Linear algebra

Linear vector spaces

Number field (K)

- The elements are scalar
- Addition and Multiplication are consistent with arithmetic law (交換律、結合律、分配律)
- Closed under addition and multiplication
- Identity element with addition and multiplication (addition: 0, multiplication: 1)
- Has inverse elements (addition: negative numbers, multiplication: reciprocal numbers)

Vector space (V)

- The elements are vectors
- Closed under addition and multiplication
  - $\forall x, y \in V,\ z = x + y \text{ where } z \in V$
  - $\forall a \in K \text{ and } x \in V,\ z = ax \text{ where } z \in V$
- Identity element with addition and multiplication (addition: 0)

V is a vector space on K if V satisfy:

- A1 加法結合律：$\forall u, v, w \in V, (u + v) + w = u + (v + w)$
- A2 交換律：$\forall u, v \in V, u + v = v + u$
- A3 零向量(加法單位元素)：$\forall u \in V, \exists 0 \in u + 0 = u$

- A4 逆向量：$\forall u \in V, \exists -u \in V, u+(-u)=0.$
- A5 分配律1：$\forall a \in K \;\&\; u, v \in V, a(u+v)=au+av$
- A6 分配律2：$\forall a, b \in K \;\&\; u \in V, (a+b)u=au+bu$
- A7 乘法結合律$\forall a, b \in K \;\&\; u \in V, (ab)u=a(bu)$
- A8 單位純量：$\forall u \in V, 1u=u \text{ where } 1 \in K$

For example: n次多項式的集合

$$a_0 \;+\; a_1 t \;+\; a_2 t^2 \;+\; \ldots \;+\; a_n t^n,\; a_i \;\in\; K$$

Linear combinations and Linear spans

vector x is the linear combination of $v_1, \ldots, v_m \;\in\; V$. Hence, all possible x could form a subset S (also vector space), which is the linear spans of V.

Linear independence

For

$$a_1 v_1 \;+\; a_2 v_2 \;+\; \ldots \;+\; a_n v_n = 0,\; a_n \;\in\; K$$

If their only solution is $a_i = 0,\; \forall i$, then $v_i, \ldots, v_n$ is a set of linearly independent vectors

Dimension, basis and components/coordinates

n-dimentional vector space V $(\dim V = n)$ can be generated from a set of linearly independent vectors $e_1, \ldots, e_n \implies \{e_1, \ldots, e_n\}$ is the basis of V

That is, $\forall v \in V,\; v = x_1 e_1 + \cdots + x_n e_n \;\implies\; \{x_1, \ldots, x_n\}$ is called the components or coordinates of $\boldsymbol{v}$ relative to the basis

For example: Fourier series

$$f(x) = \sum_{0}^{\infty} f_n e^{inx}$$

its basis is e^{inx}, component is f_n and $\dim = \infty$

Linear maps

V and U are vector spaces, $x \in V$ is the domain, and $y \in U$ is the codomain

Definition of mapping: $x \to y$, or $Ax = y$

Linear mapping properties:

- $A(x_1 + x_2) = Ax_1 + Ax_2, \text{where } x_1, x_2 \in V$
- $A(ax) = aAx, \text{ where } x \in V, a \in K$

Matrices

For an $n \times n$ matrix

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$Det(A) = |A| = \sum_{i,j,k\ldots} \epsilon_{i,j,k\ldots} a_{1i}\, a_{2j}\, a_{3k} \ldots$

- ϵ is Levi-Civita tensor (recall 胡德邦)

- If any two columns or rows of A are swapped, then Det(A) changes sign
- $\mathrm{Det}(A) = \mathrm{Det}(A^T)$
- $\mathrm{Det}(AB) = \mathrm{Det}(A)\mathrm{Det}(B)$

