## **ANOMALIES**

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### **Synopsis**

Anomalies are the breaking of classical symmetries by quantum mechanical radiative corrections, which arise when the regularizations needed to evaluate small fermion loop Feynman diagrams conflict with a classical symmetry of the theory. They have important implications for a wide range of issues in quantum field theory, mathematical physics, and string theory.

#### Chiral Anomalies, Abelian and Non-Abelian

Consider quantum electrodynamics, with the Lagrangian density

$$\mathcal{L} = \overline{\psi}(i\gamma^{\mu}\partial_{\mu} - e_0\gamma^{\mu}B_{\mu} - m_0)\psi \quad , \tag{1a}$$

where  $\overline{\psi} = \psi^{\dagger} \gamma^{0}$ , where  $e_{0}$  and  $m_{0}$  are the bare charge and mass, and where  $B_{\mu}$  is the electromagnetic gauge potential. (We reserve the notation A for axial-vector quantities.)

Under a chiral transformation

$$\psi \to e^{i\lambda\gamma_5}\psi$$
 , [1b]

with constant  $\lambda$ , the kinetic term in eqn [1a] is invariant (because  $\gamma_5$  commutes with  $\gamma^0\gamma^\mu$ ), whereas the mass term is not invariant. Therefore, naive application of Noether's theorem would lead one to expect that the axial-vector current

$$j_{\mu}^{5} = \overline{\psi} \gamma_{\mu} \gamma_{5} \psi \quad , \tag{1c}$$

obtained from the Lagrangian density by applying a chiral transformation with spatially varying  $\lambda$ , should have a divergence given by the change under chiral transformation of the mass term in eqn [1a]. To tree approximation, this is indeed true, but when one computes

the AVV Feynman diagram with one axial-vector and two vector vertices (see **Figure 1**), and insists on conservation of the vector current  $j_{\mu} = \overline{\psi}\gamma_{\mu}\psi$ , one finds that to order  $e_0^2$  the classical Noether theorem is modified to read

$$\partial^{\mu} j_{\mu}^{5}(x) = 2im_{0} j^{5}(x) + \frac{e_{0}^{2}}{16\pi^{2}} F^{\xi\sigma}(x) F^{\tau\rho}(x) \epsilon_{\xi\sigma\tau\rho} \quad ,$$
 [2]

with  $F^{\xi\sigma}(x) = \partial^{\sigma}B^{\xi}(x) - \partial^{\xi}B^{\sigma}(x)$  the electromagnetic field-strength tensor. The second term in eqn [2], which would be unexpected from application of the classical Noether theorem, is the Abelian axial-vector anomaly, often called the Adler-Bell-Jackiw (or ABJ) anomaly after the seminal papers on the subject. Since vector current conservation together with the axial-vector current anomaly implies that the left and right handed chiral currents  $j_{\mu} \pm j_{\mu}^{5}$  are also anomalous, the axial-vector anomaly is frequently called the *chiral anomaly*, and we shall use the terms interchangeably in this article.

There are a number of different ways to understand why the extra term in eqn [2] appears.

(1) If one works through the formal Feynman diagrammatic Ward identity proof of the Noether theorem, one finds that there is a step where the closed fermion loop contributions are eliminated by a shift of the loop integration variable. For Feynman diagrams that are convergent this is no problem, but the AVV diagram is linearly divergent. The linear divergence vanishes under symmetric integration, but the shift then produces a finite residue, which gives the anomaly. (2) If one defines the AVV diagram by Pauli-Villars regularization with regulator mass  $M_0$  that is allowed to approach infinity at the end of the calculation, one finds a classical Noether theorem in the regulated theory,

$$\partial^{\mu} j_{\mu}^{5}|_{m_{0}} - \partial^{\mu} j_{\mu}^{5}|_{M_{0}} = 2im_{0}j^{5}|_{m_{0}} - 2iM_{0}j^{5}|_{M_{0}} , \qquad [3a]$$

with the subscripts  $m_0$  and  $M_0$  indicating that fermion loops are to be calculated with fermion

