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Leptogenesis: A non-relativistic study

Leptogenese: Eine nicht-relativistische Betrachtung

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One way to describe the observed baryonic asymmetry is by a postulating that the universe has been in an asymmetric state just from the beginning and that the matter and antimatter is concentrated in big domains throughout the universe, which come into contact just at their outer borders. Techincally, there is no reason for the universe not to have started in an asymmetric state, but in that case one would measure high gamma rates due to the matter-antimatter-annihilation right between these distinct regions.
Since there is no such radiation observed, patches of different kinds of matter have to be as big as the presently observable universe. Because this doesn't seem very plausible, the baryonic asymmetry had to arise dynamically from an universe where matter and antimatter existed in the same amount.
Actually, in 1967 the Sovjetian physicist Andrei Sakharov postulated the criteria, which have to be met in order for an excess of baryons over anti-baryons to be generated out of a fully symmetrical universe.

2.1. Sakharov Conditions

As mentioned above, there are three crucial properties of nature, the Sakharov conditions, which are required to produce a net baryon number greater than zero. These three conditions are:

1. B-violating process(es)
2. C and CP violation
3. Departure from or loss of thermal equilibrium

For a general insight of these three conditions the first one will be skipped, since it is quite obvious that in a totally symmetric universe there has to be at least one B-violating process in order to cause an imbalance in matter and antimatter.
The general importance of the other two will be discussed in the following.

2.1.1. C and CP-violation

Charge conjugation (C), parity (P) and their combination (CP) are two or more specifically three basic symmetries of the universe. C symmetry states that physical processes are the same, even after exchanging particles for their respective anti-particles, while P-symmetry guarantees invariance under the transformation $\vec{r} \to -\vec{r}$. CP symmetry then simply is an sequence of a C followed by a P transformation.
To explain why C has to be violated for baryogenesis beeing possible, consider the B-violating reaction

$$X \to Y + B$$

with X and Y particles with B=0 and B representing the excess baryons. This reaction happens with a certain rate, which is, using C as a symmetry, just the same as the reation rate for the conjugate process.

$$\Gamma(X \to Y + B) = \Gamma(\bar{X} \to \bar{Y} + \bar{B}) \tag{2.1}$$

Eq. 2.1 implies that under C exactly the same amount of baryons and anti-baryons will be produced and therefore no excess baryons are left after the annihilaton. This means C must be violated.
But additionally to this CP violation is essential for baryogenesis. To illustrate why, take a closer look at the also clearly B violating X decay with its two channels:

$$X \to q_L q_L$$
$$X \to q_R q_R$$

with q an arbitrary quark. The subscripts L and R denote the left- or right-handedness chirality of the decay products. CP then effects each particle as follows

$$tX \underset{CP}{\longrightarrow} \bar{X}$$
$$q_L \underset{CP}{\longrightarrow} \bar{q}_R$$
$$q_R \underset{CP}{\longrightarrow} \bar{q}_L$$

So CP does not just change matter for anti-matter, but the handedness of the particles as well. So if CP holds as a symmetry the consequences for the reaction rates are:

$$\Gamma(X \to q_L q_L) = \Gamma(\bar{X} \to \bar{q}_R \bar{q}_R) \qquad\qquad \Gamma(X \to q_R q_R) = \Gamma(\bar{X} \to \bar{q}_L \bar{q}_L)$$

Adding these two results in

$$\Gamma(X \to q_L q_L) + \Gamma(X \to q_R q_R) = \Gamma(\bar{X} \to \bar{q}_R \bar{q}_R) + \Gamma(\bar{X} \to \bar{q}_L \bar{q}_L) \tag{2.2}$$

Equation 2.2 implies that as long as there are as many particles X as anti-particles $\bar{\mathrm{X}}$ in the initial state of the universe, which is just the starting point of the model the Sakharov conditions try to describe, there can only be an asymmetry between left- and right-handed particles be achieved, but that is not a baryon asymmetry, which is clearly needed for baryogenesis. CP must be violated.
So the bottom line here is, that the existence of B-violating processes is not sufficient for baryogenesis, but that there also has to be C and as well as CP violation, since without this kind of symmetry breaking any baryonic excess would be washed out by the corresponding C or CP conjugated process, as shown with the simple examples above.

2.1.2. Departure from thermal equilibrium

The last condition to be met in order for baryogenesis to be achievable is that the the B, C and CP violating processes must occur outside the thermal equilibrium. To illustrate this we first consider the phase space distribution of a species X of quantum particles

$$f(E_X) = \frac{1}{e^{\frac{E_X - \mu_X}{T}} \pm 1} \tag{2.3}$$

The energy E_X and the momentum $\vec{p}_X$ are related via the relativistic energy-momentum-relation $E^2 = \vec{p}^2 + m^2$. μ_X describes the chemical potential of the particle species X, which is an important quantity for describing thermal equilibrium states, as the chemical potentials of two species X and Y, which are in thermal equilibrium, are related by $\mu_X = \mu_Y$ or for more species $\sum_i \mu_i = 0$.

Using eq. 2.3 to compute the particle density of a certain particle species one gets

$$n_X = g_X \int \frac{d^3p}{(2\pi)^3} f_X(E) \tag{2.4}$$

where g_X denotes the number of inner degrees of freedom of X.
In the non-relativistic limit there holds m $\gg$ E-$\mu \gg$ T. With this approximation the denominator of the exponential function in eq. 2.3 gets small compared to the numerator so the exponential itself gets so big that the ± 1 can be neglected, in the non-relativistic limit, you get the same particle density for fermions and bosons. By dividing E_X into the rest energy m_X and the kinetic energy E_{kin} and after approximating

$$E_{\text{kin}} \approx \frac{p^2}{2m} \tag{2.5}$$

for non-relativistic particles, integrating according to 2.4 yields

$$n_X = g_X \frac{4\pi}{(2\pi)^3} \int dp\, p^2 e^{\frac{\mu - m_X}{T}} e^{-\frac{p^2}{2m_X T}} = g_X \left(\frac{m_X T}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m_X - \mu_X}{T}} \tag{2.6}$$

Analogously you get the number density for the corresponding anti-particle $\bar{X}$

$$n_{\bar{X}} = g_{\bar{X}} \left(\frac{m_{\bar{X}} T}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m_{\bar{X}} - \mu_{\bar{X}}}{T}} \tag{2.7}$$

Now suppose X and its anti-particle $\bar{X}$ with $B_X = -B_{\bar{X}} \neq 0$ are in thermal equilibrium then the condition $\mu_X = \mu_{\bar{X}}$ holds. Comparing eq. 2.6 and 2.7 one sees that the chemical potential is the only property that could differ for particles and antiparticles. Now using the equilibrium condition for chemical equilibrium, one finally gets

$$n_X = n_{\bar{X}} \tag{2.8}$$

Looking at eq 2.8 it is quite obvious that even with B, C and CP violating any produced excess baryon number B will be washed out in equilibrium by other processes happening in equilibrium. This illustrates the final Sakharov Condition, that next to B, C and CP violation a departure from equilibrium is needed for a dynamic production of excess baryons.
Interesting to note is that there is quite an easy way of approximately determining if reactions take place in thermal equilibrium, namely by comparing the reaction rate with the expansion of the universe, described by the Hubble constant H, which is actually not a constant but changes with time. So if the relation

$$\Gamma \gtrsim H \tag{2.9}$$

holds, the reactions take place fast enough for them to be in equilibrium. This can be made understandable, by looking at this from the rest frame of the particles taking part in the reactions. Then the particles do not notice any expansion of the universe since they move and react too fast with each other, therefore the expansion do not really affect the equilibrium state.
Otherwise, if the reactions occur slower than the universe expands, meaning

$$\Gamma < H \tag{2.10}$$

is valid, then the expansion happens fast enough that particles get separated too far from each other, so they cannot react anymore and the reactions fall out of equilibrium.

2.2. Baryogenesis in the Standard Modell

Although nowadays there are no records or experimental proofs of baryon number violating processes, that does not mean there is a need for physics beyond the Standard Modell (SM) of particle physics, at least on a qualitative level.

2.2.1. The $SU(2)_L x U(1)_Y$ symmetry of the SM

As it turns out, the electroweak part of the SM with its $SU(2)_L \times U(1)_Y$ symmetry groups suits best for describing baryogenesis. But before the way this is achieved in the SM is displayed, this section will give a short rundown on the $SU(2)_L \times U(1)_Y$ symmetry found in the SM.
The $U(1)_Y$ symmetry can be represented by the following transformations

$$\Psi \longrightarrow e^{i\frac{Y}{2}\alpha(x)}\Psi$$
$$\bar{\Psi} \longrightarrow e^{-i\frac{Y}{2}\alpha(x)}\bar{\Psi}$$

with $\alpha(x)$ being an arbitrary function. Y denotes the $U(1)_Y$ quantum number, the hypercharge. Because of the exponential nature of this transformation to check for $U(1)_Y$-symmetry, the hypercharges of all appearing particles in an Lagrangian must add up to zero.
The $SU(2)_L$ symmetry however acts just on left-handed particles with the transformation looking like

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \longrightarrow U(x) \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

where $U(x)=e^{i\theta^i(x)T^i}$ with $T^i=\sigma^i/2$ the isospin and its third component T_3, which is the $SU(2)_L$ quantum number. It should be noticed that u and d in the transformation above stand for all up-type particles (ν,u,c,t) respectively all down-type particles (e,μ,τ,d,s,b). The right-handed particles on the other hand transform as singuletts under $SU(2)_L$

$$X_R \longrightarrow X_R$$

where X_R stands for any right-handed SM particle. Using this simple transformation one can easily deduce that for right-handed particles $T=T_3=0$.
Also there is a simple relation that connects the hypercharge Y, the electrical charge Q and the third component of the isospin T_3.

$$Q = T_3 + \frac{Y}{2} \qquad (2.11)$$

2.2.2. Electroweak baryogenesis

As stated above, the electroweak sector of the SM has every ingredient needed for successfull baryogenesis. The following discussions will illustrate how the SM satisfies all three Sakharov conditions.

C and CP violation

It is already proven theoretically und experimentally by numerous well-known experiments, for example the Wu experiment in 1956, showing that C symmetry is maximally violated by the weak interaction in the leptonic as well as in the hadronic sector. As shown by Kobayashi and Maskawa through expanding the Cabibbo hypothesis and experimentally confirmed, weak interactions in the hadronic sector also violate CP invariance, which manifests as an complex phase in the CKM quark mixing matrix. In the leptonic sector however the CP violation through a complex phase only got postulated in the PMNS neutrino mixing matrix trying to describe

neutrino oscillations, but this phase still needs to be measured.
Nevertheless the elektroweak part of the SM, more precise the weak interactions, since electromagnetism does not violate C or even P, satisfies at least one of the three Sakharov conditions.

B violation

Although the first Sakharov condition, the necessity of baryon number violating processes, seems to be the most obvious, the way these are realised in the SM is a bit more difficult than it seems. Since at the first glance the baryonic and, as it is going to play an important role during the following discussion, the leptonic current are conserved

$$\partial^\mu J_\mu^B = 0 \tag{2.12}$$
$$\partial^\mu J_\mu^L = 0 \tag{2.13}$$

one would assume there is no way the SM could produce an baryon asymmetry. However, by considering quantum fluctuation meaning orders higher than just tree level one finds that the currents for the left- and right-handed parts f_L and f_R respectively, where stands for quarks and leptons equally, are not conserved and furthermore not the same [1]

$$\partial^\mu \bar{f}_L \gamma_\mu f_L = -c_L \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \tag{2.14}$$
$$\partial^\mu \bar{f}_R \gamma_\mu f_R = +c_R \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \tag{2.15}$$

where g denotes the gauge coupling, $F^{a\mu\nu}$ the field tensor, $\tilde{F}^{a\mu\nu}$ the dual field tensor and c_L and c_R depend on the representation of f_L and f_R.This behaviour of the currents at quantum levels is known as Adler-Bell-Jackiw or chiralty anomaly. Since $SU(2)_L$ gauge boson only couples with left-handed particles c_R^W=0, while the $U(1)_Y$ gauge boson couples to both handednesses, but with different strength, therefore $c_R^Y \neq c_L^Y$. Although this section only focuses on electroweak baryogenesis, it is mentionable that with the $SU(3)_c$ gauge bosons of the strong interactions do not produce any chiralty anomaly because they couple with left as well as right-handed particles with the same strenght, so $c_R^c = c_L^c$ and both currents in (2.14) and (2.15) cancel each other out in the case of strong interactions.
Putting this and equations 2.12 - 2.15 together, gives a fairly interesting result

$$\partial^\mu J_\mu^B = \partial^\mu J_\mu^L = \frac{n_F}{32\pi^2} \left(-g_w^2 W^a_{\mu\nu} \tilde{W}^{a\mu\nu} g'^2 G^a_{\mu\nu} \tilde{G}^{a\mu\nu} \right) \tag{2.16}$$

with $W^{a\mu\nu}$ and $B^{a\mu\nu}$ the field strenght tensors of the $SU(2)_L$ and $U(1)_Y$ gauge groups and n_F=3 the number of particle families.
Analyzing eq. 2.16 one easily figures out that although baryon and lepton number are not conserved separately the difference B-L of these numbers is very well conserved. Integrating both sides of eq. 2.16, as shown in [1, pp. 15f.], results in

$$\Delta B = \Delta L = n_F \Delta N_{CS} \tag{2.17}$$

where ΔN_{CS} is the difference of the so called Chern-Simons numbers. How exactly these numbers are derived and what their integral representation is can also be looked up in [1, 2, 3], but is not of great interest for this thesis. However, one property of these numbers is quite relevant for baryon asymmetry, namely that each integer valued Chern-Simons number describes one distinct vacuum state of the infinite electroweak vacua with minimal energy, which are separated by an potential barrier. The difference of these numbers of two vacuum states right next to each other

is $\Delta N_{CS} = \pm 1$, so changing from one vacuum state N_i to another N_f results in $\Delta N_{CS} \neq 0$ and therefore a change in baryon and lepton number is induced. Also interesting to notice is, since the number of particle families n_F=3 baryon and lepton numbers change at least by three units each.
The last question regarding B violation in the SM is about how such a transition between two vacuum states can be accomplished. One way is through a quantummechanical effect called the instaton, where the system simply tunnels through the barrier between two vacuum states with different Cher-Simons numbers. However 't Hooft, the one showing B vioaltion can happen through the chiral anomaly, also showed [1, Ref. 22,24] that the cross section for such a tunneling process is about

$$\sigma \propto e^{-\frac{4\pi}{\alpha_w}} \sim 10^{-164} \tag{2.18}$$

with $\alpha_w = \frac{g^2}{4\pi} \cong \frac{1}{30}$. This cross section is so small that such a instaton transition between two vacua probably did not happen even once during the whole lifetime of the universe.
A second way such a change of vacua can be induced is through the so called sphaleron processes. The requirement for these processes to take place is that the system has enough energy to go over the potential barrier instead of tunneling through. The minimum energy needed, known as the spaleron energy, is about[1, 2]

$$E_{sph} = \frac{4\pi}{g_2} v(T) \cong 8 - 13\text{TeV} \tag{2.19}$$

where the temperature dependent quantity v(T) denotes the vacuum expectation value of the Higgs field at the temperature T, which will be important later on.
In fact this kind of processes are quite possible for temperatures above around 100 GeV, however, below this temperature the rate of spaleron processes is exponentially suppressed by a Boltzmann factor. It is also mentionable that comparing the sphaleron rate for temperatures above 100 GeV, which are proportional to the fourth power of the temperature [1, p. 19], with the Hubble constant, gives information about when these processes are in thermal equilibrium and numerical evaluations yield that the sphaleron processes are in thermal equilibrium for

$$100\text{GeV} \lesssim T \lesssim 10^{12}\text{GeV}$$

So as shown in section 2.1.2, even though the SM provides the necessary tools for C, CP and B violation, below the temperature of around 10^{12}GeV any produced net baryon number will be washed out and below 100 GeV the temperature is far below the temperature needed to induce sphaleron processes.

