

Stability Analysis and Optimal Control Synthesis via Convex Optimization

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Introduction

Quick recap

The main topic so far:

$$\begin{aligned}
 v(x_0) := \min_{u,x} & \int_0^\infty l(x(t), u(t)) dt \\
 \text{subject to } & x(0) = x_0 \\
 & \dot{x}(t) = f(x(t), u(t)), \quad \text{for all } t \in [0, \infty) \\
 & u(t) \in U, \quad \text{for all } t \in [0, \infty)
 \end{aligned}$$

- Pontryagin's minimum principle (**analytical**):
 - necessary conditions for optimality as a two-point boundary value problem
 - closed-form solution only in very few special cases
- Numerical optimization (**numerical**):
 - in general, a nonconvex program
 - for linear systems, convex objective and constraints, a convex program

Working with time is not the only option

A classical example: **Lyapunov stability** is much easier in state space than in time

- An ODE such as

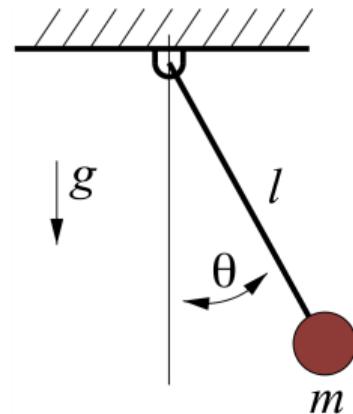
$$\ddot{\theta}(t) + \dot{\theta}(t) + \sin(\theta(t)) = 0$$

does not have closed-form solution $\theta(t)$

- Asymptotic stability ($\lim_{t \rightarrow \infty} \theta(t) = 0$) can be easily proved by showing that the energy

$$v(\theta, \dot{\theta}) := \frac{1}{2}\dot{\theta}^2 - \cos(\theta)$$

converges to zero



Lyapunov theorem

“ $\dot{x} = f(x)$ is stable if there exists $v(x) \geq 0$ such that $\dot{v}(x) = \frac{\partial v}{\partial x}(x)f(x) \leq 0$ ”

Dynamic programming

The state-space approach to optimal control is called **Dynamic Programming** (DP), and at its core we have the **Hamilton-Jacobi-Bellman** (HJB) equation

$$\min_{u \in U} \left\{ I(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \right\} = 0, \quad \text{for all } x$$

- Recall that $v(x_0)$ is the minimum of the optimal control problem for $x(0) = x_0$

DP : optimal control = Lyapunov theorem : stability? Not quite...

- In time, optimal control has similar issues to stability (hard ODEs)
- In state space:
 - HJB is a nasty **nonlinear** partial differential **equation**
 - Lyapunov conditions ($v(x) \succeq 0$, $\dot{v}(x) \preceq 0$) are nice **linear** differential **inequalities**

Quoting the authors



Bellman,
“Dynamic Programming” (1957)

The problem is not to be considered solved in the mathematical sense until the structure of the optimal policy is understood.

Put another way, in place of determining the optimal sequence of decisions from some *fixed* state of the system, we wish to determine the optimal decision to be made at *any* state of the system. Only if we know the latter, do we understand the intrinsic structure of the solution.

The conceptual advantage of thinking in terms of policies is very great. It affords us a means of thinking about and treating problems which cannot be profitably discussed in any other terms. If we were to hazard a guess as to which direction of research would achieve the greatest success in the future of multi-dimensional processes, we would unhesitatingly choose this one.



Pontryagin et al.,
“The Mathematical Theory of Optimal Processes” (1962)

Starting from the assumption that the synthesizing control (12) does exist, and that the corresponding functional (4), which is now a function of the point x :

$$J = J(x) = J(x^1, \dots, x^n) \quad (13)$$

is a continuously differentiable function of the variables x^1, \dots, x^n , the American mathematician R. Bellman constructed a partial differential equation for the functional (13). This equation of Bellman's gives rise to another approach to the solution of the optimal control problem (see §9). It is different from the one given in this book, but is closely related to it. It must be noted that the assumption on the continuous differentiability of the functional (13) does not hold in the simplest cases. Thus, Bellman's considerations yield a good heuristic method, rather than a mathematical solution of the problem. The maximum principle, in addition to its complete mathematical validity, also has the advantage that it results in a system of ordinary differential equations, whereas Bellman's approach requires the solution of a partial differential equation.