mass  $m_0$  and  $M_0$  respectively. Taking the vacuum to two photon matrix element of eqn [3a], one finds that the matrix element  $\langle 0|j^5|_{M_0}|\gamma\gamma\rangle$ , which is unambiguously computable after imposing vector current conservation, falls off only as  $M_0^{-1}$  as the regulator mass approaches infinity. Thus, the product of  $2iM_0$  with this matrix element has a finite limit, which gives the anomaly. (3) If one defines the gauge invariant axial-vector current by point-splitting,

$$j_{\mu}^{5}(x) = \overline{\psi}(x + \epsilon/2)\gamma_{\mu}\gamma_{5}\psi(x - \epsilon/2)e^{-ie_{0}\epsilon^{\sigma}B_{\sigma}(x)} , \qquad [3b]$$

with  $\epsilon \to 0$  at the end of the calculation, one finds that the divergence of eqn [3b] contains an extra term with a factor of  $\epsilon$ . On careful evaluation one finds that the coefficient of this factor is an expression that behaves as  $\epsilon^{-1}$ , which gives the anomaly in the limit of vanishing  $\epsilon$ . (4) Finally, if one defines the field theory by a functional integral over the classical action, the standard Noether analysis shows that the classical action is invariant under the chiral transformation of eqn [1b], apart from the contribution of the mass term, which gives the naive axial-vector divergence. However, as pointed out by Fujikawa, one must also apply the chiral transformation to the functional integration measure, and since the measure is an infinite product it must be regularized to be well defined. Careful calculation shows that the regularized measure is not chiral invariant, but contributes an extra term to the axial-vector Ward identity that is precisely the chiral anomaly.

A key feature of the anomaly is that it is *irreducible*: one cannot add any local polynomial counter term to the AVV diagram that preserves vector current conservation, and which eliminates the anomaly. More generally, one can show that there is no way of modifying quantum electrodynamics so as to eliminate the chiral anomaly, without spoiling either vector current conservation (that is, electromagnetic gauge invariance), renormalizability, or unitarity. Thus the chiral anomaly is a new physical effect in renormalizable quantum field

theory, that is not present in the pre-quantization classical theory.

The Abelian chiral anomaly is the simplest case of the anomaly phenomenon. It was extended to non-Abelian gauge theories by Bardeen using a point-splitting method to compute the divergence, followed by adding polynomial counter terms to remove as many of the residual terms as possible. The resulting irreducible divergence is the non-Abelian chiral anomaly, which in terms of Yang-Mills field strengths for vector and axial-vector gauge potentials  $V^{\mu}$  and  $A^{\mu}$ ,

$$F_{V}^{\mu\nu}(x) = \partial^{\mu}V^{\nu}(x) - \partial^{\nu}V^{\mu}(x) - i[V^{\mu}(x), V^{\nu}(x)] - i[A^{\mu}(x), A^{\nu}(x)] \quad ,$$

$$F_{A}^{\mu\nu}(x) = \partial^{\mu}A^{\nu}(x) - \partial^{\nu}A^{\mu}(x) - i[V^{\mu}(x), A^{\nu}(x)] - i[A^{\mu}(x), V^{\nu}(x)] \quad ,$$

$$(4a)$$

is given by

 $\partial^{\mu} j_{5\mu}^{a}(x)$  =normal divergence term

$$+(1/4\pi^{2})\epsilon_{\mu\nu\sigma\tau}\mathrm{tr}\lambda_{A}^{a}[(1/4)F_{V}^{\mu\nu}(x)F_{V}^{\sigma\tau}(x) + (1/12)F_{A}^{\mu\nu}(x)F_{A}^{\sigma\tau}(x)$$

$$+(2/3)iA^{\mu}(x)A^{\nu}(x)F_{V}^{\sigma\tau}(x) + (2/3)iF_{V}^{\mu\nu}(x)A^{\sigma}(x)A^{\tau}(x) + (2/3)iA^{\mu}(x)F_{V}^{\nu\sigma}(x)A^{\tau}(x)$$

$$-(8/3)A^{\mu}(x)A^{\nu}(x)A^{\sigma}(x)A^{\tau}(x)] \quad .$$
[4b]

In eqn [4b], tr denotes a trace over internal degrees of freedom, and  $\lambda_A^a$  is the internal symmetry matrix associated with the axial-vector external field. In the Abelian case, where there is no internal symmetry structure, the terms involving two or four factors of  $A^{\mu}$ ,  $A^{\nu}$ , ... vanish by antisymmetry of  $\epsilon_{\mu\nu\sigma\tau}$ , and one recovers the AVV triangle anomaly, as well as a kinematically related anomaly in the AAA triangle diagram. In the non-Abelian case, with non-trivial internal symmetry structure, there are also box and pentagon diagram anomalies.

