Non-relativistic leptogenesis

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Abstract

In many phenomenologically interesting models of thermal leptogenesis the heavy neutrinos are non-relativistic when they decay and produce the baryon asymmetry of the Universe. We propose a non-relativistic approximation for the corresponding rate equations in the non-resonant case, and a systematic way for computing relativistic corrections. We determine the leading order coefficients in these equations, and the first relativistic corrections. The non-relativistic approximation works remarkably well. It appears to be consistent with results obtained using a Boltzmann equation taking into account the momentum distribution of the heavy neutrinos, while being much simpler. We also compute radiative corrections to some of the coefficients in the rate equations. Their effect is of order 1% in the regime favored by neutrino oscillation data. We obtain the correct leading order lepton number washout rate in this regime, which leads to large ($\sim 20\%$) effects compared to previous computations.

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1 Introduction and motivation

Leptogenesis [1] (for reviews, see e.g. Refs. [2, 3]) is an appealing scenario which can explain the baryon asymmetry of the Universe. It requires the existence of sufficiently heavy right-handed, or sterile neutrinos N_i in addition to the particles of the Standard Model. Their Yukawa couplings with the Higgs and the Standard Model leptons have to contain CP-violating phases. When their interactions occur out of thermal equilibrium, they can produce a lepton asymmetry L, which, due to $B + L$ violating sphaleron processes, also leads to a baryon asymmetry B.

During leptogenesis most interactions are either much faster or much slower than the expansion of the Universe. Therefore most physical quantities are either very close to their corresponding thermal equilibrium values, or constant. A few quantities evolve roughly at the same rate as the Hubble rate, and can therefore deviate significantly from their equilibrium values. Among them are the number density of the heavy neutrinos and the $B-L$ asymmetry, because they are only changed by the heavy neutrino Yukawa interaction.

However, not only the number density of heavy neutrinos, but also their momentum distribution is out of equilibrium, because the kinetic equilibration is also due to the Yukawa interactions. Nevertheless, in the standard approach [4] kinetic equilibrium is assumed. In addition, Maxwell-Boltzmann statistics is used. In order to account for the correct momentum distribution Boltzmann equations for the distribution function have been used in Refs. [5, 6, 7], together with quantum statistics, and significant differences compared to the standard approach were reported.

In the standard approach both $1 \leftrightarrow 2$ decays and inverse decays of the right-handed neutrinos, as well as $2 \leftrightarrow 2$ scattering processes are taken into account, and both were found to be important. Recently the production rate of right-handed neutrinos of mass M_N, which is one ingredient of the rate equations describing leptogenesis, were studied at next-to-leading order in the Standard Model couplings [8, 9, 10]. It was pointed out that the $2 \leftrightarrow 2$ processes are only part of the next-to-leading order contribution and that $1 \leftrightarrow 3$ scattering and virtual corrections to the (inverse) decays contribute at the same order. If they are taken into account the large infrared contributions cancel [8]. Therefore a complete next-to-leading order calculation would be important. Results for the production rate were obtained in the limit $T \ll M_N$, together with T/M_N corrections in [8, 9], and more recently also in the regime $T \sim M_N$ [10].

A measure for the interaction rate of right-handed neutrinos relative to the Hubble

rate H at the time when they become non-relativistic is the washout factor

$$K \equiv \frac{\Gamma_0}{H}\bigg|_{T=M_N}, \tag{1}$$

where Γ_0 is the total tree level decay rate. In the so-called strong washout regime $K \gg 1$ the right-handed neutrinos are close to equilibrium around $T \sim M_N$, any pre-existing asymmetry is washed out, and the final asymmetry is mainly created when T has dropped below M_N. At this point the right-handed neutrinos are non-relativistic, since they are close to equilibrium. It turns out that the strong washout regime corresponds to a mass range of the light neutrinos which is favored by atmospheric and solar neutrino oscillation data [2]. Therefore the non-relativistic regime is of particular interest for leptogenesis. A first principle treatment of leptogenesis, which could allow for a complete next-to-leading order calculation can be expected to be simpler in this regime than in the fully relativistic case.

In this paper we obtain the rate equations for leptogenesis in the non-relativistic limit, and we propose a systematic expansion to compute relativistic corrections. This way one can systematically control the accuracy of the non-relativistic approximation, and one can obtain information about the momentum distribution and thus about the deviation from kinetic equilibrium. The rate equations are valid to all orders in the Standard Model couplings, and a complete next-to-leading order calculation in these couplings should be feasible. We take a first step in this direction by computing the corrections for some coefficients in the rate equations, thereby consistently including temperature dependent gauge and top Yukawa interaction in a leptogenesis calculation for the first time. We restrict ourselves to the case of hierarchical Majorana masses for the right-handed neutrinos, and we only consider the leading order in the right-handed neutrino Yukawa couplings.

This paper is organized as follows. In Sect. 2 we obtain the equations for leptogenesis in the non-relativistic limit and to all orders in the Standard Model couplings. In Sect. 3 we compute the coefficients in these equations at leading order in the Standard Model couplings. The conditions for thermal (non-)equilibrium are discussed in Sect. 4. The first relativistic corrections to the equations of Sect. 2 are obtained in Sect. 5. In Sect. 6 we use the results of [9] to compute some of the coefficients at next-to-leading order in the Standard Model couplings. Numerical results are shown in Sect. 7, and the conclusions are in Sect. 8.

2   Leptogenesis in the non-relativistic limit

We assume that the mass of the right-handed neutrinos is hierarchical and that we only need to take into account the lightest one which we denote by N. We work at leading order in the Yukawa couplings of the right-handed neutrinos. We also assume that the interactions of the right-handed neutrinos are much less frequent than gauge interactions. [3] Then on time scales which are relevant to leptogenesis the Standard Model particles are in kinetic equilibrium. Finally we assume that the Standard Model Yukawa interactions are either much faster than the right-handed neutrino reaction rates so that they are in thermal equilibrium, or that they are much slower. [4]

We assume that only one lepton flavor is involved. This is the case, for instance, at high temperature, when all Yukawa interactions of the Standard Model leptons can be neglected. At lower temperature the lepton Yukawa interactions become important, and in general one has to deal with several asymmetries [14]. However, for certain values of the Yukawa coupling matrix one may still be approximately in a one-flavor regime.

