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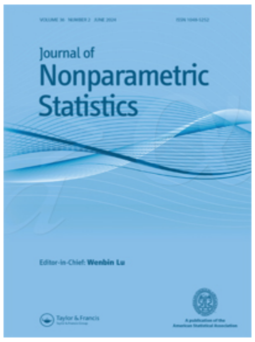
3 authors:

Brian Pidgeon
Georgia State University

Pangpang Liu
Purdue University West Lafayette

Yichuan Zhao
Georgia State University



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Jackknife empirical likelihood for the correlation coefficient with multiplicative distortion measurement errors

Brian Pidgeon[a], Pangpang Liu[b] and Yichuan Zhao[a]

[a]Department of Mathematics & Statistics, Georgia State University, Atlanta, GA, USA; [b]Mitchell E. Daniels, Jr. School of Business, Purdue University, West Lafayette, IN, USA

ABSTRACT

In this paper, we consider the estimation problem of a correlation coefficient between two unobserved variables of interest that are distorted in a multiplicative way by some unobserved confounding variable. We investigate the direct plug-in estimator of the correlation coefficient. We propose using jackknife empirical likelihood (JEL) and its variations to construct confidence intervals for the correlation coefficient based on the estimator. The proposed JEL statistic is shown to be asymptotically a standard chi-squared distribution. We compare our methods to the previous empirical likelihood (EL) techniques of Zhang et al. (2014, 'A Revisit to Correlation Analysis for Distortion Measurement Error Data', *Journal of Multivariate Analysis*, 124, 116–129) and show the JEL possesses better small sample properties. Simulation studies are conducted to examine the performance of the proposed estimator, and we also use our proposed methods to analyse the Boston housing data for illustration.

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Correlation coefficient; distortion error; jackknife empirical likelihood; multiplicative error

1. Introduction

Measurement errors are common in many areas of mathematics and statistics. Their existence is inevitable and ignoring them could lead to inflated biases in estimation procedures and inference. Confounding variables often lead to measurement errors and they commonly distort measurements in either an additive or multiplicative way. These distortions could either overestimate or underestimate the true correlation coefficient between two variables of interest and the classical estimation and inference for this quantity can become very challenging. There are several papers on the subject of measurement errors. Liang and Ren (2005) proposed generalised partially linear measurement error models. Further, Liang and Li (2009) introduced variable selection for partially linear models with measurement errors. Delaigle, Fan, and Carroll (2009) proposed a local polynomial estimator for variables with errors. Schafer (2001) proposed an EM algorithm for semiparametric likelihood analysis of linear, generalised linear and nonlinear regression

CONTACT Yichuan Zhao yichuan@gsu.edu Department of Mathematics & Statistics, Georgia State University, Atlanta, 30303 GA, USA

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models with measurement errors in explanatory variables. Wang and Hsiao (2011) considered method of moments estimation of the semiparametric nonlinear errors-in-variables models. Taupin (2001) proposed a consistent estimator of the unknown parameter in the semiparametric estimation for the nonlinear structural errors-in-variables model. Schennach (2004, 2007) introduced methods of estimation of nonlinear models with measurement errors. Li and Hsiao (2004) proposed a robust estimator of generalised linear models with measurement errors. Carroll, Ruppert, Stefanski, and Crainceanu (2006) presented various linear and nonlinear models used with data involving measurement data. Many of the currently published works on measurement errors involve errors that are additive. While work has been done studying multiplicative errors, little has been done regarding estimating the correlation coefficient between variables with this type of distortion. Sentürk and Müller (2005) introduced a moment-based estimator for the correlation coefficient between two unknown variables with multiplicative measurement errors. Cui, Guo, Lin, and Zhu (2009) proposed a direct plug-in estimator of the correlation coefficient for two unknown variables with measurement errors. More recently, Zhang, Yang, and Feng (2021) used weight least-squares methods to estimate the correlation coefficient with data that has been multiplicatively distorted. Lastly, Zhang, Feng, and Zhou (2014) employed bootstrap and empirical likelihood methods to make inference on the correlation coefficient proposed by Sentürk and Müller (2005) and Cui et al. (2009). In this paper, we propose jackknife empirical likelihood methods based on these two estimators.

Owen (1988, 1990) introduced empirical likelihood (EL), which is a nonparametric method for small samples with a superior performance over traditional parametric methods. Cheng, Zhao, and Li (2012) proposed empirical likelihood inference for the semiparametric additive isotonic regression model. Jinnah and Zhao (2017) introduced empirical likelihood inference for the bivariate survival function under univariate censoring. Huang and Zhao (2018) proposed empirical likelihood for the bivariate survival function under univariate censoring based on the influence function, and established the Wilks' Theorem. Wang, Wu, and Zhao (2019) proposed the penalised empirical likelihood for the sparse Cox regression model. Yu and Zhao (2019) introduced empirical likelihood inference for semiparametric transformation models with length-biased sampling. The method of adjusted empirical likelihood (AEL) was introduced by Chen, Variyath, and Abraham (2008). Liang, Dai, and He (2019) proposed mean empirical likelihood (MEL) by constructing a pseudo dataset through the pairwise means of the observed values for situations involving small sample sizes which is common in many areas of statistics. Furthermore, Fan, Liang, and Shen (2016) proposed penalised empirical likelihood to study asymptotic distributions of a corrected empirical log-likelihood ratio function for high-dimensional partially linear varying coefficient models where covariates contain additive distortion measurement errors. Empirical likelihood calculations can be computationally intense when working with the complicated nonlinear statistics. Thus, jackknife empirical likelihood (JEL) was introduced by Jing, Yuan, and Zhou (2009) to simplify the application of empirical likelihood in those situations. In recent years, there have been many publications involving JEL methods. Alemdjrodo and Zhao (2019) proposed jackknife empirical likelihood for comparing two correlated Gini indices. Yang and Zhao (2017) introduced the idea of smoothed jackknife empirical likelihood for the one-sample difference of quantiles. Wang, Zhao, and Gilmore (2015) used jackknife empirical likelihood methods for constructing confidence intervals for the Gini index. Lin, Li, Wang, and

Zhao (2017) proposed jackknife empirical likelihood for the error variance in linear models. The method of adjusted jackknife empirical likelihood (AJEL) was proposed by Zhao, Meng, and Yang (2015), Chen and Ning (2016). According to Zhao et al. (2015), AJEL preserves the properties of JEL, but provides slightly longer confidence intervals with better coverage probability. Most recently, Xu, Fan, and Wang (2022) proposed jackknife empirical likelihood for constructing confidence intervals for the error variance in linear models with measurement errors and missing data and Huang, Zhang, and Zhao (2023) proposed jacknife empirical likelihood for the lower-mean ratio. For recent advances in EL, refer to Lazar (2021) and Liu and Zhao (2023).

In this paper, we consider multiplicative distortion measurement errors. Both the response and the predictors are unobservable and distorted by general multiplicative effects of some observable confounding variable. Let

$$\begin{cases} \tilde{X} = \psi(U)X, \\ \tilde{Y} = \phi(U)Y, \end{cases}$$

where $(\tilde{X}, \tilde{Y})$ are the observable variables while (X, Y) are the corresponding unobservable true values of interest. $\phi(U)$ and $\psi(U)$ are unknown functions of an observed confounding variable U with identifiability condition $E[\phi(U)] = E[\psi(U)] = 1$. Different models of $\phi(\cdot)$ and $\psi(\cdot)$ have been studied under the scenario of multiplicative distortion measurement errors. Sentürk and Müller (2009) investigated linear and generalised linear models. Zhang et al. (2014) studied multiplicative regression models with distortion measurement errors and proposed three estimators for the correlation coefficient between X and Y, denoted as $\rho(X, Y)$. We will consider two of those estimators. We apply jackknife empirical likelihood to make inference on $\rho(X, Y)$. The confidence intervals based on mean and jackknife empirical likelihood are proposed. Furthermore, the confidence interval from the adjusted jackknife empirical likelihood is constructed. Our simulation studies use a bivariate normal distribution with various values of ρ to simulate X and Y. The simulation studies use sample sizes ranging from 25 to 100 to show the performance of our proposed methods from small to large sample size cases. In their original analysis, Zhang et al. (2014) only used a sample size of $n = 600$ which is unrealistic in many situations. In a real data analysis, the Boston housing data, as used in Zhang et al. (2014), is analysed using JEL methods to estimate the correlation coefficient between crime rate and housing prices. We compare with the results obtained by Zhang et al. (2014).

The organisation of the paper is as follows. In Section 2, we first review the two correlation coefficient estimators proposed by Zhang et al. (2014). In Section 3, we propose jackknife empirical likelihood, mean jackknife empirical likelihood and adjusted jackknife empirical likelihood for $\rho(X, Y)$. We also develop adjusted mean jackknife empirical likelihood (AMJEL) and mean adjusted jackknife empirical likelihood (MAJEL), which combine the methods of MEL, JEL and AJEL to improve the performance for small sample size situations. In Section 4, a simulation study is conducted using the bivariate normal distribution to examine the performance of the proposed methods. In Section 5, a real data analysis using the new methods is performed for illustrative purposes. In Section 6, we make a conclusion.

2. Review of correlation coefficient with multiplicative errors

In this section, we introduce the moment-based estimator and the direct plug-in estimator to estimate the correlation coefficient between the unobservable X and Y, $\rho(X, Y)$, as found in Zhang et al. (2014) and review their method used to make inference. We use similar notations, which are used in Zhang et al. (2014).

2.1. Moment-based estimator

We first introduce the moment-based estimator of $\rho(X, Y)$, denoted $\hat{\rho}(X, Y)$. Sentürk and Müller (2005) introduced that

$$E[\psi(U)] = 1, \quad E[\phi(U)] = 1.$$

This assumption is similar to that of the classical measurement error where $E(e) = 0$, for the model $W = X + e$. The moment-based estimator is based on $\rho(X, Y) = \rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})}/\Delta$ for some unknown constant Δ (see the Appendix for derivation), where $\rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})}$ is the correlation coefficient between $e_{\tilde{Y}U} = \tilde{Y} - E[\tilde{Y} \mid U]$ and $e_{\tilde{X}U} = \tilde{X} - E[\tilde{X} \mid U]$. This estimator $\hat{\rho}_{(e_{\tilde{Y}U}, e_{\tilde{X}U})}$ is called the residual estimator. We define the residuals of X and Y as

$$\begin{cases} \hat{e}_{i\tilde{X}U} = \tilde{X}_i - \hat{E}_h(\tilde{X}_i \mid U = U_i), \\ \hat{e}_{i\tilde{Y}U} = \tilde{Y}_i - \hat{E}_h(\tilde{Y}_i \mid U = U_i), \end{cases}$$

where

$$\hat{E}_h(\tilde{X} \mid U = u) = \frac{n^{-1}\sum_{j=1}^{n} K_h(U_j - u)\tilde{X}_j}{n^{-1}\sum_{j=1}^{n} K_h(U_j - u)},$$

$$\hat{E}_h(\tilde{Y} \mid U = u) = \frac{n^{-1}\sum_{j=1}^{n} K_h(U_j - u)\tilde{Y}_j}{n^{-1}\sum_{j=1}^{n} K_h(U_j - u)},$$

with kernel function $K_h(\cdot) = h^{-1}K(\cdot/h)$, where $h = \hat{\sigma}_U n^{-1/3}$ and $\hat{\sigma}_U$ is the sample standard deviation of U. Then, the estimator is calculated as (see page 118 in Zhang et al. 2014):

$$\hat{\rho}_{(e_{\tilde{Y}U}, e_{\tilde{X}U})} = \frac{\widehat{Cov}(e_{\tilde{Y}U}, e_{\tilde{X}U})}{\sqrt{\hat{\sigma}^2_{e_{\tilde{Y}U}}\hat{\sigma}^2_{e_{\tilde{X}U}}}}. \tag{1}$$

We define the following quantities

$$\hat{\psi}(U_i) = \frac{\hat{E}_{h_1}(\tilde{Y}_i \mid U_i)}{\bar{\tilde{Y}}}, \quad \hat{\phi}(U_i) = \frac{\hat{E}_{h_1}(\tilde{X}_i \mid U_i)}{\bar{\tilde{X}}},$$

where $\bar{\tilde{Y}} = n^{-1}\sum_{i=1}^{n} \tilde{Y}_i$, $\bar{\tilde{X}} = n^{-1}\sum_{i=1}^{n} \tilde{X}_i$. We choose another bandwidth h_1 because the under-smoothing of h_1 is needed such that the bias of $\hat{\Delta}$ is small. On Zhang et al. (2014, p. 119) define $\hat{E}[\psi(U)\phi(U)]$, $\hat{E}[\psi^2(U)]$, and $\hat{E}[\phi^2(U)]$. Thus, the estimator of Δ can be defined as

$$\hat{\Delta} = \frac{\hat{E}[\psi(U)\phi(U)]}{\sqrt{\hat{E}[\psi^2(U)]\hat{E}[\phi^2(U)]}}.$$

The estimator of $\rho(X, Y)$ can be constructed as

$$\hat{\rho}(X, Y) = \frac{\hat{\rho}_{(e_{\tilde{Y}U}, e_{\tilde{X}U})}}{\hat{\Delta}}.$$

$\hat{\rho}(X, Y)$ is used in the simulation study and real data analysis for comparison purposes.