In addition to coupling to spin 1 gauge fields, fermions can also couple to spin 2 gauge fields, associated with the graviton. When the coupling of fermions to gravitation is taken into account, the axial-vector current  $\overline{\psi}T\gamma_{\mu}\gamma_{5}\psi$ , with T an internal symmetry matrix, has

an additional anomalous contribution to its divergence proportional to

$$\operatorname{tr} T \epsilon_{\xi \sigma \tau \rho} R^{\xi \sigma \alpha \beta} R^{\tau \rho}_{\alpha \beta} , \qquad [4c]$$

where  $R_{\xi\sigma\tau\rho}$  is the Riemann curvature tensor of the gravitational field.

#### Chiral Anomaly Nonrenormalization

A salient feature of the chiral anomaly is the fact that it is not renormalized by higher order radiative corrections. In other words, the one loop expressions of eqns [2] and [4b] give the exact anomaly coefficient, without modification in higher orders of perturbation theory. In gauge theories such as quantum electrodynamics and quantum chromodynamics, this result (the Adler-Bardeen theorem) can be understood heuristically as follows. Write down a modified Lagrangian in which regulators are included for all gauge boson fields. Since the gauge boson regulators do not influence the chiral symmetry properties of the theory, the divergences of the chiral currents are not affected by their inclusion, and so the only sources of anomalies in the regularized theory are small single-fermion loops, giving the anomaly expressions of eqns [2] and [4b]. Since the renormalized theory is obtained as the limit of the regularized theory as the regulator masses approach infinity, this result applies to the renormalized theory as well.

The above argument can be made precise, and extends to non-gauge theories such as the  $\sigma$ -model as well. For both gauge theories and the  $\sigma$ -model, cancellation of radiative corrections to the anomaly coefficient has been explicitly demonstrated in fourth order calculations. Nonperturbative demonstrations of anomaly renormalization have also been given using the Callan-Symanzik equations. For example, in quantum electrodynamics, Zee, and Lowenstein and Schroer, showed that a factor f that gives the ratio of the true anomaly to

its one-loop value obeys the differential equation

$$\left(m\frac{\partial}{\partial m} + \alpha\beta(\alpha)\frac{\partial}{\partial \alpha}\right)f = 0 \quad .$$
 [5]

Since f is dimensionless it can have no dependence on the mass m, and since  $\beta(\alpha)$  is nonzero this implies  $\partial f/\partial \alpha = 0$ . Thus f has no dependence on  $\alpha$ , and so f = 1.

#### **Applications of Chiral Anomalies**

Chiral anomalies have numerous applications in the standard model of particle physics and its extensions, and we describe here a few of the most important ones.

(1) Neutral pion decay  $\pi^0 \to \gamma \gamma$ . As a result of the Abelian chiral anomaly, the partially conserved axial-vector current (PCAC) equation relevant to neutral pion decay is modified to read

$$\partial^{\mu} \mathcal{F}_{3\mu}^{5}(x) = \left(f_{\pi} \mu_{\pi}^{2} / \sqrt{2}\right) \phi_{\pi}(x) + S \frac{\alpha_{0}}{4\pi} F^{\xi\sigma}(x) F^{\tau\rho}(x) \epsilon_{\xi\sigma\tau\rho} \quad , \tag{6a}$$

with  $\mu_{\pi}$  the pion mass,  $f_{\pi} \simeq 131\,\mathrm{MeV}$  the charged pion decay constant, and with S a constant determined by the constituent fermion charges and axial-vector couplings. Taking the matrix element of eqn [6a] between the vacuum state and a two photon state, and using the fact that the left hand side has a kinematic zero (the Sutherland-Veltman theorem), one sees that the  $\pi^0 \to \gamma\gamma$  amplitude F is completely determined by the anomaly term, giving the formula

$$F = -(\alpha/\pi)2S\sqrt{2}/f_{\pi} \quad . \tag{6b}$$

For a single set of fractionally charged quarks, the amplitude F is a factor of three too small to agree with experiment; for three fractionally charged quarks (or an equivalent Han-Nambu

triplet), eqn [6b] gives the correct neutral pion decay rate. This calculation was one of the first pieces of evidence for the color degree of freedom of quarks.