Equality: $A = B \Leftrightarrow a_{ij} = b_{ij}$

Addition: $A + B = C \Leftrightarrow a_{ij} + b_{ij} = c_{ij}$

Multiplication: trivival, we skip

Direct product:

$$C_{(mn\times mn)} = A_{(m\times m)} \otimes B_{(n\times n)} = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_nB & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}$$

column vector: $|x\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$ **row vector** $\langle x| = (x_1 \quad x_2 \quad \cdots)_{1\times n}$

unit matrix:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Null matrix:

$$O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Diagonal matrices:

$$I = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

if $A,\ B$ are diagonal $\Rightarrow AB = BA$

Trace: 矩陣對角元素的和 $Trace(A) = \sum_i a_{ii}$

- $Tr(AB) = Tr(BA)$
- Cyclic properties:
  $Tr(ABCD) = Tr(DABC) = Tr(CDAB) = Tr(BCDA)$
- If $AB = BA$, $A^2 = B^2 = I$ then $Tr(A) = Tr(B) = 0$
- $Tr(A \otimes B) = Tr(A)Tr(B)$

Minor of a_{ij}: $|M_{ij}| =$ delete i^{th} row and j^{th} column, compute determinant

Cofactor of a_{ij} : $c_{ij} = (-1)^{i+j}|M_{ij}|$

Classical adjoint:

$$\mathrm{adj}\ A = \begin{pmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{pmatrix}$$

Inversion: $A^{-1} \Rightarrow AA^{-1} = A^{-1}A = I$

Transpose: $A = a_{ij} \Rightarrow A^T = \widetilde{A} = a_{ji}$

A is symmetric: $A = A^T$
A is anti-symmetric: $A = -A^T$

Orthogonal matrices

定義一座標轉換

$$\begin{cases} \hat{e}_1' = |e_1'\rangle = a_{11}\hat{e}_1 + a_{12}\hat{e}_2 + a_{13}\hat{e}_3 \\ \hat{e}_2' = |e_2'\rangle = a_{21}\hat{e}_1 + a_{22}\hat{e}_2 + a_{23}\hat{e}_3 \\ \hat{e}_3' = |e_3'\rangle = a_{31}\hat{e}_1 + a_{32}\hat{e}_2 + a_{33}\hat{e}_3 \end{cases}$$

可以進一步定義一個 M

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

假設一向量 v ，在原座標中的表示為

$$v = v_1e_1 + v_2e_2 + v_3e_3$$

則其在新座標系下的表示即為

$$v = v_1' e_1' + v_2' e_2' + v_3' e_3' = \begin{array}{c} v_1'(a_{11}\hat{e}_1 + a_{12}\hat{e}_2 + a_{13}\hat{e}_3) + \\ v_2'(a_{21}\hat{e}_1 + a_{22}\hat{e}_2 + a_{23}\hat{e}_3) + \\ v_3'(a_{31}\hat{e}_1 + a_{32}\hat{e}_2 + a_{33}\hat{e}_3) \end{array}$$

注意到

$$v_1'(a_{11}\hat{e}_1) + v_2'(a_{21}\hat{e}_1) + v_3'(a_{31}\hat{e}_1) = v_1\hat{e}_1$$

寫成以下關係式

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_{11}v_1' + a_{12}v_2' + a_{13}v_3' \\ a_{21}v_1' + a_{22}v_2' + a_{23}v_3' \\ a_{31}v_1' + a_{32}v_2' + a_{33}v_3' \end{pmatrix} = M^T v' \Leftrightarrow v = M^T v'$$

同時我們也知道 $v' = Mv$ ，與上式作比較後得 orthogonal matrix 的條件為 $M^T = M^{-1}$ ，在所有類型矩陣之中，只有旋轉矩陣與鏡射矩陣為 orthogonal matrices。

- 旋轉矩陣：$Det(A) = +1$
- 鏡射矩陣：$Det(A) = -1$

旋轉矩陣 (Rotations)

向量旋轉太簡單，跳過

坐標系旋轉：在 3D space 當中，只能分別依 x, y, z 軸旋轉，因其 number of degrees of freedom (dof) =3，所以只需要 3 個 parameter 來完成座標系的旋轉