We consider non-relativistic right-handed neutrinos with a characteristic velocity $v \ll 1$. In a first approximation their motion can be neglected. Then the out-of-equilibrium state is fully specified by the number densities n_N and n_{B-L}. In a non-expanding Universe the time derivative of these densities would only depend on their deviations from their equilibrium values and on the temperature. These considerations imply that the time evolution of this system in an expanding Universe is governed by the equations

$$\left(\frac{d}{dt} + 3H \right) n_N \;\; = \; - \, \Gamma_N \, (n_N - n_N^{\text{eq}}) + \Gamma_{N,B-L} \, n_{B-L}, \qquad (2)$$

$$\left(\frac{d}{dt} + 3H \right) n_{B-L} = \; \Gamma_{B-L,N}(n_N - n_N^{\text{eq}}) - \Gamma_{B-L} \, n_{B-L}. \qquad (3)$$

The coefficients Γ_i only depend on the temperature of the ambient plasma. Γ_N describes how quickly the number density of heavy neutrinos approaches its equilibrium value, and Γ_{B-L} describes the dissipation, or washout, of $B - L$. $\Gamma_{B-L,N}$ parametrizes how the deviation of n_N from equilibrium creates an asymmetry of $B - L$. This reaction, as well as the one described by $\Gamma_{N,B-L}$ violate CP. We stress that these equations are

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[3]The rate for large angle scattering via gauge interactions is of order $g^4 T$ modulo logarithms, where g denotes a gauge coupling. When a non-relativistic neutrino with mass M_N decays, its decay products originally have energy of approximately $M_N/2 \gg T$. They lose their energy within a stopping time of order t_{stop} with $t_{\text{stop}}^{-1} \sim \sqrt{T/M_N} \, g^4 T$ modulo logarithms [11, 12].

[4]If they occur at roughly the same rate they both have to be included in the equations for leptogenesis [13].

valid to all orders in the Standard Model couplings. The only corrections are due to relativistic effects of the motion of N. [5]

For leptogenesis one is usually interested in an initial state with vanishing n_{B-L}, which is then generated by non-vanishing $n_N - n_N^{\text{eq}}$ as described by Eq. (3). The coefficient $\Gamma_{B-L,N}$ contains a small parameter ϵ (cf. Eq. (8)) resulting in $n_{B-L} \ll n_N - n_N^{\text{eq}}$. We expect the coefficient $\Gamma_{N,B-L}$ to contain a similarly small factor, so that the second term on the right-hand-side of Eq. (2) can be safely neglected.

3   Leading order coefficients

We would like to determine the coefficients in Eqs. (2), (3) at leading order in the Standard Model couplings. We expect that at this order they can be obtained from a Boltzmann equation containing the $1 \leftrightarrow 2$ scattering processes.

We work at leading order in the Yukawa coupling of right handed neutrinos, and we therefore neglect processes which change lepton number by two units via the exchange between Standard Model particles. The corresponding rates are suppressed by an additional factor h^2, where h is a generic Yukawa coupling of the right handed neutrinos (see below).

Since leptons and Higgs bosons are in kinetic equilibrium, their phase space distributions can be parametrized by the temperature and by chemical potentials. In the Boltzmann-equation for $f_N(t, p)$ with $p \equiv |\mathbf{p}|$ we can neglect the chemical potentials of Higgs and leptons because they are of the order of the $B - L$ asymmetry (recall that $n_{B-L} \ll n_N - n_N^{\text{eq}}$). We use Boltzmann statistics instead of Fermi- or Bose-statistics for leptons and Higgs, because they have energies of order $M_N/2 \gg T$ when they produce an N. Furthermore, we neglect the Pauli blocking and Bose enhancement factors. The corresponding errors are suppressed with $\exp[-M_N/T]$. After performing the phase space integral the Boltzmann equation becomes

$$(\partial_t - Hp\partial_p) f_N = \frac{M_N \Gamma_0}{E_N} \left(e^{-E_N/T} - f_N \right), \tag{4}$$

where Γ_0 is the total tree-level decay rate of the heavy neutrinos, and $E_N = (\mathbf{p}^2 + M_N^2)^{-1/2}$ is their energy. We consider the Yukawa interaction

$$\mathcal{L}_{N\text{Yuk}} = h_{ij} \overline{N_i} \widetilde{\varphi}^\dagger \ell_j + \text{ h.c.} \tag{5}$$

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[5]For large deviations from equilibrium there could also be non-linear terms. These are expected to be higher order in the N-Yukawa couplings.

with the left-handed Standard Model lepton doublets ℓ_j and the isospin conjugate $\widetilde{\varphi} \equiv i\sigma^2\varphi^*$ of the Higgs doublet φ. Then the decay rate of the lightest right-handed neutrino $N \equiv N_1$ is

$$\Gamma_0 = \frac{|h_{11}|^2 M_N}{8\pi},\tag{6}$$

if ℓ_1 is the lepton produced in the N-decays (see Sect. 2).

One can obtain an equation for $n_N = (2s_N + 1)(2\pi)^{-3}\int d^3p\, f_N$, where $s_N = 1/2$ is the spin of N, by integrating Eq. (4) over $\mathbf{p}$. However, the loss term on the right-hand side would not only depend on the number density n_N, but also on the momentum spectrum of the right-handed neutrinos. In most papers on leptogenesis this problem is circumvented by assuming that the N's are in kinetic equilibrium. Then the deviation from thermal equilibrium is completely specified by n_N. Here we do not assume kinetic equilibrium. Instead, we make use of the fact that the N's are non-relativistic, and approximate the factor $1/E_N$ in front of the bracket by $1/M_N$. For the first coefficient in Eq. (2) we thus obtain

$$\Gamma_N = \Gamma_0 \tag{7}$$

at leading order in the Standard Model couplings. Corrections to Eq. (7) will be discussed in Sect. 6.

