3.2 SVM is a Convex OP

Machine Learning 1: Foundations

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Making the SVM Convex

Theorem

The linear hard-margin SVM from last week, that is,

$$\max_{\gamma, b \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^d} \gamma \quad \text{s.t.} \quad y_i(\mathbf{w}^\top \mathbf{x}_i + b) \geq \|\mathbf{w}\| \gamma$$

can be equivalently rewritten in convex form as given below:

Linear hard-margin SVM in convex form

$$\min_{b \in \mathbb{R}, \mathbf{w} \in \mathbb{R}^d} \quad \frac{1}{2} \|\mathbf{w}\|^2$$

convex, quadr.

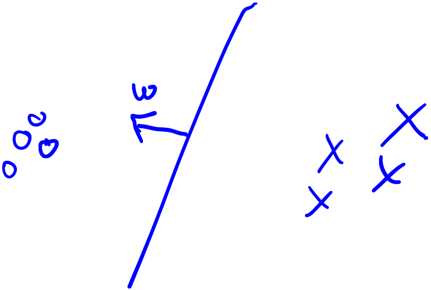
$$\text{s.t.} \qquad 1 - y_i(\mathbf{w}^\top \mathbf{x}_i + b) \leq 0 \quad \forall i = 1, \ldots, n$$

lin: ineqn

Core Idea of the Proof

The SVM results in a linear classifier $f(\mathbf{x}) = \mathrm{sign}(\mathbf{w}^\top \mathbf{x} + b)$, the parameters $(\mathbf{w}, b)$ of which are not unique:

$$\forall \lambda > 0 : \quad \mathrm{sign}(\mathbf{w}^\top \mathbf{x} + b) = \mathrm{sign}\big(\underbrace{\lambda \mathbf{w}}_{=:\mathbf{w}_\lambda}{}^\top \mathbf{x} + \underbrace{\lambda b}_{=:b_\lambda}\big). \quad (\star)$$



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For reasons that will become clear below, we choose the restriction $\|\mathbf{w}\| = 1/\gamma$, by setting $\lambda := \frac{1}{\|\mathbf{w}\|\gamma}$.

Proof (*) $\|\tilde{w}\| = \left\|\frac{w}{\|w\|\gamma}\right\| = \frac{\|w\|}{\|w\|\cdot\gamma} = \frac{1}{\gamma}$ (1/2)

$$\max_{\gamma, w, b} \gamma \quad \text{s.t.} \quad y_i(w^T x_i + b) \geq \|w\| \cdot \gamma$$

$$0 \quad 0 \quad 0 \quad 0 \qquad\qquad 0 + 0 \geq 0 \cdot 0 \checkmark$$

$$\Leftrightarrow \min_{\gamma \geq 0, w, b} \frac{1}{\gamma} \quad \text{s.t.} \quad y_i(w^T x_i + b) \geq \|w\| \cdot \gamma \quad | \cdot \lambda$$

$\forall \lambda > 0$

$$\Leftrightarrow \min_{\gamma > 0, w, b} \frac{1}{2\gamma^2} \quad \text{s.t.} \quad y_i(\lambda w^T x_i + \underbrace{\lambda b}_{=:\tilde{b}}) \geq \lambda \|w\| \cdot \gamma$$

$$\Leftrightarrow \min_{\gamma > 0, \tilde{b}, w} \frac{1}{2\gamma^2} \quad \text{s.t.} \quad y_i(\lambda w x_i + \tilde{b}) \geq \lambda \|w\| \gamma$$

$\lambda = \frac{1}{\|w\| \cdot \gamma}$

$$\Leftrightarrow \min_{\gamma > 0, \tilde{b}, w} \frac{1}{2}\|\tilde{w}\|^2 \quad \text{s.t.} \quad y_i\left(\underbrace{\frac{w^T}{\|w\|\gamma}}_{=:\tilde{w}} x_i + \tilde{b}\right) \geq 1$$

$$\Leftrightarrow \min_{\gamma > 0, \tilde{b}, \tilde{w}} \frac{1}{2}\|\tilde{w}\|^2 \quad \text{s.t.} \quad y_i(\tilde{w}^T x_i + b) \geq 1 \quad \square$$

Proof (2/2)

Soft-margin SVM

Similar, we can formulate a soft-margin SVM as convex optimization problem:

Linear soft-margin SVM in convex form

$$
\begin{aligned}
\min_{b\in\mathbb{R},\mathbf{w}\in\mathbb{R}^d,\boldsymbol{\xi}\in\mathbb{R}^n} \quad & \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{n}\xi_i \\
\text{s.t.} \quad & 1 - \xi_i - y_i(\mathbf{w}^\top \mathbf{x}_i + b) \leq 0, \quad -\xi_i \leq 0 \quad \forall i = 1, \ldots n
\end{aligned}
$$

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$$\text{s.t.} \quad 1-\xi_i - y_i(\mathbf{w}^\top\mathbf{x}_i + b) \leq 0, \quad -\xi_i \leq 0 \quad \forall i = 1,\dots n$$

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We could just put our (convex) SVM OP into one of the many solvers out there for convex optimization problems.

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Problem: CVXOPT is very powerful, but rather slow
$\Rightarrow$ For big data we need a **faster** solution