2.2. Direct plug-in estimator

Zhang et al. (2014) propose a direct plug-in estimator for $\rho(X, Y)$. If $\Delta \approx 0$, then the moment-based estimator $\hat{\rho}(X, Y)$ cannot be implemented directly. Firstly, the kernel smoothing method in (1) is used to estimate the unknown distortion function, $\psi(\cdot)$ and $\phi(\cdot)$, with $\hat{\psi}(U_i)$ and $\hat{\phi}(U_i)$. The unobserved X and Y are calibrated such that

$$\hat{Y}_i = \frac{\tilde{Y}_i}{\hat{\phi}(U_i)}, \quad \hat{X}_i = \frac{\tilde{X}_i}{\hat{\psi}(U_i)}.$$

These calibrated values $(\hat{X}, \hat{Y})$ can be used to calculate Pearson's correlation coefficient $\rho(X, Y)$ directly. Thus, the direct plug-in estimator proposed by Zhang et al. (2014) is defined as

$$\hat{\rho}^*(X, Y) = \frac{\sum_{i=1}^n [\hat{Y}_i - \bar{\hat{Y}}][\hat{X}_i - \bar{\hat{X}}]}{\sqrt{\sum_{i=1}^n [\hat{Y}_i - \bar{\hat{Y}}]^2 \sum_{i=1}^n [\hat{X}_i - \bar{\hat{X}}]^2}},$$

where $\bar{\hat{Y}} = n^{-1} \sum_{i=1}^n \hat{Y}_i$, $\bar{\hat{X}} = n^{-1} \sum_{i=1}^n \hat{X}_i$.

3. Main results

3.1. JEL for correlation coefficients with multiplicative errors

Let $\hat{\rho}_i^*(X, Y)$ represent the estimated correlation coefficient of $\rho(X, Y)$ calculated with the i^{th} observation deleted using the direct plug-in estimator $\hat{\rho}^*(X, Y)$ for $\rho(X, Y)$. Let $\hat{V}_i$ denote the jackknife pseudo-value, which is obtained by

$$\hat{V}_i = n\hat{\rho}^*(X, Y) - (n-1)\hat{\rho}_i^*(X, Y); \quad i = 1, \ldots, n.$$

The jackknife estimator $\hat{\rho}_J(X, Y)$ is defined as

$$\hat{\rho}_J(X, Y) = n^{-1} \sum_{i=1}^n \hat{V}_i.$$

The jackknife empirical likelihood ratio at $\rho(X, Y)$ can be then defined as

$$J(\rho(X, Y)) = \sup_{\mathbf{p}=(p_1,\ldots,p_n)} \left\{ \prod_{i=1}^n np_i; p_i \geq 0; \sum_{i=1}^n p_i = 1; \sum_{i=1}^n p_i(\hat{V}_i - \rho(X, Y)) = 0 \right\}.$$

We can now calculate the −2log of the empirical likelihood ratio as

$$-2\log J(\rho(X,Y)) = 2\sum_{i=1}^{n}\log\{1+\lambda((\hat{V}_i-\rho(X,Y))\},$$

where λ is the solution of the following equation

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{V}_i-\rho(X,Y)}{1+\lambda(\hat{V}_i-\rho(X,Y))}=0. \tag{2}$$

Per Zhang et al. (2014), five conditions are needed to obtain the asymptotic results.

(C1) The density function $f_U(U)$ of the confounding variable U is bounded away from 0 and satisfies the Lipschitz condition of order 1 on $\mathcal{U}$, where $\mathcal{U}$ is a compact support set of U.
(C2) Both $\psi(U)$ and $\phi(U)$ have three bounded and continuous derivatives on $\mathcal{U}$. The absolute values of $\psi(U)$ and $\phi(U)$ are greater than a positive constant on $\mathcal{U}$. Also, $E[\psi^4(U)] < \infty$ and $E[\phi^4(U)] < \infty$.
(C3) $E[Y]$ and $E[X]$ are bounded away from 0. Moreover, $E[Y^4] < \infty$ and $E[X^4] < \infty$.
(C4) The kernel function $K(\cdot)$ is a symmetric density function about zero and has bounded derivatives. In addition, $K(\cdot)$ satisfies a Lipschitz condition on $\mathbb{R}$.
(C5) as $n \to \infty$, the bandwidths h and h_1 satisfy
  (i) (i)$h \to 0$, $nh^8 \to 0$, $\frac{nh^2}{\log^2 n} \to \infty$,
  (ii) (ii)$h_1 \to 0$, $nh_1^4 \to 0$, $\frac{nh_1^2}{\log^2 n} \to \infty$.

Condition (C1) guarantees that $f_U(U)$ is greater than 0, which ensures that the denominators within the nonparametric estimators are bounded away from 0. Condition (C2) imposes a mild smoothness condition on the functions. Condition (C3) is necessary in the study of covariate-adjusted models and ensures that we have finite fourth moments. Condition (C4) is commonly found in nonparametric statistical analysis. Condition (C5) is required for our asymptotic results (cf. Zhang et al., 2014).

We can derive the Wilks' theorem for the empirical log-likelihood ratio as follows:

Theorem 3.1: *Assume that conditions (C1)–(C5) hold. Let $\rho_0(X,Y)$ be the true value of $\rho(X,Y)$. When $n \to \infty$, we have*

$$-2\log J(\rho_0(X,Y)) \xrightarrow{\mathcal{D}} \chi_1^2.$$

Following the theorem, the JEL confidence interval for $\rho(X,Y)$ is obtained by

$$I^J_{\rho(X,Y)} = \{\rho(X,Y) : -2\log J(\rho_J(X,Y)) \le \chi^2_{1-\alpha}(1)\},$$

where $\chi^2_{1-\alpha}(1)$ is the $1-\alpha$ quantile of the χ_1^2 distribution.

3.2. Adjusted and mean JEL for correlation coefficients with multiplicative errors

Many simulation studies have shown that under-coverage issues still exist when the sample size is smaller than 25. Thus, we use adjusted jackknife empirical likelihood to improve the performance of JEL. In order to construct an adjusted jackknife empirical likelihood ratio for $\rho(X, Y)$, we first define $\hat{W}_i$ as

$$\hat{W}_i(\rho(X, Y)) = \hat{V}_i - \rho(X, Y), \quad i = 1, \dots, n, \tag{3}$$

and then add one more pseudo value $\hat{W}_{n+1}$ to $\hat{W}_i$

$$\hat{W}_{n+1}(\rho(X, Y)) = -\frac{a_n}{n} \sum_{i=1}^{n} \hat{W}_i(\rho(X, Y)), \tag{4}$$

where $a_n = \max(1, \log(n)/2)$. Thus, we can calculate the AJEL estimator by implementing the adjustment to the JEL estimator

$$\hat{\rho}_A(X, Y) = \hat{\rho}_J(X, Y) + \frac{1}{n+1} \sum_{i=1}^{n+1} \hat{W}_i(\hat{\rho}_J(X, Y)).$$

The adjusted jackknife empirical likelihood ratio at $\rho(X, Y)$ is defined as

$$J_A(\rho(X, Y)) = \sup_{\mathbf{p}=(p_1,\dots,p_{n+1})} \left\{ \prod_{i=1}^{n+1} (n+1)p_i : p_i \geq 0; \sum_{i=1}^{n+1} p_i = 1; \sum_{i=1}^{n+1} p_i \hat{W}_i(\rho(X, Y)) = 0 \right\}.$$

Hence, the adjusted jackknife empirical log-likelihood ratio at $\rho(X, Y)$ is

$$l_A(\rho(X, Y)) = -\sum_{i=1}^{n+1} \log(1 + \lambda_a \hat{W}_i(\rho(X, Y))),$$

where λ_a is a solution to the following equation,

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{\hat{W}_i(\rho(X, Y))}{1 + \lambda_a \hat{W}_i(\rho(X, Y))} = 0.$$

The -2log of adjusted jackknife empirical likelihood ratio can be obtained by

$$-2 \log J_A(\rho(X, Y)) = 2 \sum_{i=1}^{n+1} \log\{1 + \lambda_a \hat{W}_i\}.$$

The Wilks' theorem also holds for the adjusted jackknife empirical log-likelihood ratio:

Theorem 3.2: *Suppose that $\rho_0(X, Y)$ is the true value of $\rho(X, Y)$. Under the same assumptions as in Theorem 3.1, when $n \to \infty$,*

$$-2 \log J_A(\rho_0(X, Y)) \xrightarrow{\mathcal{D}} \chi_1^2.$$

Thus, following the theorem, the $100(1-\alpha)\%$ AJEL confidence interval is as follows:

$$I^A_{\rho(X,Y)} = \{\rho(X,Y) : -2\log J_A(\rho(X,Y)) \le \chi^2_{1-\alpha}(1)\}.$$

By using AJEL, the length of AJEL confidence intervals are usually longer than JEL, but the coverage probability of AJEL is better in small sample cases. To combine the methods of mean and jackknife empirical likelihood, first we let M denote the pseudo vector calculated from $\hat{W}_i$, where

$$M = \left\{ \frac{\hat{W}_i + \hat{W}_j}{2} : 1 \le i \le j \le n \right\}. \tag{5}$$

Through the equation above, the original $\hat{W}_i$ is expanded into a vector of size $N = n(n+1)/2$. Meanwhile, M maintains the same mean as $\hat{W}_i$. The expected value of the new M is close to 0. Similar to the adjusted jackknife estimator, the mean jackknife estimator can be defined as follows by adding an adjustment term to the jackknife estimator:

$$\hat{\rho}_M(X,Y) = \hat{\rho}_J(X,Y) + \frac{1}{N}\sum_{i=1}^{N} M_i(\hat{\rho}_J(X,Y)).$$

Now, we can construct the empirical likelihood based on the new vector M. The mean empirical likelihood ratio, denoted as $\hat{R}^M(\rho(X,Y))$, is defined as:

$$\hat{R}^M(\rho(X,Y)) = \max_{(p_1,\ldots,p_N)} \left\{ \prod_{i=1}^{N} Np_i; p_i \ge 0; \sum_{i=1}^{N} p_i = 1; \sum_{i=1}^{N} p_i M_i(\rho(X,Y)) = 0 \right\}.$$

The log-likelihood $\hat{l}^M(\rho(X,Y))$ can then be calculated as:

$$\begin{aligned}\hat{l}^M(\rho(X,Y)) &= \frac{-2\log \hat{R}^M(\rho(X,Y))}{n+1} \\ &= \frac{2}{n+1}\sum_{i=1}^{N} \log\{1 + \lambda M_i(\rho(X,Y))\},\end{aligned}$$

where λ is the solution of the following equation

$$N^{-1}\sum_{i=1}^{N} \frac{M_i(\rho(X,Y))}{1+\lambda M_i(\rho(X,Y))} = 0.$$

To construct the confidence interval of $\rho(X,Y)$, we obtain Wilks' theorem as follows:

Theorem 3.3: *Assume the same conditions that we did in Theorem 3.1. We have that*

$$\hat{l}^M(\rho_0(X,Y)) \xrightarrow{\mathcal{D}} \chi^2_1.$$

Then, the mean jackknife empirical likelihood confidence interval is defined as follows:

$$I^M_{\rho(X,Y)} = \{\rho(X,Y) : \hat{l}^M(\rho(X,Y)) \le \chi^2_{1-\alpha}(1)\}.$$

3.3. Adjusted mean and mean adjusted JEL for correlation coefficients with multiplicative errors

Combining the methods of MJEL and AJEL, we introduce the methods of adjusted mean jackknife empirical likelihood (AMJEL) and mean adjusted jackknife empirical likelihood (MAJEL). In AMJEL, we calculate the vector M from Equation (5) and then add one more point to the vector as used in Equation (4). For MAJEL, we first obtain the vector $\hat{W}_i$ from Equation (4) and then expand the vector using the equation similar to Equation (5).

For AMJEL, M is obtained by using Equation (5) and has $N = n(n+1)/2$ elements. We add one additional point to M using

$$M_{N+1} = -\frac{a_N}{N}\sum_{i=1}^{N} M_i,$$

where $a_N = \max(1, \log(N)/2)$. The adjusted mean jackknife estimator is then defined as

$$\hat{\rho}_{AM}(X,Y) = \hat{\rho}_M(X,Y) + \frac{1}{N+1}\sum_{i=1}^{N+1} M_i(\hat{\rho}_M(X,Y)).$$

The adjusted mean jackknife empirical likelihood ratio is then defined as:

$$J_{AM}(\rho(X,Y)) = \max\left\{\prod_{i=1}^{N+1}(N+1)p_i; p_i \geq 0; \sum_{i=1}^{N+1} p_i = 1; \sum_{i=1}^{N+1} p_i M_i(\rho(X,Y)) = 0\right\}.$$

The log-likelihood of AMJEL can be then calculated by

$$l_{AM}(\rho(X,Y)) = \frac{2}{n+1}\sum_{i=1}^{N+1}\log(1+\lambda M_i(\rho(X,Y))),$$

where λ is a solution to the following equation:

$$\frac{1}{N+1}\sum_{i=1}^{N+1}\frac{M_i(\rho(X,Y))}{1+\lambda M_i(\rho(X,Y))} = 0.$$

The Wilks' theorem holds for AMJEL as follows:

Theorem 3.4: *Under the same assumptions in Theorem 3.1, when $n \to \infty$,*

$$-2\log J_{AM}(\rho_0(X,Y)) \xrightarrow{\mathcal{D}} \chi_1^2.$$

Following the theorem, the $100(1-\alpha)\%$ AMJEL confidence interval is as follows

$$I^{AM}_{\rho(X,Y)} = \{\rho(X,Y) : -2\log J_{AM}(\rho(X,Y)) \le \chi^2_{1-\alpha}(1)\}.$$

For MAJEL, we use $\widehat{W}_i$ obtained from Equations (3) to (4) and calculate M_a as follows:

$$M_a = \left\{ \frac{\widehat{W}_i + \widehat{W}_j}{2} : 1 \le i \le j \le n+1 \right\}.$$

The expectation of M_a remains close to 0 and M_a has $N_a = \dfrac{(n+1)(n+2)}{2}$ number of values. The mean adjusted jackknife estimator is then defined as

$$\hat{\rho}_{MA}(X,Y) = \hat{\rho}_A(X,Y) + \frac{1}{N_a}\sum_{i=1}^{N_a} M_a(\hat{\rho}_A(X,Y)).$$

The mean adjusted jackknife empirical likelihood ratio is then defined as follows:

$$J_{MA}(\rho(X,y)) = \max\left\{ \prod_{i=1}^{N_a} N_a p_i; p_i \ge 0; \sum_{i=1}^{N_a} p_i = 1; \sum_{i=1}^{N_a} p_i M_{ai} = 0 \right\}.$$

The log-likelihood of MAJEL can be calculated by the following equation:

$$l^{MA}(\rho(X,Y)) = \frac{2}{n+2}\sum_{i=1}^{N_a} \log(1 + \lambda M_{ai}(\rho(X,Y))),$$

where λ is the solution of the following equation:

$$\frac{1}{N_a}\sum_{i=1}^{N_a} \frac{M_{ai}(\rho(X,Y))}{1 + \lambda M_{ai}(\rho(X,Y))} = 0.$$

We can also obtain the Wilks' theorem for MAJEL.