Our goal today

Use **convex optimization** to design functions in state space:

- SemiDefinite Programming (SDP)
- Sums-of-Squares (SOS) optimization

We'll focus on **two main applications**:

- Design of Lyapunov functions
 - Not really optimal control, but of fundamental importance
 - Convex optimization will be a game changer here
- Approximate dynamic programming
 - Convex optimization will still be pretty effective here

Convex-optimization background

Convex optimization recap

With f_0, \dots, f_n convex functions

$$\min_x f_0(x)$$

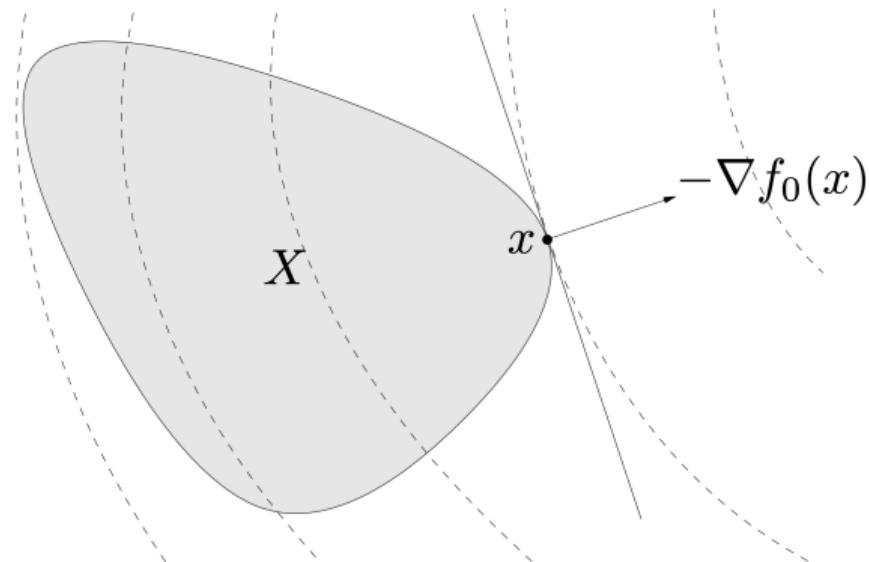
subject to $x(0) = x_0$

$$f_i(x) \leq 0, \quad i = 1, \dots, m$$

$$a_i^T x = b_i, \quad i = 1, \dots, p$$

Fundamental property:

- every local minimum is a global minimum



Boyd, Vandenberghe - "Convex Optimization"

Proof of “every local minimum is global”

- If x^* is a local minimum, there exists $r > 0$ such that

$$x^* = \arg \min_x \{f_0(x) : x \text{ is feasible}, \|x - x^*\| \leq r\}$$

- If x^* was not a global minimum, there would exist y feasible such that $f_0(y) < f_0(x^*)$
- Necessarily, $\|y - x^*\| > r$
- Take a point z on the line connecting x^* and y , distant $r/2$ from x^* :

$$z = x^* + \theta(y - x^*) = (1 - \theta)x^* + \theta y, \quad \theta = \frac{r}{2\|y - x^*\|}$$

- By convexity of the feasible set, z is feasible
- By convexity of the objective,

$$f_0(z) \leq (1 - \theta)f_0(x^*) + \theta f_0(y) < (1 - \theta)f_0(x^*) + \theta f_0(x^*) = f_0(x^*)$$

which contradicts the local optimality of x^*

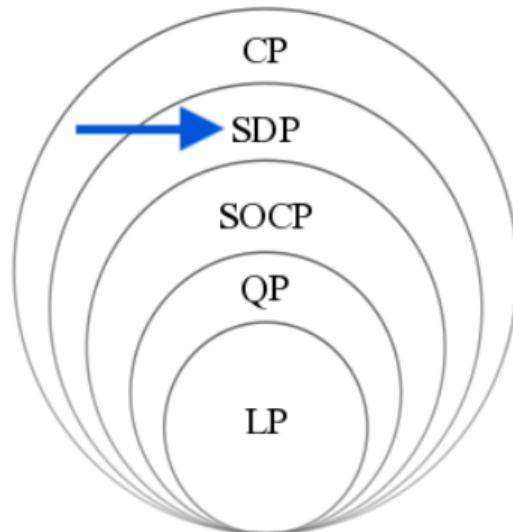
Classes of convex optimizations

Many classes of Convex Programs (CPs):

- Linear Programs (LPs) are the easiest
- SemiDefinite Programming (SDP) is a broad class of CPs that can be solved efficiently (our main tool today)
- Some CPs outside the class of SDP can be quite hard

What do we mean by “hard convex optimizations” ?