Now we determine the coefficients in Eq. (3). In many cases [6] the Boltzmann equation for a given particle species contains collision terms which are much larger than the collision terms contributing to leptogenesis itself. The processes which give these large contributions are called spectator processes [15]. Their role is to bring the quantum numbers which are not conserved into thermal equilibrium. Important examples are scattering by gauge boson exchange, the anomalous quark-chirality changing QCD-sphalerons and the anomalous $B+L$ changing electroweak sphaleron processes. Which spectators are in equilibrium depends on the temperature (cf. Table 1). All spectator processes conserve $B-L$. Thus, after adding the Boltzmann equations for all quarks and antiquarks, weighted by their respective baryon number, subtracting the equations for leptons and antileptons, weighted by their lepton number, and then integrating over momentum the spectator collision terms drop out. The only violations of $B-L$ are due to the interactions of the right-handed neutrinos.

The decays of the right-handed neutrinos contribute to the coefficient $\Gamma_{B-L,N}$. The asymmetry arises because the decay rate of N into matter differs from the one into

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[6]Eq. (4) is an exception.

antimatter, which is quantified by a non-zero value of

$$\epsilon \equiv \frac{\Gamma(N \to \varphi\ell) - \Gamma(N \to \overline{\varphi}\overline{\ell})}{\Gamma(N \to \varphi\ell) + \Gamma(N \to \overline{\varphi}\overline{\ell})}. \tag{8}$$

Therefore

$$\Gamma_{B-L,N} = \epsilon\Gamma_0 \tag{9}$$

at leading order in the Standard Model couplings.

Inverse decays of the heavy neutrinos contribute to the washout term in Eq. (3) at leading order. At higher orders in the h_{ij} there are also $2 \leftrightarrow 2$ scattering processes with a right handed neutrino exchange which change lepton number by two units. Unlike the inverse decays these are not exponentially suppressed in the non-relativistic regime (see below) which may compensate the suppression by the Yukawa couplings and powers of T/M_N. The washout factor (1) is of order $K \sim h_{ij}^2 m_{\rm Pl}/M_N$, where $m_{\rm Pl}$ denotes the Planck mass. Thus at fixed K, h_{ij} increases with increasing M_N. Therefore at very large M_N these processes can become important, but we do not consider them here.

Like in the computation of Γ_N one can use Maxwell-Boltzmann statistics and neglect Pauli blocking and Bose enhancement terms, which gives

$$\Gamma_{B-L}\, n_{B-L} = \int \prod_{a=N,\ell,\varphi} \frac{d^3 p_a}{2E_a(2\pi)^3}(2\pi)^4\delta(p_\ell + p_\varphi - p_N)\,(f_\ell f_\varphi - f_{\overline{\ell}} f_{\overline{\varphi}})\sum|\mathscr{M}_0|^2. \tag{10}$$

Here $\sum|\mathscr{M}_0|^2 = 16\pi M_N\Gamma_0$ (see Eq. (6)) is the tree level matrix element squared for the process $\ell\varphi \to N$ summed over all spins of N and all isospin components of ℓ. Since we consider small deviations from equilibrium we can expand in the chemical potentials μ_ℓ and μ_φ and keep only the linear order,

$$f_\ell f_\varphi - f_{\overline{\ell}} f_{\overline{\varphi}} \simeq 2e^{-E_N/T}(\mu_\ell + \mu_\varphi)/T. \tag{11}$$

The chemical potentials [7] are proportional to n_{B-L}. Their precise relation is temperature dependent, because it depends on which spectator processes are in effect. One can introduce coefficients c_ℓ and c_φ to parametrize the relation of lepton and Higgs asymmetries to $B-L$, [8]

$$n_\ell - n_{\overline{\ell}} = -\,c_\ell\, n_{B-L}, \tag{12}$$

$$n_\varphi - n_{\overline{\varphi}} = -\,c_\varphi\, n_{B-L}. \tag{13}$$

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[7]Oftentimes the chemical potential of the Higgs bosons is ignored.

[8]We thank Sacha Davidson for clarifying remarks regarding these relations.

Results for c_ℓ and c_φ for various temperature regimes can be found in Table 1 of Ref. [3]. To evaluate the left-hand sides of Eqs. (12) and (13) one can expand the Fermi- and Bose- distributions for leptons and Higgses up to linear order in the chemical potentials, and then integrate over momenta. Note that the momentum integrals are saturated at $p \sim T$. That means that, unlike above, one cannot use the Maxwell-Boltzmann distribution. Using the Fermi- and Bose-distributions instead one finds

$$\mu_\ell = \frac{3c_\ell}{T^2} n_{B-L},\tag{14}$$

$$\mu_\varphi = \frac{3c_\varphi}{2T^2} n_{B-L}.\tag{15}$$

Combining Eqs. (10), (11), (14), and (15) one finally obtains for the washout rate

$$\Gamma_{B-L} = \frac{3}{\pi^2}\left(c_\ell + \frac{c_\varphi}{2}\right) z^2 K_1(z)\Gamma_0.\tag{16}$$

Here K_1 is the modified Bessel function of the second kind and

$$z \equiv \frac{M_N}{T}.\tag{17}$$

Note that we did not make a non-relativistic approximation for Γ_{B-L}. Our result differs from Ref. [16] (where $c_\ell = 1$, $c_\varphi = 0$ was used) and [3] by a factor $12/\pi^2$. This is because we have used quantum statistics to obtain Eqs. (14) and (15). It is suggestive to use classical statistics for Eqs. (10) and (11), because the lepton number washout is due to leptons and Higgs bosons with energy of order $M_N/2 \gg T$. However, the kinetic equilibration rate (see footnote 3 on page 4) is much larger than the interaction rate of the heavy neutrinos. Therefore the relation between the asymmetry and the chemical potentials is determined by the bulk of kinetically equilibrated leptons and Higgs bosons, which have momenta of order T.