Theorem 3.5: *Under the same assumption in Theorem 3.1, when $n \to \infty$,*

$$-2\log J_{MA}(\rho_0(X,Y)) \xrightarrow{\mathcal{D}} \chi^2_1.$$

From the theorem, we can construct the $100(1-\alpha)\%$ MAJEL confidence interval as follows:

$$I^{MA}_{\rho(X,Y)} = \{\rho(X,Y) : -2\log J_{MA}(\rho(X,Y)) \le \chi^2_{1-\alpha}(1)\}.$$

4. Simulation study

For the simulation study, the confounding variable U is generated from Uniform(0, 1) and our true unobserved data (X, Y) is generated from a bivariate normal distribution with mean vector $\mu = (4, 4)$ and $\sigma^2_X = \sigma^2_Y = 1$. The correlation coefficient $\rho(X, Y)$ is set to be $-0.9, -0.5, 0, 0.5$ and 0.9. We present three cases for distorting functions:

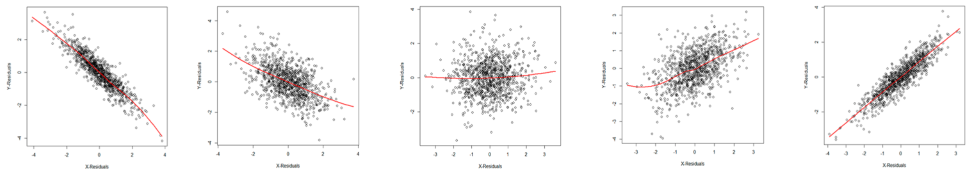

Figure 1. The local smoothing curve of $e_{\tilde{X}U}$ and $e_{\tilde{Y}U}$ against confounding variable U for $\rho = -0.9, -0.5, 0, 0.5, 0.9$ in case 1.

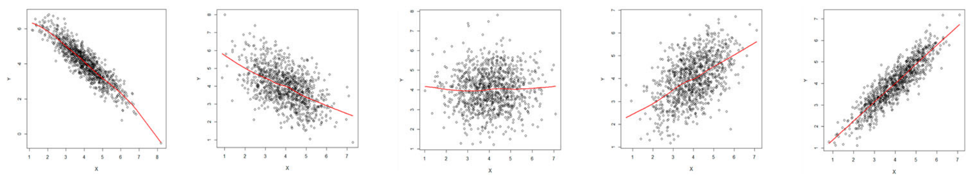

Figure 2. The local smoothing curve of X and Y against confounding variable U for $\rho = -0.9, -0.5, 0, 0.5, 0.9$ in case 1.

Case 1: The distorting functions $\psi(U) = \phi(U) = 3(U^2 + 1)/4$. $(\Delta = 1)$

Case 2: The distorting functions $\psi(U) = 3(U^2 + 1)/4, \phi(U) = 1.5 - U$. $(\Delta = 0.8790)$

In the simulations, we use the Epanechnikov kernel $K(t) = 0.75(1 - t^2)^+$ as done in Zhang et al. (2014). The bandwidth is chosen as suggested by Silverman (1986) such that $h_1 = \hat{\sigma}_U n^{-1/3}$, where $\hat{\sigma}_U$ is the sample standard deviation of U. To select bandwidth h, we use the popular leave-one-out cross validation method. Each simulation was repeated 5000 times, while Zhang et al. (2014) only did it 500 times. The sample sizes $n = 25$, 50 and 100 are chosen to show how the proposed methods work for small to large sample sizes but we also present $n = 600$ for EL and JEL to compare with the EL inference methods for $\hat{\rho}^*(X, Y)$ done by Zhang et al. (2014) since that is the only sample size that they presented in their paper. We did not perform simulations for AJEL, MJEL, AMJEL and MAJEL for $n = 600$ due to computational limitations. However, due to the large sample size, it is reasonable to assume that all four methods will see that coverage probabilities are close to the nominal level. In the simulation study, we calculate the estimator $\hat{\rho}(X, Y)$ and the direct plug-in estimator $\hat{\rho}^*(X, Y)$ and calculate the 95% confidence intervals for these estimators. EL, JEL, AJEL, MJEL, AMJEL and MAJEL are compared in terms of coverage probability (CP) and average lengths (AL) of 95% confidence intervals. Zhang et al. (2014) perform EL inference to construct 95% confidence intervals on the direct plug-in estimator $\hat{\rho}^*(X, Y)$ and they use bootstrap methods to construct 95% confidence intervals on the estimator $\hat{\rho}(X, Y)$. Our goal is to show that JEL methods outperform EL and bootstrap methods. The confidence interval results are found in Tables 1 and 2 for under both distorting scheme cases for $\hat{\rho}^*(X, Y)$. We propose JEL methods for $\hat{\rho}(X, Y)$ and the results for both distorting scheme cases are found in Tables 3 and 4. The local smoothing curve of X against Y and the local smoothing curve for $e_{\tilde{X}U}$ against $e_{\tilde{Y}U}$, the residuals used in calculating $\hat{\rho}(X, Y)$, are displayed in Figures 1–4. They are shown to highlight the linearity in the relationships indicating that the use of $\hat{\rho}(X, Y)$ and $\hat{\rho}^*(X, Y)$ may be appropriate to measure the correlation between X and Y in these inference procedures.

Table 1. Comparison of all methods using $\hat{\rho}^*(X,Y)$ in case 1.

		EL				JEL				AJEL				MJEL				AMJEL				MAJEL			
$\rho(X,Y)$	n	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP
−0.9	25	−0.932	−0.755	0.176	0.812	−1	−0.756	0.296	0.912	−1	−0.741	0.324	0.950	−1	−0.739	0.335	0.931	−1	−0.732	0.349	0.942	−1	−0.721	0.371	0.945
	50	−0.963	−0.809	0.117	0.860	−0.985	−0.813	0.172	0.945	−0.988	−0.808	0.180	0.959	−0.996	−0.806	0.191	0.961	−0.996	−0.805	0.191	0.956	−1	−0.801	0.199	0.965
	100	−0.922	−0.842	0.080	0.893	−0.950	−0.846	0.104	0.951	−0.951	−0.845	0.106	0.963	−0.954	−0.844	0.110	0.956	−0.954	−0.844	0.110	0.969	−0.955	−0.843	0.112	0.963
	600	−0.911	−0.880	0.030	0.951	−0.918	−0.885	0.034	0.949																
−0.5	25	−0.701	−0.168	0.534	0.858	−0.962	−0.053	0.910	0.901	−1.003	−0.019	0.984	0.943	−1.028	0.001	1.028	0.919	−1.044	0.010	1.054	0.929	−1.078	−0.038	1.116	0.942
	50	−0.662	−0.263	0.399	0.901	−0.782	−0.218	0.564	0.941	−0.798	−0.207	0.591	0.948	−0.809	−0.202	0.607	0.947	−0.812	−0.192	0.621	0.946	−0.829	−0.190	0.638	0.962
	100	−0.624	−0.336	0.288	0.928	−0.679	−0.321	0.358	0.945	−0.684	−0.317	0.368	0.957	−0.686	−0.309	0.377	0.952	−0.688	−0.310	0.379	0.954	−0.693	−0.310	0.384	0.951
	600	−0.556	−0.437	0.120	0.949	−0.564	−0.437	0.197	0.948																
0	25	−0.331	0.342	0.673	0.850	−0.576	0.582	1.158	0.897	−0.625	0.639	1.264	0.940	−0.663	0.666	1.329	0.923	−0.661	0.679	1.340	0.930	−0.691	0.735	1.426	0.942
	50	−0.257	0.259	0.516	0.894	−0.362	0.377	0.739	0.936	−0.390	0.382	0.773	0.947	−0.409	0.408	0.816	0.950	−0.406	0.404	0.810	0.947	−0.430	0.414	0.844	0.958
	100	−0.190	0.190	0.380	0.919	−0.242	0.234	0.476	0.948	−0.243	0.243	0.485	0.954	−0.253	0.243	0.497	0.961	−0.248	0.251	0.499	0.964	−0.257	0.255	0.511	0.959
	600	−0.080	0.080	0.160	0.946	−0.083	0.086	0.178	0.952																
0.5	25	0.181	0.715	0.534	0.854	0.051	0.961	0.909	0.894	0.024	1.022	0.998	0.924	0.023	1.065	1.042	0.930	0.013	1.040	1.054	0.929	0.039	1.080	1.120	0.930
	50	0.265	0.666	0.402	0.893	0.220	0.784	0.564	0.937	0.209	0.797	0.589	0.945	0.191	0.812	0.622	0.947	0.188	0.819	0.631	0.954	0.184	0.834	0.650	0.957
	100	0.337	0.628	0.291	0.942	0.322	0.681	0.359	0.950	0.320	0.687	0.368	0.956	0.309	0.690	0.381	0.953	0.310	0.691	0.381	0.961	0.312	0.701	0.389	0.970
	600	0.437	0.557	0.120	0.943	0.431	0.559	0.157	0.950																
0.9	25	0.792	0.948	0.157	0.853	0.782	1	0.257	0.893	0.775	1	0.272	0.917	0.764	1	0.293	0.905	0.763	1.060	0.297	0.911	0.756	1	0.315	0.927
	50	0.830	0.939	0.109	0.890	0.828	0.978	0.150	0.934	0.821	0.980	0.159	0.947	0.820	0.985	0.165	0.942	0.819	0.986	0.166	0.939	0.820	0.990	0.170	0.944
	100	0.853	0.930	0.077	0.924	0.854	0.947	0.093	0.932	0.845	0.949	0.095	0.952	0.851	0.950	0.099	0.948	0.851	0.950	0.099	0.952	0.852	0.951	0.100	0.956
	600	0.883	0.914	0.031	0.949	0.886	0.918	0.034	0.951																

Table 2. Comparison of all methods using $\hat{\rho}^*(X,Y)$ in case 2.

		EL				JEL				AJEL				MJEL				AMJEL				MAJEL			
$\rho(X,Y)$	n	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP
−0.9	25	−0.935	−0.768	0.167	0.828	−1	−0.753	0.290	0.919	−1	−0.744	0.314	0.941	−1	−0.740	0.325	0.928	−1.067	−0.735	0.332	0.941	−1	−0.726	0.354	0.950
	50	−0.929	−0.816	0.113	0.871	−0.982	−0.811	0.170	0.950	−0.989	−0.811	0.105	0.962	−0.993	−0.806	0.187	0.959	−0.992	−0.805	0.187	0.951	−0.997	−0.803	0.194	0.957
	100	−0.922	−0.843	0.079	0.892	−0.949	−0.846	0.103	0.957	−0.951	−0.846	0.105	0.962	−0.953	−0.844	0.109	0.964	−0.953	−0.843	0.110	0.959	−0.954	−0.844	0.111	0.961
	600	−0.911	−0.881	0.031	0.932	−0.916	−0.882	0.037	0.946																
−0.5	25	−0.703	−0.175	0.528	0.844	−0.955	−0.058	0.897	0.900	−0.998	−0.019	0.979	0.938	−1.025	−0.016	1.040	0.911	−1.037	−0.006	1.044	0.935	−1.067	−0.033	1.100	0.932
	50	−0.661	−0.261	0.399	0.895	−0.784	−0.224	0.560	0.934	−0.790	−0.212	0.578	0.950	−0.815	−0.198	0.617	0.954	−0.809	−0.189	0.620	0.947	−0.824	−0.187	0.638	0.952
	100	−0.624	−0.335	0.289	0.920	−0.679	−0.322	0.357	0.945	−0.683	−0.316	0.367	0.960	−0.688	−0.314	0.375	0.952	−0.688	−0.311	0.377	0.958	−0.696	−0.313	0.383	0.953
	600	−0.556	−0.436	0.120	0.954	−0.562	−0.437	0.143	0.947																
0	25	−0.335	0.335	0.670	0.860	−0.585	0.587	1.172	0.915	−0.627	0.627	1.254	0.941	−0.647	0.675	1.322	0.922	−0.068	0.675	1.359	0.933	−0.707	0.724	1.431	0.942
	50	−0.262	0.252	0.513	0.907	−0.361	0.371	0.731	0.937	−0.388	0.377	0.766	0.959	−0.399	0.413	0.812	0.953	−0.403	0.408	0.810	0.948	−0.409	0.433	0.842	0.958
	100	−0.190	0.190	0.380	0.922	−0.238	0.234	0.472	0.959	−0.236	0.247	0.483	0.954	−0.248	0.250	0.498	0.962	−0.250	0.249	0.500	0.959	−0.251	0.259	0.511	0.951
	600	−0.080	0.080	0.159	0.942	−0.091	0.077	0.174	0.953																
0.5	25	0.167	0.708	0.541	0.842	0.067	0.984	0.917	0.900	0.007	1.014	1.007	0.929	0.001	1.058	1.058	0.921	0.001	1.049	1.049	0.927	0.031	1.095	1.129	0.936
	50	0.267	0.668	0.402	0.892	0.217	0.785	0.568	0.926	0.202	0.794	0.596	0.944	0.206	0.824	0.618	0.944	0.199	0.826	0.628	0.951	0.176	0.829	0.653	0.961
	100	0.334	0.627	0.292	0.927	0.326	0.683	0.357	0.943	0.317	0.687	0.370	0.957	0.313	0.693	0.380	0.949	0.312	0.693	0.381	0.957	0.307	0.693	0.387	0.955
	600	0.436	0.556	0.120	0.941	0.444	0.570	0.135	0.950																
0.9	25	0.768	0.942	0.174	0.850	0.771	1	0.283	0.892	0.766	1	0.299	0.939	0.758	1	0.318	0.916	0.765	1	0.309	0.905	0.746	1	0.350	0.932
	50	0.821	0.936	0.115	0.919	0.828	0.981	0.154	0.931	0.821	0.984	0.163	0.946	0.819	0.990	0.171	0.936	0.822	0.991	0.169	0.935	0.812	0.994	0.182	0.949
	100	0.849	0.928	0.079	0.926	0.853	0.948	0.095	0.940	0.853	0.950	0.097	0.946	0.850	0.951	0.101	0.944	0.850	0.951	0.101	0.952	0.850	0.953	0.102	0.950
	600	0.883	0.914	0.031	0.950	0.885	0.917	0.042	0.951																

Table 3. Comparison of all methods using $\hat{\rho}(X, Y)$ in case 1.