- Interior-point methods require a function to repel a point from the boundary of the feasible set
- Some convex sets are very hard to describe mathematically!
- We'll come back to this point...



Definite symmetric matrices

Equivalent statements for a symmetric A :

- A is **positive semidefinite** (denoted as $A \succeq 0$)
- $x^T A x \geq 0$ for all x
- The eigenvalues $\lambda_1, \dots, \lambda_n$ of A are nonnegative
- There exists L such that $A = L^T L$

What's the **relation with convex optimization?**

- let

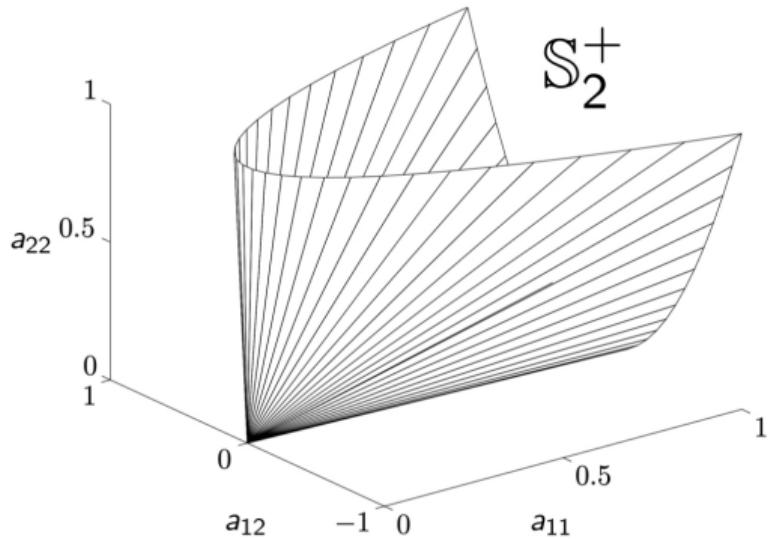
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- the set $\mathbb{S}_n^+ := \{(a_{11}, \dots, a_{nn}) : A \succeq 0\}$ is convex

Proof of “ \mathbb{S}_n^+ is convex”

Just use the definition:

- assume $A_1 \succeq 0$ and $A_2 \succeq 0$
- let $A := \theta A_1 + (1 - \theta)A_2$ with $\theta \in [0, 1]$
- then $x^T A x = \theta x^T A_1 x + (1 - \theta)x^T A_2 x \geq 0$
for all x



Boyd, Vandenberghe - “Convex Optimization”

Semidefinite program in standard form

$$\min_X \text{tr}(CX)$$

$$\begin{aligned} \text{subject to } & \text{tr}(A_i X) = b_i, \quad i = 1, \dots, p \\ & X \succeq 0 \end{aligned}$$

- recall that, e.g., $\text{tr}(CX) = \sum_{i,j=1}^n C_{ij} X_{ij}$
- linear objective function (convex)
- p linear equality constraints (convex)
- one semidefinite constraint (convex)

Why is SDP efficient?

- What's a nice function to repel a point (a_{11}, \dots, a_{nn}) from the boundary of the feasible set?
- Interior of the feasible set is $\{(a_{11}, \dots, a_{nn}) : \lambda_1 > 0, \dots, \lambda_n > 0\}$
- Natural candidate:

$$-\sum_{i=1}^n \ln(\lambda_i) = -\ln\left(\prod_{i=1}^n \lambda_i\right) = -\ln(\det A)$$

Sums-of-squares optimization

Nonlinear parameterization of the function space

- Ultimately, we want to use convex optimization to design functions $v(x) : \mathbb{R}^n \mapsto \mathbb{R}$
- **First step:** parameterize $v(x)$ with a finite number of coefficients (optimization variables)

Nonlinear parameterization

$$v(x) = \psi(x, \alpha)$$

- $\alpha \in \mathbb{R}^r$ is a vector of coefficients
- ψ is a given nonlinear scalar function

Say we want $v(0) = 0$:

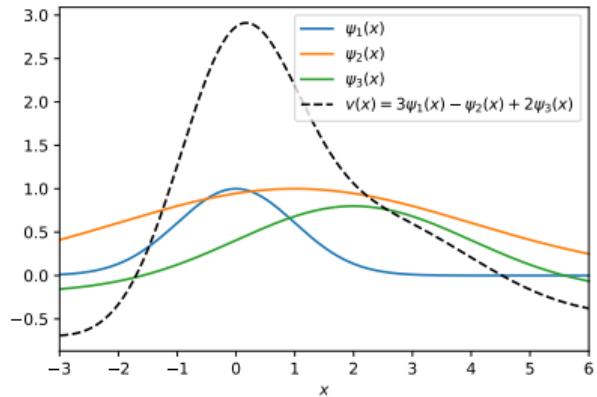
- $\psi(0, \alpha) = 0$ is a **nonlinear equality constraint** in α (not convex!)

Linear parameterization

Linear parameterization

$$v(x) = \alpha^T \psi(x)$$

- $\alpha \in \mathbb{R}^r$ is a vector of coefficients
- ψ is an r -vector of the basis functions



Let's try again $v(0) = 0$:

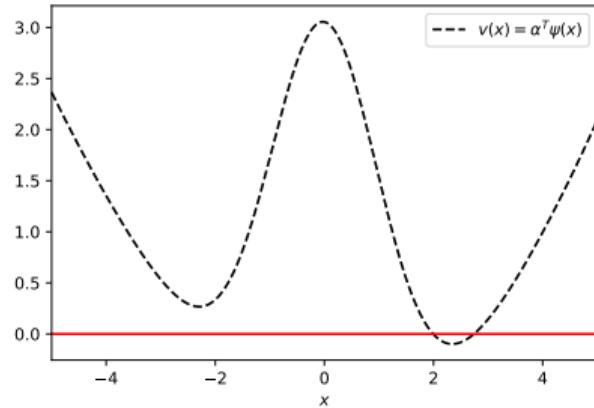
- $\alpha^T \psi(0) = 0$ is a **linear equality constraint** in α (convex)

Other ways we might want to constrain $v(x)$?

Nonnegativity constraints

For example, a Lyapunov function must be nonnegative

$$v(x) = \alpha^T \psi(x) \geq 0 \text{ for all } x$$



Let's analyze the set

$$\{\alpha : \alpha^T \psi(x) \geq 0 \text{ for all } x\} \subseteq \mathbb{R}^r$$

Nonnegativity constraints and linear parameterizations

A seemingly positive observation

The set $\{\alpha : \alpha^T \psi(x) \geq 0\}$ is convex

- Assume $v_1(x) = \alpha_1^T \psi(x) \geq 0$ and $v_2(x) = \alpha_2^T \psi(x) \geq 0$
- Then, for $\theta \in [0, 1]$,

$$v(x) = [\theta \alpha_1 + (1 - \theta) \alpha_2]^T \psi(x) = \theta v_1(x) + (1 - \theta) v_2(x) \geq 0$$

Is this enough to make the optimization over nonnegative $v(x)$ easy?

- No, in general, no
- Except for a few cases, for a given α , determining if $v(x) = \alpha^T \psi(x) \geq 0$ is NP-hard!
- We don't have a nice function to repel a point α from the boundary of the feasible set

Sums-of-squares parameterization

What about a parameterization that is nonnegative by construction?

Sums Of Squares (SOS)

$$v(x) = \psi^T(x)Q\psi(x) = \sum_{i,j=1}^r Q_{ij}\psi_i(x)\psi_j(x), \quad Q \succeq 0$$

- Q is an r -by- r matrix of coefficients (entries of Q are our optimization variables)
- ψ is an r -vector of basis functions

Remarks:

- v is still linear in Q (e.g. $v(0) = 0$ is a **linear constraint** on the entries of Q)
- $Q \succeq 0$ is a **convex constraint**

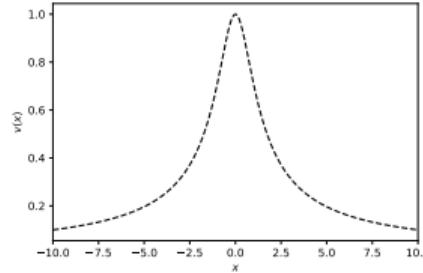
SOS constraints

What if we want to certify that a given function $v(x)$ is nonnegative?

- We can try to find a SOS decomposition $v(x) = \psi^T(x)Q\psi(x)$, with $Q \succeq 0$
- The issue however is how to pick $\psi(x)$...