Departure from thermal equilibrium and elecroweak phase transistion

The final question to answer regarding baryogenesis in the SM is how the last Sakharov condition, the departure from thermal equilibrium is realized. The most common way is by using the electroweak phase transition.
This phenomenon heavily relies on the vacuum expectation value (VEV) of the $SU(2)_L$ Higgs doublett and its behaviour during the early times of the universe. At the present day the VEV is greater than zero, which leads to a gauge symmetry breaking and therefore masses of every massive particle. But it has already been shown [1, Ref. 32] that for high temperatures the VEV of the universe equals zero and the $SU(2)_L \times U(1)_Y$ gauge symmetry is still intact, even at the ground states. This obviously means that at some point during the evolution of the universe and at some critical temperature $T=T_c$ the VEV changed from zero to non-zero, or in other words a phase transition from a totally symmetrical phase to a phase with broken symmetry

happened at some point. In order to generate a departure from thermal equilibrium for the B violating reaction this transition must be strongly of first order, meaning at T=T_c the VEV changes discontinuously from zero to non-zero.
Just as with cooling steam this process can be imagined with bubbles of phases with broken symmetries forming and expanding inside the phase of unbroken symmetry, just as droplets of water form in the vapor and expand, until they connect and finally cover all space. Now, the way this phase transition leads to a baryon asymmetry is as follows.
First of all, consider a thin wall, so that the area where quarks and fermions interact with the walls can be approximated as a step function. Also, to simplify matters, assume that the expansion of the bubbles of broken symmetry is spherical symmetric, therefore this problem can be reduced to one dimension.
At the start of this baryon asymmetry generating process there is the same amount of particles and anti-particles.

Bild zur Veran-schaulichun einfügen

While the bubble expands, left- and right-handed quarks and anti-quarks from the unbroken phase hit the bubble wall, get reflected under CP violating processes and change their handedness due to angular momentum conservation and since charge conservation holds (anti-)quarks are only allowed to scatter into (anti-)quarks. The scattering processes are the following

$$\begin{aligned} q_L &\to q_R \\ q_R &\to q_L \\ \bar{q}_L &\to \bar{q}_R \\ \bar{q}_L &\to \bar{q}_R \end{aligned}$$

Since these scattering processes are not CP conserving the reflection coefficents are not the same for all of the reactions above.

$$\Delta R = R_{\bar{L}\to\bar{R}} - R_{R\to L} = R_{\bar{R}\to\bar{L}} - R_{L\to R} \tag{2.20}$$

Using CPT invariance yields

$$R_{\bar{L}\to\bar{R}} = R_{L\to R} \tag{2.21}$$
$$R_{\bar{R}\to\bar{L}} = R_{R\to L} \tag{2.22}$$

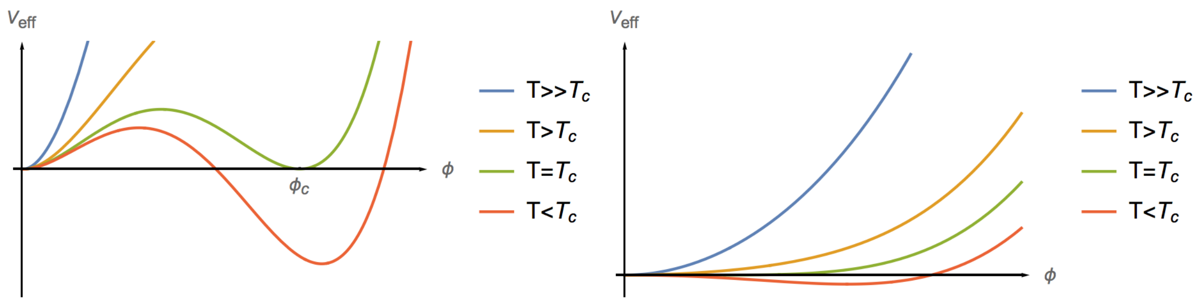
These relations alone imply that there still is no net baryon number, since the differences J_q^L of the fluxes of $\bar{q}_R$ and q_L and the J_q^R of q_R and $\bar{q}_L$ reflected back into the symmetric phase are the same and cancel each other out. But considering that the (B+L) violating sphaleron processes, because of their electroweak origin, only interact with left-handed quarks and right-handed anti-quarks, J_q^L changes while J_q^R stays the same, since it only takes right-handed quarks and left-handed antiquarks into account. This leads to a non-zero baryon number and especially if $J_q^L>0$ then there are more left-handed quarks than right-handed anti-quarks and therefore ΔB>0 in the symmetric phase away from the wall. If the bubble then expands over the region of a net baryon number greater zero this B gets frozen in, since in the broken phase the (B+L) violating processes that could wash out the asymmetry are strongly supressed by the Boltzmann factor as stated above.
Taking into account that particles from the broken phase can transmit into the symmetric phase and evaluating this quantitative as shown in [1, pp. 36-37] yields the result mentioned above. For a net baryon number greater than zero the CP violating processes at the bubble wall have to act in such way that the current J_q^L is greater than zero as well.

2.2.3. Failures of the SM

Since the SM offers everything needed to describe baryogenesis in the early universe one could naively say that the only thing left is the experimental proof to be delivered.
Having said this, recent experiments have shown that the SM alone, despite containing possible B, C and CP violating processes, is not able to provide a phase transition of strong enough first order or more precisely a phase transition of first order at all. This will be shown in the following.
According to the Landau theory phase transitions are described by the behaviour of a so called order parameter. So for a first order phase transition the order parameter has to change dicontinuously at the cirtical point, while for a second order transition the order parameter has to change drastically as well, but this change occurs continuously. In this case of the electroweak phase transition the order parameter is the expectation value of the Higgs field, denoted as ϕ. In addtion to this, one needs to describe the temperature dependent free energy of the system as a function of the parameter which result in the effective potential $V_{\text{eff}}(\phi, T)$. This potential describes the energy of a system in a state with the Higgs expectation value ϕ. Since in general this state is not one of minimal energy and because every system prefers to minimize its energy, it changes into a state described by the minimum of the potential where the expectation value of the Higgs is by definition the Higgs VEV.
As it is already known, the Higgs VEV had to change from zero at the big bang to a non-zero value while the universe cooled down to the temperature measured nowadays, so it is just natural to look at the change of this value with temperature. The change of the VEV can now happen continunously in which case the system undergoes a second order phase transition or discontinuously what is needed for a first order transition and especially for elektroweak baryogenesis. Both cases are shown in figure 2.1 for the effective Higgs potential including one-loop corrections [3]

$$V_{\text{eff}} = D(T^2 - T_0)^2\phi^2 - ET\phi^3 - \frac{1}{4}\lambda\phi^4 \tag{2.23}$$

for different non-zero temperature regimes. D and E are constant factors which are not of great interest for this discussion and λ describes the already mentioned 4-Higgs self coupling.



(a) Effective Higgs field for first order transition (b) Effective Higgs field for second order transition

Figure 2.1.: Effective Higgs field for different phase transitions

Analyzing figure 2.1a clearly shows the first order characteristics of the phase transition. While for high temperatures the VEV equals zero for the highly symmetric phase, the potential slowly develops a second minimum for decreasing temperature until at the critical temperature T=T$_c$ there are two energetically degenerated mimima, one at ϕ_1=0 and one at ϕ_2=ϕ_c, which are separated by an energy barrier. However, the system can change from the minimum at ϕ_1 to the one at ϕ_2 via tunneling through the barrier resulting in a discontinuous change of the VEV and therefore inducing a first order phase transition. While the universe keeps on cooling down because of the universe's expanding, the new minimum gets energetically lowered while the orig-

inal one stays at $V_{\text{eff}}=0$ and thus becomes a maximum, leaving an unstable state where once was a symmetric vacuum state.
How this can be imagined in the early universe is that at some point in space and time the universe tunneled from one minimun to another thus breaking SM symmetry locally and producing a local bubble of broken phase. These bubbles get bigger with time and combine with other bubbles whose generation gets much more likely as the lower temperatures lowers the barrier between the minima and increase the tunneling probability.
On the other hand, figure 2.1b shows how the universe would develop in case of a second order transition. In this case even at $T=T_c$ there are no two degenerated minima but the new one develops gradually while the orignial minimum more and more becomes the unstable maximum you also get in figure 2.1a for low temperatures. Since there are no two minima the universe can choose between, there is also no bubble formation but instead a continuous condensation throughout the universe which is not enough to induce baryogenesis.
Now that the two possibilities of the electroweak phase transition were represented the question how and why the SM fails to provide a strong enough first order phase transition still needs to be answered.
To do this it is useful to define a quantity that corresponds to the strenght of the phase transistion, which for this cause will be

$$\frac{v_{T_c}}{T_c} \gtrsim 1 \tag{2.24}$$

The reason why 2.24 is a good way to represent the strength of the phase transition is that by using equation 2.19 and the fact that the B+L violating sphaleron processes are exponentially surpressed by a Boltzmann factor inside of the phases with broken symmetry one gets for the rate of these spalerons at the critical temperature

$$\Gamma_{\text{sphaleron}} \propto e^{-E_{\text{sph}}(T_c)/T_c} \propto e^{-v_{T_c}/T_c} \tag{2.25}$$

So equation **??** really shows that the spaleron rates inside the bubbles with a Higgs VEV greater than zero are suppressed exactly by the quantity given in 2.24. So in order for these processes to be suppressed adequately has to be at least 1, which results in a suppression factor of roughly 0.36, for the phase transition to be strong enough to cause baryogenesis.
There are various methods to use the condition in 2.25 in order to calculate the Higgs mass and what the biggest mass is the Higgs particle can have in order for a first order phase transition to be possible which results in about m_H<70 GeV [4, pp. 3f.].
This theoretical result together with the experimental discovery that the Higgs mass is greater than 114 GeV [6, pp. 100ff.] clearly shows that the electroweak phase transition and therefore the SM as a whole is not able to explain how the observed baryonic asymmetry arose during the early times of the universe.
A solution for this problem is expanding the SM in such a way that the new elements are able to explain problems the SM was not able to. One of these expansions results in leptogenesis, what will be the topic of the following sections.