		Bootstrap				JEL				AJEL				MJEL				AMJEL				MAJEL			
$\rho(X, Y)$	n	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP
−0.9	25	−0.992	−0.740	0.252	0.904	−1	−0.775	0.283	0.919	−1	−0.763	0.308	0.949	−1	−0.748	0.337	0.941	−1	−0.745	0.342	0.955	−1	−0.732	0.372	0.962
	50	−0.979	−0.815	0.164	0.920	−0.988	−0.823	0.164	0.933	−0.991	−0.817	0.174	0.953	−0.999	−0.817	0.182	0.942	−0.998	−0.812	0.186	0.954	−1.002	−0.809	0.192	0.944
	100	−0.958	−0.862	0.097	0.941	−0.953	−0.851	0.102	0.949	−0.955	−0.850	0.105	0.948	−0.957	−0.846	0.111	0.952	−0.958	−0.848	0.110	0.948	−0.958	−0.846	0.112	0.951
	600	−0.921	−0.886	0.035	0.951																				
−0.5	25	−0.800	−0.019	0.781	0.932	−0.981	−0.061	0.920	0.908	−1.025	−0.019	1.005	0.937	−1.053	0.700	1.072	0.921	−1.068	−0.022	1.091	0.926	−1.106	−0..056	1.112	0.928
	50	−0.721	−0.233	0.488	0.932	−0.793	−0.213	0.581	0.936	−0.809	−0.202	0.607	0.951	−0.853	−0.184	0.651	0.960	−0.828	−0.188	0.640	0.945	−0.845	−0.165	0.680	0.954
	100	−0.652	−0.314	0.338	0.952	−0.693	−0.315	0.378	0.949	−0.695	−0.309	0.386	0.954	−0.709	−0.299	0.410	0.959	−0.711	−0.305	0.406	0.943	−0.706	−0.284	0.422	0.957
	600	−0.570	−0.436	0.134	0.953																				
0	25	−0.478	0.479	0.957	0.950	−0.584	0.603	1.187	0.913	−0.638	0.637	1.276	0.934	−0.676	0.700	1.375	0.939	−0.674	0.707	1.381	0.932	−0.744	0.742	1.486	0.947
	50	−0.323	0.305	0.628	0.946	−0.376	0.389	0.765	0.941	−0.401	0.396	0.797	0.956	−0.435	0.408	0.843	0.951	−0.430	0.426	0.856	0.944	−0.444	0.444	0.888	0.966
	100	−0.209	0.224	0.433	0.944	−0.247	0.252	0.499	0.952	−0.256	0.255	0.511	0.959	−0.264	0.271	0.535	0.949	−0.260	0.275	0.535	0.953	−0.274	0.281	0.555	0.962
	600	−0.086	0.087	0.173	0.955																				
0.5	25	0.058	0.794	0.736	0.940	0.065	0.970	0.905	0.898	0.013	1.002	0.989	0.925	0.005	1.048	1.044	0.924	0.003	1.058	1.061	0.919	0.038	1.088	1.096	0.950
	50	0.234	0.711	0.477	0.944	0.226	0.801	0.575	0.931	0.210	0.811	0.601	0.948	0.191	0.826	0.635	0.952	0.199	0.839	0.640	0.937	0.167	0.842	0.675	0.961
	100	0.321	0.649	0.329	0.948	0.320	0.691	0.371	0.951	0.312	0.695	0.383	0.956	0.300	0.702	0.402	0.958	0.300	0.705	0.404	0.960	0.293	0.709	0.416	0.952
	600	0.443	0.561	0.117	0.952																				
0.9	25	0.756	0.968	0.212	0.938	0.787	1	0.251	0.888	0.785	1	0.261	0.925	0.771	1	0.286	0.911	0.772	1.053	0.281	0.904	0.764	1.064	0.300	0.925
	50	0.818	0.948	0.129	0.914	0.830	0.979	0.149	0.927	0.824	0.982	0.155	0.927	0.824	0.988	0.163	0.922	0.820	0.989	0.169	0.932	0.820	0.990	0.170	0.943
	100	0.852	0.936	0.085	0.944	0.854	0.950	0.096	0.946	0.853	0.951	0.098	0.949	0.849	0.954	0.105	0.943	0.849	0.952	0.104	0.948	0.849	0.955	0.105	0.955
	600	0.891	0.912	0.021	0.951																				

Table 4. Comparison of all methods using $\hat{\rho}(X,Y)$ in case 2.

		Bootstrap				JEL				AJEL				MJEL				AMJEL				MAJEL			
$\rho(X,Y)$	n	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP	Lower	Upper	AL	CP
−0.9	25	−0.960	−0.758	0.202	0.870	−1	−0.710	0.399	0.915	−1	−0.697	0.437	0.957	−1	−0.687	0.456	0.938	−1	−0.669	0.478	0.951	−1	−0.656	0.497	0.958
	50	−0.951	−0.818	0.133	0.904	−1	−0.775	0.252	0.948	−1	−0.772	0.265	0.964	−1	−0.760	0.281	0.965	−1	−0.761	0.282	0.970	−1	−0.758	0.288	0.959
	100	−0.933	−0.844	0.089	0.916	−0.985	−0.818	0.167	0.946	−1.134	−0.697	0.437	0.957	−0.991	−0.816	0.176	0.947	−0.990	−0.812	0.178	0.965	−0.992	−0.808	0.184	0.958
	600	−0.931	−0.882	0.049	0.935																				
−0.5	25	−0.866	−0.080	0.786	0.938	−0.966	−0.061	0.905	0.935	−0.994	−0.018	0.976	0.952	−1.043	−0.004	1.039	0.954	−1.044	−0.010	1.044	0.953	−1.075	−0.002	1.073	0.950
	50	−0.741	−0.250	0.492	0.932	−0.784	−0.225	0.558	0.954	−0.796	−0.220	0.577	0.955	−0.815	−0.197	0.617	0.960	−0.809	−0.197	0.612	0.951	−0.830	−0.201	0.630	0.967
	100	−0.663	−0.335	0.328	0.956	−0.682	−0.327	0.355	0.948	−0.687	−0.320	0.367	0.967	−0.697	−0.319	0.378	0.963	−0.689	−0.312	0.377	0.961	−0.696	−0.311	0.385	0.966
	600	−0.531	−0.433	0.098	0.955																				
0	25	−0.505	0.458	0.962	0.930	−0.574	0.565	1.139	0.933	−0.606	0.612	1.218	0.955	−0.656	0.655	1.311	0.957	−0.661	0.649	1.311	0.958	−0.707	0.684	1.391	0.961
	50	−0.302	0.311	0.613	0.950	−0.356	0.347	0.703	0.958	−0.368	0.363	0.731	0.961	−0.384	0.383	0.767	0.965	−0.383	0.390	0.773	0.956	−0.406	0.396	0.802	0.976
	100	−0.217	0.195	0.411	0.940	−0.224	0.228	0.452	0.963	−0.235	0.229	0.464	0.964	−0.239	0.232	0.472	0.960	−0.241	0.238	0.480	0.964	−0.243	0.244	0.486	0.965
	600	−0.082	0.081	0.163	0.957																				
0.5	25	0.035	0.821	0.786	0.936	0.057	0.962	0.905	0.940	0.015	0.990	0.975	0.964	0.018	1.021	1.030	0.952	0.008	1.032	1.024	0.951	0.024	1.082	1.058	0.960
	50	0.236	0.715	0.479	0.942	0.228	0.782	0.554	0.941	0.214	0.789	0.575	0.958	0.194	0.805	0.611	0.974	0.203	0.816	0.613	0.965	0.181	0.819	0.637	0.967
	100	0.319	0.657	0.338	0.949	0.322	0.678	0.356	0.961	0.301	0.679	0.378	0.962	0.314	0.691	0.377	0.970	0.314	0.688	0.374	0.967	0.309	0.697	0.389	0.960
	600	0.443	0.557	0.114	0.950																				
0.9	25	0.692	0.995	0.303	0.938	0.707	1	0.383	0.916	0.694	1	0.410	0.954	0.683	1.118	0.435	0.937	0.685	1	0.434	0.940	0.672	1	0.465	0.956
	50	0.793	0.986	0.193	0.952	0.780	1	0.228	0.948	0.774	1	0.239	0.954	0.771	1	0.250	0.954	0.769	1	0.251	0.953	0.765	1	0.266	0.956
	100	0.847	0.949	0.102	0.955	0.825	0.969	0.145	0.950	0.814	0.986	0.172	0.959	0.821	0.974	0.153	0.959	0.819	0.973	0.154	0.952	0.817	0.975	0.158	0.965
	600	0.872	0.924	0.052	0.956																				

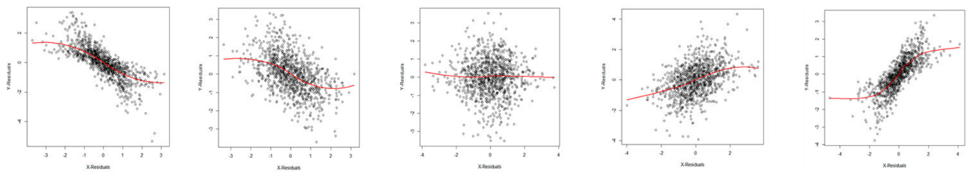

Figure 3. The local smoothing curve of $e_{\tilde{X}U}$ and $e_{\tilde{Y}U}$ against confounding variable U for $\rho = -0.9, -0.5, 0, 0.5, 0.9$ in case 2.

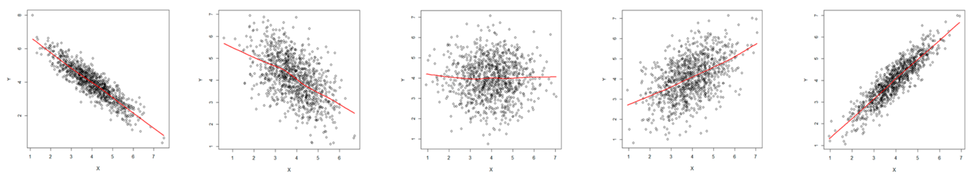

Figure 4. The local smoothing curve of X and Y against confounding variable U for $\rho = -0.9, -0.5, 0, 0.5, 0.9$ in case 2.

Replicating the EL and bootstrap methods from Zhang et al. (2014), we can see that the average lengths of all five jackknife methods are longer than those of EL and bootstrap. In addition, both EL and bootstrap methods had poor performance for smaller sample sizes. The average length of the AJEL confidence intervals is longer than that of JEL. MJEL produces longer intervals for smaller values of n. All new methods give better performance as the sample size increases for both estimators $\hat{\rho}^*(X,Y)$ and $\hat{\rho}(X,Y)$. The AMJEL and MAJEL methods have over-coverage when the sample size is greater than 50 in many of the simulations. All five jackknife methods improve the estimators of $\hat{\rho}^*(X,Y)$ and $\hat{\rho}(X,Y)$ as their coverage probabilities increase towards the nominal level over the EL and bootstrap methods. However, it is important to address that since we are estimating the correlation coefficient, values above 1 and below -1 are not possible. Many of the JEL intervals produce lower and upper bounds outside of this range when $n = 25$. Therefore, it may be important to truncate the intervals at -1 and 1. Based on the simulations, it would appear that the direct plug-in estimator $\hat{\rho}^*(X,Y)$ is the better measure of correlation since it produced confidence intervals that had overall smaller average lengths and better coverage probabilities in both cases for both small and large sample sizes. This is the same conclusion made from Zhang et al. (2014). The JEL methods greatly improved the inference over their EL methods in terms of average lengths and coverage probabilities. While we added the simulation results using JEL methods based on $\hat{\rho}(X,Y)$, the rigorous proofs of the theoretical results are not given in the Appendix. Indeed, the proofs are similar.