1. Rotation about z by α
2. Rotation about y by β
3. Rotation about z by γ

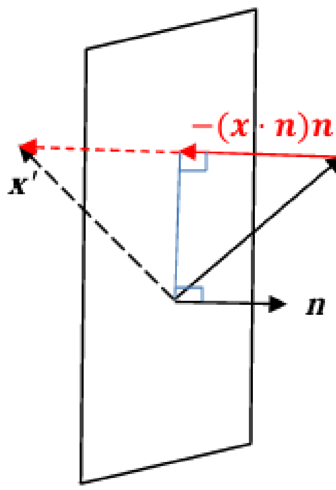
$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} R_z(\gamma) =$$

且必定能寫成以下形式

$$R = R_z(\gamma)R_y(\beta)R_z(\alpha)$$

這就是標準的 O_3^+ (SO(3)) group，稱為正交純旋轉群

鏡射矩陣 (Reflections)



注意到 $x' = x - 2n(x \cdot n)$ ，我們可以將這操作化為矩陣形式

$$x' = Ox,\ O = I - 2nn^T$$

再次注意到 $O^T = O$ ，且 $OO = I$ 。由兩條件推得 $O^{-1} = O^T$ 為 orthogonal matrix

，$|O| = -1$

Similarity Transformation

考慮一矩陣 $A \Rightarrow v_1 = Av$ ，A 為旋轉或鏡射的任一種，另考慮一正交座標旋轉矩陣 $M \Rightarrow v' = Mv$ ，則有以下關係：

$$v_1' = Mv_1 = MAv = (MAM^{-1})(Mv) = (MAM^{-1})v' = A'v'$$

就酷酷的

Linear independence

Wronskian 判別法：

If $\{u_i(x)\} \equiv \{u_1(x), u_2(x), \ldots, u_n(x)\}$ are $n^{th} - \text{order}$ differentiable function set, consider

$$W(\{u_i\}) = \begin{vmatrix} u_1 & u_2 & \ldots & u_n \\ u_1' & u_2' & \ldots & u_n' \\ \vdots & & & \\ u_1^{(n-1)} & u_2^{(n-1)} & \ldots & u_n^{(n-1)} \end{vmatrix}$$

If it's doesn't equal to 0, then $\{u_i(x)\}$ is linearly independent. Notice that if it's equal to 0, we can not determine whether it's linear dependent or independent.

Hermitian matrices & Unitary matrices

定義一些新性質：

Complex conjugates: $A^* = (a_{ij})$
Hermitian conjugate matrix: $A^+ \Rightarrow a_{ij}^+ = a_{ji}^*$, $|x\rangle^+ = \langle x|$

- $(y^+ABCx)^+ = x^+C^+B^+A^+y$
  Hermiatian matrix: $A^+ = A$
  Anti-Hermitian matrix: $A^+ = -A$

Unitary matirx: $U^+ = U^{-1}$

M is normal $\Leftrightarrow$ M commute with M^+: $[M, M^+] = 0$

- $[A, B] = AB - BA$
- $[A, B]^+ = [B^+, A^+]$
- $Tr([A, B]) = Tr(AB) - Tr(BA) = 0$
- Hermitian, anti-Hermitian, and unitary matrices are all normal

If $[A, B] = C$, A and B are Hermitian, then C is anti-Hermitian, that is, $[A, B]^+ = -[A, B]$

Inner product (scalar product)

General Definition of inner product

First we define basis e_i in order to evaluate $\langle x|y\rangle$. Thus, we have $|x\rangle \equiv x_i e_i,\ |y\rangle \equiv y_i e_i$.
Let $\langle x|y\rangle = G_{ij}$, where G is a matrix tensor and G_{ij} is metric coefficients. Its value will depend on i and j, then

$$\langle x|y\rangle = \langle x_i e_i | y_j e_j\rangle = x_i^* \langle e_i|e_j\rangle y_j = x_i^* G_{ij} y_j = \mathbf{x}^\dagger \mathbf{G} \mathbf{y}$$

We formaly define the inner product. Notice that when e_i is orthogonal and normalized, then

$$\langle e_i|e_j\rangle = G_{ij} = \delta_{ij} \implies \langle x|y\rangle = x_i^* y_i$$