Let us finally comment on the washout due to $\Delta L = 2$ processes mediated by the exchange of a right-handed Majorana neutrino. Unlike Eq. (16) the corresponding rate is not Boltzmann suppressed with $e^{-M_N/T}$ for $T \ll M_N$. On the other hand, the $\Delta L = 2$ rate is of order h^4. For a given value of K large h^2 means large M_N. Thus the $\Delta L = 2$ processes can be important for very heavy right-handed neutrinos. They were found to be neglible for $M_N \lesssim 10^{13}$ GeV [3].

4   Interaction rates and thermal (non-)equilibrium

The right-handed neutrinos are out of equilibrium if their interaction rate is smaller than the Hubble rate. When $T \lesssim M_N$, the interaction rate can be estimated as Γ_0.
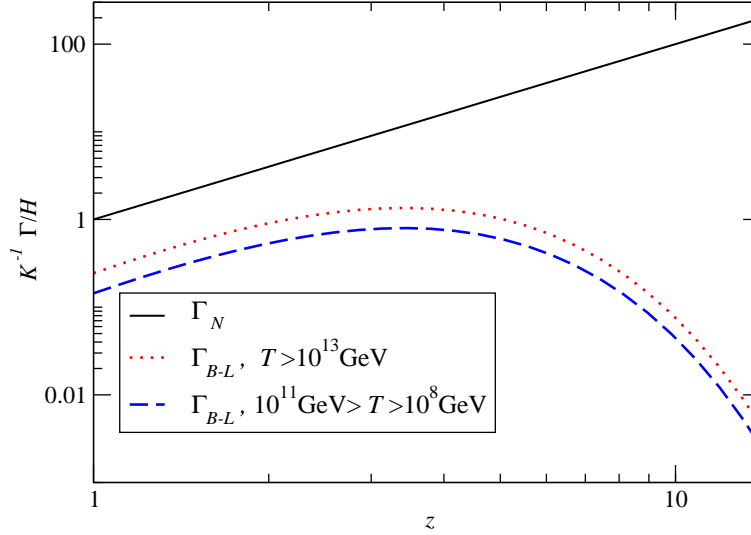
Figure 1: Ratio of reaction rates in Eqs. (2), (3) and the Hubble rate, normalized to the washout factor K. Γ_N/H (full black line) increases with z, implying that abundance of right-handed neutrinos gets closer and closer to equilibrium with increasing time, or, since the equilibrium density drops exponentially, gets closer and closer to zero. On the other hand, Γ_{B-L}/H (red dotted and blue dashed line) first increases, and then drops exponentially, implying that $B - L$ freezes out.

Therefore the ratio of the two rates is

$$\frac{\Gamma_N}{H} \simeq \frac{\Gamma_0}{(T^2/M_N^2)H|_{T=M_N}} = \left(\frac{M_N}{T}\right)^2 K, \tag{18}$$

i.e., the deviation from equilibrium becomes smaller when the temperature decreases. It also means that the deviation from kinetic equilibrium becomes smaller. On the other hand, the washout rate is of the same order as Γ_N for $T \sim M_N$, while for $T \ll M_N$

$$\frac{\Gamma_{B-L}}{H} \sim \left(\frac{M_N}{T}\right)^{7/2} e^{-M_N/T} K. \tag{19}$$

Thus Γ_{B-L}/H should first increase when the temperature falls below M_N, and should then drop sharply due to the exponential. Fig. 1 shows the ratios Γ_N/H and Γ_{B-L}/H, normalized to the washout factor K. The ratio Γ_{B-L}/H indeed has a maximum between $z = 3$ and $z = 4$. Thus one should expect that this is the most important z-range for leptogenesis. Futhermore, for $K \gtrsim 1$ the deviations from chemical and kinetic equilibrium should be small in this regime. Note that for $z \gg 1$ there is a hierarchy between the scale for $B - L$ dissipation and of the heavy neutrino equilibration, $\Gamma_{B-L} \ll \Gamma_N$.

5 Relativistic corrections

There are corrections to Eqs. (2), (3) due to the motion of the right-handed neutrinos. In general their momentum distribution will also deviate from thermal equilibrium. We expect the first correction to be related to the kinetic energy density. There can also be relativistic corrections related to the spin of the right-handed neutrinos, which we do not consider here. [9] At leading order in the gauge and Yukawa couplings we expect that the correct coefficients in the kinetic equations can be obtained from Boltzmann equations which contain only the decay and inverse decay processes.

We have $E_N = M_N[1 + O(v^2)]$ where v is a typical velocity of the N. In thermal equilibrium $\mathbf{p}^2/M_N \sim T$ and therefore $v^2 \sim T/M_N \sim 1/z$. We expand the factor $1/E_N$ in front of the bracket in Eq. (4) up to order v^2, which gives

$$\left(\frac{d}{dt} + 3H\right) n_N = \Gamma_N \left(n_N^{\mathrm{eq}} - n_N\right) + \Gamma_{N,u} \left(u - u^{\mathrm{eq}}\right). \tag{20}$$