5. Real data analysis

In this section, we analyse the Boston housing price data introduced in Harrison and Rubinfeld (1978) and analysed by Zhang et al. (2014) using EL methods. The Boston housing data contains 506 census tracts of Boston from the 1970 census with 14 variables. The variables are defined below:

Table 5. 95% confidence intervals using $\hat{\rho}^*(X,Y)$ using lstat (left) and ptratio (right) as confounding variable.

	Estimate	Lower	Upper		Estimate	Lower	Upper
EL	−0.017	−0.152	0.200	EL	−0.335	−0.405	−0.271
JEL	−0.044	−0.329	0.242	JEL	−0.322	−0.389	−0.242
AJEL	−0.043	−0.331	0.244	AJEL	−0.317	−0.390	−0.241
MJEL	−0.043	−0.384	0.297	MJEL	−0.317	−0.390	−0.236
AMJEL	−0.043	−0.384	0.297	AMJEL	−0.317	−0.390	−0.236
MAJEL	−0.044	−0.386	0.299	MAJEL	−0.322	−0.390	−0.236

(1) *crim* – per capita crime rate by town
(2) *zn* – proportion of residential land zoned for lots over 25,000 sq.ft
(3) *indus* – proportion of non-retail business acres per town
(4) *chas* – Charles River dummy variable
(5) *nox* – nitric oxides concentrations (parts per 10 million)
(6) *rm* – average number of rooms per dwelling
(7) *age* – proportion of owner-occupied units built prior to 1940
(8) *dis* – weighted distances to five Boston employment centres
(9) *rad* – index of accessibility to radial highways
(10) *tax* – full-value property-tax rate per USD 10,000
(11) *ptratio* – Pupil-teacher ratio by town
(12) *b* – proportion of African Americans by town
(13) *lstat* – proportion of the population with lower education status
(14) *medv* – median value of owner-occupied homes in USD per 1000

Each of the first 13 variables may affect the price of a house. In this analysis, we are interested in the correlation between crime rate (CR) and median value of the home (HP). There are two potential confounding variables – *lstat* and *ptratio*. These confounding variables were both analysed by Zhang et al. (2014) and used to estimate the correlation between HP and CR using $\hat{\rho}(X,Y)$ and $\hat{\rho}^*(X,Y)$. In their analysis, Zhang et al. (2014) used EL and bootstrap methods to construct 95% confidence intervals for these two estimators. We will repeat the EL and bootstrap methods, but also perform the JEL methods to construct 95% confidence intervals and make a comparison. In their analysis, it appears that Zhang et al. (2014) removed a few outliers from their datasets before performing inference and making graphs. Since it is not clear which values were omitted from their calculations, we analyse the full dataset without removing the outliers. Our inferences are similar with small differences in estimators and confidence intervals. In this real data analysis, we investigate the estimate of Δ. For comparison, we use the notation $\hat{\Delta}_L$, $\hat{\Delta}_P$ for the estimated Δ using the two different confounding variables *lstat* and *ptratio*, respectively, The estimators are obtained as $\hat{\Delta}_L = 0.4125$ and $\hat{\Delta}_P = 0.4281$. The results are displayed in Tables 5 and 6.

Figure 5 shows the patterns of the distorting functions $\hat{\psi}(U)$ and $\hat{\phi}(U)$ using the confounding variable *lstat*. Figure 6 shows the patterns of the distorting functions $\hat{\psi}(U)$ and $\hat{\phi}(U)$ using the confounding variable *ptratio*. These four plots indicate that $\psi(U)$ and $\phi(U)$ are not constant, which indicates that both variables have an effect on HP and CR. To ensure the identifiability condition is satisfied in this real data analysis, we computed the average of $\hat{\psi}(U)$ and $\hat{\phi}(U)$ using *lstat* as the confounding variable and found them to be

Table 6. 95% confidence intervals using $\hat{\rho}(X, Y)$ using lstat (left) and ptratio (right) as confounding variable.

	Estimate	Lower	Upper		Estimate	Lower	Upper
Bootstrap	−0.415	−0.603	−0.222	Bootstrap	−0.525	−0.652	−0.323
JEL	−0.411	−0.611	−0.200	JEL	−0.510	−0.648	−0.377
AJEL	−0.406	−0.612	−0.199	AJEL	−0.504	−0.648	−0.376
MJEL	−0.406	−0.620	−0.187	MJEL	−0.504	−0.655	−0.365
AMJEL	−0.406	−0.620	−0.187	AMJEL	−0.504	−0.655	−0.365
MAJEL	−0.411	−0.621	−0.185	MAJEL	−0.510	−0.655	−0.364

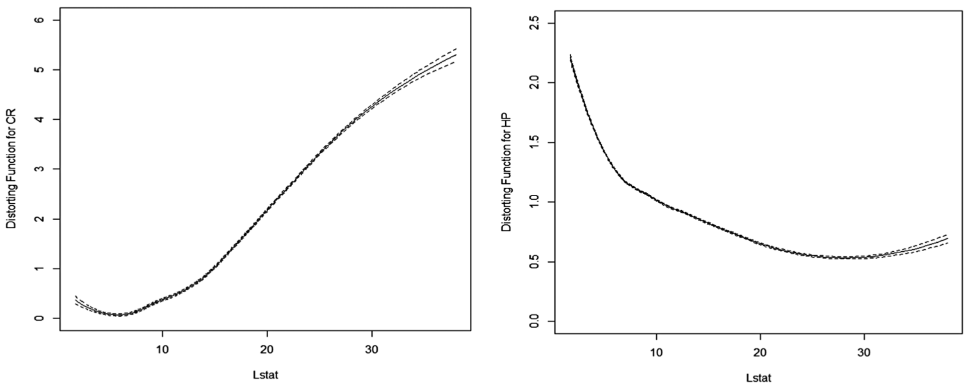

Figure 5. The estimated curve of distorting functions of housing price (HP) and crime rate (CR) against confounding variable lstat along with 95% pointwise confidence intervals.

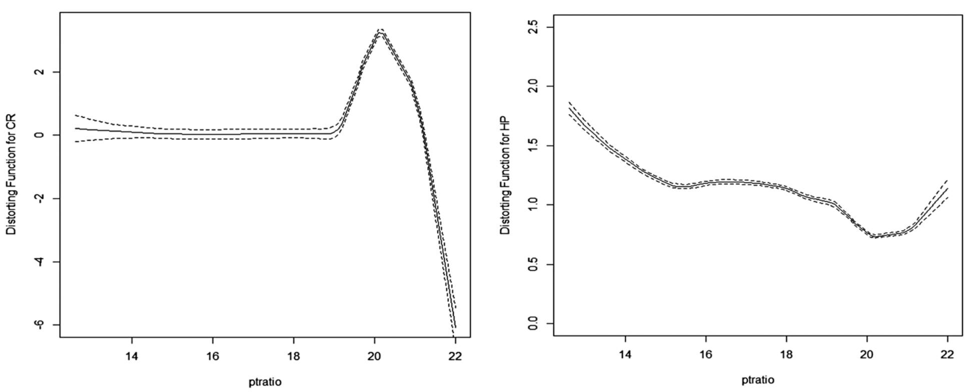

Figure 6. The estimated curve of distorting functions of housing price (HP) and crime rate (CR) against confounding variable ptratio along with 95% pointwise confidence intervals.

0.9944356 and 0.9999811, respectively. We also computed the average of $\hat{\psi}(U)$ and $\hat{\phi}(U)$ using *ptratio* as the confounding variable and found them to be 1.006414 and 0.9973814, respectively. Thus, we are confident that the identifiability condition is met. When using *lstat* as the confounding variable, EL and JEL show an uncorrelated relationship between HP and CR. This happened when Zhang et al. (2014) analysed this data set. In Figure 7, we show the local smoothing curve of predicting HP from CR using *lstat* as the confounding

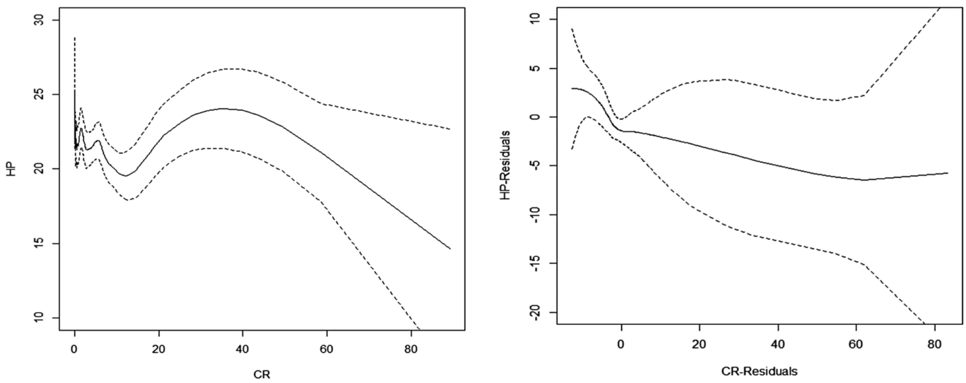

Figure 7. Confounding variable – lstat. The estimated local smoothing curve of estimated crime rate against estimated housing price with the associated 95% pointwise confidence intervals (dotted lines) (left) and the estimated local smoothing curve of the $e_{\tilde{C}RU}$ and $e_{\tilde{HP}U}$ with the associated 95% pointwise confidence intervals (dotted lines) (right).

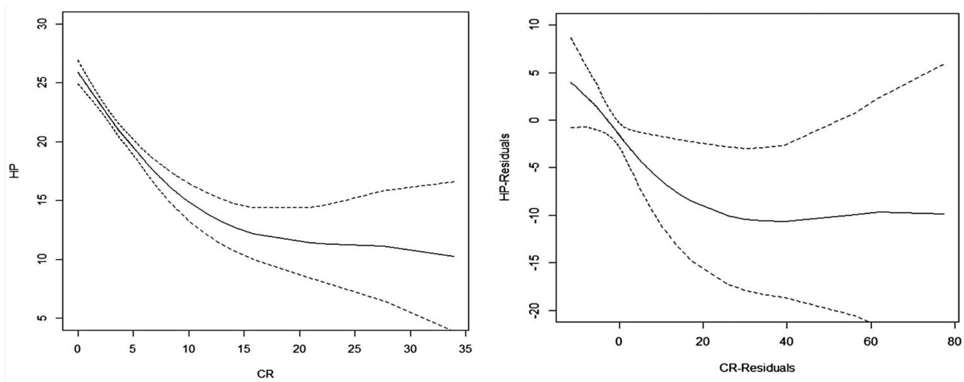

Figure 8. Confounding variable – ptratio. The estimated local smoothing curve of estimated crime rate against estimated housing price with the associated 95% pointwise confidence intervals (dotted lines) (left) and the estimated local smoothing curve of the $e_{\tilde{C}RU}$ and $e_{\tilde{HP}U}$ with the associated 95% pointwise confidence intervals (dotted lines) (right).

variable and we can see that there appears to exist a complex structure between the two variables. It is known that $\rho(X, Y)$, Pearson's correlation coefficient, is a measure of the linear correlation between the variables X and Y. The nonlinear pattern in Figure 7 highlights the fact that $\hat{\rho}^*(X, Y)$ may not be an appropriate measure to estimate the correlation and future simulation studies may need to be performed with more complex distorting schemes to study this. It is likely for this reason that the direct plug-in estimator, $\hat{\rho}^*(X, Y)$, yields a different conclusion from the moment-based estimator, $\hat{\rho}(X, Y)$. The estimated smoothing curve of $e_{\tilde{C}RU}$ against $e_{\tilde{HP}U}$ is decreasing which is likely why the confidence intervals calculated show a negative correlation between CR and HP. In Figure 8, we show the estimated smoothing curve of predicting HP from CR with *ptratio* as the confounding variable. It shows decreasing functions, which explains why all the methods produce 95% confidence intervals entirely below 0 for this confounding variable.

6. Conclusions

In this paper, we proposed five methods for the correlation coefficient with multiplicative distortion measurement errors. By the nature of JEL, MJEL and AJEL, all confidence intervals are larger than those of empirical likelihood by Zhang et al. (2014). AJEL and MJEL have longer confidence intervals than JEL. All new methods provide better coverage probability compared to the conventional empirical likelihood but the performance varies case by case. MJEL, AJEL, AMJEL and MAJEL could have over-coverage when the sample size is greater than or equal to 50. The overcoverage for AMJEL and MAJEL with the larger sample sizes needs further investigation. When applied to a real data set, the new methods obtain the same results as previously proposed EL methods but simulation studies show that JEL methods provide confidence intervals that are closer to the nominal level. None of the methods resulted in conflicting inferences but based on the simulation results, it is important to utilise JEL methods for inference. Further, more complex distorting schemes need to be studied to assess the performance of $\hat{\rho}^*(X, Y)$. For future research, jackknife empirical likelihood can be extended to measure correlation coefficients for measurement errors that are both additive and multiplicative. In addition, we will use transformed JEL and transformed AJEL to make inference for $\rho(X, Y)$ after Jing, Tsao, and Zhou (2017) proposed transformed EL methods.

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Disclosure statement

No potential conflict of interest was reported by the author(s).

References

Alemdjrodo, K., and Zhao, Y. (2019), 'Reduce the Computation in Jackknife Empirical Likelihood for Comparing Two Correlated Gini Indices', *Journal of Nonparametric Statistics*, 31, 849–866.

Carroll, R.J., Ruppert, D., Stefanski, L.A., and Crainceanu, C.M. (2006), *Nonlinear Measurement Error Models, A Modern Perspective*, London: Chapman and Hall.

Chen, Y., and Ning, W. (2016), 'Adjusted Jackknife Empirical Likelihood', arXiv:1603.04093.

