Machine Learning I: Foundations
Exercise Sheet 3

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1) (MANDATORY) 10 Points
Suppose that $k_1, \dots, k_n : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are kernels. Let $c_1, \dots, c_n \in \mathbb{R}^+$ and $p \in \mathbb{N}$. Prove that the following functions k are also kernels.

a) **Scaling**: $k(\mathbf{x}, \mathbf{x}') := c_1 k_1(\mathbf{x}, \mathbf{x}')$

b) **Sum**: $k(\mathbf{x}, \mathbf{x}') := k_1(\mathbf{x}, \mathbf{x}') + k_2(\mathbf{x}, \mathbf{x}')$

c) **Linear combination**: $k(\mathbf{x}, \mathbf{x}') := \sum_{i=1}^{n} c_i k_i(\mathbf{x}, \mathbf{x}')$

d) **Product**: $k(\mathbf{x}, \mathbf{x}') := k_1(\mathbf{x}, \mathbf{x}') k_2(\mathbf{x}, \mathbf{x}')$

e) **Power**: $k(\mathbf{x}, \mathbf{x}') := k_1(\mathbf{x}, \mathbf{x}')^p$

2) In the lecture a few kernels were proposed, and here we will prove them to be kernels. Prove the following statements:

a) **Polynomial kernel**: $k(\mathbf{x}, \mathbf{x}') := (\mathbf{x}^T \mathbf{x}' + c)^d$ is a kernel.

b) **Limits**: If $k_i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $i \in \mathbb{N}$, are kernels and $k(\mathbf{x}, \mathbf{x}') := \lim_{n\to\infty} k_n(\mathbf{x}, \mathbf{x}')$ exists for all $\mathbf{x}, \mathbf{x}'$, then $k(\mathbf{x}, \mathbf{x}')$ is a kernel. Use the definition of positive semi-definiteness.

c) **Exponents**: If $\tilde{k}$ is a kernel, then $k(\mathbf{x}, \mathbf{x}') := \exp(\tilde{k}(\mathbf{x}, \mathbf{x}'))$ is a kernel.

d) **Functions**: If $\tilde{k}$ is a kernel and $f : \mathbb{R}^d \to \mathbb{R}$ then $k(\mathbf{x}, \mathbf{x}') := f(\mathbf{x}) \tilde{k}(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')$ is a kernel.

e) **Gaussian RBF kernel**: $k(\mathbf{x}, \mathbf{x}') := \exp(-\frac{\|\mathbf{x}-\mathbf{x}'\|}{2})$ is a kernel.

Hint: Use the results from Exercise 1 above.

3) Prove the following lemma:

Lemma 1 *Let V be a vector space and I a set. Let $f_i : V \to \mathbb{R}$ be a collection of functions indexed by $i \in I$. If f_i is convex for all i, then the function*

$$f(x) = \max_{i \in I} f_i(x)$$

is also convex.

4) Solve programming task 3.