Here

$$u \equiv \frac{1}{M_N}(2s_N + 1) \int \frac{d^3 p}{(2\pi)^3} \frac{\mathbf{p}^2}{2M_N} f_N \tag{21}$$

is the kinetic energy density of the heavy neutrinos devided by their mass. At lowest order

$$\Gamma_{N,u} = \Gamma_0. \tag{22}$$

The equation for u is obtained by multiplying Eq. (4) with $\mathbf{p}^2$ and integrating over $\mathbf{p}$. At leading order in v this gives

$$\left(\frac{d}{dt} + 5H\right) u = \Gamma_u \left(u^{\mathrm{eq}} - u\right), \tag{23}$$

with $\Gamma_u = \Gamma_0$ at leading order in T/M_N and in the Standard Model couplings. The equation for the asymmetry is modified accordingly,

$$\left(\frac{d}{dt} + 3H\right) n_{B-L} = \Gamma_{B-L,N} \left(n_N - n_N^{\mathrm{eq}}\right) + \Gamma_{B-L,u} \left(u - u^{\mathrm{eq}}\right) - \Gamma_{B-L} n_{B-L}. \tag{24}$$

Using the same expansion as for Eq. (20) one finds at leading order (cf. Eq. (9))

$$\Gamma_{B-L,u} = \epsilon \Gamma_0. \tag{25}$$

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[9]We thank M. Laine for pointing this out.

6 Radiative corrections

At leading order only the decays and inverse decays of right-handed neutrinos contribute to the coefficients in Eqs. (2) and (3). In many papers $2 \leftrightarrow 2$ scattering processes which involve Standard Model interactions have been included as well, and their effects were found to be substantial. Recently [8] it was pointed out, that such calculations are not complete because they only contain $2 \leftrightarrow 2$ particle scattering, while ignoring $1 \leftrightarrow 3$ processes and virtual corrections to the leading order $1 \leftrightarrow 2$ scattering. Infrared contributions which were found earlier were shown to cancel in the complete answer.

Here we show how radiative corrections can be incorporated into our approach. The equilibrium densities n_N^{eq} and u^{eq} do not receive radiative corrections from Standard Model interactions. Therefore they only affect the coefficients Γ_i in Eqs. (20), (23), and (24).

Radiative corrections to the production rate dn_N/dt at vanishing n_N have been computed in Refs. [8, 9]. In [9] also the radiative corrections to the differential rate $\partial f_N/\partial t$ at $n_N = 0$ have been computed. It can be written as

$$\left.\frac{\partial f_N(t, \mathbf{p})}{\partial t}\right|_{f_N=0} = f_{\mathrm{F}}(E_N)\Gamma_0 \frac{M_N}{E_N}\left\{a + \frac{\mathbf{p}^2}{M_N^2}b + O\left(\frac{\mathbf{p}^4}{M_N^4}\right)\right\}. \tag{26}$$

The coefficients a, b are temperature-dependent and can be expanded in T/M_N.

Assuming that Eq. (2) is still valid for $n_N = 0$, one can read off the radiative corrections to the coefficients in the kinetic equations for n_N (Eq. (20)), and for u (Eq. (23)) from Eq. (26). Expanding the factor $1/E_N$ as in Eq. (4), and integrating over $\mathbf{p}$ we obtain

$$\Gamma_N = \Gamma_u = a\Gamma_0, \qquad \Gamma_{N,u} = (a - 2b)\Gamma_0. \tag{27}$$

Radiative corrections that would enter the coefficients in Eq. (24) for the asymmetry have not been computed so far.

The coefficients in Eq. (26) are [10]

$$a = 1 - \frac{\lambda T^2}{M_N^2} - |h_t|^2\left[\frac{21}{2(4\pi)^2} + \frac{7\pi^2}{60}\frac{T^4}{M_N^4}\right] + (g_1^2 + 3g_2^2)\left[\frac{29}{8(4\pi)^2} - \frac{\pi^2}{80}\frac{T^4}{M_N^4}\right]$$
$$+ O\left(g^2\frac{T^6}{M_N^6}, g^3\frac{T^2}{M_N^2}\right), \tag{28}$$

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[10] The $O(g^2)$ and the $O(\lambda T^2/M_N^2)$ corrections in a were first computed in Ref. [8]. The $O(g^2)$ contributions in Refs. [8] and [9] agree, but the $O(\lambda T^2/M_N^2)$ in Ref. [8] is 4 times larger than the one in Ref. [9] (see footnote 11 on page 12). For our numerical results in Sect. 7 we have used the expression in Ref. [9]. The other terms in a and b were computed in Ref. [9].

$$b = - \left[|h_t|^2 \frac{7\pi^2}{45} + (g_1^2 + 3g_2^2)\frac{\pi^2}{60} \right] \frac{T^4}{M_N^4} + O\left(g^2 \frac{T^6}{M_N^6}, g^3 \frac{T^2}{M_N^2} \right), \qquad (29)$$

where h_t is the top Yukawa coupling, g_2, g_1 are the weak SU(2) and U(1) gauge couplings, $\lambda = m_{\text{Higgs}}^2 G_{\text{F}}/\sqrt{2}$ is the Higgs self-coupling [11] and G_{F} is the Fermi constant. The neutrino Yukawa couplings are defined in the $\overline{\text{MS}}$ scheme with renormalization scale M_N.

The leading temperature-dependent term in a is of order $\lambda T^2/M_N^2 \sim \lambda v^4$, where v is a typical thermal velocity of the heavy neutrinos (cf. Sect. 5). If we think of our approach as an expansion in v, and if we assume λ to be small, then this term is of higher order than the one we included in Sect. 5. The other temperature dependent radiative corrections are of even higher order ($\sim g^2 v^8$), where g denotes a generic Standard Model gauge or Yukawa coupling. It would be straightforward to extend the equations in Sect. 5 to higher orders in v as well. We did not do so, because, as we show in Sect. 7, the order v^2 corrections are quite small in the parameter region of interest.