(2) Anomaly cancellation in gauge theories. In quantum electrodynamics the gauge particle (the photon) couples to the vector current, and so the anomalous conservation properties of the axial-vector current have no effect. The same statement holds for the gauge gluons in quantum chromodynamics, when treated in isolation from the other interactions. However, in the electroweak theory that embeds quantum electrodynamics in a theory of the weak force, the gauge particles (the  $W^{\pm}$  and Z intermediate bosons) couple to chiral currents, which are left or right handed linear combinations of the vector and axial-vector currents. In this case, the chiral anomaly leads to problems with the renormalizability of the theory, unless the anomalies cancel between different fermion species. Writing all fermions as left handed, the condition for anomaly cancellation is

$$\operatorname{tr}\{T_{\alpha}, T_{\beta}\}T_{\gamma} = \operatorname{tr}(T_{\alpha}T_{\beta} + T_{\beta}T_{\alpha})T_{\gamma} = 0 , \quad \text{all } \alpha, \beta, \gamma ,$$
 [7]

with  $T_{\alpha}$  the coupling matrices of gauge bosons to left handed fermions. These conditions are obeyed in the standard model, by virtue of three non-trivial sum rules on the fermion gauge couplings being satisfied (four sum rules if one includes the gravitational contribution to the chiral anomaly, given in eqn [4c], which also cancels in the standard model.) Note that anomaly cancellation in the locally gauged currents of the standard model does not imply anomaly cancellation in global flavor currents. Thus the flavor axial-vector current anomaly that gives the  $\pi^0 \to \gamma \gamma$  matrix element remains anomalous in the full electroweak theory. Anomaly cancellation imposes important constraints on the construction of grand unified models that combine the electroweak theory with quantum chromodynamics. For

instance, in SU(5) the fermions are put into a  $\overline{5}$  and 10 representation, which together, but not individually, are anomaly free. The larger unification groups SO(10) and  $E_6$  satisfy eqn [7] for all representations, and so are automatically anomaly free.

(3) Instanton physics and the theta vacuum. The theory of anomalies is intimately tied to the physics associated with instanton classical Yang-Mills theory solutions. Since the instanton field strength is self-dual, the nonvanishing instanton Euclidean action

$$S_E = \int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = 8\pi^2$$
 [8a]

implies that the integral of the pseudoscalar density  $F_{\mu\nu}F_{\lambda\sigma}\epsilon^{\mu\nu\lambda\sigma}$  over the instanton is also nonzero,

$$\int d^4x F_{\mu\nu} F_{\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} = 64\pi^2 \quad . \tag{8b}$$

Referring back to eqn [4b], we see that this means that the integral of the non-Abelian chiral anomaly for fermions in the background field of an instanton is an integer, which in the Minkowski space continuation has the interpretation of a topological winding number change produced by the instanton tunneling solution. This fact has a number of profound consequences. Since a vacuum with definite winding number  $|\nu\rangle$  is unstable under instanton tunneling, careful analysis shows that the non-Abelian vacuum that has correct clustering properties is a Fourier superposition

$$|\theta\rangle = \sum_{\nu} e^{i\theta\nu} |\nu\rangle \quad , \tag{8c}$$

giving rise to the  $\theta$ -vacuum of quantum chromodynamics, and a host of issues associated with (the lack of) strong CP-violation, the Peccei-Quinn mechanism, and axion physics. Also, the fact that the integral of eqn [8b] is nonzero means that the U(1) chiral symmetry of quantum chromodynamics is broken by instantons, which as shown by 't Hooft resolves the longstanding "U(1) problem" of the strong interactions, that of explaining why the flavor singlet pseudoscalar meson  $\eta'$  is not light, as are its flavor octet partners.