3. Outline of leptogenesis

As stated in the section before the SM alone is not quite enough to describe the observed baryon abundance, so the SM has to be expanded to that effect that it can describe such phenomena. Although efforts are made to explain direct baryogenesis using GUT theories, there is a much more favored alternative, namely the so called barygenesis via leptogenesis. The key idea here is to introduce a heavy, right-handed Majorana neutrino N, whose decays violate lepton number conservation.
This and the following chapter will cover this concept more detailed

3.1. Expanding the SM

There are experimental reasons why the SM does not tell the whole story about our universe, namely the results of neutrino oscillation experiments. Before the discovery of these neutrino oscillations it was accepted that neutrinos are massless and therefore their left-handedness is well defined. But being able to oscillate between different flavours implies that neutrinos are not massless and therefore are not purely left-handed and even more, right-handed neutrinos exist. The easiest way to implement right-handed neutrinos into the SM would be to show, that neutrinos are so called Majorana particles, which, in contrast to Dirac particles, are their own anti-particles. This would mean that the right-handed neutrinos are right-handed antineutrinos at the same time, but it was already shown that latter exist. Theoretically, the way to describe Majorana masses would be to exchange the usual mass term including the Higgs field for a Majorana mass term, that can be written in the following way [8, p. 18] to the SM Lagrangian.

$$\mathcal{L}_M = -\frac{1}{2}\overline{\Psi^C} M^M \Psi$$

where the superscript C stands for the charge conjugated neutrino field defined by

$$\Psi^C \equiv C\gamma_0 \Psi^*$$

with the marix C, which is dependant on the representation of the gamma matrices. The Majorana mass M^M is an $n_F \times n_F$ matrix with n_F again the number of particle families.
Using the representation of the $U(1)_Y$ symmetry group given in the previous section, it can easily be seen that in general by using a mass term like in 3.1 to the SM Lagrangian this symmetry is no longer viable.

$$\overline{\Psi^C}\Psi \xrightarrow{U(1)Y} \overline{(e^{i\frac{Y}{2}\alpha}\Psi)^C} e^{i\frac{Y}{2}\alpha}\Psi = \overline{e^{-i\frac{Y}{2}\alpha}\Psi^C} e^{i\frac{Y}{2}\alpha}\Psi = e^{i\frac{Y}{2}\alpha}\overline{\Psi^C} e^{i\frac{Y}{2}\alpha}\Psi \neq \overline{\Psi^C}\Psi$$

The relation above shows that by using Majorana masses of particles with hypercharge Y≠0, like the left-handed neutrinos, one cannot preserve the SM Lagrangians $SU(2)_L \times U(1)_Y$ symmetry. That being said, right-handed Dirac mass terms for neutrinos have to be added to the Lagrangian in order for it to still preserve its $SU(2)_L \times U(1)_Y$ symmetry. Since, like all other right-handed particles in the SM, the right-handed neutrinos form a $SU(2)_Y$ singlet, which we will call N. Also using the isospin conjugate of the Higgs doublet

$$\tilde{\phi} \equiv i\sigma^2\phi^*$$

the Yukawa term can be written as

$$\mathcal{L}_{\text{N,Yuk}} = h_{ij}\overline{N_i}\tilde{\phi}^{\dagger}l_j + h^*_{ij}\overline{l_i}\tilde{\phi}N_j \tag{3.1}$$

This interaction term will be analyzed in more detail in Appendix A.1.
The h_{ij} describe the Yukawa couplings and the l_i the left-handed lepton SU(2) doublets of the Standard model. It can be shown that this additional term does not violate the symmetries of the SM Lagrangian.
However, as explained above adding left-handed Majorana neutrino mass terms to the SM Lagrangian breaks its symmetry, but since right-handed neutrinos have to be added anyways one can also try to add a right-handed Majorana mass term, too.

$$\mathcal{L}_{N,M} = -\frac{1}{2}\overline{N^C}M^M N \tag{3.2}$$

The mass term in equation 3.2 however does not violate the Lagrangian's $SU(2)_L \times U(1)_Y$ symmetry, because $T=T^3=Y=0$ for right-handed neutrinos and therefore the transformations given in the previous section become the trivial identity transfomation. Anyways, the Dirac Lagrangian, so the SM Langrangian without any Majorana mass terms, is obviously invariant under any U(1) transformation, not only under $U(1)_Y$ transformations. The Majorana mass terms on the other hand are only invariant under exactly this $U(1)_Y$, especially only for particles with $T=T^3=Y=0$ like the right-handed neutrinos while violating other U(1) symmetries. And according to the Noether theorem, every symmetry of a theory results in a conserved current or quantum number, so breaking the U(1) symmetry not assigned to the hypercharge by using Majorana mass terms one certain quantum number, in this case breaking the following $U(1)_l$ symmetry results in a non-conservation of the lepton number.

$$\Psi \overset{U(1)_L}{\longrightarrow} e^{i\ell\alpha(x)}\Psi \tag{3.3}$$

with the lepton number ℓ of the field Ψ. This seems rather obvious because if Majorana particles are particles and anti-particles at the same time one cannot assing them a distinct lepton number and therefore it is not conserved.
Finally, after putting the Dirac and Majorana mass terms together one ends up with

$$\mathcal{L}_{M+D} = \left(\overline{\nu}, \overline{N}^C\right)\begin{pmatrix} M_L^M & m^D \\ m^D & M_R^M \end{pmatrix}\begin{pmatrix} \nu^C \\ N \end{pmatrix} + h.c. \tag{3.4}$$

The matrix m^D contains the masses of the Dirac neutrinos, so the neutrinos found in nature up until now, while M_L^M and M_R^M describe the masses of the left as well as the right-handed Majorana neutrinos. All of these matrices are of the dimension $n_F \times n_F$ with n_F the number of neutrino flavours. Also, as explained above, since it is not possible to introduce left-handed Majorana neutrinos to the SM without violating its fundemental symmetry M_L^M has to be equal to zero.

3.2. The seesaw Mechanism

Although the addition of neutrino masses can be described using the mass term 3.4 or rather

$$\mathcal{L}_{M+D} = \left(\overline{\nu}, \overline{N}^C\right)\begin{pmatrix} 0 & m^D \\ m^D & M_R^M \end{pmatrix}\begin{pmatrix} \nu^C \\ N \end{pmatrix} + h.c. \tag{3.5}$$

there is still a problem, namely why the neutrino masses are many orders of magnitude smaller than those of the other SM particles. This however can be described using the so called seesaw mechanism. In this discussion only the so called type I seesaw mechanism will be presented. By doing so the following two assumptions have to be made:

1. The Dirac masses arise directly from the Higgs mechanism that gives mass to all SM particles, introducing the electroweak mass scale of order $\sim 10^2 - 10^3$ GeV.

2. The Majorana masses are much bigger than the Dirac masses, $m^D \ll M^M$. This mass scale arises from GUT's and is of order $\sim 10^{10} - 10^{16}$ GeV.

The subscript R will be dropped from now on since there is no non-zero left-handed Majorana mass and therefore no further distinction is needed.
Now, after diagonalising the mass matrix in 3.5 [9, pp. 2-3], one gets two mass eigenvalues, in particular

$$M_1 \simeq -m^D \left(M^M\right)^{-1} \left(m^D\right)^T$$
$$M_2 \simeq M^M$$

or for just one neutrino family

$$M_1 \simeq -\frac{m_D^2}{M^M} \tag{3.6}$$
$$M_2 \simeq M^M \tag{3.7}$$

The negative sign for M_1 comes from the fact that these are just the eigenvalues of the mass matrix given in 3.5, the physical masses are the absolute values of these eigenvalues.
Now finding the corresponding eigenstates to these eigenvalues results in linear combinations of ν and N.

$$n_1 = \nu \cos\phi - N^C \sin\phi \tag{3.8}$$
$$n_2 = \nu \sin\phi + N^C \cos\phi \tag{3.9}$$

with their charge conjugated forms

$$n_1^C = \nu^C \cos\phi - N \sin\phi \tag{3.10}$$
$$n_2^C = \nu \sin\phi + N \cos\phi \tag{3.11}$$

Since the mixing angle ϕ is of order m_D/M and $m_D \ll$ M $\phi \ll 1$ and therefore one can expand the sines and cosines in 3.8-3.11 and gets

$$n_1 \approx \nu \tag{3.12}$$
$$n_1^C \approx \nu^C \tag{3.13}$$
$$n_2 \approx N^C \tag{3.14}$$
$$n_2^C \approx N \tag{3.15}$$

Putting all this back into the mass term of the Lagrangian yields at leading order

$$\mathcal{L}_{mass} \approx -\overline{n_1} M_1 n_1^C - \overline{n_2} M_2 n_2^C + h.c. \tag{3.16}$$

Seeing that the first as well as the second term of this Lagrangian only consist of left- and right-handed neutrinos respectively implies that these are Majorana mass terms and therefore

by applying the seesaw mechanism to the SM one gets two Majorana neutrinos, n_1 and n_2.
Since $\mathrm{n}_1 \approx \nu$ at leading order in ϕ this mass eigenstate mostly left-handed and with a mass M_1 very light, suggesting that this is the light neutrino observable today.
On the other hand because $\mathrm{n}_2 \approx N^C$ this mass eigenstate is mostly right-handed and therefore, while very heavy with a mass of M_2, sterile, so it cannot be detected other as via gravitational effects. This heavy right-handed neutrino is also the prime candidate for enabling baryogenesis via leptogenesis.
Interesting to note as well is how these two masses behave under finetuning. It is quite obvious from equation 3.6 that by raising the large mass scale and as a consequence thereof raising the mass of the heavy neutrino the mass of the light neutrinos gets even lower and vice versa, hence the name seesaw mechanism.

3.3. Leptogenesis and the Sakharov conditions

After the necessary expansion of the SM was performed in the previous section, this section will focus on how the right-handed, heavy neutrinos are able to produce a net, non-zero baryon number, that is how the Sakharov conditions can be fulfilled using this expanded SM.
In the following discussions the assumption that three heavy right-handed neutrinos exist will be made. In addition, it is required that the masses of these neutrions are hierarchical in the sense that $\mathrm{M}_1 \ll \mathrm{M}_{2,3}$ and that only the lightest of these neutrinos actually play a significant role for leptogenesis.
The key ingredient for baryogenesis via leptogenesis is the decay of the heavy, right-handed neutrinos introduced above, that is described by the Yukawa interaction in 3.1.The Feynman diagrams for both decay channels are depicted in figure 3.1. The right-handed neutrinos being Majorana particles as a result of the seesaw mechanism do not preserve lepton number and because of this they can decay into leptons as well as anti-leptons, as it can be seen in figure 3.1.

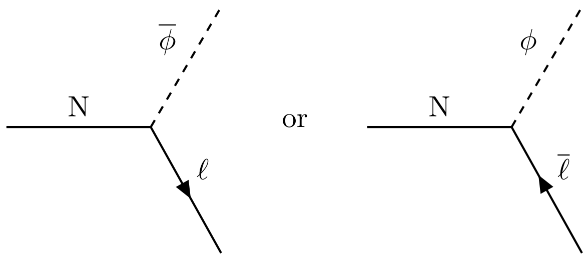


Figure 3.1.: Feynman diagrams for the N-decay

B violation

The B violation in the frame of leptogenesis is achieved in the same way as in direct electroweak baryogenesis via the B+L violating sphaleron processes. Because of the eventually by N decays produced net lepton number this means that the lepton abundance is converted into a baryon abundance by these processes.

C and CP violation

As in direct baryogenesis C violation is already maximally violated SM electroweak interaction, so this is also the case in this slightly extended model. The CP violation for the N_1 decay however is a bit more complicated, since for it to be calculated one has to take the one-loop Feynman diagrams additional to the tree level decay into account. The interference between these diagramms gives then rise to the CP violation needed for successfull baryogenesis. These one-loop diagrams are depicted in figure 3.2.

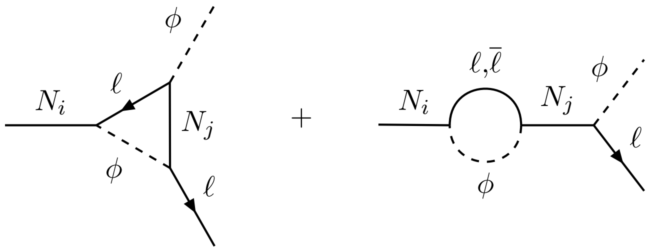


Figure 3.2.: One-loop diagrams for the N-decay

By comparing this figure to [12, Fig. 5.1] one sees that in figure 3.2 one diagram is missing. This missing diagram however does not contribute to the CP violation and therefore it was neglected. The distinction between $\mathrm{N_i}$ and $\mathrm{N_j}$ has to be made since the virtual neutrinos transmissioned during these processes are not of the same flavour as the neutrino in the initial state. This means that in order for CP to be violated there has to exist at least an extra neutrino next to the lightest one.
This being said, one can calculate the CP asymmetry[12, pp. 24ff.], which is in general defined as

$$\epsilon = \frac{\Gamma(N_1 \to \overline{\phi}\ell) - \Gamma(N_1 \to \phi\bar{\ell})}{\Gamma(N_1 \to \overline{\phi}\ell) + \Gamma(N_1 \to \phi\bar{\ell})} \tag{3.17}$$

by interfering the tree level decay with the one-loop decays. This results in [12, p. 26]

$$\epsilon \gtrsim 10^{-6} \tag{3.18}$$

Although this value is rather small it greater than zero and therefore the CP symmetry is violated by the decay of heavy right-handed Majorana neutrinos.

Deviation from the equilibrium

The last condition that has to be met for successfull baryogenesis is that the decay of the heavy neutrinos must occur outside of equilibrium. For high enough temperatures, namely for $\mathrm{T} \gtrsim \mathrm{M_1}$, the decay of the neutrino is in equilibrium with the inverse decay $\phi\ell \to N$ and no net lepton number can be produced even if the other two conditions hold as shown in sec. 2.1.2. Even if during inflation an abundance of N was produced the lepton asymmetry would be washed out by equilibrium processes as soon as the temperature rises up to at least the mass of the heavy neutrino during the reheating phase of the early universe.
However, if the temperature drops below $\mathrm{M_1}$ the heavy neutrino inverse decay is exponentially suppressed by a Boltzmann factor because with falling temperature it becomes exponentially more unprobable for a lepton and Higgs to have enough energy to form a heavy neutrino, while the neutrinos themselves can still decay into lepton and Higgs. Due to this the equilibrium density of the neutrinos is also heavily Boltzmann suppressed and if the decay rate is small enough the actual neutrino density can be greater than the equilibrium density and stay greater for an considerable amount of time. This means that decays happening before the actual neutrino density converges to the equilibrium density actually happen outside of equilibrium and a net lepton number can be produced, that, because of suppression of inverse decays, will not be washed out and as a consequence thereof will be transformed into a baryon abundance through the B+L violating electroweak sphaleron processes, that will still be active as the heavy neutrino mass at least of order of the upper limit of the temperature range in which sphaleron processes

act.
The requirement for the neutrino decay to be slow enough to sustain its out-of-equilibrium state is the following [8, p. 30].

$$\Gamma_D < H \tag{3.19}$$

Γ_D denotes the total decay rate of the neutrinos while H is the expansion rate of the universe. This relation is equivalent to the one given in 2.10 as a requirement for processes to be out of equilibrium.

