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Tutorial exercises

These exercises are to be done in class. By no means are you expected to solve all of them during class. The exercises marked $(*)$ are the most important.

Problem 1 $(*)$**.**

1. Classify all 1-dimensional complex Lie algebras.
2. Classify all 2-dimensional complex Lie algebras.
3. Construct an infinite family of pairwise non-isomorphic 3-dimensional complex Lie algebras.

   Hint Consider the bracket

   $$\begin{aligned} [x,y] &= 0 \\ [x,z] &= x \\ [y,z] &= cy \end{aligned} \tag{1}$$

   where $c \in \mathbb{C}$.

Problem 2 $(*)$**.** Let k be a field. Let A be a k-algebra. Show that the k-linear derivations

$$\mathrm{Der}_k(A) \coloneqq \{D \in \mathrm{End}_k(A) \mid \forall a,b \in A \colon D(ab) = D(a)b + aD(b)\} \tag{2}$$

equipped with the bracket

$$[D, D'] \coloneqq D \circ D' - D' \circ D \tag{3}$$

form a Lie algebra over k.

Problem 3. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. We define the product $\mathfrak{g}_1 \times \mathfrak{g}_2$ as the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, together with

$$[(x,y),(x',y')] \coloneqq ([x,x'],[y,y']) \tag{4}$$

for all $x, x' \in \mathfrak{g}_1$ and $y, y' \in \mathfrak{g}_2$.

1. Show that this is again a Lie algebra.
2. Show that it satisfies the universal property of the product.

Problem 4. Let $U \subseteq \mathbb{R}^n$ be an open subset, with coordinate functions $x_1, \ldots, x_n$, and let $A = C^\infty(U;\mathbb{R})$ be the $\mathbb{R}$-algebra of $\mathbb{R}$-valued smooth functions on U. Denote by $\mathfrak{g} = C^\infty(U;\mathbb{R}^n)$ the $\mathbb{R}$-vector space of $\mathbb{R}^n$-valued smooth functions (which form the vector fields on U). Define an operation on $\mathfrak{g}$ by

$$[X,Y] \coloneqq \sum_{i=1}^n \sum_{j=1}^n X_j \frac{\partial Y_i}{\partial x_j} - \sum_{j=1}^n Y_j \frac{\partial X_i}{\partial x_j}. \tag{5}$$

Define

$$\mathfrak{g} \times A \to A : (X,f) \mapsto \mathrm{Lie}_X(f) \coloneqq X(f) \coloneqq \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i}. \tag{6}$$

1. Show that $[-,-]$ equips $\mathfrak{g}$ with the structure of a real Lie algebra.
2. Show that the morphism

$$\mathfrak{g} \to \mathrm{Der}_{\mathbb{R}}(A) : X \mapsto \mathrm{Lie}_X(-) \tag{7}$$

   is a well-defined injective homomorphism of real Lie algebras.

Problem 5. Let $\mathfrak{g}$ be a Lie algebra. Define

$$\widetilde{\mathfrak{g}} \coloneqq \mathfrak{g} \oplus kc \tag{8}$$

where c is a formal basis vector. Let

$$\kappa \colon \mathfrak{g} \times \mathfrak{g} \to k \tag{9}$$

be a bilinear map such that for all $x, y \in \mathfrak{g}$ we have that

1. $\kappa(x, x) = 0$;
2. $\kappa(x, [y, z]) + \kappa(y, [z, x]) + \kappa(z, [x, y]) = 0$.

We then say that κ is a *2-cocycle.*

Show that $\widetilde{\mathfrak{g}}$ is a Lie algebra, when it is equipped with the Lie bracket

$$[x + \lambda c, y + \mu c] \coloneqq [x, y] + \kappa(x, y)\lambda\mu c. \tag{10}$$

This is a *1-dimensional central extension.*