7 Numerical results

In this section we solve the equations for leptogenesis in the non-relativistic limit (Sect. 2), and we add relativistic (Sect. 5) and radiative (Sect. 6) corrections. We parametrize our results by the effective light neutrino mass [17]

$$\widetilde{m}_1 \equiv \frac{\left(m_{\text{D}}^\dagger m_{\text{D}} \right)_{11}}{M_N}, \qquad (30)$$

where m_{D} is the Dirac mass matrix of the neutrinos. It is related to the washout parameter K by [16]

$$\widetilde{m}_1 = K m_*. \qquad (31)$$

We have used the Standard Model value

$$m_* \simeq 1.08 \times 10^{-3}\text{eV}. \qquad (32)$$

It can be shown [18] that $\widetilde{m}_1$ is larger than the smallest light neutrino mass. The range $(\Delta m_{\text{solar}}^2)^{1/2} < \widetilde{m}_1 < (\Delta m_{\text{atmospheric}}^2)^{1/2}$ corresponds to $7.4 < K < 46$.

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[11]This definition of λ is the same as the one used in Ref. [9]. Ref. [8] writes the rate in terms of $\lambda_h \equiv m_{\text{Higgs}}^2/v_h^2$ with $v_h = 174\text{GeV}$ ($= v$ in the notation of Ref. [8]) which is 4 times larger than λ. In Ref. [8] the rate depends on λ_h the same way as it does depend on λ in Ref. [9].
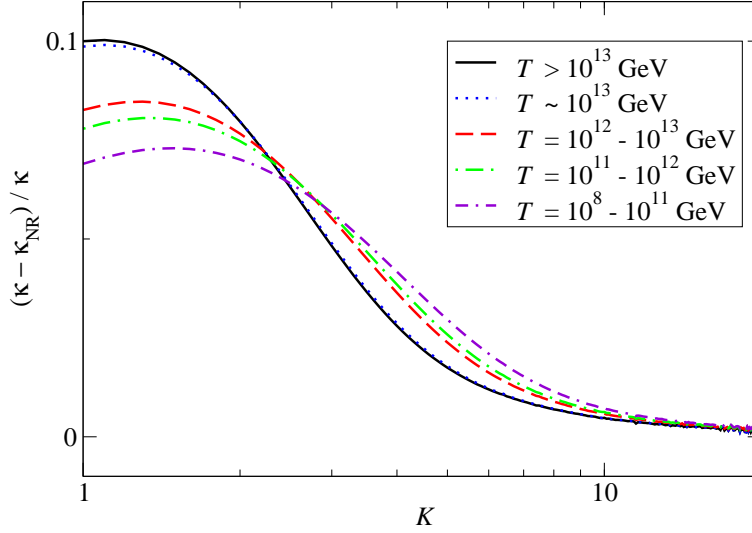
Figure 2: Relative size of the relativistic corrections to the $B - L$ asymmetry. Here we used thermal initial conditions. The temperature ranges correspond to the different values of c_ℓ and c_φ. For $T \lesssim 10^{12}$GeV these values correspond to the right-handed neutrinos decaying into τ-leptons.

We have started the evolution of n_N, n_{B-L}, and u at $z = 1$ with vanishing $B - L$ asymmetry. Since $v \sim 1$ for $z \sim 1$, the non-relativistic expansion cannot be expected to work for smaller z. Following common practice we express the final value of $B - L$ asymmetry in terms of the final efficiency factor κ defined by

$$\lim_{z \to \infty} \frac{n_{B-L}}{n_\gamma^{\text{eq}}} \equiv \frac{3}{4} \epsilon \kappa. \tag{33}$$

Here $n_\gamma^{\text{eq}} = 2\zeta(3)T^3/\pi^2$ is the photon equilibrium density, and ϵ is defined in Eq. (8).

In Fig. 2 we show the size of the $O(v^2)$ relativistic corrections normalized to the $O(v^2)$ corrected result for several temperature ranges by varying c_ℓ and c_φ according to Table 1 of Ref. [3]. We find that for $K \gtrsim 5$ the size of the relativistic corrections is already smaller than 3%. A detailed overview of the results for $K = 8$ can be found in Table 1.

Next we study the size of the relativistic corrections for different initial conditions for the temperature range $10^8 \lesssim T/\text{GeV} \lesssim 10^{11}$. We compare thermal initial conditions with the extreme case of zero initial values for n_N and u at $z = 1$. The latter initial condition is somewhat unphysical, because for $K \gtrsim 1$ a substantial number of right handed neutrinos will have been thermally produced by the time when $z = 1$. In Fig. 3 we compare the results for these two cases. For $K \lesssim 4$ the non-relativistic approximation clearly breaks down for zero initial densities. For $K \gtrsim 5$ the relativistic corrections remain small ($\lesssim 1.6\%$ for the parameters in Fig. 3) for zero initial values.

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T (GeV)	spectators	c_ℓ	c_φ	κ_{NR}	κ	$(\kappa - \kappa_{\mathrm{NR}}) / \kappa$
$\gg 10^{13}$	h_t, gauge	1	$2/3$	0.0253	0.0255	+0.7%
$\sim 10^{13}$	+ QCD sphalerons	1	$14/23$	0.0260	0.0262	+0.7%
10^{12} - 10^{13}	+ h_b, h_τ	$3/4$	$1/2$	0.0363	0.0366	+0.9%
10^{11} - 10^{12}	+ h_c, h_s, h_μ	$78/115$	$56/115$	0.0403	0.0406	+1.0%
10^{8} - 10^{11}	+ EW sphalerons	$344/537$	$52/179$	0.0495	0.0500	+1.1%
-	-	1	0	0.0363	0.0366	+0.9%

Table 1: Final efficiency factors in the non-relativistic approximation (κ_{NR}) and including $O(v^2)$ corrections (κ). Here we have used $K = 8$ and thermal initial conditions. The values of c_ℓ and c_φ are taken from Ref. [3]. We also include the case $c_\ell = 1$, $c_\varphi = 0$ which has been used in many papers.

It is remarkable that the corrections are so small. This clearly indicates that the non-relativistic approximation works very well, and that it is not necessary to use a Boltzmann equation to take into account the momentum distribution of the heavy neutrinos.

