{Hom}_k(A^{\otimes 2}, A)$ which turns (1.146) into an associative product on $A[t]/(t^3)$, which we will call a *second-order deformation*. So we have proven the following proposition.

Proposition 98. Let $\mu_0 + \mu_1 t$ be a first-order deformation. Then it can be extended to a second-order deformation if and only if $[\mu_1, \mu_1] = 0$ in $\mathrm{HH}^3(A)$.

Definition 99. We call $[\mu_1, \mu_1]$ the *obstruction* to extending the first-order deformation $\mu_0 + \mu_1 t$ to a second-order deformation. If it vanishes we call the deformation *unobstructed*, otherwise we call it an *obstructed* deformation.

In particular, if $\mathrm{HH}^3(A) = 0$ then all obstructions automatically vanish. On the other hand it is possible that $\mathrm{HH}^3(A) \neq 0$ but a first-order deformation still extends to a second-order deformation. For this we can consider the following example, which explains this behaviour.

Example 100. Let $A = k[x, y, z]/(xy - z, x^2, y^2, z^2)$. Then A is a commutative 4-dimensional algebra, which we can also express as $k[x]/(x^2) \otimes_k k[y]/(y^2)$. We will use the basis $(1, x, y, z)$ for A, and the induced basis of tensor products for $A \otimes_k A$. Consider the following two infinitesimal deformations, or 2-cocycles:

1. $f\colon A \otimes_k A \to A$ is the cocycle $f(y \otimes x) = z$, and 0 for other basis vectors;
2. $g\colon A \otimes_k A \to A$ is the cocycle $g(x \otimes x) = y$, and 0 for other basis vectors.

Then one sees that

1. f defines an unobstructed noncommutative first-order deformation of A;
2. g defines an unobstructed commutative first-order deformation of A.

On the other hand, $f + g$ defines another first-order deformation of A, and one can check that

$$(1.149)\quad [f + g, f + g] \neq 0$$

in $\mathrm{HH}^3(A)$, so this is an obstructed deformation.

[15] Or rather three, but we know that μ_0 is associative.

Remark 101. Observe that, if the obstruction vanishes, there is actually a choice of μ_2, all of which gives an equivalent deformation. But for the purpose of extending it to higher-order deformations, the choice might matter.

Extending to higher order deformations Let us now generalise the previous discussion to arbitrary order, i.e. given an nth order deformation of A, when can we extend it to an $(n+1)$th order deformation of A? So we start with an assocative multiplication

$$\mu_0 + \mu_1 t + \ldots + \mu_n t^n \tag{1.150}$$

on $A[t]/(t^{n+1})$. We wish to extend it to an associative multiplication on $A[t]/(t^{n+2})$. The analysis as before gives us a hierarchy of associativity conditions at t^i, for $i = 1, \ldots, n+1$, with only the condition at t^{n+1} being new. By isolating the terms involving μ_0 and μ_{n+1} in this expression we obtain the equality

$$\sum_{i=1}^{n} \mu_i \left(\mu_{n+1-i}(a \otimes b) \otimes c\right) - \mu_i \left(a \otimes \mu_{n+1-i}(b \otimes c)\right) = \mathrm{d}(\mu_{n+1})(a \otimes b \otimes c). \tag{1.151}$$

As the left-hand side can be interpreted as the sum over $[\mu_i, \mu_{n+1-i}]$ this is a 3-cocycle. So by the same reasoning as before we get the following, where we denote left-hand side by D_{n+1} we have that

Proposition 102. Let $\mu_0 + \mu_1 t + \ldots + \mu_n t^n$ be an nth order deformation. Then it can be extended to an $(n+1)$th order deformation if and only if $D_{n+1} = 0$ in $\mathrm{HH}^3(A)$.

Observe that, again, there is a choice of μ_{n+1} at this point, and this choice might lead you into trouble when you want to continue this process: for some choices the following step might be obstructed, whilst for others it isn't. See also exercise 115.