(4) Anomaly matching conditions. The anomaly structure of a theory, as shown by 't Hooft, leads to important constraints on the formation of massless composite bound states. Consider a theory with a set of left handed fermions  $\psi^{if}$ , with i a "color" index acted on by a non-Abelian gauge force, and f an ungauged family or "flavor" index. Suppose that the family multiplet structure is such that the global chiral symmetries associated with the flavor index have nonvanishing anomalies  $\operatorname{tr}\{T_{\alpha}, T_{\beta}\}T_{\gamma}$ . Then the 't Hooft condition asserts that if the color forces result in the formation of composite massless bound states of the original completely confined fermions, and if there is no spontaneous breaking of the original global flavor symmetries, then these bound states must contain left handed spin-1/2 composites with a representation structure S that has the same anomaly coefficient as that in the underlying theory. In other words, we must have

$$\operatorname{tr}\{S_{\alpha}, S_{\beta}\} S_{\gamma} = \operatorname{tr}\{T_{\alpha}, T_{\beta}\} T_{\gamma} \quad .$$
 [9]

To prove this, one adjoins to the theory a set of right-handed spectator fermions  $\psi^f$  with the same flavor structure as the original set, but which are not acted on by the color force. These right handed fermions cancel the original anomaly, making the underlying theory at zero color coupling anomaly-free; since dynamics cannot spontaneously generate anomalies, the theory when the color dynamics is turned on must also have no global chiral anomalies. This implies that the bound state spectrum must conspire to cancel the anomalies associated with the right-handed spectators, in other words, the bound state anomaly structure must match

that of the original fermions. This anomaly matching condition has found applications in the study of the possible compositeness of quarks and leptons. It has also been applied to the derivation of non-perturbative dynamical results in whole classes of supersymmetric theories, where the combined tools of holomorphicity, instanton physics, and anomaly matching have given incisive results.

#### Global Structure of Anomalies

We noted above that the chiral anomalies are irreducible, in that they cannot be eliminated by adding a local polynomial counter-term to the action. However, anomalies can be described by a non-local effective action, obtained by integrating out the fermion field dynamics, and this point of view proves very useful in the non-Abelian case. Starting with the Abelian case for orientation, we note that if  $A^{\mu}$  is an external axial-vector field, and we write an effective action  $\Gamma[A]$ , then the axial-vector current  $j_{\mu}^{5}$  associated with  $A^{\mu}$  is given (up to an overall constant) by the variational derivative expression

$$j_{\mu}^{5}(x) = \frac{\delta\Gamma[A]}{\delta A^{\mu}(x)} \quad , \tag{10a}$$

and the Abelian anomaly appears as the fact that the expression

$$\partial^{\mu} j_{\mu}^{5} = X\Gamma[A] = G \neq 0 \quad ,$$

$$X = \partial^{\mu} \frac{\delta}{\delta A^{\mu}(x)} \quad ,$$
[10b]

is non-vanishing even when the theory is classically chiral invariant. Turning now to the non-Abelian case, the variational derivative appearing in eqns [10a,b] must be replaced by an appropriate covariant derivative. In terms of the internal symmetry component fields  $A^a_{\mu}$ 

and  $V_{\mu}^{a}$  of the Yang-Mills potentials of eqn [4a], one introduces operators

$$-X^{a}(x) = \partial^{\mu} \frac{\delta}{\delta A_{\mu}^{a}(x)} + f_{abc} V_{\mu}^{b} \frac{\delta}{\delta A_{\mu}^{c}(x)} + f_{abc} A_{\mu}^{b} \frac{\delta}{\delta V_{\mu}^{c}(x)} ,$$

$$-Y^{a}(x) = \partial^{\mu} \frac{\delta}{\delta V_{\mu}^{a}(x)} + f_{abc} V_{\mu}^{b} \frac{\delta}{\delta V_{\mu}^{c}(x)} + f_{abc} A_{\mu}^{b} \frac{\delta}{\delta A_{\mu}^{c}(x)} ,$$
[11a]

with  $f_{abc}$  the antisymmetric non-Abelian group structure constants. The operators  $X^a$  and  $Y^a$  are easily seen to obey the commutation relations

