Machine Learning I: Foundations
Math Placement Test

Prof. Marius Kloft Billy Joe Franks

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- This test will not be graded and is not relevant for your admission to any exam!
- This test is meant to give you an idea of what types of math fundamentals you should possess for ML1.
- You may use any free space on this sheet or any extra sheet of paper of your choosing.
- Please do not consult a friend, neighbour, or the internet while you are solving these questions, as this would falsify your experience.
- If you are unable to solve any one of the following questions consider whether you should do ML1 at this point, or whether you should catch up on math by visiting the following website AFTER you have concluded the test: `https://ml.informatik.uni-kl.de/teaching/`

1. Find A_{23}.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

2. Let $B \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{d \times n}$, $D \in \mathbb{R}^{k \times d}$. Which of the following can be computed? For any computable multiplication state the dimension of the resulting matrix.
   a) BD
   b) BC
   c) $B^T C^T$
   d) DC^T

3. Is the following matrix singular?
$$D := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

4. Let $A \in \mathbb{R}^{k \times k}$, $B \in \mathbb{R}^{k \times d}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^k$, $\mathbf{a} \in \mathbb{R}^d$. With what assumption on A can we solve the following equation for $\mathbf{x}$? Solve the following equation for $\mathbf{x}$.
$$A\mathbf{x} + B\mathbf{a} = \mathbf{b}$$

5. Find an eigenvalue and eigenvector for the following matrix:
$$E := \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

6. Let $A \in \mathbb{R}^{k \times k}$, and $\mathbf{x}, \mathbf{a} \in \mathbb{R}^k$. Derive the gradient of $J(\mathbf{x})$, i.e. $\nabla_{\mathbf{x}} J(\mathbf{x}) = ?$
$$J(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{x}^T\mathbf{a} + \log\left(\mathbf{x}^T\mathbf{x}\right)$$

7. Calculate the intersection of the following two hyperplanes, h and g.
$$h := \left\{ \mathbf{x} \in \mathbb{R}^2 \middle| \begin{pmatrix} 3 \\ 0 \end{pmatrix}^T \mathbf{x} - 6 = 0 \right\}, g := \left\{ \mathbf{x} \in \mathbb{R}^2 \middle| \begin{pmatrix} 2 \\ 2 \end{pmatrix}^T \mathbf{x} - 8 = 0 \right\}$$

8. Prove the following:
$$\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}$$