7.3 Regularization

Machine Learning 1: Foundations

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What is Regularization?

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SVM, LR, and ANN employ regularization:

$$\min_{[W_l] \; b,\mathbf{w}} \underbrace{\frac{1}{2}\|\mathbf{w}\|^2 \left[+\frac{1}{2}\sum_{l=1}^{L}\|W_l\|^2\right]}_{\text{regularizer } R(\theta)} + \underbrace{C}_{\substack{\text{regularization}\\\text{constant}}} \underbrace{\sum_{i=1}^{n} \ell\left(y_i\left(\langle \mathbf{w}, \phi(\mathbf{x}_i)\rangle + b\right)\right)}_{\text{loss } L(\theta)},$$

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Why does it work?

Why It Works: Example of Polynomial Kernel

Recall: prediction functions are degree-m polynomials:

$$f(\mathbf{x}) = \langle \mathbf{w}, \phi(\mathbf{x}) \rangle = \sum_{i=(i_1,\ldots,i_d)\in\mathbb{N}_0^d : \sum_{j=1}^d i_j \leq m} w_i c_i x_1^{i_1} \cdots x_d^{i_d}$$

+b +b

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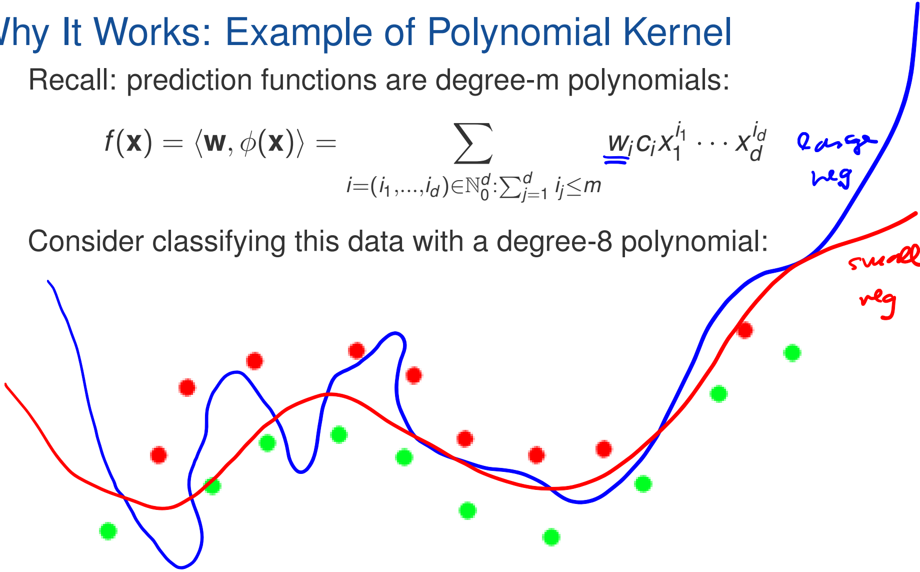
Consider classifying this data with a degree-8 polynomial:



The more regularization, the smaller the coefficients

- leads to smoother functions

Why Regularization Works: Regularizer in Constraint

Using a Lagrangian argument, one can show that

$$\min_{\theta} \quad \underbrace{R(\theta)}_{\text{regularizer}} + \underbrace{C}_{\substack{\text{regularization}\\ \text{constant}}} \underbrace{L(\theta)}_{\text{loss}}$$

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$$\min_{\theta} \underbrace{R(\theta)}_{\text{regularizer}} + \underbrace{C}_{\substack{\text{regularization}\\ \text{constant}}} \underbrace{L(\theta)}_{\text{loss}}$$

is equivalent to

$$\begin{aligned} \min_{\theta} \quad & L(\theta) \\ \text{s.t.} \quad & R(\theta) \leq \tilde{C} \end{aligned}$$

for some adequate choice of $\tilde{C}$.

Proof

$$\min_{\theta} \quad L(\theta) \quad \text{s.t.} \quad R(\theta) \leq \tilde{C}$$

$$\overset{(L)}{=} \max_{\lambda \geq 0} \min_{\theta} \quad L(\theta) + \lambda \cdot \left(R(\theta) - \tilde{C} \right)$$

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$$= \min_{\theta} \quad L(\theta) + \lambda^* \left(R(\theta) \cancel{- \tilde{C}} \right)$$

$$\hat{=} \min_{\theta} \quad L(\theta) + \underbrace{\lambda^*}_{=: C} R(\theta)$$

Denote (θ^*, λ^*) the optimum of min-max problem

$\square$

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- By changing $\tilde{C}$ we can control the size of the set

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Bottom line: The **larger** the set, the more likely the algorithm will pick a function

- that describes the training data well
- but does not generalize well