Linear Algebra Notes

School of Mathematics Students

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Preface

This is a shared collection of notes for Linear Algebra. Please visit https://github.com/UoB-Mathematics-Students/Year-2-LA to find out more, and to see other modules. You can contribute to this document by:

- Editing the LaTeX document you wish to contribute to, then submit a pull request to https://github.com/UoB-Mathematics-Students/Year-2-LA.
- Creating a new chapter, placing the LaTeX file in the `tex` folder, and adding a line to `Notes.tex` such as `\input{./tex/MY_CHAPTER.tex}`

Here are some points to follow:

- For the purposes of version control, please try to put each sentence on a new line. (LaTeX treats a single new line as a space, so inserting these extra spaces won't affect the display of your document).
- Place any package imports in `Notes.sty`.
- If you wish to contribute, try to make fairly small changes, and then submit a pull request.
- Use hyphens instead of spaces in your file names, e.g. `My-File.tex` instead of `My File.tex`
- Follow the current naming convention for files/chapters. For example, if the current file names are `1-Alpha`, `2-Beta`, then you should name your file `n-FILENAME`.

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Chapter 1

Definitions

1.1 Vector Spaces and Fields

1.1.1 Fields

Definition 1. Let X, Y be sets. The Cartesian product of X and Y is the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$

Definition 2. Let $f, g : X \to Y$, then $f = g \Leftrightarrow f(x) = g(x), \forall x \in X$

Definition 3. A binary operation on a set X is a function from $X \times X$ to X.

Definition 4. A field is a set $\mathbb{F}$ equipped with two binary operations:

- $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$
- $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

satisfying the following axioms:

$$\forall a, b, c \in \mathbb{F} : (a + b) + c = a + (b + c) \tag{F1}$$

$$\forall a, b \in \mathbb{F} : a + b = b + a \tag{F2}$$

$$\exists 0 \in \mathbb{F} : \forall a \in \mathbb{F} : a + 0 = a \tag{F3}$$

$$\forall a \in \mathbb{F}, \exists (-a) \in \mathbb{F} : a + (-a) = 0 \tag{F4}$$

$$\forall a, b, c \in \mathbb{F} : a \cdot (b \cdot c) = (a \cdot b) \cdot c \tag{F5}$$

$$\forall a, b \in \mathbb{F} : a \cdot b = b \cdot a \tag{F6}$$

$$\exists 1 \in \mathbb{F} : 1 \neq 0 \text{ and } \forall a \in \mathbb{F} : 1 \cdot a = a \tag{F7}$$

$$\forall a \in \mathbb{F} : a \neq 0, \exists a^{-1} \in \mathbb{F} : a \cdot a^{-1} = 1 \tag{F8}$$

$$\forall a, b, c \in \mathbb{F} : a \cdot (b + c) = (a \cdot b) + (a \cdot c) \tag{F9}$$

Notice that F1 and F5 demonstrate associativity; F2 and F6—commutitivity; F3 and F7—identity existence; F4 and F8—inverse existence; and F9 demonstrates distributivity with respect to addition.

Theorem 1. Let $\mathbb{F}$ be a field. Then

1. the element 0 satisfying property (F3) is unique.
2. for each $a \in \mathbb{F}$, the element $(-a)$ satisfying (F4) is unique.
3. the element 1 satisfying (F7) is unique.
4. for each $a \in \mathbb{F} : a \neq 0$, the element a^{-1} satisfying (F8) is unique.