- G is Hermitian

In addition, for a non-orthogonal basis, we could turn it into orthogonal one for easy calculation

Consider e_i is orthogonal basis, $e_i' \equiv Me_i$, where M is arbitrary matrix, so that

$$|x\rangle = (M^\dagger)^{-1}|x\rangle$$

then

$$\langle x'|y'\rangle = x'^\dagger G y' = x^\dagger (M^{-1}) G (M^{-1})^\dagger y \equiv \langle x|y\rangle = x^\dagger y$$

$$\implies M^{-1} G (M^\dagger)^{-1} = I \implies G = MM^\dagger$$

Recall orthogonal matrices, but we only consider real number there

Some useful skill

- $\langle x|y\rangle = \langle x, y\rangle = (x, y)$ is **bilinear** (Be linear on each variable)
- $\langle x|y\rangle = \langle y|x\rangle^*$
- $\langle x|x\rangle \geq 0$
- $\langle x, y\rangle^{1/2} = |x| = ||x||$ is the length or norm of the $|x\rangle$
- $\langle x|y\rangle = |x||y|\cos\theta$
- If $\langle x|y\rangle = 0$ (in any vector space) the vectors are said to be orthogonal

The standard (Euclidean) inner product on $\mathbb{R}^n$ is called **dot product**

$$\langle x, y\rangle = \sum x_i y_i$$

which is generalized to $\mathbb{C}^n$ as

$$\langle x, y\rangle = \sum x_i^* y_i$$

Cauchy-Schwarz inequality

$$\langle x|x\rangle\langle y|y\rangle \geq |\langle x|y\rangle|^2$$

Adjoint operator

The adjoint of a linear operator A satisfy with

$$\begin{aligned}\langle A^\dagger x|y\rangle &\stackrel{\text{def}}{=} \langle x|Ay\rangle \quad \forall x,y\\ &\Leftrightarrow (A_{ji})^{\dagger *}x_j^*y_j \stackrel{\text{def}}{=} x_i^*A_{ij}y_j\\ &\Leftrightarrow (A_{ji})^{\dagger *} = A_{ij} \Leftrightarrow (A_{ji})^\dagger = A_{ji}^*\end{aligned}$$

That is, A is Hermitian

Unitary Transformation

在 Similarity transformation 中，如果進行座標轉換的 matrix 是 unitary，則這個轉換叫做 unitary transformation，在內積空間中保持向量長度和角度不變，其有以下關係：

$$A' = UAU^{-1} = UAU^\dagger$$

若 U_1 和 U_2 皆為 unitary $\Rightarrow U_3 = U_1U_2$ 也是 unitary

Pauli matrices and Dirac matrices

Pauli matrices

To describe a particle of spin 1/2 (in a non-relativistic theory):

$$\sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

- $(\sigma_i)^2 = I$
- Anti-commutation: $(\sigma_i\sigma_j + \sigma_j\sigma_i) = 2\delta_{ij}I$
- Cyclic permutation of indices: $\sigma_i\sigma_j = i\sigma_k$

Dirac matrices

首先就是 Dirac 覺得 Pauli 的矩陣不夠用，在他的 relativistic electron theory 中，需要 4 個 anti-commuting matrices，因此他來了

出發點，任一 2×2 的矩陣必可被表示為

$$M = C_0 I + C_1 \sigma_1 + C_2 \sigma_2 + C_3 \sigma_3$$

故 I, σ_1, σ_2, σ_3 形成一組 basis，但其中只有 σ_i 是 anti-commutative，所以我們要以這四個為基礎來建構一組 4×4 的矩陣，使用以下算式

$$\begin{cases} \sigma_{i,\text{Dirac}} = I \otimes \sigma_{i,\text{Pauli}} \\ \rho_{i,\text{Dirac}} = \sigma_{i,\text{Pauli}} \otimes I \end{cases}$$

