

Sheet 4

Solutions to be handed in before class on Friday (!) May 3

Problem 19 (Jordan–Chevalley decomposition). Let k be an algebraically closed field, and V a finite-dimensional vector space over it. We call an element $\varphi \in \text{End}_k(V)$ *semisimple* if the roots of its minimal polynomial are all distinct, or equivalently if φ is diagonalisable.

1. If φ and ψ are commuting semisimple endomorphisms, then so are $\varphi + \psi$ and $\varphi - \psi$. (1 point)
2. For $\varphi \in \text{End}_k(V)$ there exists (unique!) φ_s and φ_n such that $\varphi = \varphi_s + \varphi_n$, with φ_s semisimple, φ_n nilpotent and φ_s and φ_n commute. (3 points)

Hint Decompose V into eigenspaces V_i on which φ has characteristic polynomial $(t - a_i)^{m_i}$. We wish to construct a polynomial $f(t)$ which you will evaluate on φ , such that this polynomial satisfies the congruences

$$\begin{aligned} f(t) &\equiv a_i \pmod{(t - a_i)^{m_i}} \\ f(t) &\equiv 0 \pmod{t}. \end{aligned} \tag{32}$$

Which properties does $f(\varphi)$ have, and how to continue from here?

Problem 20. Let D be a (not necessarily inner) derivation of a Lie algebra \mathfrak{g} . Construct a semidirect product $\mathfrak{g} \rtimes_f k$ (where k is the one-dimensional Lie algebra), such that D is the restriction to \mathfrak{g} of an inner derivation on the semidirect product. (2 points)

Problem 21. Let \mathfrak{g} be a Lie algebra. Let U, V, W be \mathfrak{g} -modules.

1. Give an isomorphism of \mathfrak{g} -modules

$$\text{Hom}_k(U \otimes_k V, W) \xrightarrow{\sim} \text{Hom}_k(U, \text{Hom}_k(V, W)). \tag{33}$$

(2 points)

2. Show that taking \mathfrak{g} -invariants gives an isomorphism of vector spaces

$$\text{Hom}_{\mathfrak{g}}(U \otimes_k V, W) \xrightarrow{\sim} \text{Hom}_{\mathfrak{g}}(U, \text{Hom}_k(V, W)). \tag{34}$$

(2 points)

Problem 22. A *Borel subalgebra* of a Lie algebra is a maximal solvable Lie subalgebra. Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

1. Show that the set \mathfrak{b} of upper triangular matrices forms a Borel subalgebra. (2 points)

2. Show that given any other Borel subalgebra \mathfrak{b}' of \mathfrak{g} there exists an element $g \in \mathrm{SL}_2(\mathbb{C})$ such that $g\mathfrak{b}'g^{-1} = \mathfrak{b}$. (2 points)
3. Let \mathfrak{b} now denote any Borel subalgebra of \mathfrak{g} . Consider the natural representation of \mathfrak{b} on \mathbb{C}^2 . Show that there exists a unique one-dimensional subrepresentation $U_{\mathfrak{b}}$ of \mathbb{C}^2 , such that the assignment $\mathfrak{b} \mapsto U_{\mathfrak{b}}$ defines a bijection

$$\{\text{Borel subalgebras of } \mathfrak{g}\} \leftrightarrow \mathbb{P}^1(\mathbb{C}). \quad (35)$$

(2 points)