$$[X^{a}(x), X^{b}(y)] = f_{abc}\delta(x - y)Y_{c}(x) ,$$

$$[X^{a}(x), Y^{b}(y)] = f_{abc}\delta(x - y)X_{c}(x) ,$$

$$[Y^{a}(x), Y^{b}(y)] = f_{abc}\delta(x - y)Y_{c}(x) .$$

$$[11b]$$

Let  $\Gamma[V, A]$  be the effective action as a functional of the fields  $V^{\mu}$ ,  $A^{\mu}$ , constructed so that the vector currents are covariantly conserved, as expressed formally by

$$Y^a\Gamma[V,A] = 0 . [12a]$$

Then the non-Abelian axial-vector current anomaly is given by

$$X^a\Gamma[V,A] = G^a \quad . \tag{12b}$$

From eqns [12a,b] and the first line of eqn [11b], we have

$$X^{b}G^{a} - X^{a}G^{b} = (X^{b}X^{a} - X^{a}X^{b})\Gamma[V, A] \propto f_{abc}Y^{c}\Gamma[V, A] = 0$$
 , [12c]

which is the Wess-Zumino consistency condition on the structure of the anomaly  $G^a$ . It can be shown that this condition uniquely fixes the form of the non-Abelian anomaly to be that of eqn [4b], up to an overall constant, which can be determined by comparison with the simplest anomalous AVV triangle graph. A physical consequence of the consistency condition is that the  $\pi^0 \to \gamma \gamma$  decay amplitude determines uniquely certain other anomalous amplitudes, such as  $2\gamma \to 3\pi$ ,  $\gamma \to 3\pi$ , and a five pseudoscalar vertex.

Although the action  $\Gamma[V,A]$  is necessarily nonlocal, Wess and Zumino were able to write down a local action, involving an auxiliary pseudoscalar field, that obeys the anomalous Ward identities and the consistency conditions. Subsequently, Witten gave a new construction of this local action, in terms of the integral of a fifth rank antisymmetric tensor over a five dimensional disk which has four dimensional space as its boundary. He also showed that requiring  $e^{i\Gamma}$  to be independent of the choice of the spanning disk requires, in analogy with Dirac's quantization condition for monopole charge, the condition that the overall coefficient in the non-Abelian anomaly be quantized in integer multiples. Comparison with the lowest order triangle diagram shows that in the case of  $SU(N_c)$  gauge theory, this integer is just the number of colors  $N_c$ . Thus, global considerations tightly constrain the non-Abelian chiral anomaly structure, and dictate that up to an integer proportionality constant, it must have the form given in eqns [4a,b].

#### Trace Anomalies

The discovery of chiral anomalies inspired the search for other examples of anomalous behavior. First indications of a perturbative trace anomaly obtained in a study of broken scale invariance by Coleman and Jackiw were shown by Crewther, and by Chanowitz and Ellis, to correspond to an anomaly in the three point function  $\theta_{\sigma}^{\sigma}V_{\mu}V_{\nu}$ , where  $\theta_{\mu}^{\mu}$  is the energy-momentum tensor. Letting  $\Delta_{\mu\nu}(p)$  be the momentum space expression for this three-point function, and  $\Pi_{\mu\nu}$  the corresponding  $V_{\mu}V_{\nu}$  two-point function, the trace anomaly equation in quantum electrodynamics reads

$$\Delta_{\mu\nu}(p) = \left(2 - p_{\sigma} \frac{\partial}{\partial p_{\sigma}}\right) \Pi_{\mu\nu}(p) - \frac{R}{6\pi^2} (p_{\mu}p_{\nu} - \eta_{\mu\nu}p^2) \quad ,$$
 [13a]

with the first term on the right hand side the naive divergence, and the second term the

trace anomaly, with anomaly coefficient R given by

$$R = \sum_{i,\text{spin}\frac{1}{2}} Q_i^2 + \frac{1}{4} \sum_{i,\text{spin}0} Q_i^2 \quad .$$
 [13b]

The fact that there should be a trace anomaly can readily be inferred from a trace analog of the Pauli-Villars regulator argument for the chiral anomaly given in eqn [3a]. Letting  $j = \overline{\psi}\psi$  be the scalar current in Abelian electrodynamics, one has

$$\theta^{\mu}_{\mu}|_{m_0} - \theta^{\mu}_{\mu}|_{M_0} = m_0 j|_{m_0} - M_0 j|_{M_0}$$
 [13c]

Taking the vacuum to two photon matrix element of this equation, and imposing vector current conservation, one finds that the matrix element  $\langle 0|j|_{M_0}|\gamma\gamma\rangle$  is proportional to  $M_0^{-1}\langle 0|F_{\lambda\sigma}F^{\lambda\sigma}|\gamma\gamma\rangle_{M_0}$  for large regulator mass, and so makes a non-vanishing contribution to the right hand side of eqn [13c], giving the lowest order trace anomaly.