4. Analytic approximations and calculations

4.1. Rate equations for leptogenesis

Now, in order to qualitatively describe leptogenesis one has to consider rate equations for the lepton number and B-L number densities. In a static universe without lepton number violating processes the rate equation would trivially be

$$\frac{d}{dt}n = 0 \tag{4.1}$$

If one now condsiders a universe expanding with the rate H the rate equation above changes to

$$\left(\frac{d}{dt} + 3H\right) n = 0 \tag{4.2}$$

Finally including lepton number violating processes the rate equation one obtains for the neutrino number density is

$$\left(\frac{d}{dt} + 3H\right) n_N = -\Gamma_N \left(n_N - n_N^{eq}\right) + \Gamma_{N,B-L} n_{B-L} \tag{4.3}$$

Applying this reasoning to the B-L number density yields

$$\left(\frac{d}{dt} + 3H\right) n_{B_L} = \Gamma_{B-L,N} \left(n_N - n_N^{eq}\right) + \Gamma_{B-L} n_{B-L} \tag{4.4}$$

The coefficient Γ_N denotes how fast the neutrino density equalizes with its equilibrium density, while Γ_{B-L} describes the washout of a net B-L number. $\Gamma_{B-L,N}$ describes how the B-L asymmetry is affected by the deviation of the neutrino density from its equilibrium value and together with $\Gamma_{N,B-L}$ these two coefficients characterize CP violating processes [13, p. 4]. Since both these coefficients describe CP violating processes they must depend on the CP violating parameter ϵ introduced in the section before and it was also shwon there that this parameter is small for heavy neutrino decays and therefore the second term in equation 4.3 can be neglegted. The goal now is to determine these coefficients at least at leading order and the first one will be Γ_N. To get this coefficient one has to integrate equation 4.3 over phase space, resulting in

$$\left(\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right) f_N = \Gamma_N \left(e^{E_N/T} - f_N\right) \tag{4.5}$$

The whole calculation on how to perform the phase space integral can be found in appendix C.1.
One might now notice that equation 4.5 differs from equation 4 in [13] by a factor of $\mathrm{M}_N/\mathrm{E}_N$, that origins in a different normaliziation of the phase space. However, since we operate in a non-relativistic regime $\mathrm{E_N} \approx \mathrm{M_N}$ and therefore this factor is ~ 1 and negligible. Using this argumentation one can also see that $\Gamma_N = \Gamma_0$ with Γ_0 the total decay rate of the heavy neutrinos, which is governed by the Yukawa interaction term 3.1. It has to be mentioned that for the equilibrium distribution the Boltzmann statistic with neglegted chemical potential was used since the energy of a neutrino $\mathrm{E_N} \approx \mathrm{M_N} \gg \mathrm{T}$ during the phase where the decay happen outside

equilibrium and therefore quantum mechanical effects play an insignificant role and can be neglected. Going back to the decay rate, that can be calculated as done in appendix A.2, one gets

$$\Gamma_N = \Gamma_0 = \frac{|h_{11}|^2 M_N}{8\pi} \tag{4.6}$$

As explained above $\Gamma_{B-L,N}$ connects the B-L asymmetry with the deviation of $\mathrm{n_N}$ from its equilibrium value and because the only processes able to produce a B-L asymmetry are the decays of the heavy neutrino and therefore $\Gamma_{B-L,N}$ is directly connected to its decay rate. On the other hand though, any B-L asymmetry can only arise through CP violation, meaning the neutrino has to decay into leptons and anti-leptons at different rates. The parameter describing this difference is ϵ, as it is given in 3.17. Putting all this together results in

$$\Gamma_{B-L,N} = \epsilon\Gamma_0 \tag{4.7}$$

The last coefficient left to determine is Γ_{B-L}. This one however refers to the washout of the B-L asymmetry hence it arises from the inverse decay $\mathrm{l}\phi \to \mathrm{N}$. It can be calculated using the quatum field theoretical means for obtaining decay rates and taking into account that particle distributions and neutrino momenta have to be considered because of non-zero temperatures.

$$\Gamma_{B-L} n_{B-L} = \int \prod_{a=N,\ell,\phi} \frac{d^3p_a}{2E_a(2\pi)^3}(2\pi^4)\delta(p_\ell + p_\phi - p_N)(f_\ell f_\phi - f_{\bar{\ell}} f_{\bar{\phi}}) \sum |M_0|^2 \tag{4.8}$$

$\sum |M_0|^2 = 16\pi M_N \Gamma_0$ describes the tree level matrix element for exactly this inverse decay summed over all spins of N and isospin components of ℓ. In order to evaluate this integral properly one has to first describe the term $(f_l f_\phi - f_{\bar{l}} f_{\bar{\phi}})$ more precisely. Since we operate at rather high temperatures one can use Boltzmann statistics for lepton and Higgs regardless and expanding in the chemical potentials up to first order yields.

$$f_\ell f_\phi - f_{\bar{\ell}} f_{\bar{\phi}} \simeq 2e^{-\frac{E_N}{T}} \frac{\mu_l + \mu_\phi}{T} \tag{4.9}$$

As mentioned in [13, p. 7] the chemical potentials are proportional to the B-L number density and by using the coefficients c_l and c_ϕ in order to avoid introducing the exact temperature dependant relations one can also connect n_{B-L} to the lepton and Higgs asymmetries through [13, p. 7]

$$n_\ell - n_{\bar{\ell}} = -c_\ell n_{B-L} \tag{4.10}$$
$$n_\phi - n_{\bar{\phi}} = -c_\phi n_{B-L} \tag{4.11}$$

Now by expanding the number distribution for leptons and Higgs and their respective anti particles in the chemical potentials up to first order one can finally put the chemical potential and n_{B-L} in relation to each other. It is important to note that one has to use the Fermi-Dirac- and Bose-Einstein distribution for leptons and Higgses respectively. The reason for this is that these two particle species are in equilibrium at the temperature T. Because of this and the fact that they are relativistic particles, their kinetic energy and therefore momenta are of order T and quantum effects cannot be neglected. On the other hand it is possible to use Boltzmann statistics for leptons and Higgs bosons above since particles taking part in the production of a heavy neutrinos having energies or momenta of order $\mathrm{M_N}/2 \gg T$ and quantum effects can safely be neglected in this case. Finally using quantum statistics results in

$$\mu_\ell = \frac{3c_l}{T^2} n_{B-L} \tag{4.12}$$

$$\mu_\phi = \frac{3c_l}{2T^2} n_{B-L} \tag{4.13}$$

Using all these results and putting them into relation 4.8 one gets the following result for the washout rate $\Gamma_{\text{B-L}}$

$$\Gamma_{\text{B-L}} = \frac{3}{\pi^2}\left(c_\ell + \frac{c_\phi}{2}\right) z^2 K_1(z)\Gamma_0 \tag{4.14}$$

with K_1(z) the modified Bessel function of the second kind and

$$z \equiv \frac{M_N}{T} \tag{4.15}$$

This can be seen as an unitless measure for time, since the temperature of the universe decreases over time and therefore z increases. The exact derivation of these results can be looked up in appendix C.2.
Using the relations given in 2.9 and 2.10 it is usefull to define the so called wash out factor K as follows

$$K \equiv \left.\frac{\Gamma_0}{H}\right|_{T=M_{\text{N}}} \tag{4.16}$$

Using this and equation 3 of [14]

$$H(z) = H|_{\text{T=M}_\text{N}} \cdot \frac{1}{z^2} \tag{4.17}$$

one can now compare the rates calculated above to the Hubble constant H.
For T $\lesssim$ M_N one can approximate Γ_N as Γ_0 resulting in

$$\frac{\Gamma_\text{N}}{H} \simeq \frac{\Gamma_0}{\frac{1}{z^2}\, H|_{\text{T=M}_\text{N}}} = z^2 K \tag{4.18}$$

Since z increases with time, equation 4.18 clearly shows that the rate with which the neutrino number density approaches its equilibrium value gets more and more greater than the expansion of the universe and therefore the deviation from equilibrium vanishes over time.
Looking at 4.14 one can easily see that for T $\sim$ M_N, or z $\approx$ 1, $\Gamma_{\text{B-L}}$ is of the same order as Γ_N, since $K_1(1)=\mathcal{O}(1)$. On the other hand however, for really low temperatures, meaning T $\ll$ M_{arg} or z very big, one can see the quantitative behaviour of $\Gamma_{\text{B-L}}$ by using the asymptotic expansion of the modified Bessel function given by

$$K_\text{n}(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{\infty}\frac{a_\text{k}(n)}{z^k}$$

with a_k(n) some coefficients that are not of any interest for this matter. Now using this and dropping all but the first term of the sum above, which is justified because z is large, one gets

$$\frac{\Gamma_{\text{B-L}}}{H} \sim \sqrt{\frac{1}{z}}e^{-z}z^2\frac{\Gamma_0}{\frac{1}{z^2}\, H|_{\text{T=M}_\text{N}}} = z^{\frac{7}{2}}e^{-z}K \tag{4.19}$$

So because Γ_N increases with z for z $\approx$ 1 so does $\Gamma_{\text{B-L}}$. However, as equation 4.19 suggests, $\Gamma_{\text{B-L}}$ gets more and more suppressed by the exponential factor for increasing z and therefore has to have some maximum. Additionally for sufficiently low temperatures, like the extremely low 2.7K of the present universe, the washouth rate $\Gamma_{\text{B-L}}$ is effectively zero and any symmetry produced in earlier times is frozen out. All this can be seen in figure 4.1 where the previously calculated ratios are plotted against z and are normalized to the washout factor K.

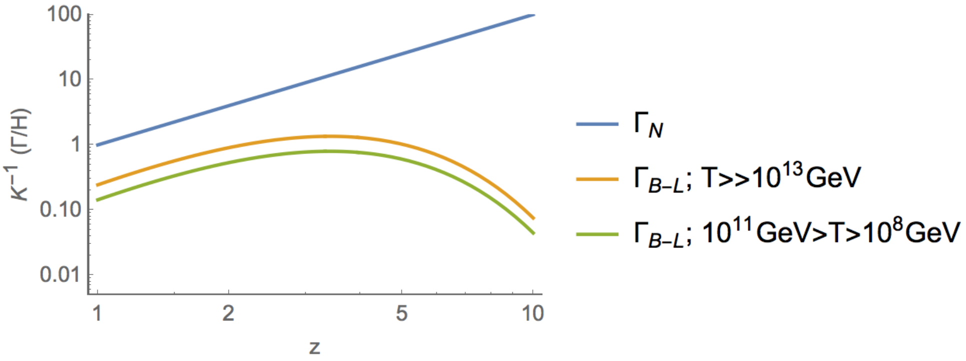


Figure 4.1.: The ratios Γ_{N}/H and $\Gamma_{\text{B-L}}$/H plotted against H

4.2. Classical versus quantum mechanical computation

4.3. Relativistic corections

Although this thesis focuses on a non-relativistic scenario for leptogenesis it might be interesting to see how much relativistic corrections influence the previous results. For this matter one has to reintroduce the factor $\text{M}_{\text{N}}/\text{E}_{\text{N}}$ seen in equation 4 of [13], that was dropped in equation 4.5 since it simplifies to 1 in the non-relativistic limit, so this equation reads

$$\left(\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right) f_N = \frac{M_N \Gamma_0}{E_N}\left(e^{E_N/T} - f_N\right) \tag{4.20}$$

Now, using the relativistic energy-momentum relation

$$E = \sqrt{M^2 + p^2} \tag{4.21}$$

one can expand the factor $1/\text{E}_N$ on the right-hand side up to order p^2 and therefore get the next-to-leading order relativisitic corrections for the rate equation 4.3

$$\left(\frac{d}{dt} + 3H\right) n_N = \Gamma_N\left(n_N^{eq} - n_N\right) + \Gamma_{N,u}\left(u - u^{eq}\right) \tag{4.22}$$

with u the kinetic energy density of the heavy neutrinos divided by their mass

$$u = \frac{g_N}{M_N}\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{2M_N} f_N \tag{4.23}$$

with g_N the the internal degrees of freedom of the neutrinos.
By multiplying 4.22 with p^2 and integrating over p one can easily get a rate equation for u, which at leading order in p, so for E_N=M_N, reads as

$$\left(\frac{d}{dt} + 5H\right) u = \Gamma_u\left(u^{eq} - u\right) \tag{4.24}$$

It is important to note that for the rates appearing in above the following relations are holding

$$\Gamma_{N,u} = \Gamma_0 \tag{4.25}$$
$$\Gamma_u = \Gamma_0 \tag{4.26}$$

Now expanding the factor $1/E_N$ in the Lorentz invariant integration measure in n_N one gets an equivalent expansion for equation 4.4. Since the B-L asymmetry density is not directly affected by any relativistic effects, only the term containing the neutrino density has to be expanded, resulting in

$$\left(\frac{d}{dt} + 3H\right) n_{B_L} = \Gamma_{B-L,N}\left(n_N - n_N^{eq}\right) - \frac{1}{2}\Gamma_{B-L,u}\left(u - u^{eq}\right) + \Gamma_{B-L} n_{B-L} \tag{4.27}$$

At lowest order in relativistic corrections, in equivalence to 4.7, one gets

$$\Gamma_{B-L,u} = \epsilon\Gamma_0 \tag{4.28}$$

How exactly these corrected rate equations are obtained can be looked up in appendix C.3.

