Understanding $\mathbf{D}^{\mathrm{b}}(kQ)$ using moduli spaces

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Goals

Bernhard Keller and Sarah Scherotzke, Graded quiver varieties and derived categories, `arXiv:1303.2318v2`:

1. connect $\mathbf{D}^{\mathrm{b}}(kQ)$ to a moduli variety $\mathfrak{M}_0(w)$;
2. describe the moduli variety in terms of $\mathbf{D}^{\mathrm{b}}(kQ)$ and vice versa.

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My feeble goal:

3. generalise $\mathbf{D}^{\mathrm{b}}(kQ)$ to a derived moduli stack $\mathbf{R}\mathrm{Perf}_Q$;
4. describe the derived moduli stack using the moduli variety.

Context

It generalises

1. Hiraku Nakajima, Quiver varieties and finite-dimensional representations of quantum affine algebras, `arXiv:math/9912158`
2. Hiraku Nakajima, Quiver varieties and cluster algebras, `arXiv:0905.0002v5`
3. Yoshiyuki Kimura and Fan Qin, Graded quiver varieties, quantum cluster algebras and dual canonical bases, `arXiv:1205.2066v2`
4. Bernard Leclerc and Pierre-Guy Plamondon, Nakajima varieties and repetitive algebras, `1208.3910v2`

Conventions

1. k algebraically closed
2. Q a finite acyclic quiver
3. take Q connected for ease of statements

A reminder on derived categories

Construction

1. A a ring;

A reminder on derived categories

Construction

1. A a ring;
2. Mod-A abelian category of A-modules;

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Motivation

Natural location to do homological algebra.

A reminder on moduli spaces

Philosophy

A moduli “space” is an geometric object parametrising “families of objects”.

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1. moduli space of curves (= Riemann surfaces) $\mathcal{M}_g$, $\dim \mathcal{M}_g = 3g - 3$
2. moduli space of algebra structures on finite-dimensional vectorspace Alg_r

Repetition quivers

We need a (technical) construction. . .

Repetition quivers

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Definition

The *repetition quiver* $\mathbb{Z}Q$ has as vertices

$$Q_0 \times \mathbb{Z} = \{(i,p) \mid i \in Q_0, p \in \mathbb{Z}\}$$

and edges

$$\bigcup_{\alpha\colon i \to j} \{(\alpha,p)\colon (i,p) \to (j,p); \sigma(\alpha,p)\colon (j,p-1) \to (i,p)\}\,.$$

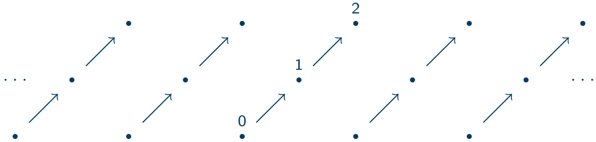
Translations in repetition quivers

1. in the definition: $\sigma \colon \mathbb{Z}Q_1 \to \mathbb{Z}Q_1$;
2. translation to the left: τ, both on $\mathbb{Z}Q_0$ and $\mathbb{Z}Q_1$;
3. we have $\sigma^2 = \tau$.

Examples of repetition quivers

$$A_3 \qquad \begin{tikzcd}[row sep=small, column sep=small] & & \overset{2}{\bullet} \\ & \overset{1}{\bullet} \arrow[ur] & \\ \overset{0}{\bullet} \arrow[ur] & & \end{tikzcd}$$

Examples of repetition quivers



Examples of repetition quivers

$$
\begin{tikzcd}
 & & & \bullet & & \overset{\tau(2)}{\bullet} & & \overset{2}{\bullet} & & \overset{\tau^{-1}(2)}{\bullet} & & \bullet \\
\cdots & & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \overset{1}{\bullet} \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & \cdots \\
 & \bullet \arrow[ur] & & \bullet \arrow[ur, "\tau(\beta)"'] & & \overset{0}{\bullet} \arrow[ur, "\beta"'] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & &
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\mathbb{Z}\mathsf{A}_3 & \cdots & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \overset{1}{\bullet} \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[ur] & \cdots \\
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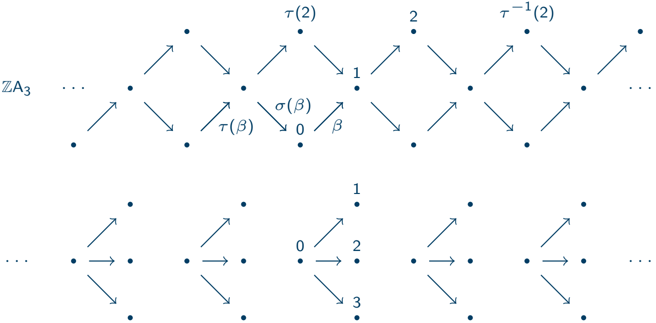
$$