這樣我們可以造出6個 4×4 的矩陣，滿足：

- Anti-commutation: $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} I, \quad \rho_i \rho_j + \rho_j \rho_i = 2\delta_{ij} I$
- commutation: $[\sigma_i, \rho_j] = 0$
- Cyclic permutation of indices: $\sigma_i \sigma_j = i\sigma_k, \quad \rho_i \rho_j = i\rho_k$

接著建構一個乘法表，選我們要的東東

E_{ij}	**j=**	I_0	σ_1	σ_2	σ_3
i=					
I_0		I	σ_1	σ_2	σ_3
ρ_1		$\rho_1, -Y_5$	α_1	α_2	α_3
ρ_2		ρ_2, α_5	Y_1	Y_2	Y_3
ρ_3		$\rho_3, \alpha_4, Y_4, \beta$	δ_1	δ_2	δ_3

狄拉克選了 α_i $(i = 1, 2, 3, 4)$ ，作為他 relativistic electron theory 中的一組anti-commutative matrices.

最後我們來討論一下 E_{ij} 的性質

- $Det(E_{ij}) = 1$
- $E_{ij}^2 = I$
- E_{ij} is Hemitian and Unitary
- $Tr(E_{ij}) = 0$ 除了 $Tr(E_{00}) = 4$
- 任二個 E_{ij} 的乘積必等於另一個 E_{ij} 乘上 $\{-1, \pm i\}$ 三者其中之一
- 彼此線性獨立
- 任一 4×4 的矩陣必可寫成這 16 個 E_{ij} 的線性和：
  $M = \sum_{i,j=0}^{3} c_{ij} E_{ij}$

Eigenvalues, Eigenvectors & Diagonalisation

Introduction

牽涉到矩陣運算的物理問題中，常藉由將矩陣對角化來簡化計算或釐清其中的物理觀念，像是剛體轉動問題，在數學上這個步驟是將一個給定的 matrix 做 orthogonal similarity transformation (if real) 或 unitary transformation 來轉換成一個 diagonal matrix。

範例：Inertial matrix

Eigenvalues and eigenvectors

If $A|v\rangle = \lambda|v\rangle$ where A is $n \times n$, $\lambda \in C^1$, $|v\rangle \neq 0$, then

$$\lambda : \text{eigenvalue}$$
$$|v\rangle : \text{eigenvector}$$

That is, $(A-\lambda I)|v\rangle = 0$

且注意到 $A-\lambda I$ 中的每一列都和 $|v\rangle$ 正交，因此這 n 個向量不可能全部彼此正交，以致該矩陣的 Det 必為零：

$$\mathrm{Det}(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = P_A(\lambda) = \prod_{i=1}^{n}(\lambda_i - \lambda) =$$

- $P_A(\lambda)$ 為 λ 的 n 次多項式，他的所有解即為所有 eigenvalue，其集合 $\{\lambda_i\}$ 稱為 spectrum of A
- 對於對稱矩陣，由於其解得的 eigenvectors 必互相正交，因此我們一般選取 $\langle v_i|v_i\rangle = 1$ ，這樣 $\{|v_i\rangle\}$ 形成 orthonormal basis 也叫做 eigenvector basis

一些解題重要性質

- $\mathrm{Det}(A) = \prod_{i=1}^{n} \lambda_i$
- If $\mathrm{Det}(A) = 0 \implies \exists \lambda_i = 0 \implies (A-\lambda I)|v\rangle = A|v\rangle = 0$
- $Tr(A) = \sum_{i=1}^{n} \lambda_i$
- For a real $\boldsymbol{A}$, its complex λ_i (if any) must appear in complex-conjugate pairs.
- If λi are the eigenvalues of $\boldsymbol{A}$, then:
  - (a) the eigenvalues of $\boldsymbol{A}$† are λ_i^*
  - (b) the eigenvalues of A^{-1} are λ_i^{-1}
  - (c) $A, A^{\dagger}, A^{-1}$ have the same eigenvectors
- Eigenvalues remain the same under orthogonal/unitary transformation of coordinate system
- If A is Hermitian, then: (skip proof)
  - λ_i is real number