Deformation functors To make the link with the formalism used in algebraic geometry we want to consider algebras more general than $k[t]/(t^{n+1})$. These are the *test algebras*, which are commutative artinian local k-algebras with residue field k. In particular, the maximal ideal $\mathfrak{m}$ is nilpotent. We will denote them by R, so that there is no confusion with the algebras that we are deforming (which are denoted A). Then generalising the notion of an nth order deformation we have the following.

Definition 103. Let $(R, \mathfrak{m})$ be a test algebra. An *R-deformation* of A is an associative and R-bilinear multiplication $- * -$ on $A \otimes_k R$ such that modulo $\mathfrak{m}$ it reduces to the multiplication on A.

In other words, the square

$$\begin{tikzcd} (A \otimes_k R) \otimes_R (A \otimes_k R) \arrow[r] \arrow[d, "- * -"] & A \otimes_k A \arrow[d, "\mu_0"] \\ A \otimes_k R \arrow[r] & A \end{tikzcd} \tag{1.152}$$

commutes.

Then the equivalence relation for first-order deformations is generalised as follows.

Definition 104. Let $- *_1 -$ and $- *_2 -$ be two R-deformations. We say that they are *(gauge) equivalent* if there exists an automorphism f of the R-module $A \otimes_k R$ which is the identity modulo $\mathfrak{m}$, such that

$$f(a *_1 b) = f(a) *_2 f(b). \tag{1.153}$$

A morphism of test algebras $f\colon R \to S$ allows us to base change a deformation $A \otimes_k R$ to $A \otimes_k R \otimes_R S$, and hence we have a covariant functor

$$\operatorname{Def}(A,-)\colon \mathcal{R} \to \mathrm{Set} : R \mapsto \operatorname{Def}(A,R) \tag{1.154}$$

where $\operatorname{Def}(A,R)$ is the set of R-deformations of A up to equivalence. In particular, $\operatorname{Def}(A,k) = \{A\}$ and $\operatorname{Def}(A,k[t]/(t^2)) = \mathrm{HH}^2(A)$.

Axiomatising the properties that this functor has, we obtain the following definition.

Definition 105. A functor $F\colon \mathcal{R} \to \mathrm{Set}$ is a *deformation functor* if for every cartesian diagram

$$\begin{tikzcd} R' \times_R R'' \arrow[r] \arrow[d] & R'' \arrow[d] \\ R' \arrow[r] & R \end{tikzcd} \tag{1.155}$$

the induced morphism

$$\eta\colon F(R' \times_R R'') \to F(R') \times_{F(R)} F(R'') \tag{1.156}$$

is

1. bijective, if $R \cong k$;
2. surjective, if $R' \to R$ is surjective.

One can check that $\operatorname{Def}(A,-)$ is a deformation functor in this general sense of the word. Later on we will see a way of describing the deformation functor using Hochschild cohomology, and explain the role dg Lie algebras play in describing deformation functors.

We will say that $F(k[t]/(t^2))$ is the *tangent space* to a deformation functor, so we see that the second Hochschild cohomology is the tangent space to the deformation functor $\operatorname{Def}(A,-)$.

1.5.2 Formal deformations

The intuition from deformation theory in algebraic geometry tells us that deformations over

$$k[[t]] = \varprojlim k[t]/(t^{n+1}) \tag{1.157}$$

are supposed to describe deformations in a sufficiently small (indeed: infinitesimally small) open neighbourhood around the algebra that we are interested in. These are precisely the deformations one obtains when taking the limit of the process with the $k[t]/(t^{n+1})$ in the previous section. Similarly we will discuss later in this section how the step from $k[t]/(t^{n+1})$ to local artinian k-algebras has an analogue, going from $k[[t]]$ to complete augmented k-algebras.

One-parameter formal deformations Let A be a k-algebra. We can consider the $k[[t]]$-module $A[[t]]$. We are interested in new algebra structures on the module $A[[t]]$ in the following sense.