1.1.2 Vector Spaces

Definition 5. Let $\mathbb{F}$ be a field. A vector space over $\mathbb{F}$ is a set V together with two operations: $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ satisfying the following axioms:

$$\forall u, v, w \in V : u + (v + w) = (u + v) + w \tag{VS1}$$

$$\forall u, v \in V : u + v = v + u \tag{VS2}$$

$$\exists 0 \in V \text{ s.t. } \forall u \in V : u + 0 = u \tag{VS3}$$

$$\forall u \in V, \exists (-u) \in V : u + (-u) = 0 \tag{VS4}$$

$$\forall a, b \in \mathbb{F}, u \in V : a(bu) = (ab)u \tag{VS5}$$

$$\forall a \in \mathbb{F}, \forall u, v \in V : a(u + v) = au + av \tag{VS6}$$

$$\forall a, \in \mathbb{F}, \forall u \in V : (a + b)u = au + bu \tag{VS7}$$

$$\forall u \in V : 1 \cdot u = u \tag{VS8}$$

Notice that VS1 and VS5 refer to associativity; VS2 to commutivity; VS3 to and VS8 to identity; VS4 to inverse; and VS6 and VS7 to distributivity.

Theorem 2. Let V be a vector space over a field $\mathbb{F}$. Then

1. the element 0 satisfying (VS3) is unique
2. $\forall u \in V$, the element $(-u)$ satisfying (VS4) is unique lol.

1.1.3 Examples of Vector Spaces

Function Spaces

Let X be any set, and $\mathbb{F}$ be any field. The set $V = \text{Fun}(X, \mathbb{F})$ is defined as the set of all functions from X to $\mathbb{F}$. We equip V with the following operations of addition and scalar multiplication.

Let $f, g \in V, \lambda \in \mathbb{F}$. Then we define $f + g : X \to \mathbb{F}$ by setting

$$(f+g)(x) = f(x) + g(x), \forall x \in X$$

and $\lambda f : X \to \mathbb{F}$ by setting $(\lambda f)(x) = \lambda(f(x))$. Thus $f + g \in V, \lambda f \in V$.

With these operations, V is a vector space over $\mathbb{F}$.

Let us prove (VS2) (commutativity of addition).

Proof. Let $f, g \in V$. We'd like to prove that $f + g = g + f$. Observe that

$$\begin{aligned}\forall x \in X, (f+g)(x) &= f(x) + g(x) \\ &= g(x) + f(x), \text{ by (F2)} \\ &= (g+f)(x)\end{aligned}$$

$\therefore f + g = g + f$. $\square$

Let us prove (VS3) (existence of 0).

Proof. We need to find some $g \in V$ s.t. $\forall f \in V : f + g = f$. Define $g \in V$ by setting $g(x) = 0, \forall x \in X$. We can safely define g in this way because $0 \in \mathbb{F}$. Then $(f+g)(x) = f(x) + g(x) = f(x) + 0 = f(x), \forall x \in X$ by (F3).

$\therefore f + g = f$ $\square$

Continuous Functions

If X is a subset of $\mathbb{R}$, then $C(X)$ denotes the set of all continuous functions $f : X \to \mathbb{R}$. Then $C(X)$ is a vector space over $\mathbb{R}$ with the same kind of operations as $\mathrm{Fun}(X, \mathbb{F})$. E.g. if $X = \mathbb{R}$, then the function given by $f(x) = \sin(5x) - 3\cos(7x), \forall x \in \mathbb{R}$, is an element of $C(X) : f \in C(X)$.

Polynomials

A polynomial with coefficients in $\mathbb{F}$ in a variable t is a formal expression of the form

$$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t^1 + a_0 t^0$$

where $a_n, \ldots, a_0 \in \mathbb{F}$. We refer to a_k as the coefficient of t^k in $f(t)$. For example, $3t^3 - 5t^2 + (1/7)t + \sqrt{2}$ is a polynomial with coefficients in $\mathbb{R}$. When we say that this is a 'formal expression', we mean that t is a formal variable. In particular, we are **not** substituting a number (say) for t.

The usual conventions are used. The order in which the terms of a polynomial are written is immaterial. Also, adding $0 \cdot t^m$ terms does not change the polynomial.

Definition 6. $P_\infty = P_\infty(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$.