    - Each eigenvectors must be orthogonal
- 若$P_A(\lambda) = |A - \lambda I| = 0$ 有 m 重根, 則對應到該重根 λ 的 eigenspace 可能有 1 至m 個線性獨立的eigenvectors
- 若 A 為 normal, 則 m 重根 λ 剛好有 m 個線性獨立的 eigenvectors。
- If A is normal, each eigenvector corresponded to different λ_i are orthogonal
- If A is anti-Hermitian, λ_i will be pure imaginary number or zero
- If A is unitary, $|\lambda_i| = 1$
- The eigenvalues of a lower or upper triangular matrix are the diagonal entries of the matrix

Rank, nullity and eigenvalues

Triangular factorization $A_{n\times m} = L_{n\times n}U_{n\times m}$

- U: upper triangular matrix，可由對 A 做 Gaussian elimination 求得
- rank: U 中非零列的數量
- pivot: 每一非零列的第一個數
- L: Lower triangular matrix，規則看下面範例，這很酷

E.g.1

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{pmatrix}$$

執行Gaussian elimination:
Step 1: 將第2列扣除第1列的1倍：L_{21}=1
Step 2: 將第3列扣除第1列的0倍：L_{31}=0
Step 3: 將第3列扣除第2列的1/3倍：L_{32}=1/3

然後 L 的對角線全取1，得

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1/3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{pmatrix}$$

Rank=3, Pivots=1, 3/2, 4/3

藉此方式，我們可以快速得到 A 的 Det，因為 $Det(A) = \prod_{i=0}^{n} U_{ii}$

然而，有些時候 A 不能直接做高斯消去，因此我們需要引入一個 P 矩陣，其作用在改變 A 的行順序，式子變為 $PA = LU$

E.g.2

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 3/2 \\ 1 & 3/2 & 1/2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \implies PA = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{pmatrix}$$

這時我們發現，在寫程式的時候 U 這逼會有點麻煩，因為他長得有點醜。所以我們引入一個 $D = \text{diag}(\text{pivots})$ ，然後讓 U 變成 U' (pivots 變成1)，這樣記憶體運用效率比較高，因此 e.g.2 變成

$$PA = LDU' \implies \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

最後是 Triangular factorization 的存在性及唯一性：

存在性1：對於可逆矩陣 ，若且唯若它的所有(包含原矩陣和反矩陣) leading principal minors 都非零，則其 LU 分解存在。

存在性2：對於不可逆矩陣，rank 為 k，若其前 k 個首要 leading principal minors 都非零，則其 LU 分解存在。

存在性3：對於 $n \times n$ 方形 (square) 矩陣，若其前 $n-1$ 行為線性獨立，但至少有一個 leading principal minors 為零，則其 LU 分解不存在。

唯一性1：對於方形的可逆矩陣，若其 LU 分解存在，則為唯一。

唯一性2：對於 $n \times n$ 方形矩陣，若其前 $n-1$ 行非線性獨立，則其有無窮多個 LU 分解。

解方程式

$n \times n$ 矩陣 A 的 eigenvalues 若有 m 個 0 解，則定義 rank=n-m。若 rank < n，則 A^{-1} 不存在，也就是 $\mathrm{Det}(A) = 0$

rank 和m元一次聯立方程式之解的關係：解帶有 (m - rank) 個獨立變數。以下我們將以求解一方程式做舉例

$$\begin{cases} x+y+z & =1 \\ 2x+2y+3z & =3 \\ 2x+2y+5z & =5 \end{cases} \implies Aw=b \implies \begin{cases} x+y+z=1 \\ z=1 \\ 0=0 \end{cases}$$

我們發現在第一行式子只會得到 $x=-y$ 的結果，因此此解帶有 1 個獨立變數，與關係式 m - rank 相等，選擇 x 為獨立變數，其解為：

$$\begin{cases} x=x \\ y=-x \\ z=1 \end{cases} \implies w = w_{\text{particular}} + w_{\text{nullspace}}$$

此處即為

$$w_p = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad w_n = x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \implies w = \begin{pmatrix} x \\ -x \\ 1 \end{pmatrix}$$

