Majoranized Feynman rules

R. Kleiss[1], I. Malamos[2] and G. v.d. Oord[3]
IMAPP, FNWI
Radboud University, Nijmegen, the Netherlands

Abstract

We point out that the compact Feynman rules for Majorana fermions proposed by Denner *et al.* are in fact a convention for the complex phases of (anti)spinors, valid for both Majorana and Dirac fermions. We establish the relation of this phase convention with that common in the use of spinor techniques.

In [1], compact Feynman rules for Majorana fermions have been proposed that allow simple computations in for instance supersymmetric theories. These are based on a distinction between the *fermion number flow* along a fermion line, always taken to follow the direction of the arrow on the fermion line, and an arbitrarily assigned *fermion flow* that can be taken in the direction of the arrow or against it. Majorana fermions, lacking a definite fermion number, of course have only a fermion flow. The assignment of (anti)spinors for external fermions in any process, as well as the fermion propagators, are then based on the fermion flow. For instance, an incoming electron with momentum p is assigned the spinor $u(p)$ if the fermion flow follows the fermion number flow (into the diagrams) ; alternatively, it is assigned $\overline{v}(p)$ if the fermion flow is chosen to be opposite to the fermion number flow. The Feynman rules for the various vertices also pick up some minus signs depending on the relative orientation of fermion flow and fermion number flow. It is obvious that the possibility of formulating such Majoranized Feynman rules must be independent of whether the theory actually contains Majorana fermions or only Dirac fermions.

The method described in [1] is based on the use of the charge conjugation matrix C to relate u and v :

$$u = C\,(\overline{v})^T \quad , \tag{1}$$

where ^{T} denotes transposition. Now, two remarks are in order. In the first

[1] R.Kleiss@science.ru.nl
[2] J.Malamos@science.ru.nl
[3] vdoord@nikhef.nl

place, the *only* properties that Dirac matrices have to obey are

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu} \quad , \quad \left(\gamma^0\right)^\dagger = \gamma^0 \quad , \quad \left(\gamma^k\right)^\dagger = -\gamma^k \ \ (k = 1, 2, 3) \ \ , \tag{2}$$

and any scattering amplitude should be (up to an overall phase) be independent of the particular representation chosen for the Dirac matrices : in contrast, the precise form of C in terms of the γ^μ depends sensitively on the particular representation. This lack of elegance leads us to state that *any* result for scattering amplitudes obtained by invoking the charge conjugation matrix C must also be provable without any referral to C at all[1]. In the second place, the Dirac spinors are defined by their projection operators, which we take to be[2]

$$\begin{aligned} & u(p,s)\overline{u}(p,s) = \Pi(m,p,s) \quad , \quad v(p,s)\overline{v}(p,s) = \Pi(-m,p,s) \quad , \\ & \Pi(m,p,s) = \frac{1}{2}\left(\not{p} + m\right)\left(1 + \gamma^5 \not{s}\right) \quad . \end{aligned} \tag{3}$$

The distinction between particle and antiparticle is simply in the sign of m. The only thing left undetermined by these definitions is the complex phase of the spinors. If an incoming electron is always written as u, this overall complex phase is unimportant ; but as soon as in some other diagrams the incoming electron is also written as $\overline{v}$, the phase can no longer be ignored. The charge conjugation relation of Eq.(1) is seen to be simply a choice of complex phase. We shall investigate how this phase choice relates to the one used in the so-called spinor techniques (see, for instance, [2]). That is, we adopt two four-vectors ${k_{0,1}}^\mu$ with

$${k_0}^2 = k_0 \cdot k_1 = 0 \quad , \quad {k_1}^2 = -1 \tag{4}$$

which we use to define two basis spinors u_λ $(\lambda = \pm)$:

$$u_\lambda \overline{u}_\lambda = \frac{1}{2}\left(1 + \lambda\gamma^5\right)\not{k}_0 \quad , \quad u_+ \equiv \not{k}_1 u_- \quad . \tag{5}$$

For any massless momentum q we can then construct left- and right-handed spinors as follows :

$$u_\lambda(q) \equiv (2q\cdot k_0)^{-1/2}\, \not{q} u_{-\lambda} \quad , \quad u_\lambda(q)\overline{u}_\lambda(q) = \frac{1}{2}\left(1 + \lambda\gamma^5\right)\not{q} \quad . \tag{6}$$

Many useful properties of these conventions can be found in [2] ; here, the most important one is reversal :

$$\overline{u}_{\lambda_1}(q_1)\,\Gamma\, u_{\lambda_2}(q_2) = \lambda_1\lambda_2\; \overline{u}_{-\lambda_2}(q_2)\,\Gamma^R\, u_{-\lambda_1}(q_1) \quad , \tag{7}$$

where Γ is any string of Dirac matrices, and the superscript ^{R} denotes reversal of the order of all Dirac matrices in Γ.