We denote by . This is a vector space. Addition and multiplication by scalars are defined term-by-term, which is the usual way:

$$(a_n t^n + \cdots + a_1 t + a_0) + (b_n t^n + \cdots b_1 t + b_0) = (a_n + b_n)t^n + \cdots + (a_1 + b_1)t + (a_0 + b_0)$$

$$\lambda(a_n t^n + \cdots + a_1 t + a_0) = (\lambda a_n)t^n + \cdots + (\lambda a_1)t + (\lambda a_0)$$

Definition 7 (Degree of a polynomial)**.** The degree of a polynomial $a_n t^n + \cdots + a_1 t + a_0$ as the largest number d such that $a_d \neq 0$.

$P_\infty(\mathbb{F})$ becomes a vector space over $\mathbb{F}$ with operations

$$\left(\sum_{k=0}^{n} a_k t^k\right) + \left(\sum_{k=0}^{n} b_k t^k\right) = \sum_{k=0}^{n} (a_k + b_k)t^k$$

and

$$\lambda\left(\sum_{k=0}^{n} a_k t^k\right) = \sum_{k=0}^{n} (\lambda a_k)t^k$$

Definition 8. $P_n(\mathbb{F}) = \{f \in P_\infty(\mathbb{F}) : \deg(f) \leq n\}$

1.1.4 Basic Properties of a Vector Space

Theorem 3 (Cancellation in sums)**.** If $u, v, w \in V$ and $u + v = u + w$, then $v = w$.

Proof. Since $u+v = u+w$, we have $(-u)+(u+v) = (-u)+(u+w)$. By (VS1), this implies $((-u) + u) + v = ((-u) + u) + w \Leftrightarrow 0 + v = 0 + w$ by definition of $(-u)$. Now $0 + v = v$ and $0 + w = w$ by definition of 0. So $v = w$. □

Theorem 4. Consider a vector space V.

1. For all $u \in V, 0u = \underline{0}$.
2. For all $a \in \mathbb{F}, a0 = \underline{0}$.
3. For $a \in \mathbb{F}$ and $u \in V$, $(-a)u = -(au) = a(-u)$. In particular, $-u = (-1)u$.

Theorem 5. Consider a vector space V.

1. $\forall a \in \mathbb{F}$ and $u \in V$, if $au = \underline{0}$, then either $a = 0$ or $u = 0$.
2. (Cancellation in products)
   (a) For $0 \neq a \in \mathbb{F}$ and $u, v \in V$, if $au = av$ then $u = v$.
   (b) For $a, b \in \mathbb{F}$ and $0 \neq u \in V$, if $au = bu$ then $a = b$.

Proof.

1. If $a = 0$, there is nothing to prove, so assume $a \neq 0$. Since $au = 0$ and $a \neq 0$, we have $a^{-1}(au) = 1u = u$ by the axioms of fields and vector spaces. Hence, $u = 0$, as required.
2. (a) We have $au = av \Leftrightarrow a^{-1}(au) = a^{-1}(av) \Leftrightarrow 1 \cdot u = 1 \cdot v \Leftrightarrow u = v$

   (b) Since $au = bu$, we have $0 = au+(-bu) = au+(-b)u = (a-b)u$, where the second equality uses the previous theorem and the other equalities use the axioms of vector spaces. Now the equality $(a-b)u = 0$ implies that either $a - b = 0$ or $u = 0$ by (i). We are given that $u \neq 0$, so $a - b = 0$, whence $a = b$.

□

Theorem 6. Consider a vector space V.

1. We have $a(u_1 + \cdots + u_n) = au_1 + \cdots + au_n$ if $a \in \mathbb{F}$ and $u_1, \ldots, u_n \in V$.
2. We have $a_1 + \cdots + a_n)u = a_1u + \cdots + a_nu$ if $a_1, \ldots, a_n \in \mathbb{F}$ and $u \in V$.

1.2 Subspaces