Chen, J., Variyath, A., and Abraham, B. (2008), 'Adjusted Empirical Likelihood and Its Properties', *Journal of Computational and Graphical Statistics*, 17, 426–443.

Cheng, G., Zhao, Y., and Li, B. (2012), 'Empirical Likelihood Inference for the Semiparametric Additive Isotonic Regression', *Journal of Multivariate Analysis*, 112, 172–182.

Cui, X., Guo, W., Lin, L., and Zhu, L. (2009), 'Covariate-Adjusted Nonlinear Regression', *Annals of Statistics*, 37, 1839–1870.

Delaigle, A., Fan, J., and Carroll, R.J. (2009), 'A Design-Adaptive Local Polynomial Estimator for the Errors-in-Variables Problem', *Journal of the American Statistical Association*, 104, 348–359.

Fan, G., Liang, H., and Shen, Y. (2016), 'Penalized Empirical Likelihood for High-Dimensional Partially Linear Varying Coefficient Model with Measurement Errors', *Journal of Multivariate Analysis*, 147, 183–201.

Harrison, D.J., and Rubinfeld, D.L. (1978), 'Hedonic Housing Prices and Demand for Clean Air', *Journal of Environmental Economics and Management*, 5, 81–102.

Huang, L., Zhang, L., and Zhao, Y. (2023), 'Jackknife Empirical Likelihood for the Lower-Mean Ratio', *Journal of Nonparametric Statistics*, in press.
Huang, H., and Zhao, Y. (2018), 'Empirical Likelihood for the Bivariate Survival Function Under Univariate Censoring', *Journal of Statistical Planning and Inference*, 194, 32–46.
Jing, B., Tsao, M., and Zhou, W. (2017), 'Transforming the Empirical Likelihood Towards Better Accuracy', *The Canadian Journal of Statistics*, 45, 340–352.
Jing, B., Yuan, J., and Zhou, W. (2009), 'Jackknife Empirical Likelihood', *Journal of the American Statistical Association*, 104(487), 1224–1232.
Jinnah, A., and Zhao, Y. (2017), 'Empirical Likelihood Inference for the Bivariate Survival Function Under Univariate Censoring', *Communications in Statistics – Simulation and Computation*, 46(6), 4348–4355.
Lazar, N. (2021), 'A Review of Empirical Likelihood', *Annual Review of Statistics and Its Application*, 8, 329–344.
Li, T., and Hsiao, C. (2004), 'Robust Estimation of Generalized Linear Models with Measurement Errors', *Journal of Econometrics*, 118, 51–65.
Liang, W., Dai, H., and He, S. (2019), 'Mean Empirical Likelihood', *Computational Statistics and Data Analysis*, 138, 155–169.
Liang, H., and Li, R. (2009), 'Variable Selection for Partially Linear Models with Measurement Errors', *Journal of the American Statistical Association*, 104, 234–248.
Liang, H., and Ren, H. (2005), 'Generalized Partially Linear Measurement Error Models', *Journal of Computational and Graphical Statistics*, 14, 237–250.
Lin, H., Li, Z., Wang, D., and Zhao, Y. (2017), 'Jackknife Empirical Likelihood for the Error Variance in Linear Models', *Journal of Nonparametric Statistics*, 29, 151–166.
Liu, P., and Zhao, Y. (2023), 'A Review of Recent Advances in Empirical Likelihood', *WIREs Computational Statistics*, 15, e1599.
Owen, A.B. (1988), 'Empirical Likelihood Ratio Confidence Intervals for a Single Functional', *Biometrika*, 75, 237–249.
Owen, A.B. (1990), 'Empirical Likelihood Confidence Regions', *Annals of Statistics*, 18, 90–120.
Schafer, D.W. (2001), 'Semiparametric Maximum Likelihood for Measurement Error Model Regression', *Biometrics*, 57, 53–61.
Schennach, S.M. (2004), 'Estimation of Nonlinear Models with Measurement Error', *Econometrica*, 72, 33–75.
Schennach, S.M. (2007), 'Instrumental Variable Estimation of Nonlinear Errors-in-Variables Models', *Econometrica*, 75, 201–239.
Sentürk, D., and Müller, H.G. (2005), 'Covariate Adjusted Correlation Analysis Via Varying Coefficient Models', *Scandinavian Journal of Statistics*, 32, 365–383.
Sentürk, D., and Müller, H.G. (2009), 'Covariate-adjusted Regression', *Biometrika*, 92, 75–89.
Silverman, B.W. (1986), *Density Estimation for Statistics and Data Analysis*, London: Chapman and Hall.
Taupin, M.L. (2001), 'Semiparametric Estimation in the Nonlinear Structural Errors-in-Variables Model', *Annals of Statistics*, 29, 66–93.
Wang, L., and Hsiao, C. (2011), 'Method of Moments Estimation and Identifiability of Semiparametric Nonlinear Errors-in-Variables Models', *Journal of Econometrics*, 165, 30–44.
Wang, D., Wu, T., and Zhao, Y. (2019), 'Penalized Empirical Likelihood for the Sparse Cox Regression Model', *Journal of Statistical Planning and Inference*, 201, 71–85.
Wang, D., Zhao, Y., and Gilmore, D. (2015), 'Jackknife Empirical Likelihood Confidence Intervals for the Gini Index', *Statistics and Probability Letters*, 110, 289–295.
Xu, H., Fan, G., and Wang, J. (2022), 'Jackknife Empirical Likelihood for the Error Variance in Linear Errors-in-Variables Models with Missing Data', *Communications in Statistics – Theory and Methods*, 51(14), 4827–4840.
Yang, H., and Zhao, Y. (2017), 'Smoothed Jackknife Empirical Likelihood for the One-Sample Difference of Quantiles', *Computational Statistics and Data Analysis*, 120, 58–69.
Yu, X., and Zhao, Y. (2019), 'Empirical Likelihood Inference for Semi-Parametric Transformation Models with Length-Biased Sampling', *Computational Statistics and Data Analysis*, 132, 115–125.

Zhang, J., Chen, Q., and Zhou, N. (2017), 'Correlation Analysis with Additive Distortion Measurement Errors', *Journal of Statistical Computation and Simulation*, 87(4), 664–688.

Zhang, J., Feng, Z., and Zhou, B. (2014), 'A Revisit to Correlation Analysis for Distortion Measurement Error Data', *Journal of Multivariate Analysis*, 124, 116–129.

Zhang, J., Yang, B., and Feng, Z. (2021), 'Estimation of Correlation Coefficient Under a Linear Multiplicative Distortion Measurement Errors Model', *Communications in Statistics – Simulation and Computation*, 53, 62–93.

Zhao, Y., Meng, X., and Yang, H. (2015), 'Jackknife Empirical Likelihood Inference for the Mean Absolute Deviation', *Computational Statistics & Data Analysis*, 91, 92–101.

Appendix

A.1 Proof of moment-based estimator

We present a proof the estimator $\rho(X, Y) = \rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})}/\Delta$, for some constant Δ. We must assume that the confounding variable U is independent of both the unobservable X and Y. We first compute the residuals of the nonparametric regression by calculating

$$\begin{aligned} e_{\tilde{X}U} &= \tilde{X} - E[\tilde{X} \mid U], \\ e_{\tilde{Y}U} &= \tilde{Y} - E[\tilde{Y} \mid U]. \end{aligned}$$

By the independence of U with X and Y, we can see that

$$\begin{aligned} E[\tilde{X} \mid U] &= \psi(U)E[X \mid U] = \psi(U)E[X], \\ E[\tilde{Y} \mid U] &= \phi(U)E[Y \mid U] = \phi(U)E[Y]. \end{aligned}$$

Thus, the residuals can be calculated as

$$\begin{aligned} e_{\tilde{X}U} &= \tilde{X} - E[\tilde{X} \mid U] \\ &= \psi(U)Y - \psi(U)E[X] \\ &= \psi(U)[X - E[X]], \\ e_{\tilde{Y}U} &= \tilde{Y} - E[\tilde{Y} \mid U] \\ &= \phi(U)Y - \phi(U)E[Y] \\ &= \phi(U)[Y - E[Y]], \end{aligned}$$

Now, we can calculate the correlation coefficient $\rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})}$.

$$\begin{aligned} \rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})} &= \frac{\operatorname{cov}(e_{\tilde{X}U}, e_{\tilde{Y}U})}{\sqrt{\operatorname{Var}(e_{\tilde{Y}U})\operatorname{Var}(e_{\tilde{Y}U})}} \\ &= \frac{E[e_{\tilde{X}U}e_{\tilde{Y}U}] - E[e_{\tilde{X}U}]E[e_{\tilde{Y}U}]}{\sqrt{\operatorname{Var}(e_{\tilde{Y}U})\operatorname{Var}(e_{\tilde{Y}U})}}. \end{aligned}$$

Since U is independent of both X and Y,

$$\begin{aligned} &\rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})} \\ &= \frac{E[\psi(U)\phi(U)]E[[X - E[X]]E[Y - E[Y]] - E[\psi(U)]E[\phi(U)]E[X - E[X]]E[Y - E[Y]]}{\sqrt{\operatorname{Var}(\psi(U)[X - E[X]])\operatorname{Var}(\phi(U)[Y - E[Y]])}} \\ &= \frac{E[\psi(U)\phi(U)]E\,[[X - E[X]][Y - E[Y]]]}{\sqrt{E[\psi^2(U)]E[[X - E[X]]^2] \cdot E[\phi^2(U)]E[[Y - E[Y]]^2]}} \\ &= \frac{E[\psi(U)\phi(U)}{\sqrt{E[\psi^2 U]E[\phi^2(u)]}} \cdot \frac{E[XY] - E[X]E[Y]}{\sqrt{E[[X - E[X]]^2]E[[Y - E[Y]]^2]}}. \end{aligned}$$

Now, let

$$\Delta = \frac{E[\psi(U)\phi(U)]}{\sqrt{E[\psi^2 U]E[\phi^2(u)]}}.$$

As a result,

$$\begin{aligned}\rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})} &= \Delta \cdot \frac{E[XY] - E[X]E[Y]}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.\\ &= \Delta \cdot \rho(X, Y).\end{aligned}$$

Thus,

$$\rho(X, Y) = \frac{\rho_{(e_{\tilde{Y}U}, e_{\tilde{X}U})}}{\Delta}$$

when $\Delta = \frac{E[\psi(U)\phi(U)}{\sqrt{E[\psi^2 U]E[\phi^2(u)]}}$.

A.2 Proofs of theorems

Recall that

$$\hat{\rho}^*_{i(X,Y)} = \frac{\widehat{Cov}_{-i}(X, Y)}{\sqrt{\hat{\sigma}^2_{-i,X}\hat{\sigma}^2_{-i,Y}}},$$

where

$$\widehat{Cov}_{-i}(X, Y) = \frac{1}{n-1}\sum_{j\neq i}^{n} \hat{X}_j\hat{Y}_j - \left(\frac{1}{n-1}\sum_{j\neq i}^{n} \hat{X}_j\right)\left(\frac{1}{n-1}\sum_{j\neq i}^{n} \hat{Y}_j\right),$$

$$\hat{\sigma}^2_{-i,X} = \frac{1}{n-1}\sum_{j\neq i}^{n} \hat{X}_j^2 - \left(\frac{1}{n-1}\sum_{j\neq i}^{n} \hat{e}_j\right)^2 \quad \text{and}$$

$$\hat{\sigma}^2_{-i,Y} = \frac{1}{n-1}\sum_{j\neq i}^{n} \hat{Y}_j^2 - \left(\frac{1}{n-1}\sum_{j\neq i}^{n} \hat{Y}_j\right)^2.$$

By some basic derivation, it is straightforward to have

$$\begin{aligned}\widehat{Cov}_{-i}(X, Y) &= \frac{n}{n-1}\widehat{Cov}(X, Y) - \frac{n}{(n-1)^2}(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}}),\\ \hat{\sigma}^2_{-i,X} &= \frac{n}{n-1}\hat{\sigma}^2_X - \frac{n}{(n-1)^2}(\hat{X}_i - \bar{\hat{X}})^2,\\ \hat{\sigma}^2_{-i,Y} &= \frac{n}{n-1}\hat{\sigma}^2_Y - \frac{n}{(n-1)^2}(\hat{Y} - \bar{\hat{Y}})^2.\end{aligned}$$

Then, it follows that

$$\begin{aligned}\hat{\rho}^*_{i(Y,X)} &= \frac{\widehat{Cov}(Y, X) - \frac{1}{(n-1)}(\hat{Y}_i - \bar{\hat{Y}})(\hat{X} - \bar{\hat{X}})}{\sqrt{\hat{\sigma}^2_Y - \frac{1}{(n-1)}(\hat{Y} - \bar{\hat{Y}})^2}\sqrt{\hat{\sigma}^2_X - \frac{1}{(n-1)}(\hat{X}_i - \bar{\hat{X}})^2}}\\ &= \frac{\hat{\rho}_{(Y,X)} - \frac{1}{(n-1)}(\hat{Y}_i - \bar{\hat{Y}})(\hat{X} - \bar{\hat{X}})/\sqrt{\hat{\sigma}^2_X\hat{\sigma}^2_Y}}{\sqrt{1 - \frac{1}{(n-1)}(\hat{Y}_i - \bar{\hat{Y}})^2/\hat{\sigma}^2_Y}\sqrt{1 - \frac{1}{(n-1)}(\hat{X}_i - \bar{\hat{X}})^2/\hat{\sigma}^2_X}}. \qquad \text{(A1)}\end{aligned}$$

By the assumption (C3) with finite fourth moments and the modification of Lemma 3 in Owen (1990), we have

$$\max_{1\leq i\leq n}(Y_i - \bar{Y})^2 \leq \max_{1\leq i\leq n}\{|Y_i - E[Y]| + |E[Y] - \bar{Y}|\}^2$$

$$\leq 2 \max_{1\leq i\leq n} (Y_i - E[Y])^2 + 2 \max_{1\leq i\leq n} (\bar{Y} - E[Y])^2$$
$$= o_p(n^{1/2}) + o_p(1)$$
$$= o_p(n^{1/2}),$$

and similarly,

$$\max_{1\leq i\leq n} (X_i - \bar{X})^2 \leq \max_{1\leq i\leq n} \{|X_i - E[X]| + |E[X] - \bar{X}|\}^2$$
$$\leq 2 \max_{1\leq i\leq n} (X_i - E[X])^2 + 2 \max_{1\leq i\leq n} (\bar{X} - E[X])^2$$
$$= o_p(n^{1/2}) + o_p(1)$$
$$= o_p(n^{1/2}).$$

Using (A.21) and (A.22) in Zhang et al. (2014), we have

$$\max_{1\leq i\leq n} \frac{(\hat{Y}_i - \bar{\hat{Y}})^2}{(n-1)\hat{\sigma}_Y^2} = o_p(n^{-1/2}), \tag{A2}$$

and similarly,

$$\max_{1\leq i\leq n} \frac{(\hat{X}_i - \bar{\hat{X}})^2}{(n-1)\hat{\sigma}_X^2} = o_p(n^{-1/2}). \tag{A3}$$

Let σ_Y^2 and σ_X^2 denote the variances of Y and X, respectively. To prove Theorem 3.1, we need the following two lemmas.

