Machine Learning I: Foundations
Exercise Sheet 1

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1) (MANDATORY) 10 Points

Find the global minima of the following functions $f : \mathbb{R} \to \mathbb{R}$ and $g, h, i : \mathbb{R}^2 \to \mathbb{R}$.

a) $f(w) := aw^2 + bw + c$

b) $g(\mathbf{w}) := \mathbf{w}^T A\mathbf{w} + \mathbf{b}^T \mathbf{w} + c$

c) $h(\mathbf{w}) := aw_1^2 + bw_2 + c$

d) $i(\mathbf{w}) := w_1^2 + w_2^2 + w_1^2 w_2$

Hint: Compute the gradient of the above functions and set it to zero. Do not forget to check the necessary conditions on global minima.

2) In this question we will be exploring eigenvectors and eigenvalues. Let $A \in \mathbb{R}^{d\times d}$. Recall the following definition from linear algebra: a vector $\mathbf{v} \in \mathbb{R}^d$ is an *eigenvector* of A if and only if there exists $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. We then call λ the *eigenvalue* of A associated with the vector $\mathbf{v}$. Note that, if $\mathbf{v}$ is an eigenvector of A, then, for any $a \in \mathbb{R}$, $a\mathbf{v}$ is also an eigenvector of A. Therefore, $\mathbf{v}$ and $a\mathbf{v}$ are not considered 'distinct' eigenvectors. Prove the following:

Proposition 1 *If A has a finite number of distinct eigenvectors then each eigenvector must have a unique eigenvalue.*

3) Recall the following definition from linear Algebra: a symmetric matrix $A \in \mathbb{R}^{d\times d}$ is called positive definite, if $\mathbf{x}^\top A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^d$ with $\mathbf{x} \neq \mathbf{0}$.

a) Prove that, if all eigenvalues of A are positive, then A is positive definite.

b) Prove that, all eigenvalues of A are positive, if A is positive definite.

Now let $F : \begin{array}{lcl} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x^2 + 2y^2 + 4.97. \end{array}$

c) Find the critical point of F.

d) Compute the Hessian matrix H of F in any point $(x, y)^\top \in \mathbb{R}^2$.

e) Recall from multivariate calculus that, if H is positive definite in a critical point $(x, y)^\top$, then $(x, y)^\top$ is a local minimum. Show that the critical point of F is a local minimum. **Hint:** note that H is symmetric.

4) Solve programming task 1.