[1] Furry's theorem is an example : in [3] a proof without C is given.

[2] There is some arbitrariness in how the spin vector s ought to transform if a spinor u is transformed into an antispinor v. This depends on whether one takes s to refer to the particle's angular momentum or to its magnetic momentum. We shall simply take s to be defined from Eq.(3).

We can now define spinors for massive fermions as well. Let p^μ be the momentum, and s^μ the spin vector of such a fermion. Its mass is $|m|$, with the convention that m is positive for particles and negative for antiparticles. Momentum and spin obey

$$p^2 = m^2 \ \ , \ \ p \cdot s = 0 \ \ , \ \ s^2 = -1 \quad \Rightarrow \quad (p \pm ms)^2 = 0 \ \ . \tag{8}$$

The corresponding spinor can be defined by

$$u(\lambda; p, m, s) = (4k_0 \cdot (p - \lambda ms))^{-1/2} \ \left(\not{p} + m\right)\left(1 + \gamma^5 \not{s}\right) \ u_\lambda \ \ ; \tag{9}$$

some simple Dirac algebra shows that this is equivalent to

$$u(\lambda; p, m, s) \equiv \frac{1}{\sqrt{2}}\left(1 + \frac{1}{m}\not{p}\right) \ u_{-\lambda}(p - \lambda ms) \ \ . \tag{10}$$

It is easy to check that

$$u(\lambda; p, m, s)\overline{u}(\lambda; p, m, s) = \Pi(m, p, s) \ \ , \tag{11}$$

as required. There are thus two different phase choices for each massive fermion ; for massless fermions of definite helicity there is only one nonsingular convention[3], but at any rate for massless fermions the spinor defintion (6) is adequate.

We can now investigate the effect of reversing the fermion flow in a fermionic current. Let Γ be a basis element of the Clifford algebra, that is,

$$\Gamma \quad = \quad 1 \ \ , \ \ \gamma^\mu \ \ , \ \ \sigma^{\mu\nu} \ \ , \ \ \gamma^5\gamma^\mu \ \ , \ \ \gamma^5 \ \ . \tag{12}$$

We write for the current and its flow-reversed form :

$$\begin{aligned} J(\Gamma) &= \overline{u}(\lambda_1; p_1, m_1, s_1)\,\Gamma\, u(\lambda_2; p_2, m_2, s_2) \ \ , \\ J(\Gamma)^F &= \lambda_1\lambda_2 \ \ \overline{u}(-\lambda_2; p_2, -m_2, s_2)\,\Gamma\, u(-\lambda_1; p_1, -m_1, s_1) \ \ . \end{aligned} \tag{13}$$

Note that J^F is written using Γ and *not* Γ^R. Using the reversal identity of Eq.(7) and the spinor definition of Eq.(10), it is simple to obtain relations between J and J^F. As an example, we consider $\Gamma = 1$ with $\lambda_1 = \lambda_2 = \lambda$:

$$\begin{aligned} J(1) &= \overline{u}(\lambda; p_1, m_1, s_1)\, u(\lambda; p_2, m_2, s_2) \\ &= N\, \overline{u}_{-\lambda}(p_1 - \lambda m_1 s_1)\left(1 + \frac{1}{m_1}\not{p}_1\right)\left(1 + \frac{1}{m_2}\not{p}_2\right) u_{-\lambda}(p_2 - \lambda m_2 s_2) \\ &= N\, \overline{u}_{-\lambda}(p_1 - \lambda m_1 s_1)\left(\frac{1}{m_1}\not{p}_1 + \frac{1}{m_2}\not{p}_2\right) u_{-\lambda}(p_2 - \lambda m_2 s_2) \\ &= N\, \overline{u}_{\lambda}(p_2 - \lambda m_2 s_2)\left(\frac{1}{m_1}\not{p}_1 + \frac{1}{m_2}\not{p}_2\right) u_{\lambda}(p_1 - \lambda m_1 s_1) \\ &= -\, N\, \overline{u}_{\lambda}(p_2 - \lambda m_2 s_2)\left(1 - \frac{1}{m_2}\not{p}_2\right)\left(1 - \frac{1}{m_1}\not{p}_1\right) u_{\lambda}(p_1 - \lambda m_1 s_1) \\ &= -\overline{u}(-\lambda; p_2, -m_2, s_2)\, u(-\lambda; p_1, -m_1, s_1) = -J(1)^F \ \ , \end{aligned} \tag{14}$$

[3] For a left-handed fermion, $p^\mu + ms^\mu$ approaches zero as $m \to 0$.

where

$$N = (4k_0 \cdot (p_1 - \lambda_1 m_1 s_1))^{-1/2} (4k_0 \cdot (p_2 - \lambda_2 m_2 s_2))^{-1/2} \quad . \tag{15}$$