Example

- Consider $v(x) = \frac{1}{\sqrt{1+x^2}}$
- This function is nonnegative for all x
- I don't quite know how to pick $\psi(x)$ here...



Can you think of any class of functions for which this approach might work?

SOS polynomials

With $v(x)$ polynomial of degree $2d$, we can just fill $\psi(x)$ with all the monomials up to degree d

Example

- Given $v(x) = 2 - 2x + 3x^2 + 2x^3 + x^4$, we set $\psi(x) = (1, x, x^2)$
- We look for $Q \succeq 0$ such that

$$2 - 2x + 3x^2 + 2x^3 + x^4 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

- Comparing the coefficients we get **linear constraints** on Q , e.g., $Q_{11} = 2$
- This works even if the coefficients of v are (linear functions of) optimization variables!
- Other classes of functions could do the job, but polynomials have many other useful properties

The power of SOS optimization

► Try this in Drake

- Minimize the six-hump-camel function

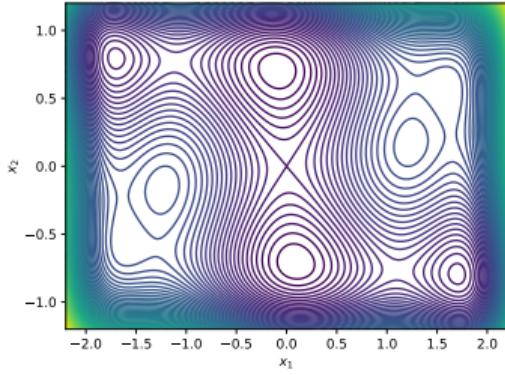
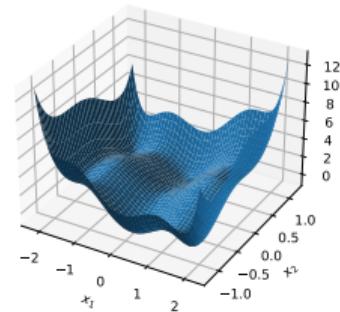
$$\min_x p(x) = 4x_1^2 + x_1 x_2 - 4x_2^2 - 2.1x_1^4 + 4x_2^4 + x_1^6/3$$

- Write it as

$$\max_{\lambda, Q} \lambda$$

subject to $p(x) - \lambda$ is SOS

- Where “is SOS” means “equal to a SOS polynomial in x parameterized by Q ”

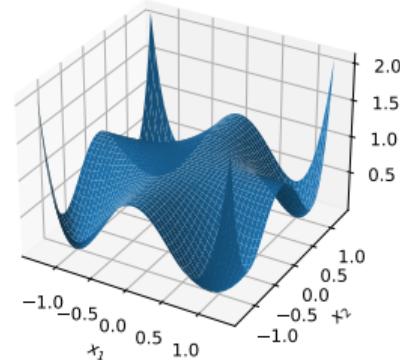


Did we just solve global optimization over polynomials?

Not all the nonnegative polynomials are SOS...

- Hilbert in 1888¹ showed that “SOS = nonnegative” only in 3 cases²:
 - Univariate polynomials (1 variable)
 - Quadratic polynomials (degree 2)
 - Bivariate quartics (2 variables, degree 4)
- Famous counterexample by Motzkin

$$x_1^4 x_2^2 + x_1^2 x_2^4 + 1 - 3x_1^2 x_2^2$$



- However, counterexamples are “rare enough that people give names to them”

¹Hilbert - “Ueber die Darstellung definiter Formen als Summe von Formenquadraten”

²See also [Hilbert's 17th problem](#)