Definition 106. A *(one-parameter) formal deformation* of A is an associative and $k[[t]]$-bilinear multiplication

$$-*-\colon A[[t]] \otimes_k A[[t]] \to A[[t]] \tag{1.158}$$

which is continuous in the t-adic topology, such that

(1.159) $a * b \equiv ab \bmod t$

for all $a, b \in A \subseteq A[[t]]$.

The continuity in the previous definition is expressed by saying that the multiplication takes on the form

(1.160) $$\left(\sum_{i\geq 0} a_i t^i\right) * \left(\sum_{j\geq 0} b_j t^j\right) = \sum_{k\geq 0}\sum_{i+j=k} (a_i * b_j) t^{i+j}.$$

Because of this the multiplication is completely determined by the restriction $- * -\colon A \otimes_k A \to A[[t]]$. For $a, b \in A$ we will write

(1.161) $a * b = \mu_0(a \otimes b) + \mu_1(a \otimes b)t + \mu_2(a \otimes b)t^2 + \dots$

where $\mu_0(a \otimes b) = \mu(a \otimes b)$ is the original multiplication, as before. So a formal deformation consists of μ_i's such that they define an associative multiplication, which is expressed by considering (1.151) for all n simultaneously.

The relationship between $k[[t]]$ and $k[t]/(t^{n+1})$ is also easily expressed by observing that setting $t^{n+1} = 0$ in a formal deformation results in an nth order deformation.

Remark 107. Although we formally speaking haven't introduced the notation for the left-hand side yet (as this involves gauge equivalence for $k[[t]]$), we have a bijection

(1.162) $\mathrm{Def}(A, k[[t]]) = \varprojlim \mathrm{Def}(A, k[t]/(t^{n+1}))$,

in other words: a formal deformation consists of a *compatible* family of nth order deformations. It is also important to observe that the obstructions we have discussed earlier tell us that the morphism

(1.163) $\mathrm{Def}(A, k[[t]]) \to \mathrm{Def}(A, k[t]/(t^{n+1})$

is necessarily surjective.

1.5.3 The Maurer–Cartan equation

We will now discuss some aspects of the general formalism of deformation theory using dg Lie algebras. The main example to keep in mind is of course $\mathrm{C}^\bullet(A)[1]$ (and variations constructed using this). Recall that the reason for the shift is to make the Gerstenhaber bracket of degree 0. In this way the relevant cohomology groups will be H^0, H^1 and H^2.

We will now assume that k is of characteristic 0.

Definition 108. Let $\mathfrak{g}^\bullet$ be a dg Lie algebra. The *Maurer–Cartan equation* for $\mathfrak{g}^\bullet$ is

(1.164) $\mathrm{d}(f) + \frac{1}{2}[f, f] = 0$

where $f \in \mathfrak{g}^1$. Elements of $\mathfrak{g}^1$ satisfying the Maurer–Cartan are *Maurer–Cartan elements*, and we will denote

(1.165) $\mathrm{MC}(\mathfrak{g}) := \left\{f \in \mathfrak{g}^1 \mid \mathrm{d}(f) + \frac{1}{2}[f, f] = 0\right\} \subseteq \mathfrak{g}^1$

the space of Maurer–Cartan elements.

The dg Lie algebras that we will be using are constructed from $\mathrm{C}^\bullet(A)[1]$ as follows: let $(R, \mathfrak{m})$ be a complete augmented k-algebra, then $\mathfrak{m}\widehat{\otimes}_k \mathrm{C}^\bullet(A)[1]$ is again a dg Lie algebra. As before, the main examples to which we will apply this are $(k[t]/(t^{n+1}), (t))$ and $(k[[t]], (t))$. For the latter we will use the shorthand notation $t\,\mathrm{C}^\bullet(A)[1][[t]]$.

With these definitions we have the following theorem, explaining the importance of the Maurer–Cartan equation in our situation.