In our approach a deviation from kinetic equilibrium enters the difference $u - u^{\mathrm{eq}}$. Thus the fact that the relativistic corrections are so small also indicates that kinetic equilibrium is a good approximation. We have checked this by comparing the non-relativistic approximation with the kinetic equilibrium approximation. The difference between the two is indeed quite small, and similar in size as the $O(v^2)$ corrections. What turns out to be even smaller is the difference between our non-relativistic approximation plus $O(v^2)$ corrections and the kinetic equilibrium approximation. For example, for $K = 8$ and $T = 10^{10}$ GeV the kinetic equilibrium approximation deviates from the non-relativistic one by 0.87%, and it deviates from the $O(v^2)$ corrected one by only 0.3%.

It is interesting to compare our results with those of Refs. [5, 6, 7]. There Boltzmann equations were used to compute the momentum spectrum of the heavy neutrinos without assuming kinetic equilibrium. Ref. [5] compares results obtained with quantum statistics and the momentum dependent Boltzmann equation with results obtained with classical statistics and kinetic equilibrium. For $K > 5$ a discrepancy of about 15% was found. Ref. [6] considers the same setup and confirms a discrepancy of 20%
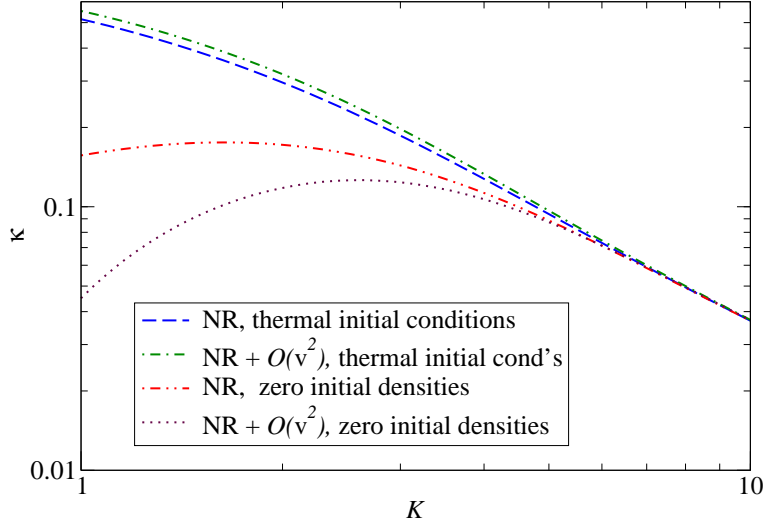
Figure 3: Final efficiency factors for two different initial conditions imposed at $z = 1$. For thermal initial conditions the non-relativistic approximation remains fairly reliable down to very small K. For vanishing initial densities it breaks down for $K \lesssim 4$. For larger K the dependence on the initial conditions is rather weak. The temperature is in the range $10^8 \lesssim T/\mathrm{GeV} \lesssim 10^{11}$.

for $K > 1$. Our results, on the other hand, indicate that the deviation from kinetic equilibrium should have a much smaller effect, which would mean that the discrepancy found in Ref. [5, 6] is due to the statistics. We have checked this switching from quantum to Boltzmann-statistics for Eqs. (14), (15) in the non-relativistic approximation. Results are shown in Fig. 4, where we see that using Boltzmann distributions would underestimate the asymmetry by at least 20% in the phenomenologically interesting region, confirming that the discrepancy is due to the statistics. Let us stress again that for Eqs. (14) and (15) the correct quantum statistics has to be used even in the non-relativistic regime, because the corresponding momentum integrals are for relativistic particles and they are saturated at momenta of order T.

The radiative corrections discussed in Sect. 6 depend not only on the washout factor, but, through the running of the Standard Model couplings, also on the neutrino mass M_N. In Fig. 5 we show results for the corrected efficiency factors, normalized to the non-relativistic approximation without radiative corrections κ_{LO}, for $M_N = 10^{10}$ and 10^8 GeV. All curves contain the relativistic $O(v^2)$ corrections. They were obtained with Γ_N, $\Gamma_{N,u}$, and Γ_u computed at different orders in the Standard Model couplings and in $v^2 \sim T/M_N$. In the favored regime $7.4 \lesssim K \lesssim 46$ the effects of radiative corrections are smaller than the $O(v^2)$ corrections. For smaller values of K the effect of the $O(g^2)$ and the $O(\lambda v^4)$ corrections to the Γ's remain small. The fact that the
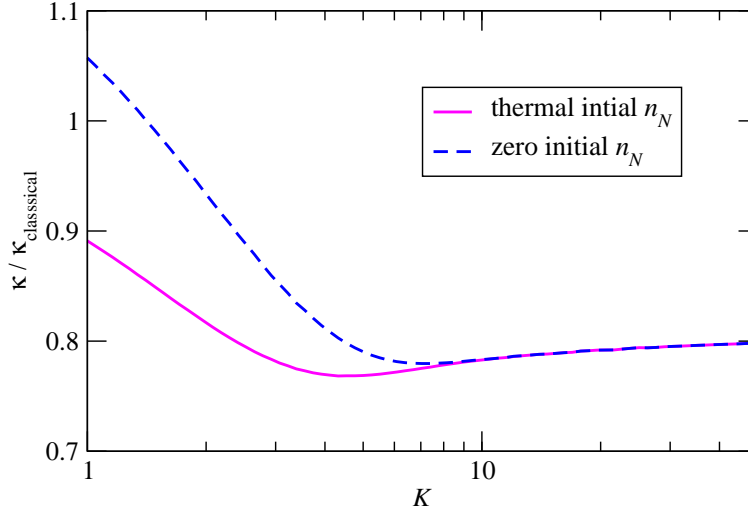
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Figure 4: Final efficiency factor κ computed with Bose- and Fermi-distributions in the washout term, divided by the final efficiency factor with the classical Maxwell-Boltzmann statistics. In both cases we have used the non-relativistic approximation. We have chosen $c_\ell = 1$, $c_\varphi = 0$ to facilitate comparison with Ref. [5]. This figure shows that it is important to use the correct quantum statistics. Even at very large K the error caused by using classical statistics is of order 20%.