One more trick: S-procedure

- To check if the polynomial $v(x)$ is nonnegative over the set

$$\{x : g(x) = 0\}$$

with $g(x)$ polynomial, we can solve

find a polynomial λ : $v(x) + \lambda(x)g(x)$ is SOS

- Similarly, for the set

$$\{x : g(x) \leq 0\}$$

we can solve

find a SOS polynomial λ : $v(x) + \lambda(x)g(x)$ is SOS

Design of Lyapunov functions

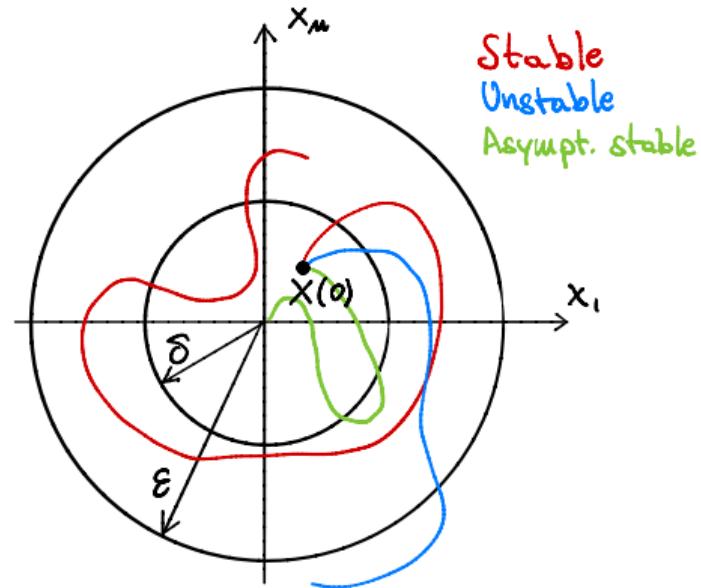
Recap on Lyapunov stability

The equilibrium $x = 0$ for the system $\dot{x} = f(x)$ is:

- **stable** if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \text{ for all } t \geq 0$$

- **asymptotically stable** if it is stable, and $\lim_{t \rightarrow \infty} \|x(t)\| = 0$



Recap on Lyapunov direct method

The equilibrium $x = 0$ for the system $\dot{x} = f(x)$ is:

- **stable** iff there exists $v(x)$ such that

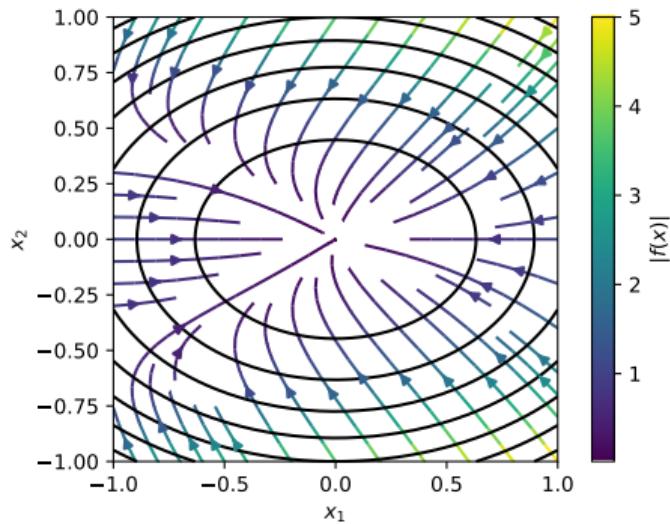
$$v(0) = 0, \quad v(x) > 0 \text{ for all } x \neq 0,$$

and

$$\dot{v}(x) = \frac{\partial v}{\partial x}(x)f(x) \leq 0 \text{ for all } x \neq 0$$

- **asymptotically stable** iff

$$\dot{v}(x) < 0 \text{ for all } x \neq 0$$



Design of Lyapunov functions via SOS

SOS parameterization of the Lyapunov function

$$v(x) = \psi^T(x)Q\psi(x), \quad Q \succeq 0$$

Filling $\psi(x)$ with monomials is the natural choice:

- It works in the linear case $f(x) = Ax$ (we always have $v(x) = x^T Px$)
- If $f(x)$ is polynomial, then

$$\dot{v}(x) = \frac{\partial v}{\partial x}(x)f(x)$$

is also polynomial, and its coefficients are linear in Q

- If $f(x)$ is not polynomial we can use, e.g., Taylor approximation

Design of Lyapunov functions via SOS

Try this in Drake

The overall SOS program

$$\text{find } v(x) \text{ SOS : } \frac{\partial v}{\partial x}(x)f(x) \text{ is SOS}$$

Miscellaneous observations:

- To avoid the trivial solution $v(x) = 0$ we can add a linear constraint like $v(1) = 1$
- Be aware: not all the stable polynomial systems admit a polynomial Lyapunov function³

³Ahmadi, Krstic, Parrilo - “A Globally Asymptotically Stable Polynomial Vector Field with no Polynomial Lyapunov Function”

