Vector calculus

因為微四教過向量微積分，所以這邊只會寫一些比較酷的

Useful information

ⓘ Info

- $[e_1, e_2, e_3] \equiv e_1 \cdot (e_2 \times e_3)$。若此式 $= +1$，代表此 basis 為 right-handed
- Kronecker delta symbol: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
- Levi-Civita permutation symbol:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

- Einstein summation convention:

$$a \cdot b = a_i b_i \quad a \times b = \epsilon_{ijk}\, e_i\, a_j\, b_k$$

- $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$ 通常我們會讓第一個 (i) 一樣

ⓘ nabla 一坨

- ∇ 基本用法

$$\nabla f = \frac{df}{dr}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{df}{dr}\frac{\mathbf{r}}{r} \quad \nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i} \quad (\nabla \times \mathbf{F})_i = \epsilon_{i}$$

- Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x_i \partial x_i}$$

- 恒等式

$$\begin{aligned}
&\nabla \cdot (\nabla \phi) = \nabla^2 \phi \\
&\nabla \times (\nabla \phi) = 0 \\
&\nabla \cdot (\nabla \times \mathbf{F}) = 0 \\
&\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}
\end{aligned}$$

- 其他恒等式

$$\begin{aligned}
&\nabla(\phi\Psi) = \Psi\nabla\phi + \phi\nabla\Psi \\
&\nabla(F \cdot G) = (G \cdot \nabla)F + G \times (\nabla \times F) + (F \cdot \nabla)G + F \times (\nabla \times \\
&\nabla \cdot (\phi F) = (\nabla\phi) \cdot F + \phi\nabla \cdot F \\
&\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G) \\
&\nabla \times (\phi F) = (\nabla\phi) \times F + \phi\nabla \times F \\
&\nabla \times (F \times G) = (G \cdot \nabla)F - G(\nabla \cdot F) - (F \cdot \nabla)G + F(\nabla \cdot G)
\end{aligned}$$

Jacobian

Consider three sets of n variables $\alpha_i, \beta_i, \gamma_i$ with $1 \le i \le n$

$$\frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(\gamma_1, \ldots, \gamma_n)} = \frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(\beta_1, \ldots, \beta_n)} \frac{\partial(\beta_1, \ldots, \beta_n)}{\partial(\gamma_1, \ldots, \gamma_n)}$$

$$\frac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(\beta_1, \ldots, \beta_n)} = \left[\frac{\partial(\beta_1, \ldots, \beta_n)}{\partial(\alpha_1, \ldots, \alpha_n)}\right]^{-1}$$

Nabla

因為我真的看不懂他的推導(有緣再補)，所以直接列結論

$$
\begin{aligned}
\nabla\phi &= \frac{e_1}{h_1}\frac{\partial\phi}{\partial q_1}+\frac{e_2}{h_2}\frac{\partial\phi}{\partial q_2}+\frac{e_3}{h_3}\frac{\partial\phi}{\partial q_3} \\
\nabla\cdot\mathbf{F} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3F_1)+\frac{\partial}{\partial q_2}(h_3h_1F_2)+\frac{\partial}{\partial q_3}(h_1h_2F_3)\right] \\
\nabla\times\mathbf{F} &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_2 & h_3e_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \\
\nabla^2\phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial q_2}\right)\right] \\
&\quad+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)
\end{aligned}
$$

這是廣義下的形式，由此可以推導出柱座標和球座標時的形式

柱座標

$$
\begin{aligned}
\nabla\phi &= e_\rho\frac{\partial\phi}{\partial\rho}+\frac{e_\phi}{\rho}\frac{\partial\phi}{\partial\phi}+e_z\frac{\partial\phi}{\partial z} \\
\nabla\cdot\mathbf{F} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho)+\frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z} \\
\nabla\times\mathbf{F} &= \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\phi & e_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \\
\nabla^2\phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2}+\frac{\partial^2\phi}{\partial z^2}
\end{aligned}
$$

球座標

$$
\begin{aligned}
\nabla\Phi &= e_r\frac{\partial\Phi}{\partial r}+\frac{e_\theta}{r}\frac{\partial\Phi}{\partial\theta}+\frac{e_\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\\
\nabla\cdot\mathbf{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta)+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}\\
\nabla\times\mathbf{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} e_r & re_\theta & r\sin\theta e_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ F_r & rF_\theta & r\sin\theta F_\phi\end{vmatrix}\\
\nabla^2\phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\,\partial
\end{aligned}
$$

Some Derivation

Directional derivative

考慮一純量場

$$\Phi(x,y,z)=\Phi(\mathbf{r})$$

用泰勒定理扁它

$$\Phi(x+\delta x,y+\delta y,z+\delta z)=\Phi(x,y,z)+\frac{\partial\Phi}{\partial x}\delta x+\frac{\partial\Phi}{\partial y}\delta y+\frac{\partial\Phi}{\partial z}\delta z+O(x^2,\ldots$$