$O(\lambda v^4)$ has a small effect is partly due to the smallness of the Higgs self-coupling at these scales. The effect of the $O(g^2v^8)$ contributions grows significantly with decreasing K (see also Table 2). This indicates that in the regime where v-dependent corrections are important, the expansion of radiative corrections in powers of v^2 does not converge (cf. Ref. [10]).

8   Summary and discussion

We have obtained the rate equations for single flavor leptogenesis in the non-relativistic limit which are valid to all orders in the Standard Model couplings. We proposed a systematic expansion around the non-relativistic limit. It can be used to improve the non-relativistic approximation, and to assess its range of validity. The coefficients in the rate equations and the first relativistic corrections were determined at leading order in the Standard Model couplings. We computed the $B - L$ asymmetry from these equations, and we found that the relativistic corrections are quite small in the parameter region favored by atmospheric and solar neutrino oscillations, which lies in the so-called strong washout regime. They are much smaller than differences between the non-relativistic approximation and the standard result which uses the assumption
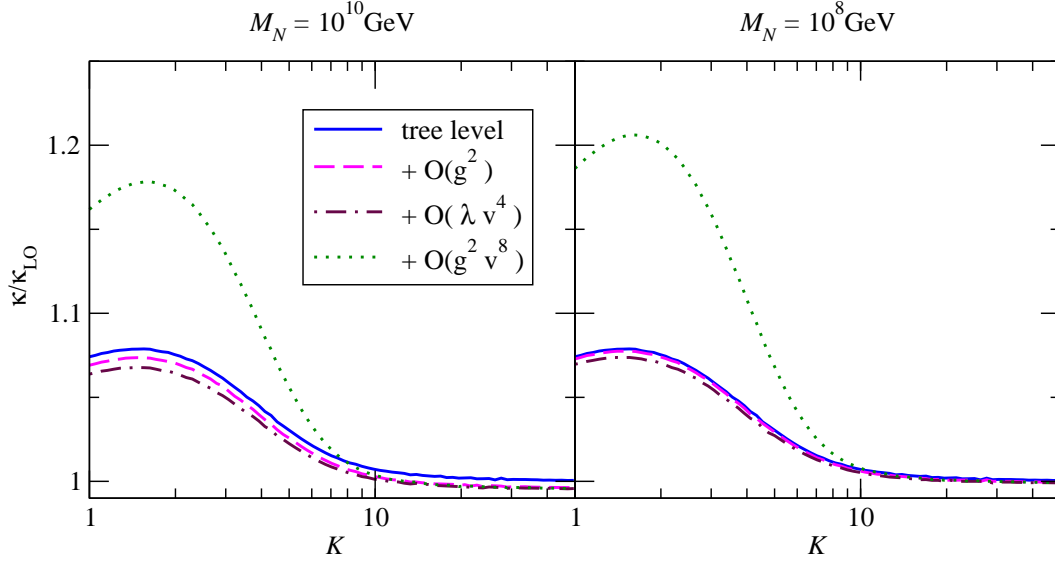
Figure 5: Final efficiency factors containing the relativistic $O(v^2)$ corrections, normalized to the non-relativistic approximation without radiative corrections κ_{LO}. The different curves correspond to different orders included in the coefficients Γ_N, $\Gamma_{N,u}$, and Γ_u. The renormalization scale of the Standard Model coupling is chosen as the mass of the heavy neutrinos. Thermal initial conditions for n_N and u were used.

of kinetic equilibrium for the heavy neutrinos as well as classical statistics for all particle species. We found that this difference is mainly due to the statistics used for computing the washout term, and not so much due to the assumption of kinetic equilibrium.

The smallness of relativistic corrections should be very useful, because one can obtain precise results without having to compute the momentum spectrum of the right-handed neutrinos, and also because it significantly simplifies the task to take into account Standard Model radiative corrections. We have taken a step in this direction by including the production rate of right handed neutrinos at next-to-leading order in the Standard Model couplings. We find that the radiative corrections are smaller than the tree-level relativistic corrections in the favored region of the washout factor K. For smaller values of K they become larger than the tree-level relativistic corrections. This is due to temperature dependent gauge and top Yukawa interactions which we consistently included in a leptogenesis calculation for the first time.

To conclude, the non-relativistic expansion proposed in this paper is a convenient tool for computing the lepton asymmetry, and to assess the validity of the non-relativistic approximation. In case the relativistic corrections are important, also radiative corrections become important, and then the non-relativistic expansion breaks down. In this case one has to use equations which are valid in the fully relativistic

K	$(\kappa_{\mathrm{NLO}} - \kappa_{\mathrm{LO}})/\kappa_{\mathrm{LO}}$	
	$M_N = 10^8$ GeV	$M_N = 10^{10}$ GeV
5	$+3.7\%$	$+2.5\%$
8	$+0.5\%$	0.0%
10	$+0.1\%$	-0.4%
20	-0.1%	-0.5%
50	-0.1%	-0.5%

Table 2: Influence of the radiative corrections on the final efficiency factor for two different values of M_N. κ_{NLO} contains all relativistic corrections discussed in Sects. 5 and 6 and corresponds to the dotted lines in Fig. 5. Thermal initial conditions for n_N and u were used.

regime. This conclusion is based on the radiative corrections to the equilibration rate of right-handed neutrinos. It would be interesting to see whether they are also valid if radiative corrections to the other terms in the kinetic equations are included.

Note added: After this paper was submitted for publication, we were informed by the authors of Ref. [19] that they have confirmed the results of Ref. [9] for the radiative corrections to the production rate of right-handed neutrinos.

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