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & \overset{1}{\bullet} \\
\mathsf{D}_4 & \overset{0}{\bullet} \arrow[ur] \arrow[r] \arrow[dr] & \overset{2}{\bullet} \\
 & & \overset{3}{\bullet}
\end{tikzcd}
$$

Examples of repetition quivers

$\mathbb{Z}\mathsf{A}_3$



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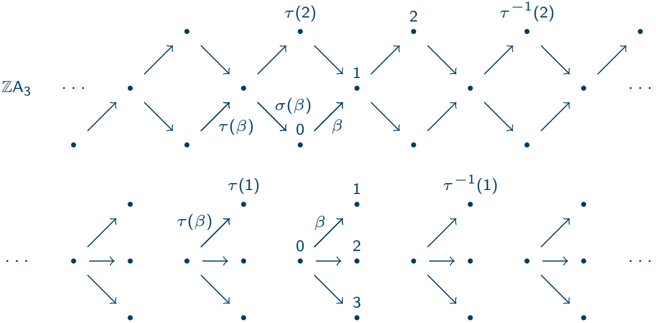

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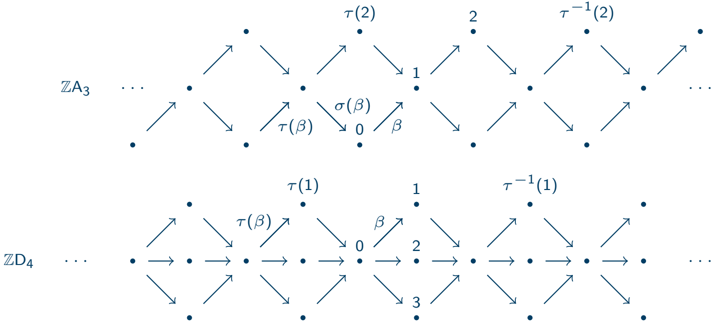

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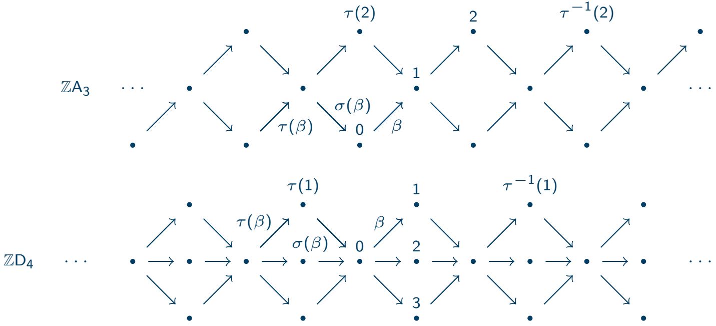

Framed quivers

Definition

The *framed quiver* $\widetilde{Q}$ of Q has vertices Q_0 and $Q_0' = \{i' \mid i \in Q_0\}$, and edges Q_1 and $\{i \to i' \mid i \in Q_0\}$. The vertices i' are the *frozen vertices*.

Examples of framed quivers

$$A_3 \qquad \begin{tikzcd} & & \overset{2}{\bullet} \\ & \overset{1}{\bullet} \arrow[ur] & \\ \overset{0}{\bullet} \arrow[ur] & & \end{tikzcd}$$

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$\widetilde{A_3}$

$$
\begin{tikzcd}
 & & \overset{2}{\bullet} \arrow[r] & \overset{2'}{\blacksquare} \\
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\overset{0}{\bullet} \arrow[ur] \arrow[r] & \overset{0'}{\blacksquare} & &
\end{tikzcd}
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\end{tikzcd}$$

D_4

$$\begin{tikzcd}
 & \overset{1}{\bullet} \\
\overset{0}{\bullet} \arrow[ur] \arrow[r] \arrow[dr] & \overset{2}{\bullet} \\
 & \overset{3}{\bullet}
\end{tikzcd}$$