4.4. Radiative corrections

Until now only the decays and inverse decays of heavy neutrinos, so $1 \leftrightarrow 2$ processes, but naturally there are more processes that affect the neutrino density and therefore the asymmetry n_{B-L} like $2 \leftrightarrow 2$ scattering, $1 \leftrightarrow 3$ decays and virtual corrections to the already treated $1 \leftrightarrow 2$ decays.
Since the density n_{B-L} does not describe actual particles but only the B-L asymmetry relativistic corrections will only affect the neutrino densities or rather the rates associated with these densities. Furthermore it can be shown that the radiative corrections to the number distribution f_N to the change over time, so f_N/dt, for n_N=0 has the following form [15]

$$\left.\frac{\partial f_N}{\partial t}\right|_{f_N=0} = f^{eq}\,\Gamma_0 \frac{M_N}{E_N}\left[a + \frac{p^2}{M_N^2} b + \mathcal{O}\left(\frac{p^4}{M_N^4}\right)\right] \tag{4.29}$$

with a and b the following temperature-dependant coefficients

$$a = 1 - \frac{\lambda T^2}{M_N^2} - |h_t|^2\left[\frac{21}{2(4\pi)^2} + \frac{7\pi^2}{60}\frac{T^4}{M_N^4}\right] + \left(g_1^2 + 3g_2^2\right)\left[\frac{29}{8\,(4\pi)^4} - \frac{\pi^2}{80}\frac{T^4}{M_N^4}\right] + \mathcal{O}\left(g^2\frac{T^6}{M_N^6}, g^3\frac{T^2}{M_N^2}\right) \tag{4.30}$$

$$b = -\left[|h_t|^2\frac{7\pi^2}{45} + \left(g_1^2 + 3g_2^2\right)\frac{\pi^2}{60}\right]\frac{T^4}{M_N^4} + \mathcal{O}\left(g^2\frac{T^6}{M_N^6}, g^3\frac{T^2}{M_N^2}\right) \tag{4.31}$$

h_t here denotes the Yukawa coupling of top quarks to the Higgs field, g_1 and g_2 are the U(1) and SU(2) gauge couplings and λ again the Higgs self-coupling.
Plugging equation 4.29 into 4.27 and 4.24 one gets the radiative corrections for Γ_N, Γ_u and Γ_{Nu} by expandig the E_N factor in 4.29 just as done for obtaining the relativistic corrections and then integrating over the momentum. The results of this procedure that can be looked up in

appendix C.4, are at leading order of the respective equation,

$$\Gamma_N = \Gamma_u = a\Gamma_0 \tag{4.32}$$

$$\Gamma_{N,u} = -\frac{1}{2}(a-2b)\Gamma_0 \tag{4.33}$$

One thing interesting to note about the coefficients a and b is the following. The term of lowest order in T in these two coefficients is proportional to $\mathrm{T}^2/\mathrm{M}_N^2$, which corresponds to $\mathcal{O}(p^4)$ because $\mathrm{E}=p^2/M_N \sim \mathrm{T}$ and therefore $\mathrm{p}^2 \sim \mathrm{T}/\mathrm{M}_N$. This would imply that the relativistic corrections should be calculated up to an order of at least p^4, but, as will be obvious later, these corrections of order p^2 are already really small.

4.5. Numerical results

5. Summary

A. Appendix A

A.1. Feynman rules for the Yukawa interaction

In quantum field theories without Majorana particles the Feynman rules for evaluating Feynman diagrams are acquired from the elements of the so called scattering matrix S, given as [18, Eq. 3.26], given as

$$\lim_{t_\pm \to \pm\infty} \langle f|U(t_+, t_-)|i\rangle = \langle f|S|i\rangle \tag{A.1}$$

U(U(t_+,t_-)) is an unitary time evolution operator explicitly given by Dyson's formula

$$U(t_+, t_-) = T \exp\left(-i \int_{t_-}^{t_+} dt\, H_{int}(t)\right)$$

In the following $\psi_{x_i} \equiv \psi(x_i)$ will describe an arbitrary fermionic Dirac, so one can write the time ordering symbol T above in the following way

$$T\psi(x)\psi(y) = \begin{cases} \psi(x)\psi(y) & \text{if } x^0 > y^0 \\ \pm\psi(y)\psi(x) & \text{if } x^0 < y^0 \end{cases}$$

The sign in the lower line depends on if the number of permutations of anticommutating fermion spinors is odd or even.
Using equation 3.1 for general fields ψ and ϕ, the interaction Hamiltonian can be given as

$$H_{int} = g \int d^3x \overline{\psi}\psi\phi$$

with g the coupling constant.
Interesting to note is that, given g is small, by expanding the time evolution operator above one can use quantum mechanical pertubation theory up to an arbitratry order.
Now in order to transform the matrix element into a purely algebraic expression, one first has to reorder the fields ψ_1 to ψ_n in a way that the time order symbol T is not needed any more. This can be done using the so called Wick's theorem

$$T\left[\psi_1\psi_2\cdots\psi_n\right] =: \psi_1\psi_2\cdots\psi_n + \text{all possible contractions} :$$

The colon notation simply means that the field operators are normally ordered, meaning all creation operators are on the left side of a product, while the annihilation operators are on the right. The exact mathematical expression of the contraction of two field is not of importance here, but can be looked up for example in section 4.2 of [16]. On the other hand the notation and the final result of such a contraction are vital for the following discussion.

$$\text{Contraction of } \psi \text{ and } \overline{\psi} = \psi(x)\overline{\psi}(y) = \langle 0|T[\psi(x)\overline{\psi}(y)]|0\rangle \to S_F(p)$$

The arrow simply corresponds to the Fourier transformed, since the matrix element above lives in position-space, while it is easier to compute Feynman rules in momentum space. S$_F$(p) is the fermionic Feynman propagator

$$S_F(p) = \frac{(\not{p} + m)}{p^2 + m^2 + i\epsilon}$$

where the Feynman slash notation

$$\not{p} = \gamma^\mu p_\mu$$

was used. Now, in order to illustrate what the term 'all possible contractions' in the Wick theorem above means, this theorem will be applied to a time ordered product of two barred and two unbarred fields will explictily be given

$$\begin{aligned} T\left[\psi_1\overline{\psi}_2\psi_3\overline{\psi}_4\right] &= :\psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \\ &+ \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 := \\ &= :\psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \\ &+ \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 + \psi_1\overline{\psi}_2\psi_3\overline{\psi}_4 : \end{aligned}$$

In the last step the relations

$$\overline{\psi}(x)\overline{\psi}(y) = 0 \tag{A.2}$$

$$\psi(x)\psi(y) = 0 \tag{A.3}$$

were used, having the effect that conctracted field are always adjacend to each other in a sense, that to conctraction symbols do not intersect each other. These contractions of just ψ with $\bar{\psi}$ fields can be interpreted as a fermion number flow and therefore the conservation of fermion number. It can be thought of as an incoming particle ψ that becomes an outgoing particle $\bar{\psi}$ after interacting and therefore the fermion number stays the same.
So in order to apply Wicks theorem one has to basically connect two fields pairwise until all possible contractions are performed.
The next important step ist do note that only terms with no uncontracted fields contribute to the corellation function above. In the previous example this means that only the last three terms have to be taken into account in order to obtain the corresponding Feynman rules.
Until now only internal contractions, so contractions of two field operators resulting in the internal propagators, where considered. In order to also take the external, so incoming and outgoing, particles into account one has to take a closer look at the initial and final states $|i\rangle$ and $\langle f|$. Because in general these states are not just the vacuum, but (multi-)particle states they must consist of creation and anihilation operators acting on the vacuum state, causing contractions of such operators and the fields to arise. Asuming that the particles are not plane waves but rather localized wave packages one can write the states $|i\rangle$ and $\langle f|$ as[16]

$$\begin{aligned} |i\rangle &= a_1^\dagger a_2^\dagger \cdots b_{n-1}^\dagger b_n^\dagger |0\rangle \\ \langle f| &= \langle 0| a_1 a_2 \cdots b_{n-1} b_n \end{aligned}$$

with the annihilation operators a_i, b_i and the creation operators $a_i^\dagger, b_i^\dagger$ for particles and antiparticles, respectively. The subsript i is short for p$_i$, the particle momentum. In this notations one distinct creation or annihiliation operator can appear twice if two particle with the same momentum are created or destroyed.
Using this and equation A.1, for a scattering process with a m-particle initial state and a n-

particle final state, this results in the following matrix element [17, Eq. 2.2]

$$\left\langle 0 \left| a_1 a_2 \cdots b_{n-1} b_n T \left[(\bar{\psi}\Gamma\psi) \cdots (\bar{\psi}\Gamma\psi) \right] a^\dagger_{n+1} a^\dagger_{n+2} \cdots b^\dagger_{m-1} b^\dagger_m \right| 0 \right\rangle \tag{A.4}$$

Here the exponential function appearing in the relation above was expanded and only some arbitrary order was chosen, in order to express how the Feynman rules are obtained.
In analogy to the notation in Ref.[17] Γ describes some arbitrary fermionic interaction

$$\bar{\psi}\Gamma\psi = g^i_{abc}\bar{\psi}_a\Gamma_i\psi_b\phi_c$$

with $\Gamma_i = 1, i\gamma_5, \gamma_\mu\gamma_5, \gamma_\mu, \sigma_{\mu\nu}$, but for the next discussion $\Gamma_i = 1$ will be assumed simply for the sake of clarity.
Applying Wick's Theorem to this new matrix element not only results in the already known propagator terms, but because of the contractions of the fields directly with the creation and annihilation operators one also gets

$$\begin{aligned}
\underbracket{\psi a^\dagger_i} &= \langle 0|\psi(x)a^\dagger_i(p,s)|0\rangle \longrightarrow u(p,s) \text{ incoming particle} \\
\underbracket{a_i\bar{\psi}} &= \langle 0|a_i(p,s)\bar{\psi}(x)|0\rangle \longrightarrow \bar{u}(p,s) \text{ outgoing particle} \\
\underbracket{\bar{\psi}b^\dagger_i} &= \langle 0|\bar{\psi}(x)b^\dagger_i(p,s)|0\rangle \longrightarrow \bar{v}(p,s) \text{ incoming antiparticle} \\
\underbracket{b_i\bar{\psi}} &= \langle 0|b_i(p,s)\psi(x)|0\rangle \longrightarrow v(p,s) \text{ outgoing antiparticle}
\end{aligned}$$

Here the following expressions for fermion fields

$$\begin{aligned}
\psi &= \sum_{s=\pm 1/2} \int \frac{d^3p}{2(\pi)^3} \frac{1}{2E} \left(a(p,s)u(p,s)e^{-ipx} + b^\dagger(p,s)v(p,s)e^{ipx} \right) \\
\bar{\psi} &= \sum_{s=\pm 1/2} \int \frac{d^3p}{2(\pi)^3} \frac{1}{2E} \left(a^\dagger(p,s)\bar{u}(p,s)e^{ipx} + b(p,s)\bar{v}(p,s)e^{-ipx} \right)
\end{aligned}$$

and the anti-commutator relations

$$\begin{aligned}
\{a(p,s), a^\dagger(q,s')\} &= \{b(p,s), b^\dagger(q,s')\} = \delta(p-q)\delta_{ss'} \\
\{a(p,s), a(q,s')\} &= \{b(p,s), b(q,s')\} = 0
\end{aligned}$$

where used to obtain the results above. In this context u, v and their barred versions describe the incoming or outgoing (anti-)particles as seen above, but their exact mathematical expression is of no greater interest here and will therefore not be given.
Shortly summarized the previous discussion shows that, for Dirac particles, using the matrix element A.4 one cannot only determine the internal propagators needed in most processes but also the spinors of incoming and outgoing particles. So a full set of Feynman rules,except for internal,which will not be used, and external scalars,which is trivialy 1, was obtained and can directly be used for calculating the scattering matrix elements for various processes.
However, because the scenario of leptogenesis requires Majorana neutrinos at leas the Feynman rules for exactly this neutrino have to be reiterated. The main difference in obtaining the Feynman rules for Majorana particles in comparison to Dirac particles is that the relations A.2 and A.3 do no longer hold and therefore contractions of fields not adjacent to each other are possible. It can easy be seen that the same reasoning from above with incoming and outgoing fermions for Dirac particles can not be applied for Majorana particles because of lepton number non-conserving contractions containing just ψ or $\bar{\psi}$ fields.

This problem can be solved by introducing the charge-conjugated particle

$$\tilde{\psi} = C\bar{\psi}^T$$
$$\bar{\tilde{\psi}} = -\psi^T C^{-1}$$

where C is the charge conjugation operator. The interaction term above then becomes the reversed one

$$\bar{\psi}\Gamma\psi = g^i_{abc}\bar{\psi}_a\Gamma_i\psi_b\phi_c = (-1)g^i_{abc}\psi_b^T\Gamma_i^T\bar{\psi}_a^T\phi_c = g^i_{abc}\bar{\tilde{\psi}}_a C\Gamma_i^T C^{-1}\tilde{\psi}_b\phi_c = g^i_{abc}\bar{\tilde{\psi}}_a\eta_i\Gamma_i\tilde{\psi}_b\phi_c =: \bar{\tilde{\psi}}\Gamma'\tilde{\psi}$$

with the reversed fermion interaction

$$\Gamma' = C\Gamma C^{-1}$$

In additon the relation

$$C\Gamma_i C^{-1} = \eta_i\Gamma_i$$

with

$$\eta_i = \begin{cases} +1 & \text{for } 1, i\gamma_5, \gamma_\mu\gamma_5 \\ -1 & \text{for } \gamma_\mu, \sigma_{\mu\nu} \end{cases}$$

Because for Majorana fermions $\tilde{\psi} = \psi$, it is rather easy to notice that the vertex term and the reversed vertex term are the same, $\Gamma' = \Gamma$.
How this untangles intersecting contractions will be shown exemplarily using the following contraction.

$$\bar{\psi}_a\psi_b\ \bar{\psi}_a\psi_b\ \bar{\psi}_a\psi_b$$

According to the relation above, we now reverse the middle interaction term and therefore get

$$\bar{\psi}_a\psi_b\ \bar{\tilde{\psi}}_b\tilde{\psi}_a\ \bar{\psi}_a\psi_b$$

Now applying again the contractions, while the same indices as in the original term stay contracted, yields the desired result where all conctracted field are directly next to each other.

$$\bar{\psi}_a\psi_b\ \bar{\tilde{\psi}}_b\tilde{\psi}_a\ \bar{\psi}_a\psi_b \tag{A.5}$$