Unlike the chiral anomaly, the trace anomaly is renormalized in higher orders of perturbation theory; heuristically, the reason is that whereas boson field regulators do not affect the chiral symmetry properties of a gauge theory (which are determined just by the fermionic terms in the Lagrangian), they do alter the energy-momentum tensor, since gravitation couples to all fields, including regulator fields. An analysis using the Callan-Symanzik equations shows, however, that the trace anomaly is computable to all orders in terms of various renormalization group functions of the coupling. For example, in Abelian electrodynamics, defining  $\beta(\alpha)$  and  $\delta(\alpha)$  by  $\beta(\alpha) = (m/\alpha)\partial\alpha/\partial m$  and  $1 + \delta(\alpha) = (m/m_0)\partial m_0/\partial m$ , the trace of the energy-momentum tensor is given to all orders by

$$\theta^{\mu}_{\mu} = \left[1 + \delta(\alpha)\right] m_0 \overline{\psi} \psi + \frac{1}{4} \beta(\alpha) N[F_{\lambda\sigma} F^{\lambda\sigma}] + \dots \quad , \tag{14}$$

with N[...] specifying conditions that make the division into two terms in eqn [14] unique.

A similar relation holds in the non-Abelian case, again with the  $\beta$  function appearing as the coefficient of the anomalous  $\operatorname{tr} N[F_{\lambda\sigma}F^{\lambda\sigma}]$  term.

Just as in the chiral anomaly case, when spin-0, spin-1/2, or spin-1 fields propagate on a background spacetime, there are curvature dependent contributions to the trace anomaly, in other words, gravitational anomalies. These typically take the form of complicated linear combinations of terms of the form  $R^2$ ,  $R_{\mu\nu}R^{\mu\nu}$ ,  $R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ ,  $R_{,\mu}^{;\mu}$ , with coefficients depending on the matter fields involved.

In supersymmetric theories, the axial-vector current and the energy-momentum tensor are both components of the supercurrent, and so their anomalies imply the existence of corresponding supercurrent anomalies. The issue of how the non-renormalization of chiral anomalies (which have a supercurrent generalization given by the Konishi anomaly), and the renormalization of trace anomalies, can coexist in supersymmetric theories originally engendered considerable confusion. This apparent puzzle is now understood in the context of a perturbatively exact expression for the  $\beta$  function in supersymmetric field theories (the so-called NSVZ, for Novikov, Shifman, Vainshtein, and Zakharov,  $\beta$  function). Supersymmetry anomalies can be used to infer the structure of effective actions in supersymmetric theories, and these in turn have important implications for possibilities for dynamical supersymmetry breaking. Anomalies may also play a role, through anomaly mediation, in communicating supersymmetry breaking in "hidden sectors" of a theory, that do not contain the physical fields that we directly observe, to the "physical sector" containing the observed fields.

#### Further Anomaly Topics

The above discussion has focused on some of the principal features and applications of

anomalies. There are further topics of interest in the physics and mathematics of anomalies, that are discussed in detail in the references cited for further reading, and elsewhere in the literature. We briefly describe a few of them here.