Note that changing the sign of both λ and m preserves the value of N. The above reasoning is based on the fact that between $\overline{u}_\lambda$ and u_λ only an *odd* number of Dirac matrices survives, while between $\overline{u}_\lambda$ and $u_{-\lambda}$ only an *even* number of Dirac matrices gives a nonzero result. Moreover it must be realized that, for the basis spinors,

$$\left(u_\lambda \overline{u}_\lambda\right)^R = u_{-\lambda}\overline{u}_{-\lambda} \quad , \quad \left(u_\lambda \overline{u}_{-\lambda}\right)^R = -u_\lambda \overline{u}_{-\lambda} \quad . \tag{16}$$

Using all this, it is simple to derive the following relations :

$$\begin{array}{lcllcl} J(1) & = & -J(1)^F & J(\gamma^\mu) & = & J(\gamma^\mu)^F \\ J(\gamma^5\gamma^\mu) & = & -J(\gamma^5\gamma^\mu)^F & J(\sigma^{\mu\nu}) & = & J(\sigma^{\mu\nu})^F \\ J(\gamma^5) & = & -J(\gamma^5)^F & & & \end{array} \tag{17}$$

It must be noted that the identities (17) can also be proven directly by explicit traces, for instance for $\lambda_1 = \lambda_2$ we have

$$J(\Gamma) \quad \propto \quad \mathrm{Tr}\left(\left(1 + \lambda_1\gamma^5\right) \not{k}_0 \Pi(m_1, p_1, s_1)\, \Gamma\, \Pi(m_2, p_2, s_2)\right)$$

but proving the identities then involves using the Schouten identity[4] in many places. Now, the flow reversal implies the interchange of two fermions and ought to introduce a minus sign in the flow-reversed current. The scalar, axial and pseudoscalar currents behave 'correctly', while flow reversal must be accompanied by a sign change for vector and tensor vertices : this accords with [1]. We conclude that the correct way of implementing the effect of flow reversal for external fermions is

$$u(\lambda; p, m, s) \quad \to \quad \lambda\, \overline{u}(-\lambda; p, -m, s) \tag{18}$$

and vice versa. As to the internal Dirac structure, the rule is simply that every Dirac matrix undergoes a sign change, and the string is written in reversed order. In that case, Eq.(17) naturally arises, and also any fermionic propagator with numerator $\not{q} + m$ is transformed into $-\not{q} + m$, in agreement with the prescription that the momentum should be counted in the direction of the fermion flow. Note that applying flow reversal *twice* brings $u(\lambda; p, m, s)$ not back to itself but rather to $-u(\lambda; p, m, s)$. Since this holds for both external fermions in a fermionic current, this does not lead to any problems ; but it indicates that the charge conjugation assignment implied by the definition of Eq.(10) is non-trivial.

In closing, we want to stress that the spinor-antispinor transformation for which the charge conjugation matrix is commonly employed can also be obtained without it. Let us consider a spinor ξ for an on-shell particle, given as simply

[4] In four dimensions, $g^{\lambda\mu}\,\epsilon^{\nu\rho\alpha\beta} + g^{\lambda\nu}\,\epsilon^{\rho\alpha\beta\mu} + g^{\lambda\rho}\,\epsilon^{\alpha\beta\mu\nu} + g^{\lambda\alpha}\,\epsilon^{\beta\mu\nu\rho} + g^{\lambda\beta}\,\epsilon^{\mu\nu\rho\alpha} = 0$.

four complex numbers in a scheme where we know the vectors k_0 and k_1 that were employed to construct it using the above spinor techniques. Therefore there exist λ, m, p and s such that $\xi = u(\lambda; p, m, s)$. It is easily seen that $\bar{\xi}\xi = 2m$, $\bar{\xi}\gamma^\mu\xi = 2p^\mu$ and $\bar{\xi}\gamma^5\gamma^m u\xi = -2ms^\mu$, so that m, p and s are readily determined. Furthermore, λ can be unearthed from the fact that $u_\lambda\xi$ is always *real* while $u_{-\lambda}\xi$ is always[5] complex. We can therefore construct $u(-\lambda; p, -m, s)$ without any reference to the C matrix. Nevertheless, in practical calculations (see for instance [4]) C is probably the more computation-efficient tool.

References

[1] A. Denner, H. Eck, O. Hahn, J. Kublbeck, **Phys.Lett.B**(1992)278; **Nucl.Phys.B**387(1992)467.

[2] R. Kleiss, W.J. Stirling **Nucl.Phys.B**(1985)262.

[3] R. Kleiss, *Pictures, Paths, Particles and Processes*, lecture notes available from `http://www.hef.kun.nl/~kleiss/`

[4] R. Kleiss, G. van den Oord, *Recursive equations for Majorana currents*, arXiv:0906.0697

[5]Except in the singuklar case where either $p + ms$ or $p - ms$ happens to be parallel to k_0.