This can be applied to an arbitrarlily long string of field operators and therefore any combination of general fermion fields can be transformed in a way that is has the same structure as for Dirac fermions, with the only difference being that some fields may be exchanged with their charge conjugated counterparts. As already mentioned, these kind of interaction terms do not conserve fermion number, however, equation A.5 having the same structure as for Dirac particles the fermion number flow has to be replaced by something else, namely the so called fermiom flow. This fermion flow plays the role of an orientation of a Feynman diagram. This orientation can be chosen arbitrarily, the resulting matrix element are still the same, regardless of the chosen orientation
The Feynman rules than can be obtained exactly in the same way as for Dirac particles. Using the definition of the charge conjugated fields results in the algebraic expression for the internal propagator given as

$$\langle 0|T[\tilde{\psi}\bar{\tilde{\psi}}]|0\rangle = C\,\langle 0|T[\psi\bar{\psi}]|0\rangle^T C^{-1} \longrightarrow CS(p)^T C^{-1} = \frac{(-\not{p} + m)}{p^2 + m^2 + i\epsilon} = S(-p) =: S'(p)$$

Using the explicit expressions for the charge conjugated fields

$$\tilde{\psi} = \sum_{s=\pm 1/2} \int \frac{d^3p}{2(\pi)^3} \frac{1}{2E} \left(a^\dagger(p,s)u(p,s)e^{-ipx} + b(p,s)v(p,s)e^{ipx} \right)$$
$$\bar{\tilde{\psi}} = \sum_{s=\pm 1/2} \int \frac{d^3p}{2(\pi)^3} \frac{1}{2E} \left(a(p,s)\bar{u}(p,s)e^{ipx} + b^\dagger(p,s)\bar{v}(p,s)e^{-ipx} \right)$$

and the exact same anti-commutator relations one gets the analogous results for the external fermion fields.

$$\begin{aligned}
\tilde{\psi} b_i^\dagger &= \langle 0|\tilde{\psi}(x) b_i^\dagger(p,s)|0\rangle \longrightarrow u(p,s) \\
b_i \bar{\tilde{\psi}} &= \langle 0|b_i(p,s)\bar{\tilde{\psi}}(x)|0\rangle \longrightarrow \bar{u}(p,s) \\
\bar{\tilde{\psi}} a_i^\dagger &= \langle 0|\bar{\tilde{\psi}}(x) a_i^\dagger(p,s)|0\rangle \longrightarrow \bar{v}(p,s) \\
a_i \bar{\tilde{\psi}} &= \langle 0|a_i(p,s)\tilde{\psi}(x)|0\rangle \longrightarrow v(p,s)
\end{aligned}$$

Summarizing all these results gives a full set of Feynman rules for Majorana as well as Dirac fermions, as they are given in [17]. For the sake of completeness they will be given here as well. In the following diagramms dotted lines represent a boson field, solid lines without arrows represent Majorana fermions and solid lines with arrows represent Dirac fermions, while the arrow points in the direction of fermion number flow, the thin arrows indicate the fermion flow. The momentum always flows from left to right.

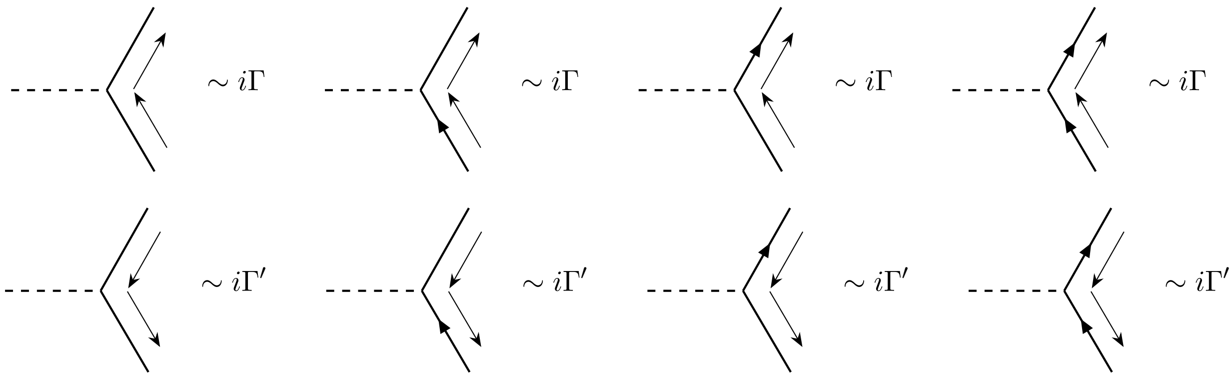


Figure A.1.: Feynman rules for fermionic vertices

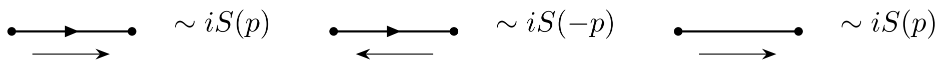


Figure A.2.: Feynman rules for internal fermionic propagators

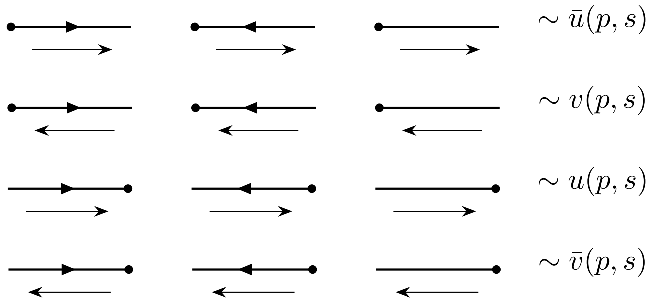
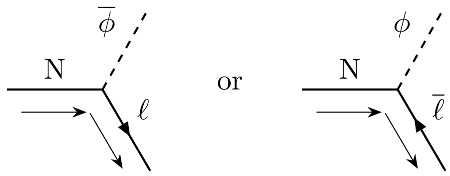


Figure A.3.: Feynman rules for external fermion lines

A.2. The tree level decay rate for heavy neutrinos

In this section the heavy neutrion decay rate given in equation 4.6 will be calculated using the tree level diagramms in figure 3.1, which, for the sake of clarity, will be given here once more, but also with arrows showing the arbitrarily chosen orientation



In general the decay rate can be obtained using

$$\Gamma = \int d\Gamma$$

with the differential decay rate given as

$$d\Gamma = \frac{1}{2M_N} |\mathcal{M}|^2 \frac{d^3p_\ell}{(2\pi)^3 2E_\ell} \frac{d^3p_\phi}{(2\pi)^3 2E_\phi} (2\pi)^4 \delta^4 \left(p_\ell + p_\phi - p_N\right)$$

$|\mathcal{M}|$ denotes the matrix element corresponding to exactly this decay, that can be read of directly from the corresponding Feynman diagram. Now, using the Feynman rules obtained in the previous section one has to first formulate this matrix element in order to be able to evaluate it. Choosing an arbitrary orientation the matrix element for the decay into $\bar{\phi}$ and a lepton reads

$$i\mathcal{M} = \bar{u}_{\ell_a}(p_{\ell_a}, s_{\ell_a}) i h_{a1} P_R u_N(p_N, s_N)$$

where the subscripts ℓ_a and N describe the lepton of flavor a and heavy neutrino, respectively and

$$P_R = \frac{1+\gamma^5}{2} \qquad P_L = \frac{1-\gamma^5}{2}$$

the chirality operators. This can be used because the heavy neutrino N, per defnition, is right-handed. Then the respective absolute square of this matrix element is

$$|\mathcal{M}|^2 = \bar{u}_{\ell_a}(p_{\ell_a}, s_{\ell_a}) h_{a1} P_R u_N(p_N, s_N)\, \bar{u}_N(p_N, s_N) P_L i h^*_{a1} u_{\ell_a}(p_{\ell_a}, s_{\ell_a})$$

However, this matrix element only describes the decay of a neutrino with a certain spin into a lepton of another certain spin. In order to generalize this for unknown spins of the lepton as well as of the spin one has to sum over the final spins of the lepton and average over the initial spin of the neutrino, resulting in

$$|\mathcal{M}|^2 = 2 \cdot \frac{1}{2} \sum_{s_N, s_{\ell_a}} \bar{u}_{\ell_a}(p_{\ell_a}, s_{\ell_a}) h_{a1} P_R u_N(p_N, s_N)\, \bar{u}_N(p_N, s_N) P_L i h^*_{a1} u_{\ell_a}(p_{\ell_a}, s_{\ell_a})$$

The factor of 2 arises from the fact that general leptons are part of a weak isospin doublet, so there are two possible values for the isospin, or more precise, its third component, namely T_3=-1/2 for the charged leptons e, μ, τ and T_3=1/2 for the corresponding light neutrinos $\nu_{e,\mu,\tau}$. If we just looked at the decay into for example the light neutrinos this factor of 2 has to be omitted since there is only one isospin component T_3 possible. Additionally, the heavy neutrino doesn't contribute to this factor because it is a weak isospin singulett and therefore only T_3=0 is possible.
In the following we only consider the decay into one lepton flavor a=1, taking more flavors into account simply means summing the following result with different coupling constants h_{a1} for each lepton flavor. This results in

$$\begin{aligned}
|\mathcal{M}|^2 &= \sum_{s_N, s_\ell} \bar{u}_\ell(p_\ell, s_\ell) h_{11} P_R u_N(p_N, s_N)\, \bar{u}_N(p_N, s_N) P_L i h^*_{11} u_\ell(p_\ell, s_\ell) = \\
&= |h_{11}|^2 \sum_{s_N, s_\ell} \bar{u}_\ell(p_\ell, s_\ell)_\nu\, P_R^{\nu\mu}\, u_N(p_N, s_N)_\mu\, \bar{u}_N(p_N, s_N)_\alpha\, P_L^{\alpha\beta}\, u_\ell(p_\ell, s_\ell)_\beta = \\
&= |h_{11}|^2 \sum_{s_\ell} \bar{u}_\ell(p_\ell, s_\ell)_\nu\, P_R^{\nu\mu}\, (\not{p}_N + M_N)_{\mu\alpha}\, P_L^{\alpha\beta}\, u_\ell(p_\ell, s_\ell)_\beta = \\
&= |h_{11}|^2 P_R^{\nu\mu}\, (\not{p}_N + M_N)_{\mu\alpha}\, P_L^{\alpha\beta}\, (\not{p}_\ell + M_\ell)_{\beta\nu} = \\
&= |h_{11}|^2\, \mathrm{Tr}\left[P_R\, (\not{p}_N + M_N)\, P_L\, \not{p}_\ell\right]
\end{aligned}$$

with ν, μ, α, β being spinor indices.
The way this trace was obtained is commonly known as the Casimir trick, which uses the following relation

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m$$

m being the mass of the particle.
Because the leptons are relativistic their mass can be neglected, what was done in the last step above.
The trace now simplifies to

$$\begin{aligned}
\mathrm{Tr}\left[P_R\, (\not{p}_N + M_N)\, P_L\, \not{p}_\ell\right] &= \frac{1}{4}\mathrm{Tr}\left[\left(1+\gamma^5\right)\, \left(\not{p}_N + M_N\right)\, \left(1-\gamma^5\right)\, \not{p}_\ell\right] = \\
&= \frac{1}{4}\left(\mathrm{Tr}\left[\gamma^\mu (p_N)_\mu \gamma^\nu (p_\ell)_\nu\right] + \mathrm{Tr}\left[M\gamma^\mu (p_\ell)_\mu\right] + \right. \\
&+ \mathrm{Tr}\left[\gamma^5\gamma^\mu (p_N)_\mu \gamma^\nu (p_\ell)_\nu\right] + \mathrm{Tr}\left[M\gamma^5\gamma^\mu (p_\ell)_\mu\right] - \\
&- \mathrm{Tr}\left[\gamma^\mu (p_N)_\mu \gamma^5\gamma^\nu (p_\ell)_\nu\right] - \mathrm{Tr}\left[M\gamma^5\gamma^\mu (p_\ell)_\mu\right] - \\
&\left. - \mathrm{Tr}\left[\gamma^5\gamma^\mu (p_N)_\mu \gamma^5\gamma^\nu (p_\ell)_\nu\right] - \mathrm{Tr}\left[M\gamma^5\gamma^5\gamma^\mu (p_\ell)_\mu\right]\right) = \\
&= \frac{1}{4}\mathrm{Tr}\left[\gamma^\mu (p_N)_\mu \gamma^\nu (p_\ell)_\nu\right] - \mathrm{Tr}\left[\gamma^5\gamma^\mu (p_N)_\mu \gamma^5\gamma^\nu (p_\ell)_\nu\right] = \\
&= \frac{1}{2}\mathrm{Tr}\left[\gamma^\mu (p_N)_\mu \gamma^\nu (p_\ell)_\nu\right] = \frac{1}{2}\mathrm{Tr}\left[\not{p}_N \not{p}_\ell\right]
\end{aligned}$$

All of the terms, aside from the first and last one, vanish becaus of the following relations

$$\mathrm{Tr}\left[\gamma^\mu\right] = 0 \qquad \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = 0 \qquad \mathrm{Tr}\left[\text{odd number of } \gamma\text{'s}\right] = 0$$

The final relation necessary to finally evaluate the trace is

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}$$

and the Minkowski metric $\eta^{\mu\nu}$. Using this one gets the final result for the squared matrix element

$$|\mathcal{M}|^2 = \frac{1}{2}|h_{11}|^2 \mathrm{Tr}\left[\not{p}_N \not{p}_\ell\right] = 2|h_{11}|^2 (p_N)_\mu (p_\ell)_\nu \eta^{\mu\nu} = 2|h_{11}|^2 p_N p_\ell$$

