Linear Algebra Notes

School of Mathematics Students

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Preface

This is a shared collection of notes for Linear Algebra. Please visit https://github.com/UoB-Mathematics-Students/Year-2-LA to find out more, and to see other modules. You can contribute to this document by:

- Editing the LaTeX document you wish to contribute to, then submit a pull request to https://github.com/UoB-Mathematics-Students/Year-2-LA.
- Creating a new chapter, placing the LaTeX file in the `tex` folder, and adding a line to `Notes.tex` such as `\input{./tex/MY_CHAPTER.tex}`

Here are some points to follow:

- For the purposes of version control, please try to put each sentence on a new line. (LaTeX treats a single new line as a space, so inserting these extra spaces won't affect the display of your document).
- Place any package imports in `Notes.sty`.
- If you wish to contribute, try to make fairly small changes, and then submit a pull request.
- Use hyphens instead of spaces in your file names, e.g. `My-File.tex` instead of `My File.tex`
- Follow the current naming convention for files/chapters. For example, if the current file names are `1-Alpha`, `2-Beta`, then you should name your file `n-FILENAME`.

Contents

Chapter 1

Definitions

Definition 1. Let X, Y be sets. The <u>Cartesian</u> product of X and Y is the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$

Definition 2. Let $f, g : X \to Y$, then $f = g \Leftrightarrow f(x) = g(x), \forall x \in X$

Definition 3. A <u>binary operation</u> on a set X is a function from $X \times X$ to X.

1.1 Fields

Definition 4. A <u>field</u> is a set $\mathbb{F}$ equipped with two binary operations:

- $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$
- $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

satisfying the following axioms:

$$\forall a, b, c \in \mathbb{F} : (a + b) + c = a + (b + c) \tag{F1}$$

$$\forall a, b \in \mathbb{F} : a + b = b + a \tag{F2}$$

$$\exists 0 \in \mathbb{F} : \forall a \in \mathbb{F} : a + 0 = a \tag{F3}$$

$$\forall a \in \mathbb{F}, \exists (-a) \in \mathbb{F} : a + (-a) = 0 \tag{F4}$$

$$\forall a, b, c \in \mathbb{F} : a \cdot (b \cdot c) = (a \cdot b) \cdot c \tag{F5}$$

$$\forall a, b \in \mathbb{F} : a \cdot b = b \cdot a \tag{F6}$$

$$\exists 1 \in \mathbb{F} : 1 \neq 0 \text{ and } \forall a \in \mathbb{F} : 1 \cdot a = a \tag{F7}$$

$$\forall a \in \mathbb{F} : a \neq 0, \exists a^{-1} \in \mathbb{F} : a \cdot a^{-1} = 1 \tag{F8}$$

$$\forall a, b, c \in \mathbb{F} : a \cdot (b + c) = (a \cdot b) + (a \cdot c) \tag{F9}$$

Notice that F1 and F5 demonstrate associativity; F2 and F6—commutitivity; F3 and F7—identity existence; F4 and F8—inverse existence; and F9 demonstrates distributivity with respect to addition.

Theorem 1. Let $\mathbb{F}$ be a field. Then

1. the element 0 satisfying property (F3) is unique.
2. for each $a \in \mathbb{F}$, the element $(-a)$ satisfying (F4) is unique.
3. the element 1 satisfying (F7) is unique.
4. for each $a \in \mathbb{F} : a \neq 0$, the element a^{-1} satisfying (F8) is unique.

1.2 Vector Spaces

Definition 5. Let $\mathbb{F}$ be a field. A vector space over $\mathbb{F}$ is a set V together with two operations: $+ : V \times V \to V$ and $\cdot : \mathbb{F} \times V \to V$ satisfying the following axioms:

$$\forall u, v, w \in V : u + (v + w) = (u + v) + w \tag{VS1}$$

$$\forall u, v \in V : u + v = v + u \tag{VS2}$$

$$\exists 0 \in V \text{ s.t. } \forall u \in V : u + 0 = u \tag{VS3}$$

$$\forall u \in V, \exists (-u) \in V : u + (-u) = 0 \tag{VS4}$$

$$\forall a, b \in \mathbb{F}, u \in V : a(bu) = (ab)u \tag{VS5}$$

$$\forall a \in \mathbb{F}, \forall u, v \in V : a(u + v) = au + av \tag{VS6}$$

$$\forall a, \in \mathbb{F}, \forall u \in V : (a + b)u = au + bu \tag{VS7}$$

$$\forall u \in V : 1 \cdot u = u \tag{VS8}$$

Notice that VS1 and VS5 refer to associativity; VS2 to commutivity; VS3 to and VS8 to identity; VS4 to inverse; and VS6 and VS7 to distributivity.

Theorem 2. Let V be a vector space over a field $\mathbb{F}$. Then

1. the element 0 satisfying (VS3) is unique
2. $\forall u \in V$, the element $(-u)$ satisfying (VS4) is unique.