Lemma A.1: *Under the conditions (C1)–(C5), one has that as $n \to \infty$,*

$$\sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n} \hat{V}_i - \rho_0(X,Y)\right\} \xrightarrow{\mathcal{D}} N(0, \sigma_{\rho_0}^2),$$

where

$$\sigma_{\rho_0}^2 = \frac{1}{4}\rho_0(X,Y)^2 \left\{\frac{E[\{X-E(X)\}]^4}{\sigma_X^4} + 2\frac{E[\{X-E(X)\}^2\{Y-E(Y)\}^2]}{\sigma_X^2\sigma_Y^2} + \frac{E[\{Y-E(Y)\}]^4}{\sigma_Y^4}\right\} - \rho_0(X,Y)\left\{\frac{E[\{X-E(X)\}^3\{Y-E(Y)\}]}{\sigma_X^3\sigma_Y} + \frac{E[\{X-E(X)\}\{Y-E(Y)\}^3]}{\sigma_X\sigma_Y^3}\right\} + \frac{E[\{X-E(X)\}^2\{Y-E(Y)\}^2]}{\sigma_X^2\sigma_Y^2}.$$

Proof: We notice that $\sigma_{\rho_0}^2$ is the same as the $\sigma_{\rho(X,Y)}^2$ given on page 121 of Zhang et al. (2014) and on pages 667–668 of Zhang, Chen, and Zhou (2017). We first consider to get the expansion of $\hat{\rho}^*_{i(Y,X)}$. By (A2), (A3) and the Taylor's expansion, (A1) can be written as

$$\hat{\rho}^*_{i(Y,X)} = \left\{\hat{\rho}^*_{(Y,X)} - \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{(n-1)\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}}\right\}\left\{1 + \frac{1}{2}\Delta_i + \frac{3}{8}\Delta_i^2 + o_p(n^{-3/2})\right\}$$
$$= \hat{\rho}^*_{(Y,X)} - \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{(n-1)\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} + \frac{\Delta_i\hat{\rho}^*_{(Y,X)}}{2}$$

$$- \frac{1}{2}\Delta_i \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{(n-1)\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} + \frac{3}{8}\hat{\rho}^*\Delta_i^2$$

$$- \frac{3}{8}\Delta_i^2 \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{(n-1)\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} + o_p(n^{-3/2}), \tag{A4}$$

where

$$\Delta_i = \frac{(\hat{X}_i - \bar{\hat{X}})^2}{(n-1)\hat{\sigma}_X^2} + \frac{(\hat{Y}_i - \bar{\hat{Y}})^2}{(n-1)\hat{\sigma}_Y^2} - \frac{(\hat{X}_i - \bar{\hat{X}})^2(\hat{Y}_i - \bar{\hat{Y}})^2}{(n-1)^2\hat{\sigma}_X^2\hat{\sigma}_Y^2}.$$

Since $\max_{1\le i\le n}(\hat{Y}_i - \bar{\hat{Y}})^4 = o_p(n)$ and $\max_{1\le i\le n}(\hat{X}_i - \bar{\hat{X}})^4 = o_p(n)$, we have

$$\Delta_i^2 = \frac{(\hat{X}_i - \bar{\hat{X}})^4}{(n-1)^2(\hat{\sigma}_X^2)^2} + \frac{(\hat{Y}_i - \bar{\hat{Y}})^4}{(n-1)^2(\hat{\sigma}_Y^2)^2} + \frac{2(\hat{X}_i - \bar{\hat{X}})^2(\hat{Y}_i - \bar{\hat{Y}})^2}{(n-1)^2\hat{\sigma}_X^2\hat{\sigma}_Y^2} + o_p(n^{-3/2}). \tag{A5}$$

Then, according to $\max_{1\le i\le n}|\hat{Y}_i - \bar{\hat{Y}}| = o_p(n^{1/4})$, $\max_{1\le i\le n}|\hat{X} - \bar{\hat{X}}| = o_p(n^{1/4})$ and the expansion (A4), we have

$$\begin{aligned}
\hat{V}_i &= n\hat{\rho}^*_{(Y,X)} - (n-1)\hat{\rho}^*_{i(Y,X)} \\
&= \hat{\rho}^*_{(Y,X)} + \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} - \frac{(n-1)\Delta_i\hat{\rho}^*_{(Y,X)}}{2} + \frac{1}{2}\Delta_i \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} \\
&\quad - \frac{3}{8}(n-1)\hat{\rho}^*\Delta_i^2 + \frac{3}{8}\Delta_i^2 \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} + o_p(n^{-1/2}), \\
&= \hat{\rho}^*_{(Y,X)} + \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} - \frac{(n-1)\Delta_i\hat{\rho}^*_{(Y,X)}}{2} \\
&\quad + \frac{1}{2}\Delta_i \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} - \frac{3}{8}(n-1)\hat{\rho}^*\Delta_i^2 + o_p(n^{-1/2}).
\end{aligned} \tag{A6}$$

Then, it follows from (A6) that

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n \hat{V}_i &= \hat{\rho}^*_{(Y,X)} + \frac{1}{n}\sum_{i=1}^n \frac{(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} - \frac{\hat{\rho}^*_{(Y,X)}}{2n}\left\{\sum_{i=1}^n \frac{(\hat{Y}_i - \bar{\hat{Y}})^2}{\hat{\sigma}_Y^2} + \sum_{i=1}^n \frac{(\hat{X}_i - \bar{\hat{X}})^2}{\hat{\sigma}_X^2}\right\} \\
&\quad + \frac{\hat{\rho}^*_{(Y,X)}}{2(n-1)} \frac{\sum_{i=1}^n(\hat{X}_i - \bar{\hat{X}})^2(\hat{Y}_i - \bar{\hat{Y}})^2}{n\hat{\sigma}_X^2\hat{\sigma}_Y^2} + \frac{1}{2(n-1)} \frac{\sum_{i=1}^n(\hat{X}_i - \bar{\hat{X}})(\hat{Y}_i - \bar{\hat{Y}})^3}{n(\hat{\sigma}_X^2)^{1/2}(\hat{\sigma}_Y^2)^{3/2}} \\
&\quad + \frac{1}{2(n-1)} \frac{\sum_{i=1}^n(\hat{X}_i - \bar{\hat{X}})^3(\hat{Y}_i - \bar{\hat{Y}})}{n(\hat{\sigma}_X^2)^{3/2}(\hat{\sigma}_Y^2)^{1/2}} \\
&\quad - \frac{3\hat{\rho}^*_{(Y,X)}}{8(n-1)}\left\{\frac{\sum_{i=1}^n(\hat{Y}_i - \bar{\hat{Y}})^4}{n(\hat{\sigma}_Y^2)^2} + \frac{\sum_{i=1}^n(\hat{X}_i - \bar{\hat{X}})^4}{n(\hat{\sigma}_X^2)^2}\right\} \\
&\quad - \frac{3\hat{\rho}^*_{(Y,X)}}{4(n-1)} \frac{\sum_{i=1}^n(\hat{X}_i - \bar{\hat{X}})^2(\hat{Y}_i - \bar{\hat{Y}})^2}{n\hat{\sigma}_X^2\hat{\sigma}_Y^2} + o_p(n^{-1/2})
\end{aligned}$$

$$
\begin{aligned}
&= \hat{\rho}^*_{(Y,X)} - \frac{\hat{\rho}^*_{(Y,X)}}{4(n-1)} \frac{n^{-1}\sum_{i=1}^n (\hat{X}_i - \bar{\hat{X}})^2 (\hat{Y}_i - \bar{\hat{X}})^2}{\hat{\sigma}_X^2 \hat{\sigma}_Y^2} \\
&\quad - \frac{3\hat{\rho}^*_{(Y,X)}}{8(n-1)} \left\{ \frac{n^{-1}\sum_{i=1}^n (\hat{Y}_i - \bar{\hat{Y}})^4}{(\hat{\sigma}_Y^2)^2} + \frac{n^{-1}\sum_{i=1}^n (\hat{X}_i - \bar{\hat{X}})^4}{(\hat{\sigma}_X^2)^2} \right\} \\
&\quad + \frac{1}{2(n-1)} \frac{n^{-1}\sum_{i=1}^n (\hat{X}_i - \bar{\hat{X}})(\hat{Y}_i - \bar{\hat{Y}})^3}{(\hat{\sigma}_X^2)^{1/2}(\hat{\sigma}_Y^2)^{3/2}} \\
&\quad + \frac{1}{2(n-1)} \frac{n^{-1}\sum_{i=1}^n (\hat{X}_i - \bar{\hat{X}})^3(\hat{Y}_i - \bar{\hat{Y}})}{(\hat{\sigma}_X^2)^{3/2}(\hat{\sigma}_Y^2)^{1/2}} + o_p(n^{-1/2}) \\
&:= \hat{\rho}^*_{(Y,X)} - \frac{\hat{\rho}^*_{(Y,X)}}{4(n-1)} \frac{\Pi_1}{\hat{\sigma}_X^2 \hat{\sigma}_Y^2} - \frac{3\hat{\rho}^*_{(Y,X)}}{8(n-1)} \left\{ \frac{\Pi_2}{(\hat{\sigma}_Y^2)^2} + \frac{\Pi_3}{(\hat{\sigma}_X^2)^2} \right\} \\
&\quad + \frac{1}{2(n-1)} \left\{ \frac{\Pi_4}{(\hat{\sigma}_X^2)^{1/2}(\hat{\sigma}_Y^2)^{3/2}} + \frac{\Pi_5}{(\hat{\sigma}_X^2)^{3/2}(\hat{\sigma}_Y^2)^{1/2}} \right\} + o_p(n^{-1/2}).
\end{aligned}
$$

We now show that Π_1, Π_2, Π_3, Π_4 and Π_5 are at the order of $O_p(1)$. By Zhang et al. (2014), we have

$$
\begin{aligned}
&\bar{\hat{Y}} = \bar{Y} + O_p(n^{-1/2}), \quad \bar{\hat{X}} = \bar{X} + O_p(n^{-1/2}), \\
&\frac{1}{n}\sum_{i=1}^n \frac{X_i Y_i [\phi(U_i) - \hat{\phi}(U_i)][\psi(U_i) - \hat{\psi}(U_i)]}{\hat{\phi}(U_i)\hat{\psi}(U_i)} = o_p(n^{-1/2}).
\end{aligned} \tag{A7}
$$

Note $\hat{Y}_i = \frac{\tilde{Y}_i}{\hat{\phi}(U_i)} = \frac{\phi(U_i)}{\hat{\phi}(U_i)} Y_i$. By (A7), we have

$$
\begin{aligned}
\Pi_1 &= \frac{1}{n}\sum_{i=1}^n (\hat{Y}_i - \bar{\hat{Y}})^2 (\hat{X}_i - \bar{\hat{X}})^2 \\
&= \frac{1}{n}\sum_{i=1}^n \left[\frac{\phi(U_i)Y_i}{\hat{\phi}(U_i)} - \bar{Y} - O_p(n^{-1/2}) \right]^2 \left[\frac{\psi(U_i)X_i}{\hat{\psi}(U_i)} - \bar{X} - O_p(n^{-1/2}) \right]^2 \\
&= \frac{1}{n}\sum_{i=1}^n \left[\left(\frac{\phi(U_i) - \hat{\phi}(U_i)}{\hat{\phi}(U_i)} + 1 \right) Y_i - \bar{Y} - O_p(n^{-1/2}) \right]^2 \\
&\quad \left[\left(\frac{\psi(U_i) - \hat{\psi}(U_i)}{\hat{\psi}(U_i)} + 1 \right) X_i - \bar{X} - O_p(n^{-1/2}) \right]^2 \\
&= \frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})^2 (X_i - \bar{X})^2 + o_p(1).
\end{aligned}
$$