In order to rewrite the product of the 4-momentum vectors the decay will be treated in the center of mass system, meaning $p_N = (M_N, \vec{0})$. Additionally, treating the leptons as relativistic particles, so $p_\ell = (|\vec{p}_\ell|, \vec{p}_\ell)$, results in

$$|\mathcal{M}|^2 = 2|h_{11}|^2 p_N p_\ell = 2|h_{11}|^2 M_N |\vec{p}_\ell|$$

But due to the neutrino not only decaying into leptons but also into anti-leptons this is not the whole contribution to the tree level decay rate. However, looking at figure A.3, the corresponding matrix element looks exactly the the same and therefore would contribute the same term to the decay rate. This result is not surprising, remembering that CP violation only arises through interference of tree level and one-loop diagrams, thus $\Gamma(N \to \bar{\phi}\ell) = \Gamma(N \to \phi\bar{\ell})$. Plugging both these contribution back into the relation for the total decay rate results in

$$\begin{aligned}
\Gamma &= \frac{1}{2M_N}\int \frac{d^3p_\ell}{(2\pi)^3 2E_\ell}\frac{d^3p_\phi}{(2\pi)^3 2E_\phi}(2\pi)^4 4|h_{11}|^2 M_N|\vec{p}_\ell|\delta^4\left(p_\ell + p_\phi - p_N\right) = \\
&= \frac{1}{8\pi^2}\int \frac{d^3p_\ell}{E_\ell}\frac{d^3p_\phi}{E_\phi}|h_{11}|^2 M_N |\vec{p}_\ell|\delta\left(|\vec{p}_\ell| + |\vec{p}_\phi| - M_N\right)\delta^3\left(\vec{p}_\ell + \vec{p}_\phi\right) = \\
&= \frac{|h_{11}|^2}{8\pi^2}\int \frac{d^3p_\ell}{E_\ell}\frac{1}{E_\phi}|\vec{p}_\ell|\delta\left(2|\vec{p}_\ell| - M_N\right) = \\
&= \frac{|h_{11}|^2}{8\pi^2}\int \frac{d^3p_\ell}{p_\ell}\delta\left(2|\vec{p}_\ell| - M_N\right) =
\end{aligned}$$

To get to the third line the 3-momentum delta function was used, resulting in $\vec{p}_N = -\vec{p}_\ell$. Also, since the Higgs particles and leptons are relativistic their energies can be replaced by the absolute value of their momentum. To solve the last remaining integral, one has to use the final delta function.

$$\begin{aligned}
\Gamma &= \frac{|h_{11}|^2}{8\pi^2}\int \frac{d^3p_\ell}{p_\ell}\delta\left(2|\vec{p}_\ell| - M_N\right) = \\
&= \frac{|h_{11}|^2}{2\pi}\int_0^\infty dp_\ell\, p_\ell \delta\left(2|\vec{p}_\ell| - M_N\right) = \\
&= \frac{|h_{11}|^2}{2\pi}\int_0^\infty \frac{dx}{4}\, x\delta\left(x - M_N\right) = \\
&= \frac{|h_{11}|^2}{8\pi}M_N
\end{aligned}$$

B. Appendix B

B.1. The CP-Violation parameter

C. Appendix C

C.1. Integrating the rate equation over phase space

In order to be able to determine Γ_N it is usefull to perform the phase space integral over following equation.

$$\left(\frac{d}{dt} + 3H\right) n_N = -\Gamma_N \left(n_N - n_N^{eq}\right)$$

Comparing this to 4.3 one sees that the second term on the right-hand side has already been omitted since it is, as stated above, small enough to be safely neglected.
Now perfomring the phase space integral results in

$$\int \frac{d^3p}{(2\pi)^3} \left(\frac{d}{dt} + 3H\right) f_N = -\int \frac{d^3p}{(2\pi)^3} \Gamma_N \left(f_N - f_N^{eq}\right)$$
$$\int dp \left(\frac{d}{dt} + 3H\right) f_N p^2 = -\int dp\, \Gamma_N \left(f_N - f_N^{eq}\right)^2$$

Using the following partial integration to evaluate the left-hand side

$$\int dp\, p^3 \frac{\partial f_N}{\partial p} = \left[p^3 f_N\right]_0^\infty - 3\int dp\, p^2 f_N dp = -3\int p^2 f_N$$

$$3H \int dp\, f_N p^2 = -H \int dp\, p^3 \frac{\partial f_N}{\partial p}$$

it follows that

$$\int dp\, \left(\partial_t - Hp\partial_p\right) f_N p^2 = -\int dp\, \Gamma_N \left(f_N - f_N^{eq}\right) p^2$$
$$\Rightarrow \left(\partial_t - Hp\partial_p\right) f_N = \Gamma_N \left(f_N^{eq} - f_N\right)$$

C.2. Detailed calculation of Γ_{B-L}

First the rather simple derivation of 4.9 shall be displayed here. For the product of two Boltzman factors one has

$$f_\ell^{eq} f_\phi^{eq} = e^{-\frac{E_\ell - \mu_\ell}{T}} e^{-\frac{E_\phi - \mu_\phi}{T}} = e^{-\frac{E_N - \mu_\ell - \mu_\phi}{T}} = e^{-\frac{E_N}{T}} e^{\frac{\mu_\ell + \mu_\phi}{T}}$$

where the relation $\mathrm{E}_\ell + \mathrm{E}_\phi = \mathrm{E_N}$ was used.
Expanding this up to first order in the chemical potential results in

$$f_\ell f_\phi \simeq e^{-\frac{E_N}{T}} \left(1 + \frac{\mu_\ell + \mu_\phi}{T}\right)$$

Finally putting this togehter for leptons and Higgs particles and using the relation $\mu_X = -\mu_{\bar{X}}$ for the chemical potentials of a particle X and its anti particle X yields the result presented above.

$$f_l f_\phi - f_{\bar{l}} f_{\bar{\phi}} \simeq e^{-\frac{E_N}{T}} \left(1 + \frac{\mu_l + \mu_\phi}{T} - 1 - \frac{\mu_{\bar{l}} + \mu_{\bar{\phi}}}{T}\right) = 2e^{-\frac{E_N}{T}} \frac{\mu_l + \mu_\phi}{T}$$

Now, as already explained, in order to relate the chemical potentials to the density $n_{\text{B-L}}$ one has to expand the left-hand side of 4.10 and 4.11 up to first order in the chemical potentials while using Fermi-Dirac and Bose-Einstein distributions instead of Boltzmann statistics.

$$\begin{aligned}
n_\ell - n_{\bar\ell} &= g_\ell \int \frac{d^3p}{(2\pi)^3} f_\ell - f_{\bar\ell} = \\
&= g_\ell \frac{1}{(2\pi)^3} \int dp\, dcos\theta\, d\phi \left(f_\ell - f_{\bar\ell}\right) p^2 = \\
&= g_\ell \frac{1}{2\pi^2} \int dp \left(f_\ell\left(\mu_\ell = 0\right) + \frac{e^{E/T}}{\left(e^{E/T}+1\right)^2 T}\mu_\ell - f_{\bar\ell}\left(\mu_{\bar l} = 0\right) - \frac{e^{E/T}}{\left(e^{E/T}+1\right)^2 T}\mu_{\bar\ell}\right) p^2 = \\
&= g_\ell \frac{1}{2\pi^2} \int dp\, \frac{2e^{E/T}}{\left(e^{E/T}+1\right)^2 T}\mu_\ell p^2 = \\
&= g_\ell \frac{\mu_\ell}{\pi^2 T} \int_0^\infty dE\, \frac{2e^{E/T}}{\left(e^{E/T}+1\right)^2} E^2 = \\
&= \frac{\mu_\ell T^2}{3}
\end{aligned}$$

Analogous calculation for the Higgs yields:

$$n_\phi - n_{\bar\phi} = \frac{2\mu_\phi T^2}{3}$$

Here the degrees of freedom $g_\ell = 2 = g_\phi$ were used for leptons as well as for the Higgs. Solving 4.10 and 4.11 for the chemical potentials results in

$$\begin{aligned}
\mu_\ell &= \frac{3c_\ell}{T^2} n_{B-L} \\
\mu_\phi &= \frac{3c_\phi}{2T^2} n_{B-L}
\end{aligned}$$

Plugging all these results into 4.8 one gets the following relation

$$\begin{aligned}
\Gamma_{B-L} &= \frac{3}{8\pi^4}\left(c_\ell + \frac{c_\phi}{2}\right)\frac{M_N}{T^3}\Gamma_0 \int \prod_{a=N,\ell,\phi} \frac{d^3p_a}{E_a}\delta^4\left(p_\ell + p_\phi - p_N\right) e^{-\frac{E_N}{T}} = \\
&= \frac{3}{8\pi^4}\left(c_\ell + \frac{c_\phi}{2}\right)\frac{M_N}{T^3}\Gamma_0 \int \frac{d^3p_N}{E_N} e^{-\frac{E_N}{T}} \left(\int \frac{d^3p_\ell}{E_\ell}\frac{d^3p_\phi}{E_\phi}\delta^4\left(p_\ell + p_\phi - p_N\right)\right)
\end{aligned}$$

For the sake of minimizing the writing of redundant coefficients we first will evaluate the two body decay phase space in the parentheses above. For simplification the frame of reference used will be the rest frame of the neutrino $\vec{p}_N = 0$.

$$\begin{aligned}
\int \frac{d^3p_\ell}{E_\ell}\frac{d^3p_\phi}{E_\phi}\delta^4\left(p_\ell + p_\phi - p_N\right) &= \int \frac{d^3p_\ell}{E_\ell}\frac{d^3p_\phi}{E_\phi}\delta\left(E_\ell + E_\phi - M_N\right)\delta^3\left(\vec{p}_\ell + \vec{p}_\phi\right) = \\
&= \int \frac{d^3p}{E_\ell}\frac{1}{E_\phi}\delta\left(E_\ell + E_\phi - M_N\right)
\end{aligned}$$

From the first to the second line using the delta distribution for the particle momenta yields $\vec{p} \equiv \vec{p}_\ell = -\vec{p}_\phi$. Also by introducing $\mathrm{E} \equiv E_\ell + E_\phi$ and using spherical coordinates and the following change of integration variables

$$\begin{aligned} E = E_\ell + E_\phi = \sqrt{M_\ell + p^2} + \sqrt{M_\phi + p^2} \Longrightarrow \frac{dE}{dp} = \frac{p}{E_\ell} + \frac{p}{E_\phi} \\ \Longrightarrow dp = \left(\frac{p}{E_l} + \frac{p}{E_\phi}\right)^{-1} dE = \frac{E_\ell E_\phi}{pE_\ell + pE_\phi} dE = \frac{E_\ell E_\phi}{pE} dE \end{aligned}$$

one gets

$$4\pi \int dE \; \frac{p^2}{pE} \delta\left(E - M_N\right) = 4\pi \int dE \; \frac{p}{E} \delta\left(E - M_N\right) = 4\pi \frac{p}{M_N} = 2\pi \frac{M_N}{M_N} = 2\pi$$

In order to evaluate the last integral one has to use the delta distribution, resulting in E=$\mathrm{M_N}$, while the last step uses the fact that leptons and Higgs are relativistic and therefore their energy is $\mathrm{E}_{\ell,\phi} \approx$ p and in addition that the energy needed for the inverse decay to be possible in the neutrino's rest frame has to be $\mathrm{E}_{\ell,\phi} \approx p = \frac{M_\mathrm{N}}{2}$. Finally plugging this into the original relation for Γ_{B-L} yields

$$\begin{aligned} \Gamma_{B-L} &= \frac{3}{8\pi^4} \; \left(c_\ell + \frac{c_\phi}{2}\right) \; \frac{M_N}{T^3} \Gamma_0 \int \frac{d^3 p_N}{E_N} e^{-\frac{E_N}{T}} \cdot 2\pi \frac{E_N}{M_N} = \\ &= \frac{3}{\pi^2} \; \left(c_\ell + \frac{c_\phi}{2}\right) \; \frac{M_N}{T^3} \Gamma_0 \int_0^\infty \frac{dp_N}{E_N} \; p_N^2 e^{-\frac{E_N}{T}} \end{aligned}$$

And with the final change of integration variables

$$x = \sqrt{\frac{p_N^2}{M_N^2} + 1} \Longrightarrow p_N = \sqrt{x^2 - 1} M_N \Longrightarrow dp_N = \frac{x}{\sqrt{x^2 - 1}} M_N dx$$

one gets

$$\begin{aligned} \Gamma_{B-L} &= \frac{3}{\pi^2} \; \left(c_\ell + \frac{c_\phi}{2}\right) \; \frac{1}{T^3} \Gamma_0 \int_1^\infty dx \; \frac{\left(x^2 - 1\right) M_N^2}{M_N \cdot x} \cdot \frac{x}{\sqrt{x^2 - 1}} \cdot M_N e^{-zx} = \\ &= \frac{3}{\pi^2} \; \left(c_\ell + \frac{c_\phi}{2}\right) z^3 \Gamma_0 \int_1^\infty dx \; \sqrt{x^2 - 1} e^{-zx} = \\ &= \frac{3}{\pi^2} \; \left(c_\ell + \frac{c_\phi}{2}\right) z^2 K_1(z) \Gamma_0 \end{aligned}$$

where in the last step the definition of the modified Bessel function of the second kind was used

$$K_n(z) = \frac{\sqrt{\pi}}{\Gamma\left(n - \frac{1}{2}\right)} \left(\frac{1}{2} z\right)^n \int_1^\infty dx \; \left(x^2 - 1\right)^{n - \frac{1}{2}} e^{-zx}$$

With $\Gamma(n)$ the gamma function as generalization of the factorial.