其中需要注意的是 w_n 為 $Aw = 0$ 的解，所有 w 的集合構成 nullspace

最後是 rank-nullity theorem: n = rank + nullity

- $\text{kernel}(A)$: 所有滿足 $A\vec{x} = \vec{0}$ 的向量
- nullspace: kernel 的同義詞
- nullity: kernel 的維度（有幾個自由變數）

Diagonalization of a matrix

Given a $n \times n$ normal matrix A (e.g. n=3), where the orthonormal eigenvectors are $|v_1\rangle, |v_2\rangle, |v_3\rangle$. Let $U = (|v_1\rangle, |v_2\rangle, |v_3\rangle)$, which is unitary. Hence, U can be used to perform unitary transformation on A

$$\Lambda = U^\dagger A U = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Important: A and Λ are similar

A is diagonalized by U. The eigenvectors of A become the basis of new coordinates

Something you should notice:

- Normal matrices are always diagonalizable
- The eigenvectors of Λ are canonical basis

$(1,0\dots0),(0,1\dots0),\dots(0,0\dots1)$

- The eigenvalues of Λ is the same as A
- $\mathrm{Det}(A)=\mathrm{Det}(\Lambda)$
- $\mathrm{Tr}(A)=\mathrm{Tr}(\Lambda)$
- $A^m=U\Lambda^mU^\dagger$
- $\mathrm{Tr}(A^m)=\mathrm{Tr}(\Lambda^m)=\sum_{i=1}^n\lambda_i^m$
- 求反矩陣

$$A^{-1}=U\begin{pmatrix}\lambda_1^{-1} & 0 & 0\\ 0 & \lambda_2^{-1} & 0\\ 0 & 0 & \lambda_3^{-1}\end{pmatrix}U^\dagger$$

Quadratic and Hermitian forms

Homogeneous function

$$f(sx)=s^kf(x),\ \text{k: degree}$$

When k=2, f(x) is a homogeneous quadratic function, which is always the quadratic form of a symmetric matrix.

Quadratic form of a real symmetric A is

$$f(x)=Q(x)=x^TAx=(Ux')^T(U\Lambda U^T)(Sx')=x'^T\Lambda x'=\lambda_ix_i'^2$$

E.g.1: Diagonalize the quadratic form $Q=2x^2+4xy+5y^2$

$$Q=(x,y)\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=(x,y)\begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=x^TAx$$

$$\implies A=S\Lambda S^T=\begin{pmatrix}\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt5} & -\frac{1}{\sqrt5}\\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{pmatrix}$$

$$\lambda=1,6>0\Rightarrow A\text{ is positive definite}$$

$$\implies Q = x'^2 + 6y'^2 = \left[\frac{1}{\sqrt{5}}(2x-y)\right]^2 + 6\left[\frac{1}{\sqrt{5}}(x+2y)\right]^2$$

接著我們可以來複習酷酷的二次曲面

$$Q(x) = k \Leftrightarrow \lambda_i^2 x_i^2 = k$$

沒了，剩下隨便

然後我們來討論一下什麼時候 symmetric A 會是 positive definite ，只要滿足以下之一：

- $\lambda_i > 0$
- $Q(x) > 0 \ \ \forall x$
- All pivots > 0
- $A = R^T R$

求 R 的方法：

法一 Eigenvalues
$A = U\Lambda U^\dagger \implies R = U\sqrt{\Lambda}U^\dagger$

法二 Eigen decomposition
$A = U\Lambda U^\dagger \implies R = \sqrt{\Lambda}U^\dagger$

法三 Triangular factorization 複習一下
$A = LU = LDU'$ ，若 A 為 symmetric，則
$A = LDL^T \implies R = \sqrt{D}L^T$ (Cholesky decomposition)

我們也可以用這個方法來快速的找到 A 的反矩陣
$A^{-1} = R^{-1}R^{-1^T} = P^T P \quad \text{where} \quad P = R^{-1^T} = \sqrt{D^{-1}}L^{-1}$