Theorem 109. Let A be an associative algebra. Then

$$\{\text{one-parameter formal deformations of } A\} \xleftrightarrow{1:1} \mathrm{MC}(t\,\mathrm{C}^\bullet(A)[1][[t]]). \tag{1.166}$$

Proof. Consider

$$\mu \coloneqq \sum_{i=1}^{+\infty} \mu_i t^i \in t\,\mathrm{C}^1(A)[1][[t]], \tag{1.167}$$

i.e. $\mu_i \in \mathrm{C}^2(A)$. Then on $A[[t]]$ we can define the multiplication

$$a \otimes b \mapsto ab + \sum_{i=1}^{+\infty} \mu_i(a \otimes b)t^i \tag{1.168}$$

for $a, b \in A$, and then extended bilinearly to all of $A[[t]]$. We only need to check that associativity of this multiplication corresponds to μ satisfying the Maurer–Cartan equation. But this is checked in exactly the same was as for $k[t]/(t^{n+1})$. □

Remark 110. The same proof works for arbitrary formal deformations over arbitrary complete augmented k-algebras, using the dg Lie algebra $\mathfrak{m}\widehat{\otimes}_k \mathrm{C}^\bullet(A)[1]$, only adding some mild notational complexity.

Example 111. As an application of the previous remark, if we consider $R = k[t]/(t^2)$ then there is no need for the completed tensor product, and $\mathrm{MC}(\mathfrak{m} \otimes_k \mathrm{C}^\bullet(A)[1]) = \mathrm{MC}(\mathrm{C}^\bullet(A)[1])$ is the set of infinitesimal deformations (not up to equivalence, yet).

Gauge equivalence Observe that in the bijection of theorem 109 we are not considering formal deformations up to gauge equivalence. Likewise, in example 111 we are not taking the equivalence relation on first-order deformations into account. To do this, we need to introduce gauge equivalence for the Maurer–Cartan locus. For convergence reasons, we will assume that $\mathfrak{g}^\bullet$ is of the form $\mathfrak{h}^\bullet\widehat{\otimes}_k\mathfrak{m}$, for some complete augmented k-algebra $(R, \mathfrak{m})$ (as this is the situation we are interested in).

Definition 112. Let $g_1, g_2 \in \mathrm{MC}(\mathfrak{g}^\bullet)$ be Maurer–Cartan elements. We say that they are *gauge equivalent* if there exists an element $h \in \mathfrak{g}^0$ such that

$$g_2 = \exp(\mathrm{ad}\,h)(g_1) + \frac{1 - \exp(\mathrm{ad}\,h)}{h}(\mathrm{d}(h)). \tag{1.169}$$

We can now relate the two notions of gauge equivalence. Observe that by the assumption on the characteristic, a gauge equivalence ϕ between formal deformations $- *_1 -$ and $- *_2 -$ on $A[[t]]$ can be written as $\exp(h)$ for some $h \in t\,\mathrm{Hom}_k(A, A)[[t]]$. And recall that $\mathrm{Hom}_k(A, A) = \mathrm{C}^1(A)$.

Proposition 113. The formal deformations $- *_1 -$ and $- *_2 -$ are gauge equivalent if and only if the associated Maurer–Cartan elements g_1 and g_2 (see theorem 109) are gauge equivalent.

Proof. If we denote $\mu_0 \colon A \otimes_k A \to A$ the original multiplication, then for $a, b \in A$ we have that

$$
\begin{aligned}
a *_2 b &= \phi\left(\phi^{-1}(a) *_1 \phi^{-1}(b)\right) \\
&= \exp(h)\,(\exp(-h)(a) *_1 \exp(-h)(b)) \\
&= \exp(\mathrm{ad}(h))(\mu + g_1)(a \otimes b) \\
&= \left(\mu + \exp(\mathrm{ad}(h))(g_1) + \frac{1 - \exp(\mathrm{ad}(h))}{\mathrm{ad}(h)}(\mathrm{d}(h))\right)(a \otimes b)
\end{aligned}
\tag{1.170}
$$

by remark 44, and applying $\mathrm{ad}(h)$ once in all the non-constant applications to μ. So gauge equivalence for $- *_i -$ is expressed through gauge equivalence for g_i and vice versa. ◻

1.5.4 Exercises

Exercise 114. Check the claims in example 100, i.e.