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$\widetilde{A_3}$

$$
\begin{tikzcd}
 & & 2 \arrow[r] & 2' \\
 & 1 \arrow[r] \arrow[ur] & 1' & \\
0 \arrow[r] \arrow[ur] & 0' & &
\end{tikzcd}
$$

$\widetilde{D_4}$

$$
\begin{tikzcd}
 & 1 \arrow[r] & 1' \\
 & 0' & \\
0 \arrow[uur] \arrow[ur] \arrow[r] \arrow[ddr] & 2 \arrow[dr] & \\
 & & 2' \\
 & 3 \arrow[r] & 3'
\end{tikzcd}
$$

Mesh categories

Definition

The *mesh category* $k(\mathbb{Z}Q)$ is the k-linear category with $\mathrm{Obj}(k(\mathbb{Z}Q)) = \mathbb{Z}Q_0$ and

$$\mathrm{Hom}_{k(\mathbb{Z}Q)}(a,b) = \langle \text{paths from } a \text{ to } b \text{ in } \mathbb{Z}Q \rangle / (u\mathrm{r}_x v \mid x \in \mathbb{Z}Q_0)$$

where r_x is the *mesh relator* associated to x, given by

$$\mathrm{r}_x = \sum_{\beta\colon y \to x} \sigma(\beta)\beta\colon \quad \begin{tikzcd} & y_1 \arrow[dr, "\beta_1"] & \\ \tau(x) \arrow[ur, "\sigma(\beta_1)"] \arrow[dr, "\sigma(\beta_n)"'] & \vdots & x \\ & y_n \arrow[ur, "\beta_n"'] & \end{tikzcd}$$

Remarks on mesh categories

This construction finds its origins in Auslander-Reiten theory.

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Example

In the mesh category $k(\mathsf{A}_2)$ all paths of length 2 or more are identified with 0.

$$\begin{tikzcd}
 & \bullet \arrow[dr] & & \bullet \arrow[dr] & & \bullet \arrow[dr] & & \bullet \arrow[dr] & & \bullet \\
\bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] &
\end{tikzcd}$$

More interesting examples: see next, when we've introduced Nakajima categories.

Regular Nakajima categories

Definition

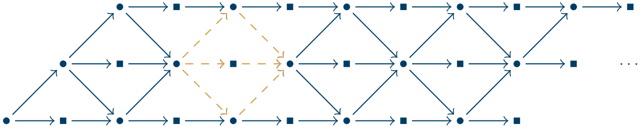
The *regular Nakajima category* $\mathcal{R}_Q$ (or just $\mathcal{R}$) is the mesh category on the framed quiver, where we only impose the mesh relators on the non-frozen vertices.

Examples of regular Nakajima categories

$$\begin{tikzcd}
& & & \bullet \arrow[r] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] & \blacksquare & \\
\mathcal{R}_{\mathsf{A}_3} \cdots & & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] & \blacksquare & & \cdots \\
& \bullet \arrow[r] \arrow[ur] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] & \blacksquare \arrow[r] & \bullet \arrow[r] \arrow[ur] & \blacksquare & & &
\end{tikzcd}$$

Examples of regular Nakajima categories

$\mathcal{R}_{\mathsf{A}_3} \cdots$



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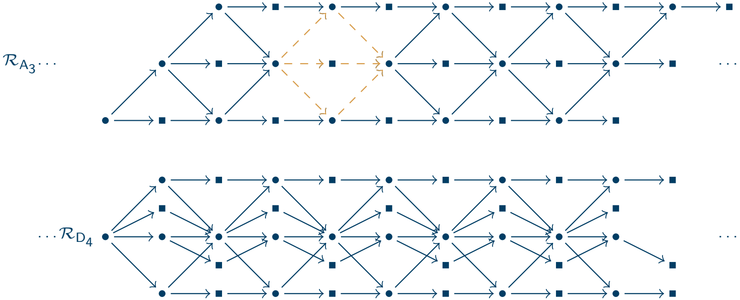

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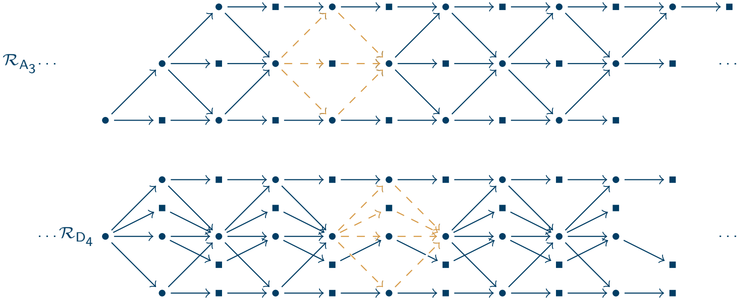

Singular Nakajima categories

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We have regular versus singular because of the related moduli varieties: one is regular, the other can be singular.