Therefore, the law of large number yields that as $n \to \infty$,

$$
\Pi_1 \xrightarrow{p} E[\{Y - E(Y)\}^2 \{X - E(X)\}^2]. \tag{A8}
$$

As for Π_2, similar to Π_1, one has that as $n \to \infty$,

$$
\begin{aligned}
\Pi_2 &= \frac{1}{n}\sum_{i=1}^n (\hat{Y}_i - \bar{\hat{Y}})^4 \\
&= \frac{1}{n}\sum_{i=1}^n \left[\frac{\phi(U_i)Y_i}{\hat{\phi}(U_i)} - \bar{Y} - O_p(n^{-1/2}) \right]^4
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{\phi(U_i)-\hat{\phi}(U_i)}{\hat{\phi}(U_i)}+1\right)Y_i-\bar{Y}-O_p(n^{-1/2})\right]^4 \\
&= \frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^4+o_p(1) \\
&\xrightarrow{p} E\{Y-E(Y)\}^4. \qquad\qquad (A9)
\end{aligned}
$$

In the similar way, as $n \to \infty$,

$$
\Pi_3 = \frac{1}{n}\sum_{i=1}^{n}(\hat{X}_i-\bar{\hat{X}})^4 \xrightarrow{p} E\{X-E(X)\}^4. \qquad (A10)
$$

On the other hand, one can derive that as $n \to \infty$,

$$
\begin{aligned}
\Pi_4 &= \frac{1}{n}\sum_{i=1}^{n}(\hat{Y}_i-\bar{\hat{Y}})^3(\hat{X}_i-\bar{\hat{X}}) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\phi(U_i)Y_i}{\hat{\phi}(U_i)}-\bar{Y}-O_p(n^{-1/2})\right]^3\left[\frac{\psi(U_i)X_i}{\hat{\psi}(U_i)}-\bar{X}-O_p(n^{-1/2})\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{\phi(U_i)-\hat{\phi}(U_i)}{\hat{\phi}(U_i)}+1\right)Y_i-\bar{Y}-O_p(n^{-1/2})\right]^3 \qquad (A11) \\
&\quad \left[\left(\frac{\psi(U_i)-\hat{\psi}(U_i)}{\hat{\psi}(U_i)}+1\right)X_i-\bar{X}-O_p(n^{-1/2})\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^3(X_i-\bar{X})+o_p(1) \\
&= E[\{Y-E(Y)\}^3\{X-E(X)\}]. \qquad (A12)
\end{aligned}
$$

And similarly, as $n \to \infty$,

$$
\Pi_5 \xrightarrow{p} E[\{Y-E(Y)\}\{X-E(X)\}^3]. \qquad (A13)
$$

Therefore, by (A8)–(A13) and Theorem 4 in Zhang et al. (2014), we have $n^{-1}\sum_{i=1}^{n}\hat{V}_i = \hat{\rho}^*_{(Y,X)} + o_p(n^{-1/2})$. As a result, as $n \to \infty$,

$$
\begin{aligned}
\sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\hat{V}_i-\rho_0(X,Y)\right\} &= \sqrt{n}\{\hat{\rho}^*_{(Y,X)}-\rho_0(X,Y)\}+o_p(1) \\
&\xrightarrow{\mathcal{D}} N(0,\sigma^2_{\rho_0}). \qquad \blacksquare
\end{aligned}
$$

Lemma A.2: *Under the conditions (C1)–(C5), we have that as $n \to \infty$,*

$$
S_n = \frac{1}{n}\sum_{i=1}^{n}\{\hat{V}_i-\rho_0(X,Y)\}^2 \xrightarrow{p} \sigma^2_{\rho_0}.
$$

Proof: Since $\max_{1\le i\le n}|\hat{Y}_i-\bar{\hat{Y}}|=o_p(n^{1/4})$, $\max_{1\le i\le n}|\hat{X}_i-\bar{\hat{X}}|=o_p(n^{1/4})$, (A2) and (A3) hold, we have $\max_{1\le i\le n}\Delta_i^2=o_p(n^{-1})$ and

$$\max_{1\le i\le n}\left|\Delta_i\frac{(\hat{Y}_i-\bar{\hat{Y}})(\hat{X}_i-\bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}}\right|=o_p(1).$$

It follows from (A6) that

$$\hat{V}_i=\hat{\rho}^*_{(Y,X)}+\frac{(\hat{Y}_i-\bar{\hat{Y}})(\hat{X}_i-\bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}}-\frac{1}{2}\hat{\rho}^*_{(Y,X)}\frac{(\hat{Y}_i-\bar{\hat{Y}})^2}{\hat{\sigma}_Y^2}-\frac{1}{2}\hat{\rho}^*_{(Y,X)}\frac{(\hat{X}_i-\bar{\hat{X}})^2}{\hat{\sigma}_X^2}+o_p(1). \tag{A14}$$

Then, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n\hat{V}_i^2&=\frac{1}{n}\sum_{i=1}^n\left\{\hat{\rho}^*_{(Y,X)}+\frac{(\hat{Y}_i-\bar{\hat{Y}})(\hat{X}_i-\bar{\hat{X}})}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}}-\frac{\hat{\rho}^*_{(Y,X)}}{2}\left[\frac{(\hat{Y}_i-\bar{\hat{Y}})^2}{\hat{\sigma}_Y^2}-\frac{(\hat{X}_i-\bar{\hat{X}})^2}{\hat{\sigma}_X^2}\right]+o_p(1)\right\}^2\\
&=\hat{\rho}^{*2}_{(Y,X)}+\frac{1}{n}\sum_{i=1}^n\frac{(\hat{Y}_i-\bar{\hat{Y}})^2(\hat{X}_i-\bar{\hat{X}})^2}{\hat{\sigma}_Y^2\hat{\sigma}_X^2}\\
&\quad+\frac{\hat{\rho}^{*2}_{(Y,X)}}{4}\left\{\frac{1}{n}\sum_{i=1}^n\frac{(\hat{Y}_i-\bar{\hat{Y}})^4}{(\hat{\sigma}_Y^2)^2}+\frac{2}{n}\sum_{i=1}^n\frac{(\hat{Y}_i-\bar{\hat{Y}})^2(\hat{X}_i-\bar{\hat{X}})^2}{\hat{\sigma}_Y^2\hat{\sigma}_X^2}+\frac{1}{n}\sum_{i=1}^n\frac{(\hat{X}_i-\bar{\hat{X}})^4}{(\hat{\sigma}_X^2)^2}\right\}\\
&\quad-\hat{\rho}^*_{(Y,X)}\left\{\frac{1}{n}\sum_{i=1}^n\frac{(\hat{Y}_i-\bar{\hat{Y}})^3(\hat{X}_i-\bar{\hat{X}})}{(\hat{\sigma}_Y^2)^{3/2}(\hat{\sigma}_X^2)^{1/2}}+\frac{1}{n}\sum_{i=1}^n\frac{(\hat{Y}_i-\bar{\hat{Y}})(\hat{X}_i-\bar{\hat{X}})^3}{(\hat{\sigma}_Y^2)^{1/2}(\hat{\sigma}_X^2)^{3/2}}\right\}+o_p(1)\\
&=\hat{\rho}^2_{(Y,X)}+\frac{\Pi_1}{\hat{\sigma}_Y^2\hat{\sigma}_X^2}+\frac{\hat{\rho}^2_{(Y,X)}}{4}\left\{\frac{\Pi_2}{(\hat{\sigma}_Y^2)^2}+\frac{2\Pi_1}{\hat{\sigma}_Y^2\hat{\sigma}_X^2}+\frac{\Pi_3}{(\hat{\sigma}_X^2)^2}\right\}\\
&\quad-\hat{\rho}_{(Y,X)}\left\{\frac{\Pi_4}{(\hat{\sigma}_Y^2)^{3/2}(\hat{\sigma}_X^2)^{1/2}}+\frac{\Pi_5}{(\hat{\sigma}_Y^2)^{1/2}(\hat{\sigma}_X^2)^{3/2}}\right\}+o_p(1).
\end{aligned}$$

From Lemma A.1, one has $n^{-1}\sum_{i=1}^n\hat{V}_i=\rho_0(X,Y)+O_p(n^{-1/2})$. According to (A8)–(A13) and Theorem 4 in Zhang et al. (2014), we have that as $n\to\infty$,

$$\begin{aligned}
S_n&=\frac{1}{n}\sum_{i=1}^n\{\hat{V}_i-\rho_0(X,Y)\}^2\\
&=\frac{1}{n}\sum_{i=1}^n\hat{V}_i^2-2\rho_0(X,Y)\frac{1}{n}\sum_{i=1}^n\hat{V}_i+\rho_0(X,Y)^2\\
&=\frac{1}{n}\sum_{i=1}^n\hat{V}_i^2-\rho_0(X,Y)^2+O_p(n^{-1/2})\\
&\xrightarrow{p}\frac{E[\{Y-E(Y)\}^2\{X-E(X)\}^2]}{\sigma_Y^2\sigma_X^2}\\
&\quad+\frac{\rho_0(X,Y)^2}{4}\left\{\frac{E\{Y-E(Y)\}^4}{\sigma_Y^4}+\frac{2E[\{Y-E(Y)\}^2\{X-E(X)\}^2]}{\sigma_Y^2\sigma_X^2}+\frac{E\{X-E(X)\}^4}{\sigma_X^4}\right\}
\end{aligned}$$

$$- \rho_0(X,Y)\left\{\frac{E[\{Y-E(Y)\}^3\{X-E(X)\}]}{\sigma_Y^3\sigma_X} + \frac{E[\{Y-E(Y)\}\{X-E(X)\}^3]}{\sigma_Y\sigma_X^3}\right\}$$
$$= \sigma_{\rho_0}^2.$$

■

Proof of Theorem 3.1: Let $W_n = \max_{1\le i\le n}|\hat{V}_i - \rho_0(X,Y)|$. We first show that $W_n = o_p(n^{1/2})$. In fact, it follows from the expansion (A14) and Lemma A.1 that for any $1 \le i \le n$,

$$\begin{aligned}
|\hat{V}_i - \rho_0(X,Y)| &\le |\hat{\rho}^*_{(Y,X)} - \rho_0(X,Y)| + \frac{|(\hat{Y}_i - \bar{\hat{Y}})(\hat{X}_i - \bar{\hat{X}})|}{\sqrt{\hat{\sigma}_Y^2\hat{\sigma}_X^2}} + \frac{1}{2}|\hat{\rho}^*_{(Y,X)}|\frac{(\hat{Y}_i - \bar{\hat{Y}})^2}{\hat{\sigma}_Y^2} \\
&\quad + \frac{1}{2}|\hat{\rho}^*_{(Y,X)}|\frac{(\hat{X}_i - \bar{\hat{X}})^2}{\hat{\sigma}_X^2} + o_p(1) \\
&= \frac{|(Y_i - \bar{Y})(X_i - \bar{X})\{1 + o_p(1)\}|}{\sqrt{\sigma_Y^2\sigma_X^2 + o_p(1)}} \\
&\quad + \frac{1}{2}\frac{(Y_i - \bar{Y})^2\{|\rho_0(X,Y)| + o_p(1)\}\{1 + o_p(1)\}}{\sigma_Y^2 + o_p(1)} \\
&\quad + \frac{1}{2}\frac{(X_i - \bar{X})^2\{|\rho_0(X,Y)| + o_p(1)\}\{1 + o_p(1)\}}{\sigma_X^2 + o_p(1)} + o_p(1),
\end{aligned}$$

Since $\{Y_i, X_i\}_{i=1}^n$ are i.i.d. samples with finite fourth moments, we have $W_n = o_p(n^{1/2})$ by the modification of Lemma 3 in Owen (1990). Note that from Equation (2), we have

$$\begin{aligned}
0 &= \frac{1}{n}\left|\sum_{i=1}^n\{\hat{V}_i - \rho_0(X,Y)\} - \lambda\sum_{i=1}^n\frac{\{\hat{V}_i - \rho_0(X,Y)\}^2}{1 + \lambda\{\hat{V}_i - \rho_0(X,Y)\}}\right| \\
&\ge \frac{|\lambda|S_n}{1 + |\lambda|W_n} - \frac{1}{n}\left|\sum_{i=1}^n\{\hat{V}_i - \rho_0(X,Y)\}\right|. \qquad \text{(A15)}
\end{aligned}$$

By Lemma A.1, the second term of the inequality (A15) is $O_p(n^{-1/2})$. Therefore, we have $|\lambda| = O_p(n^{-1/2})$ by Lemma A.2. As a result, we can write

$$\lambda = S_n^{-1}\left\{\frac{1}{n}\sum_{i=1}^n \hat{V}_i - \rho_0(X,Y)\right\} + o_p(n^{-1/2}).$$

Let $\gamma_i = \lambda\{\hat{V}_i - \rho_0(X,Y)\}$. We have

$$\begin{aligned}
-2\log J(\rho_0(X,Y)) &= 2\sum_{i=1}^n \log(1 + \gamma_i) \\
&= 2n\lambda\left\{\frac{1}{n}\sum_{i=1}^n \hat{V}_i - \rho_0(X,Y)\right\} - nS_n\lambda^2 + o_p(1) \\
&= \frac{n\left\{n^{-1}\sum_{i=1}^n \hat{V}_i - \rho_0(X,Y)\right\}^2}{S_n} + o_p(1).
\end{aligned}$$

Therefore, we have $-2\log J(\rho_0(X,Y)) \xrightarrow{\mathcal{D}} \chi_1^2$ from Slutsky's theorem. ■

Proof of Theorem 3.2: Following the similar arguments on proofs of Theorem 2 in Zhao et al. (2015), we prove Theorem 3.2. The details are escaped. ■

Proof of Theorem 3.3: Following the same lines on proofs of Theorem 3.1 in Liang et al. (2019), we prove Theorem 3.3 as in Huang et al. (2023). ■

Proof of Theorem 3.4: Combining the results of Theorems 3.2 and 3.3, we prove Theorem 3.4 as in Huang et al. (2023). ■

Proof of Theorem 3.5: Combining the results of Theorems 3.2 and 3.3, we prove Theorem 3.5 as in Huang et al. (2023). ■