1. f and g are cocycles;
2. f and g are unobstructed;
3. $f + g$ is obstructed.

Also, assuming the Künneth formula for Hochschild cohomology[16]

$$
\mathrm{HH}^\bullet(A_1 \otimes_k A_2) \cong \mathrm{HH}^\bullet(A_1) \otimes_k \mathrm{HH}^\bullet(A_2)
\tag{1.171}
$$

describe the Hochschild cohomology of A using example 31.

Exercise 115. Perform the following reality checks.

1. If the trivial deformation has $\mu_i = 0$ for all $i \geq 1$, why can't we choose μ_1 non-zero, and then take $\mu_i = 0$ for all $i \geq 2$?
2. Explain that, if $\mu_1 = 0$, then any cocycle μ_2 gives a second-order deformation.
3. Use exercise 114 to construct an unobstructed second-order deformation, which extends trivially to a third-order deformation (why can we do this?), such that the third-order deformation is obstructed.

[16]Which we haven't proven, but which holds in this case by [21, theorem 2.1.2].

Chapter 2

Differential graded categories

Chapter 3

Schemes

3.1 Polyvector fields

3.2 Atiyah classes

3.3 The Hochschild–Kostant–Rosenberg decomposition

3.4 Riemann–Roch versus Hochschild homology

3.5 Căldăraru's conjecture

Appendix A

Preliminaries

A.1 Differential graded (Lie) algebras

Definition 116. A *differential graded algebra* $A^\bullet$ is a graded algebra $A^\bullet$ together with the structure of a cochain complex $\mathrm{d}\colon A^\bullet \to A^{\bullet+1}$ satisfying the *graded Leibniz rule*, i.e. for all homogeneous $a, b \in A^\bullet$

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^{|a|} a\,\mathrm{d}(b). \tag{A.1}$$

We will abbreviate differential graded algebra to *dg algebra*.

Observe that the graded Leibniz rule implies the following.

Proposition 117. Let $A^\bullet$ be a dg algebra. Then $\mathrm{H}^\bullet(A^\bullet)$ is a graded algebra.

Definition 118. A *differential graded Lie algebra* $L^\bullet$ is a graded Lie algebra together with the structure of a cochain complex $\mathrm{d}\colon L^\bullet \to L^{\bullet+1}$ satisfying the *graded Leibniz rule*, i.e. for all homogeneous $l, m \in L^\bullet$

$$\mathrm{d}([l,m]) = [\mathrm{d}(l), m] + (-1)^{|l|}[l, \mathrm{d}(m)]. \tag{A.2}$$

We will abbreviate differential graded Lie algebra to *dg Lie algebra*, or even *dgla*.

Remark 119. For graded algebras the axioms do not pick up any signs. For graded Lie algebras there are non-trivial signs involved in the axioms: for all homogeneous $l, m \in L^\bullet$ the graded skew-symmetry is

$$[l,m] = (-1)^{|l||m|}[m,l] \tag{A.3}$$

whilst the graded Jacobi identity is

$$[l,[m,n]] = [[l,m],n] + (-1)^{|l||m|}[m,[l,n]]. \tag{A.4}$$

Observe that the graded Leibniz rule implies the following.

Proposition 120. Let $L^\bullet$ be a dg Lie algebra. Then $\mathrm{H}^\bullet(L^\bullet)$ is a graded Lie algebra.

A.2 Chevalley–Eilenberg cohomology

Appendix B

Additional topics

B.1 Kontsevich's formality theorems

B.2 Calabi–Yau algebras and Poincaré–Van den Bergh duality

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