Examples of singular Nakajima categories

These categories become really hard to draw, see Keller–Scherotzke for the case D_4 which is next to impossible to reproduce.

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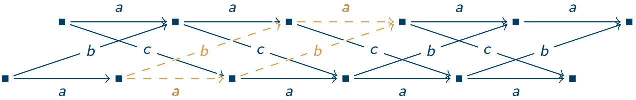
The singular Nakajima category for A_2:



with relations

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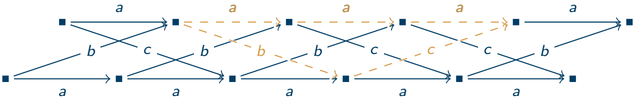

with relations

1. $ab - ba$
2. $a^3 - cb$

Graded affine quiver varieties

Definition

The *graded affine quiver variety* $\mathfrak{M}_0(w)$ for a finitely supported dimension vector $w\colon \operatorname{Obj}(\mathcal{S}) \to \mathbb{N}$ is the variety of $\mathcal{S}$-modules M, such that $M(x) \cong k^{w(x)}$.

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$$\mathfrak{M}_0(w) \cong \prod_{x,y \in \mathrm{Obj}(\mathcal{S})} \mathrm{Hom}_k \left(\mathrm{Hom}_{\mathcal{S}}(x,y), k^{w(x)w(y)} \right) / I$$

where I is an ideal of relations: a module M is described by

1. images of the morphisms in $\mathcal{S}$;
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Hence, $\mathfrak{M}_0(w)$ Zariski closed subset of an affine space!

Structure of $\mathfrak{M}_0(w)$

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Theorem (Keller–Scherotzke, 2013)

We have

$$\#\{\sigma(y) \to \sigma(x)\} = \dim \operatorname{Ext}^1_{\mathcal{S}}(\mathsf{S}_{\sigma(x)}, \mathsf{S}_{\sigma(y)})$$

and

$$\#\{\textit{relations for } \sigma(y) \textit{ to } \sigma(x)\} = \dim \operatorname{Ext}^2_{\mathcal{S}}(\mathsf{S}_{\sigma(x)}, \mathsf{S}_{\sigma(y)}).$$

Relating $\mathbf{D}^{\mathrm{b}}(kQ)$ to $k(\mathbb{Z}Q)$

Theorem (Happel, 1987)

There exists a canonical fully faithful functor

$$\mathrm{H}\colon k(\mathbb{Z}Q) \to \operatorname{ind}(\mathbf{D}^{\mathrm{b}}(kQ))$$

such that the vertex $(i, 0)$ *is sent to the indecomposable projective module* P_i*, for* $i \in Q_0$*.*

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It is moreover an equivalence if and only if Q is a Dynkin quiver.

Hence we get a relationship between the repetition quiver and the derived category!

An isomorphism of Ext's

Theorem

Let $p \geq 1$. For all $x, y \in \mathbb{Z}Q_0$ we have

$$\mathrm{Ext}^p_{\mathcal{S}}(\mathsf{S}_{\sigma(x)}, \mathsf{S}_{\sigma(y)}) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(kQ)}(\mathrm{H}(x), \Sigma^p \, \mathrm{H}(x)).$$

Moreover, if Q is not Dynkin these are zero for $p \geq 2$.

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Applying Keller–Scherotzke's result:

Corollary

For Q not Dynkin there are no relations! We have $\mathfrak{M}_0(w)$ isomorphic to affine space.

Stability and costability

Definition

An $\mathcal{R}$-module is *stable* if for all $x \in \mathbb{Z}Q_0$ non-frozen we have

$$\mathrm{Hom}_{\mathcal{R}}(\mathsf{S}_x, M) = 0.$$

Interpretation

M does not contain a non-zero submodule supported only on non-frozen vertices.