C.3. Obtaining relativistic corrections to the rate equations

Expanding the factor $1/\mathrm{E}_N$ on the right-hand side of 4.22 by using 4.21 up to order p^2, one gets

$$\left(\frac{\partial}{\partial t} - Hp\frac{\partial}{\partial p}\right) f_N = \Gamma_0 \left(f_N^{eq} - f_N\right) - \frac{\Gamma_0}{2}\left(\frac{p_N}{M_N}\right)^2 \left(f_N^{eq} - f_N\right)$$

Now integrating over p and using the definition of u given in 4.23 the right-hand side simplifies to

$$g_N \int \frac{d^3p}{(2\pi)^3}\left(\Gamma_0 \left(f_N^{eq} - f_N\right) - \frac{\Gamma_0}{2}\left(\frac{p_N}{M_N}\right)^2 \left(f_N^{eq} - f_N\right)\right) = \Gamma_N \left(n_N^{eq} - n_N\right) + \Gamma_{N,u}\left(u - u^{eq}\right)$$

and it can easily be seen that $\Gamma_{N,u} = \Gamma_0$
The left-hand side however can be evaluated by performing the steps done in C.1, but in reverse order.
The next thing to do is acquiring the rate equation for u and therefore one has to multiply equation 4.22 with p^2 and integrate over p. At leading order $\left(E_N = M_N\left[1 + \mathcal{O}(p^2)\right]\right)$ in p this yields

$$g_N \int \frac{d^3p}{(2\pi)^3 M_N}\left(\frac{\partial}{\partial t} - Hp^3\frac{\partial}{\partial p}\right) f_N = \Gamma_0\, g_N \int \frac{d^3p}{M_N}\left(e^{E_N/T} - f_N\right) p^2$$

Using the following partial integration

$$\begin{aligned}
-\int d^3p\, Hp^3 \frac{\partial}{\partial p} f_N &= -4\pi \int dp\, Hp^5 \frac{\partial}{\partial p} f_N = \\
&= -4\pi\left(\left[p^5 f_N\right]_0^\infty - 5\int dp\, p^4 f_N\right) = \\
&= 4\pi \int dp\, 5p^4 f_N = 5\int d^3p\, p^4 f_N
\end{aligned}$$

and again the definition of u one easily gets

$$\left(\frac{d}{dt} + 5H\right) u = \Gamma_u \left(u^{eq} - u\right)$$

with $\Gamma_u = \Gamma_0$ at leading order in p.
At last, the relativistic correction for the asymmetry rate equation has to be determined. As stated above in order for this to be done one has to only look at the first term on the right-hand side of 4.4. Faktor + Minus klären

$$\begin{aligned}
\Gamma_{B-L,N}\left(n_N - n_N^{eq}\right) &= \Gamma_{B-L,N}\, g_N \int \frac{d^3p}{(2\pi^2)^3\, 2E_N}\left(f_N - f_N^{eq}\right) \simeq \\
&\simeq \Gamma_{B-L,N}\, g_N \int \frac{d^3p}{(2\pi)^3}\frac{1}{2M_N}\left(1 - \frac{1}{2}\left(\frac{p}{M_N}\right)^2\right)\left(f_N - f_N^{eq}\right) = \\
&= \Gamma_{B-L,N}\left(n_N - n_N^{eq}\right) - \frac{1}{4}\Gamma_{B-L,u}\left(u - u^{eq}\right)
\end{aligned}$$

Where in g_N=2 was used in the last step.
Also easy to see is the following

$$\Gamma_{B-L,u} = \epsilon\Gamma_0$$

C.4. Obtaining radiative corrections to the rate equations

Starting from equation 4.22 one first expresses the number densities through the corresponding distributions and gets

$$\left(\frac{\partial}{\partial t} + 3H\right) \int \frac{d^3p}{(2\pi)^3} f_N = \Gamma_N \int \frac{d^3p}{(2\pi)^3} \left(f_N^{eq} - f_N\right) + \frac{\Gamma_{N,u}}{M_N} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2M_N} \left(f_N - f_N^{eq}\right)$$

Since the integration variables and range of integration are the same on both sides the integrals and phase space normalization factors can be dropped. Now assuming that these equations hold for $f_N \to 0$ and using 4.29 this simplifies to

$$\begin{aligned} f_N^{eq}\Gamma_0 \frac{M_N}{E_N}\left(a + \frac{p^2}{M_N^2} b + \mathcal{O}\left(\frac{p^4}{M_N^4}\right)\right) &= \Gamma_N f_N^{eq} - \Gamma_{N,u} \frac{p^2}{M_N^2} f_N^{eq} \\ \Gamma_0 \frac{M_N}{E_N}\left(a + \frac{p^2}{M_N^2} b + \mathcal{O}\left(\frac{p^4}{M_N^4}\right)\right) &= \Gamma_N - \Gamma_{N,u} \frac{p^2}{M_N^2} \end{aligned}$$

Now again, expanding the $1/E_N$ factor yields

$$\begin{aligned} \Gamma_0 \left(1 - \frac{p^2}{M_N^2}\right)\left(a + \frac{p^2}{M_N^2} b + \mathcal{O}\left(\frac{p^4}{M_N^4}\right)\right) &= \Gamma_N - \Gamma_{N,u} \frac{p^2}{M_N^2} \\ a\Gamma_0 - \frac{1}{2}(a - 2b)\Gamma_0 \frac{p^2}{M_N^2} + \mathcal{O}\left(\frac{p^4}{M_N^4}\right) &= \Gamma_N - \Gamma_{N,u} \frac{p^2}{M_N^2} \end{aligned}$$

Since equation 4.22 is a relation of order p^2 terms of higher order do not need to be taken into account.
Now simply comparing coefficients of the terms of different order in p yields the already known result

Faktor 2 klären

$$\begin{aligned} \Gamma_N &= \Gamma_u = a\Gamma_0 \\ \Gamma_{N,u} &= \frac{1}{2}(a - 2b)\Gamma_0 \end{aligned}$$

Using the same procedure on equation 4.24 results in

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + 5H\right) \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2M_N^2} f_N = \frac{\Gamma_u}{M_N} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2M_N} \left(f_N - f_N^{eq}\right) \\ &\overset{f_N \to 0}{\Longrightarrow} f_N^{eq}\Gamma_0 \frac{M_N}{E_N}\left(a + \mathcal{O}\left(\frac{p^2}{M_N^2}\right)\right) = \Gamma_u \frac{p^2}{M_N^2} f_N^{eq} \\ &\overset{E_N = M_N + \mathcal{O}(p^2)}{\Longrightarrow} \Gamma_u = a\Gamma_0 \end{aligned}$$

Because equation 4.24 is only of order p, one can drop all terms of order p^2 or higher to obtain this result

C.5. Obtaining quantum corrections

Although the Higgs particles and leptons are of high energy and can therefore be treated using Maxwell-Boltzmann statistics, it is interesting to see how using the correct quantum statistics, namely Fermi-Dirac and Bose-Einstein statistics respectively, affect previous results.
The only point where this difference between classical and quantum distributions comes into play is during the calculation of Γ_{B-L}, because this is the only time where the explicit form of

the equilibrium distributions is used, more precisely the factor $f_\ell f_\phi - f_{\bar{\ell}} f_{\bar{\phi}}$. How this factor will be corrected using the correct statistics will be shown in this section.
The product of these two distribution simply yields

$$f_\ell f_\phi = \frac{1}{\left(e^{(E_\ell-\mu_\ell)/T}+1\right)\left(e^{(E_\phi-\mu_\phi)/T}-1\right)}$$

Now using the following derivatives

$$\frac{\partial}{\partial\mu_\ell} f_\ell f_\phi = \frac{1}{\left(e^{(E_\ell-\mu_\ell)/T}+1\right)^2\left(e^{(E_\phi-\mu_\phi)/T}-1\right)} \frac{e^{(E_\ell-\mu_\ell)/T}}{T}$$
$$\frac{\partial}{\partial\mu_\phi} f_\ell f_\phi = \frac{1}{\left(e^{(E_\ell-\mu_\ell)/T}+1\right)\left(e^{(E_\phi-\mu_\phi)/T}-1\right)^2} \frac{e^{(E_\phi-\mu_\phi)/T}}{T}$$

this product can be expanded for small chemical potentials up to linear order.

$$f_\ell f_\phi = \frac{1}{\left(e^{E_\ell/T}+1\right)\left(e^{E_\phi/T}-1\right)}\left(1+\frac{e^{E_\phi/T}}{T\left(e^{E_\phi/T}-1\right)}\mu_\phi + \frac{e^{E_\ell/T}}{T\left(e^{E_\ell/T}-1\right)}\mu_\ell\right) + \mathcal{O}(\mu^2)$$

Using this and the relations for the chemical potentials form chapter C.2 one gets

$$f_\ell f_\phi - f_{\bar{\ell}} f_{\bar{\phi}} = \frac{6n_{B-L}}{T^3\left(e^{E_\ell/T}+1\right)\left(e^{E_\phi/T}-1\right)}\left(\frac{e^{E_\phi/T}}{e^{E_\phi/T}-1}\frac{c_\phi}{2} + \frac{e^{E_\ell/T}}{e^{E_\ell/T}-1}c_\ell\right)$$

where again the relation $\mu_x = \mu_{\bar{x}}$ was used.
It is also easy to see that by neglecting the ±1 terms in the denominators, so for high Higgs and lepton energies, exactly the result that is obtained by using Maxwell-Boltzmann statistics is reproduced.
Putting this back into 4.8 results in

$$\Gamma_{B-L,q} = \frac{3}{8\pi^4}\frac{M_N}{T^3}\Gamma_0 \int \frac{d^3p_N}{E_N} \cdot$$
$$\cdot\left(\int \frac{d^3p_\ell}{E_\ell}\frac{d^3p_\phi}{E_\phi}\frac{1}{\left(e^{E_\ell/T}+1\right)\left(e^{E_\phi/T}-1\right)}\left(\frac{e^{E_\phi/T}}{e^{E_\phi/T}-1}\frac{c_\phi}{2} + \frac{e^{E_\ell/T}}{e^{E_\ell/T}-1}c_\ell\right)\delta^4\left(p_\ell+p_\phi-p_N\right)\right)$$

The addition q to the subscript simply means that this is the quantum corrected rate Γ_{B-L}
First the integral in parentheses will be calculated using the 4-momentum delta function.

$$\int \frac{d^3p_\ell}{E_\ell}\frac{d^3p_\phi}{E_\phi}\frac{1}{\left(e^{E_\ell/T}+1\right)\left(e^{E_\phi/T}-1\right)}\left(\frac{e^{E_\phi/T}}{e^{E_\phi/T}-1}\frac{c_\phi}{2} + \frac{e^{E_\ell/T}}{e^{E_\ell/T}-1}c_\ell\right)\delta^4\left(p_\ell+p_\phi-p_N\right) =$$
$$= \int \frac{d^3p_\ell}{E_\ell}\frac{1}{E_\phi}\frac{1}{\left(e^{E_\ell/T}+1\right)\left(e^{E_\phi/T}-1\right)}\left(\frac{e^{E_\phi/T}}{e^{E_\phi/T}-1}\frac{c_\phi}{2} + \frac{e^{E_\ell/T}}{e^{E_\ell/T}-1}c_\ell\right)\delta\left(E_\ell+E_\phi-M_N\right) =$$
$$= \frac{1}{e^{E_N/T}-1}\int \frac{d^3p_\ell}{p_\ell}\left(\frac{e^{p_\ell/T}}{e^{p_\ell/T}-1}\frac{c_\phi}{2} + \frac{e^{p_\ell/T}}{e^{p_\ell/T}-1}c_\ell\right)\delta\left(2p_\ell-M_N\right)$$

The fact that Higgs particles and leptons are relativistic was used and therefore their masses can be neglected. By utilizing the remaining delta function this integral can be calculated as follows

$$\begin{aligned}
&\frac{1}{e^{E_N/T}-1}\int\frac{d^3p_\ell}{p_\ell}\left(\frac{e^{p_\ell/T}}{e^{p_\ell/T}-1}\,\frac{c_\phi}{2}+\frac{e^{p_\ell/T}}{e^{p_\ell/T}-1}c_\ell\right)\delta\left(2p_\ell-M_N\right)=\\
&=\frac{4\pi}{e^{E_N/T}-1}\int_0^\infty dp\,\left(\frac{e^{p_\ell/T}}{e^{p_\ell/T}-1}\,\frac{c_\phi}{2}+\frac{e^{p_\ell/T}}{e^{p_\ell/T}-1}c_\ell\right)\delta\left(2p_\ell-M_N\right)=\\
&=\frac{2\pi}{e^{E_N/T}-1}\int_0^\infty dx\,\left(\frac{e^{x/2T}}{e^{x/2T}-1}\,\frac{c_\phi}{2}+\frac{e^{x/2T}}{e^{x/2T}-1}c_\ell\right)\delta\left(x-M_N\right)=\\
&=\frac{2\pi}{e^{E_N/T}-1}\left(\frac{e^{M_N/2T}}{e^{M_N/2T}-1}\,\frac{c_\phi}{2}+\frac{e^{M_N/2T}}{e^{M_N/2T}-1}c_\ell\right)=\\
&=\frac{2\pi}{e^{E_N/T}-1}\left(\frac{e^{z/2}}{e^{z/2}-1}\,\frac{c_\phi}{2}+\frac{e^{z/2}}{e^{z/2}-1}c_\ell\right)
\end{aligned}$$

Plugging this back into the original relation for $\Gamma_{B-L,q}$ results in

$$\Gamma_{B-L,q}=\frac{3}{4\pi^2}\,\frac{M_N}{T^3}\left(\frac{e^{z/2}}{e^{z/2}-1}\,\frac{c_\phi}{2}+\frac{e^{z/2}}{e^{z/2}-1}c_\ell\right)\Gamma_0\int\frac{d^3p_N}{E_N}\frac{1}{e^{E_N/T}-1}$$

The fact that a term -1 arises in the denominator in the integral means that this integral cannot be evaluated analytically or even expressed in terms of some known functions as the Bessel function in the case of the uncorrected rate Γ_{B-L}. However, using the same substitution as for calculating Γ_{B-L} we get the final result for the corrections due to the correct quantum statistics

$$\Gamma_{B-L,q}=\frac{3}{\pi^2}\,z^3\left(\frac{e^{z/2}}{e^{z/2}-1}\,\frac{c_\phi}{2}+\frac{e^{z/2}}{e^{z/2}-1}c_\ell\right)I(z)\Gamma_0$$

with

$$I(z):=\int\frac{d^3p_N}{E_N}\frac{1}{e^{E_N/T}-1}=4\pi M_N^2\int_1^\infty\frac{\sqrt{x^2-1}}{e^{zx}-1}$$

Using this result one can easily see that for high z, meaning for an high neutrino mass M_N and therefore high Higgs and lepton momenta, $\Gamma_{B-L,q}$ changes over to Γ_{B-L}, where classical statistics were used.

$$\Gamma_{B-L,q}\overset{z\to\infty}{\longrightarrow}\Gamma_{B-L}$$

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