Multivariable and Vector Analysis Notes

School of Mathematics Students

September 28, 2015

Preface

This is a shared collection of notes for Multivariable and Vector Analysis. Please visit
https://github.com/UoB-Mathematics-Students/Year-2-MVA to find out more, and to see other modules. You can contribute to this document by:

- Editing the LaTeX document you wish to contribute to, then submit a pull request to https://github.com/UoB-Mathematics-Students/Year-2-MVA.
- Creating a new chapter, placing the LaTeX file in the `tex` folder, and adding a line to `Notes.tex` such as `\input{./tex/MY_CHAPTER.tex}`

Here are some points to follow:

- For the purposes of version control, please try to put each sentence on a new line. (LaTeX treats a single new line as a space, so inserting these extra spaces won't affect the display of your document).
- Place any package imports in `Notes.sty`.
- If you wish to contribute, try to make fairly small changes, and then submit a pull request.
- Use hyphens instead of spaces in your file names, e.g. `My-File.tex` instead of `My File.tex`
- Follow the current naming convention for files/chapters. For example, if the current file names are `1-Alpha`, `2-Beta`, then you should name your file `n-FILENAME`.

Contents

Chapter 1

Definitions

1.1 Functions of Several Variables

Definition 1.1.1. A function f whose domain is $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$, for $n \geq 2$ and $n \in \mathbb{N}$, is called a function of several real variables.

Definition 1.1.2. For a function $z = f(x, y)$: A vertical section is the graph of $z = f(x, c)$ or $z = f(c, y)$, for some constant c. A level curve is the curve $f(x, y) = c$, for some constant c.

1.2 Partial Differentiation

Definition 1.2.1. The function $f : \mathbb{R}^3 \to \mathbb{R}$ is said to have a partial derivative with respect to x at the point $(x_0, y_0, z_0$ if the following limit exists

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

which is called the **partial derivative** of f with respect to x at the point (x_0, y_0, z_0), denoted as

$$\frac{\partial f(x_0, y_0, z_0)}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0, z_0) - f(x_0, y_0, z_0)}{\Delta x}$$

1.3 Differentiable Function

Definition 1.3.1. The function $f(x, y, z)$ is called differentiable at (x_0, y_0, z_0) if $\Delta f = f(x, y, z) - f(x_0, y_0, z_0)$ can be expressed as

$$\Delta f = f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z + o(\rho)$$

where $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta z = z - z_0$, and $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.

Definition 1.3.2. Let f be a function of the variables $x_1, x_2, \ldots, x_n$, i.e.

$$f = f(x_1, x_2, \ldots, x_n)$$

where each x_j is a function of (some of) the variables $t_1, t_2, \ldots, t_m$, i.e.

$$x_j = x_j(t_1, t_2, \ldots, t_m), j = 1, 2, \ldots, n$$

If f and x_j are sufficiently smooth, then

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}, i = 1, 2, \ldots, m$$