

Sheet 5

Solutions to be handed in before class on Wednesday May 8

Problem 23. Let \mathfrak{g} be a semisimple complex Lie algebra.

1. Show that every finite-dimensional $U(\mathfrak{g})$ -module is semisimple. (0.5 point)
2. Give an example of a $U(\mathfrak{g})$ -module which is not semisimple. (0.5 point)

Problem 24. Give an example of a Lie algebra \mathfrak{g} and a finite-dimensional representation of \mathfrak{g} that is not semisimple. (2 points)

Problem 25. In the universal enveloping algebra $U(\mathfrak{sl}_2(k))$, express the elements ehf , $e^2h^2f^2$, hf^3 and h^3f in the standard Poincaré–Birkhoff–Witt basis given by $\{f^l h^m e^n\}_{l,m,n \in \mathbb{N}}$. (3 points)

Problem 26 (Clebsch–Gordan). Let k be algebraically closed of characteristic not 2. Decompose the $\mathfrak{sl}_2(k)$ -representation

$$V(n) \otimes_k V(m) \quad (36)$$

into irreducible representations, where $V(n)$ is the $(n+1)$ -dimensional irreducible representation of $\mathfrak{sl}_2(k)$. (4 points)

Hint If V is a representation, use the *formal character* of V

$$\text{ch}(V) := \sum_{n \in \mathbb{Z}} \dim_k V_n t^n, \quad (37)$$

where V_n denotes the weight space of weight n , and notice that $\text{ch}(V \oplus W) = \text{ch}(V) \oplus \text{ch}(W)$, and that the formal character determines the representation up to isomorphism. Then show that $\text{ch}(V \otimes_k W) = \text{ch}(V) \text{ch}(W)$.

Problem 27. Show that if A is a filtered algebra such that the associated graded is a domain, then A is itself a domain. From this one can conclude that $U(\mathfrak{g})$ is a domain. (1 point)

Problem 28. Show that if A is a filtered algebra such that the associated graded is left (resp. right) noetherian, then A is itself left (resp. right) noetherian. From this conclude that $U(\mathfrak{g})$ is left (resp. right) noetherian, provided that \mathfrak{g} is finite-dimensional. (2 points)

Hint You can define left (resp. right) noetherian as every left (resp. right) ideal being finitely generated. Consider the homogeneous left (resp. right) ideal generated by the leading term of a given left (resp. right) ideal of A .

Problem 29. Let \mathfrak{g} be a Lie algebra. Show that there are unique algebra homomorphisms

1. $\Delta: U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$
2. $\eta: U(\mathfrak{g}) \rightarrow k$

$$3. S: \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})^{\text{op}}$$

for which

$$1. \Delta(X) = X \otimes 1 + 1 \otimes X$$

$$2. \eta(X) = 0$$

$$3. S(X) = -X$$

for all $X \in \mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$.

(3 points)