上式等價於

$$\Phi(\mathbf{r}+\delta\mathbf{r})=\Phi(\mathbf{r})+(\nabla\Phi)\cdot\delta\mathbf{r}+O(|\delta\mathbf{r}|^2)$$

這時我們即可推導一多變數函數在 r 處沿著任意 unit vector t 走 s 距離的變化率

$$\frac{d}{ds}[\Phi(\mathbf{r}+ts)-\Phi(\mathbf{r})]\Big|_{s=0}=\frac{d}{ds}[(\nabla\Phi)\cdot(ts)+O(s^2)]\Big|_{s=0}=t\cdot\nabla\Phi$$

Gradient theorem

$$\int_C (\nabla\Phi)\cdot d\mathbf{r} = \int_C d\Phi = \Phi(\mathbf{r}_2) - \Phi(\mathbf{r}_1)$$

where C is any curve from r_1 to r_2

Divergence theorem

$$\int_V (\nabla\cdot\mathbf{F})\,dV = \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)dx\,dy\,dz$$

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\left[F_x(x_2,y,z)-F_x(x_1,y,z)\right]dy\,dz + \cdots = \int_{\partial V}\mathbf{F}\cdot d\mathbf{S}$$

一些延伸結果：

$$1.\ \int_V (\nabla\phi)\,dV = \int_S \phi\,dS \qquad 2.\ \int_V (\nabla\times\mathbf{F})\,dV = \int_S d\mathbf{S}\times\mathbf{F}$$

Curl theorem

$$\begin{aligned}
\int_A (\nabla\times\mathbf{F})\cdot d\mathbf{S} &= \int_{y_1}^{y_2}\int_{x_1}^{x_2}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)dx\,dy \\
&= \int_{y_1}^{y_2}\left[F_y(x_2,y)-F_y(x_1,y)\right]dy \\
&- \int_{x_1}^{x_2}\left[F_x(x,y_2)-F_x(x,y_1)\right]dx \\
&= \int_{x_1}^{x_2} F_x(x,y_1)\,dx + \int_{y_1}^{y_2} F_y(x_2,y)\,dy \\
&+ \int_{x_2}^{x_1} F_x(x,y_2)\,dx + \int_{y_2}^{y_1} F_y(x_1,y)\,dy \\
&= \int_C (F_x\,dx + F_y\,dy) \\
&= \oint_C \mathbf{F}\cdot d\mathbf{r}
\end{aligned}$$

Geometrical definition of grad, div and curl

Gradient

$$t \cdot (\nabla \Phi) = \lim_{\delta s \to 0} \frac{\delta \Phi}{\delta s}$$

它描述的是純量場對距離的變化率

Divergence

$$\nabla \cdot \mathbf{F} = \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{\delta S} \mathbf{F} \cdot d\mathbf{S}$$

- 這裡的 δ 只單純代表微小變化
- 因為我們現在考慮的體積很小，所以 $\int_{\delta V} \nabla \cdot F \, dV \to \nabla \cdot F$

它描述的是一向量場在每單位體積下的淨源及流出量

Curl

$$n \cdot (\nabla \times F) = \lim_{\delta \to 0} \frac{1}{\delta S} \int_{\delta C} F \cdot dr$$

跟上面差不多，想一下 (注意 $d\mathbf{S} = dS$)。它描述一向量場每單位面積下的旋轉程度

curvilinear coordinate

是一種非笛卡爾坐標系統，其中坐標線可以是曲線，而不是像笛卡爾坐標系中的直線，像柱座標和球座標就是其中幾種。以下是一點簡單推導，在笛卡爾座標下考慮一極小位移 $d\mathbf{r}$

$$d\mathbf{r} = e_x dx + e_y dy + e_z dz$$

在廣義的 curvilinear coordinates 下，我們有

$$d\mathbf{r} = h_1 \; dq_1 + h_2 \; dq_2 + h_3 \; dq_3$$

你會發現

$$\mathbf{h_i} = e_i h_i = \frac{\partial \mathbf{r}}{\partial q_i}$$

$h_i = |\mathbf{h}_i|$ is the scale factor with the coordinate，有了這東西我們可以量化坐標系轉換下帶來的改變 (這句話我也不知道我自己在打什麼)，也就是定義 Jacobian

$$J = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = \mathbf{h}_1 \cdot \mathbf{h}_2 \times \mathbf{h}_3$$

這與我們常見的形式等價

$$J = \frac{\partial(x, y, z)}{\partial(q_1, q_2, q_3)} = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix}$$