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Dual definition for *costable*: $\mathrm{Hom}_{\mathcal{R}}(M, \mathsf{S}_x) = 0$.

Interpretation

M does not have a non-zero quotient supported only on non-frozen vertices.

Dimension vectors

We'll denote (v, w)

$$v\colon \operatorname{Obj}(\mathcal{R}) \setminus \operatorname{Obj}(\mathcal{S}) \to \mathbb{N}$$
$$w\colon \operatorname{Obj}(\mathcal{S}) \to \mathbb{N}$$

dimension vectors for the regular Nakajima category.

A related moduli variety

Definition

The variety $\tilde{\mathfrak{M}}(v, w)$ is a moduli space for the $\mathcal{R}$-modules M such that

1. M is stable;
2. $M(x) \cong k^{v(x)}$;
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There is moreover a (free) base change action by the group

$$\mathsf{G}_v := \prod_{x \in \mathrm{Obj}(\mathcal{R}) \setminus \mathrm{Obj}(\mathcal{S})} \mathsf{GL}_{v(x)}(k)$$

Only on the non-frozen vertices!

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Using GIT this becomes a smooth quasi-projective variety, and the restriction $\mathrm{res}\colon \text{Mod-}\mathcal{R} \to \text{Mod-}\mathcal{S}$ becomes a projection map

$$\pi\colon \mathfrak{M}(v, w) \to \mathfrak{M}_0(w)$$

which is *proper* ("=" inverse images of compacts are compact).

Stratification

Goal

A stratification of $\mathfrak{M}_0(w)$.

Definition

Denote by $\mathfrak{M}^{\mathrm{reg}}(v, w)$ the open subset of $\mathfrak{M}(v, w)$ formed by isomorphism classes of $\mathcal{R}$-modules which are also costable.

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By varying the vector v (w is fixed) we can stratify $\mathfrak{M}_0(w)$ by the images of the non-empty $\mathfrak{M}^{\mathrm{reg}}(v, w)$, and each of these is isomorphic to its image in $\mathfrak{M}_0(w)$.

Theorem (Keller–Scherotzke, 2013)

There is a canonical δ-functor

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such that

1. *the simple module* $\mathsf{S}_{\sigma(x)}$ *for* $x \in \mathbb{Z}Q_0$ *is sent to* $\mathsf{H}(x)$*;*
2. $M_1, M_2 \in \mathfrak{M}_0(w)$ *lie in the same stratum if and only if* $\Phi(M_1) \cong \Phi(M_2)$ *in* $\mathbf{D}^{\mathrm{b}}(kQ)$*.*

Applications

1. generalising the following result: Desingularization of quiver Grassmannians for Dynkin quivers, Giovanni Cerulli Irelli, Evgeny Feigin and Markus Reineke, `arXiv:1209.3960`

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2. link with derived algebraic geometry and moduli spaces of derived categories: Moduli of objects in dg categories, Bertrand Toën and Michel Vaquié, `arXiv:math/0503269`

Derived moduli stacks

$$
\begin{tikzcd}
\mathrm{i}\left(\bigsqcup_w \mathfrak{M}_0(w)\right) \arrow[r] \arrow[dr] & \mathbf{R}\mathrm{Perf}_{\mathcal{S}} \arrow[d] \\
& \mathbf{R}\mathrm{Perf}_{Q}
\end{tikzcd}
$$

All of these objects are “derived”.

Questions

1. What are the geometric properties of these morphisms?
2. Do we obtain a smooth atlas for the moduli stacks?
3. Can we strengthen the results on the stratification?
4. Do these stacks have interesting intrinsic structure?

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Claim

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1. non-derived moduli stack Vect_C of vector bundles on a commutative curve C is *smooth* (no need for derivedness);
2. non-derived moduli stack Vect_S of vector bundles on a commutative surface S is *singular* (but derived smooth);

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1. non-derived moduli stack Vect_C of vector bundles on a commutative curve C is *smooth* (no need for derivedness);
2. non-derived moduli stack Vect_S of vector bundles on a commutative surface S is *singular* (but derived smooth);
3. derived moduli stack of vector bundles (associated to Q non-Dynkin) on a noncommutative curve is as nice as the commutative counterpart.