(1) Anomalies in other spacetime dimensions and in string theory. We have focused above on anomalies in four dimensional spacetime, but there anomalies of various types both in lower dimensional quantum field theories (such as theories in 2 and 3 dimensional spacetimes), and in quantum field theories in higher dimensional spacetimes (such as N=1 supergravity in 10 dimensional spacetime). Anomalies also play an important role in the formulation and consistency of string theory. The bosonic string is consistent only in 26 dimensional spacetime, and the analogous supersymmetric string is consistent only in 10 dimensional spacetime, because in other dimensions both of these theories violate Lorentz invariance after quantization. In the Polyakov path integral formulation of these string theories, these special dimensions are associated with the cancellation of the Weyl anomaly, which is the relevant form of the trace anomaly discussed above. Yang-Mills, gravitational, and mixed Yang-Mills gravitational anomalies make an appearance both in  $N=1\ 10$  dimensional supergravity and in superstring theory, and again special dimensions play a role. In these theories, only when the associated internal symmetry groups are either SO(32) or  $E_8 \times E_8$ is elimination of all anomalies possible, by cancellation of hexagon diagram anomalies with anomalous tree diagrams involving exchange of a massless antisymmetric two-form field. This mechanism, due to Green and Schwarz, requires the factorization of a sixth order trace invariant that appears in the hexagon anomaly in terms of lower order invariants, as well as two numerical conditions on the adjoint representation generator structure, restricting the allowed gauge groups to the two noted above.

(2) Covariant Versus Consistent Anomalies; Descent Equations. The non-Abelian anomaly of eqns [4a,b] is called the "consistent anomaly", because it obeys the Wess-Zumino consistency conditions of eqn [12c]. This anomaly, however, is not gauge-covariant, as can be seen from the fact that it involves not only the Yang-Mills field strengths  $F_{V,A}^{\mu\nu}$ , but the potentials  $V^{\mu}$ ,  $A^{\mu}$  as well. It turns out to be possible, by adding appropriate polynomials to the currents, to transform the consistent anomaly to a form, called the "covariant anomaly", which is gauge-covariant under gauge transformations of the potentials  $V^{\mu}$ ,  $A^{\mu}$ . This anomaly, however, does not obey the Wess-Zumino consistency conditions, and cannot be obtained from variation of an effective action functional.

The consistent anomalies (but not the covariant anomalies) obey a remarkable set of relations, called the Stora-Zumino descent equations, that relate the Abelian anomaly in 2n + 2 spacetime dimensions to the non-Abelian anomaly in 2n spacetime dimensions. This set of equations has been interpreted physically by Callan and Harvey as reflecting the fact that the Dirac equation has chiral zero modes in the presence of strings in 2n + 2 dimensions and of domain walls in 2n + 1 dimensions.

(3) Anomalies and fermion doubling in lattice gauge theories. A longstanding problem in lattice formulations of gauge field theories is that when fermions are introduced on the lattice, the process of discretization introduces an undesirable doubling of the fermion particle modes. In particular, when one attempts to put chiral gauge theories, such as the electroweak theory, on the lattice, one finds that the doublers eliminate the chiral anomalies, by cancellation between modes with positive and modes with negative axial-vector charge. Thus for a long time it appeared doubtful whether chiral gauge theories could be simulated on the lattice. However, recent work has led to formulations of lattice fermions that use a mathematical

analog of a domain wall to successfully incorporate chiral fermions, and the chiral anomaly, into lattice gauge theory calculations.

(4) Relation of anomalies to the Atiyah-Singer index theorem. The singlet  $(\lambda_A^a = 1)$  anomaly of eqn [4b] is closely related to the Atiyah-Singer index theorem. Specifically, the Euclidean spacetime integral of the singlet anomaly constructed from a gauge field can be shown to give the *index* of the related Dirac operator for a fermion moving in that background gauge field, where the index is defined as the difference between the numbers of right and left handed zero eigenvalue normalizable solutions of the Dirac equation. Since the index is a topological invariant, this again implies that the Euclidean spacetime integral of the anomaly is a topological invariant, as noted above in our discussion of instanton-related applications of anomalies.

#### Retrospect

The wide range of implications of anomalies has surprised – even astonished – the founders of the subject. New anomaly applications have appeared within the last few years, and very likely the future will see continued growth of the area of quantum field theory concerned with the physics and mathematics of anomalies.

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# Figure Captions

Figure 1 The AVV triangle diagram responsible for the Abelian chiral anomaly.

# Keywords

Anomalies
Axial-vector current
Chiral anomaly
Descent equations
Electroweak theory
Instantons
Lattice gauge theory
Non-Abelian gauge theory
Quantum Electrodynamics
Quantum Chromodynamics
String theory
Theta vacuum
Trace anomaly

## Nomenclature

MeV million electron volts