Multivariable and Vector Analysis Notes

School of Mathematics Students

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Preface

This is a shared collection of notes for Multivariable and Vector Analysis. Please visit
https://github.com/UoB-Mathematics-Students/Year-2-MVA to find out more, and to see other modules. You can contribute to this document by:

- Editing the LaTeX document you wish to contribute to, then submit a pull request to https://github.com/UoB-Mathematics-Students/Year-2-MVA.
- Creating a new chapter, placing the LaTeX file in the `tex` folder, and adding a line to `Notes.tex` such as `\input{./tex/MY_CHAPTER.tex}`

Here are some points to follow:

- For the purposes of version control, please try to put each sentence on a new line. (LaTeX treats a single new line as a space, so inserting these extra spaces won't affect the display of your document).
- Place any package imports in `Notes.sty`.
- If you wish to contribute, try to make fairly small changes, and then submit a pull request.
- Use hyphens instead of spaces in your file names, e.g. `My-File.tex` instead of `My File.tex`
- Follow the current naming convention for files/chapters. For example, if the current file names are `1-Alpha`, `2-Beta`, then you should name your file `n-FILENAME`.

Contents

Chapter 1

Definitions

1.1 Functions of Several Variables

Definition 1. A function f whose domain is $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$, for $n \geq 2$ and $n \in \mathbb{N}$, is called a function of several real variables.

Definition 2. For a function $z = f(x, y)$: A vertical section is the graph of $z = f(x, c)$ or $z = f(c, y)$, for some constant c. A level curve is the curve $f(x, y) = c$, for some constant c.

1.2 Partial Differentiation

Definition 3. The function $f : \mathbb{R}^3 \to \mathbb{R}$ is said to have a partial derivative with respect to x at the point (x_0, y_0, z_0) if the following limit exists

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

which is called the **partial derivative** of f with respect to x at the point (x_0, y_0, z_0), denoted as

$$\frac{\partial f(x_0, y_0, z_0)}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

Geometric interpretation of partial derivative The partial derivative $f_x(a, b)$ is the slope of the tangent line to the curve $f(x, b)$ at $x = a$.

Theorem 1. When a function has the second order continuous partial derivatives, the partial derivations of this function do not depend on the order with respect to the variables.

1.3 L'Hospital's rule

Theorem 2. Let lim stand for the limit of $\lim_{x\to c}, \lim_{x\to+\infty}, \lim_{x\to-\infty}$

If $\lim \frac{f'(x)}{g'(x)}$ has a finite value or if the limit is $\pm\infty$ then $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$

1.4 Order of variables

1.4.1 Big-O notation

Theorem 3. Let lim stand for the limit of: $\lim_{x\to c}, \lim_{x\to+\infty}, \lim x \to -\infty$

$$\lim \frac{f(x)}{g(x)} = k, k \neq 0, \text{ if and only if } f(x) = O(g(x))$$

1.4.2 Little-o notation

Theorem 4. Let lim stand for the limit of: $\lim_{x\to c}, \lim_{x\to+\infty}, \lim_{x\to-\infty}$

$$\lim \frac{f(x)}{g(x)} = 0 \text{ if and only if } f(x) = o(g(x))$$

1.5 Differentiable function of a single variable

Theorem 5. The function $y = f(x)$ is called differentiable at x_0 if

$$y - y_0 = f_x(x_0)(x - x_0) + o(x - x_0)$$

or

$$\Delta y = f_x(x_0)\Delta x + o(\Delta x)$$

we have

$$y - y_0 \approx f_x(x_0)(x - x_0)$$

1.6 Differentiable functions with two variables

Theorem 6. The function $z = f(x, y)$ is called differentiable at (x_0, y_0) if

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + o(\rho)$$

or

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + o(\rho)$$

It is the equation for the tangent plane of the surface $z = f(x, y)$ at (x_0, y_0)

1.7 Differentiable Function

Definition 4. The function $f(x, y, z)$ is called differentiable at (x_0, y_0, z_0) if $\Delta f = f(x, y, z) - f(x_0, y_0, z_0)$ can be expressed as

$$\Delta f = f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z + o(\rho)$$

where $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta z = z - z_0$, and $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.

As ρ is infinitely small, we have

$$df = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0) + f_z(x_0, y_0, z_0)dz$$

Definition 5. Let f be a function of the variables $x_1, x_2, \ldots, x_n$, i.e.

$$f = f(x_1, x_2, \ldots, x_n)$$

where each x_j is a function of (some of) the variables $t_1, t_2, \ldots, t_m$, i.e.

$$x_j = x_j(t_1, t_2, \ldots, t_m), j = 1, 2, \ldots, n$$

If f and x_j are sufficiently smooth, then

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}, i = 1, 2, \ldots, m$$