Verification of nonpolynomial systems via SOS

▶ Try this in Drake

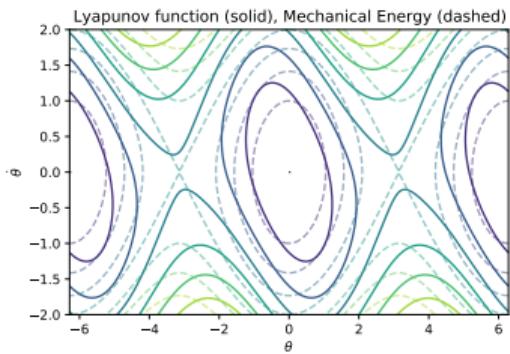
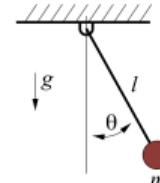
- Some non-polynomial systems can be verified exactly using SOS⁴

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - \sin(x_1)$$

- Introduce auxiliary variables $s = \sin(x_1)$, $c = \cos(x_1)$
- Substitute to get the polynomial system

$$\dot{s} = cx_2, \quad \dot{c} = -sx_2, \quad \dot{x}_2 = -x_2 - s$$

subject to the polynomial constraint $s^2 + c^2 = 1$



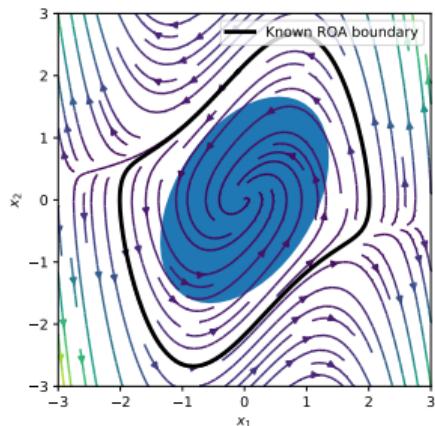
¹See also Papachristodoulou, Prajna - “Analysis of Non-polynomial Systems using the Sum of Squares Decomposition”

Bonus: approximation of region of attraction

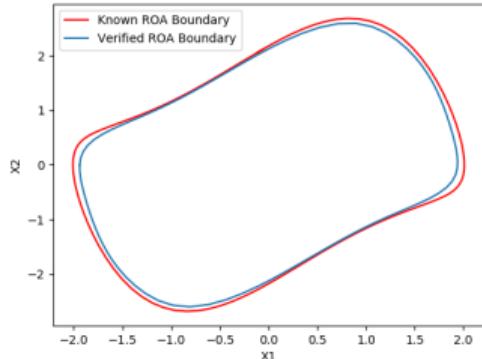
Time-reversed Van der Pol oscillator

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2\end{aligned}$$

Maximize volume of level set



Alternate between maximization of level set
and reshape Lyapunov function



Control-Lyapunov function

Can we use SOS + Lyapunov verification to design controllers? Unfortunately no...

- We let $v(x)$ be a SOS polynomial
- We use a linear parameterization for $u(x)$ (polynomial)
- For simplicity, let the system be polynomial and control-affine

$$\dot{x} = f(x) + G(x)u$$

- Lyapunov condition becomes

$$-\frac{\partial v}{\partial x}(x)[f(x) + G(x)u(x)] \text{ is SOS}$$

- Multiplication of $\frac{\partial v}{\partial x}$ and u leads to **nonlinear equality constraints**

Approximate dynamic programming

Hamilton-Jacobi-Bellman (HJB) equation

How to arrive to the HJB equation

$$\min_{u \in U} \left\{ l(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \right\} = 0, \quad \text{for all } x$$

from the Optimal Control Problem (OCP)

$$v(x_0) := \min_{u,x} \int_0^\infty l(x(t), u(t)) dt$$

subject to $x(0) = x_0$

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{for all } t \in [0, \infty)$$

$$u(t) \in U, \quad \text{for all } t \in [0, \infty)$$

HJB: informal derivation

First order approximation of the control problem with time step h :

- Cost function

$$\int_0^\infty I(x(t), u(t)) dt \approx h \sum_{t=0}^{\infty} I(x_t, u_t)$$

- Dynamics

$$\dot{x}(t) = f(x(t), u(t)) \Rightarrow x_{t+1} = x_t + hf(x_t, u_t)$$

- Overall

$$v(x_0) := \min_{u,x} h \sum_{t=0}^{\infty} I(x_t, u_t)$$

subject to x_0 given

$$x_{t+1} = x_t + hf(x_t, u_t), \quad t = 0, \dots, \infty$$

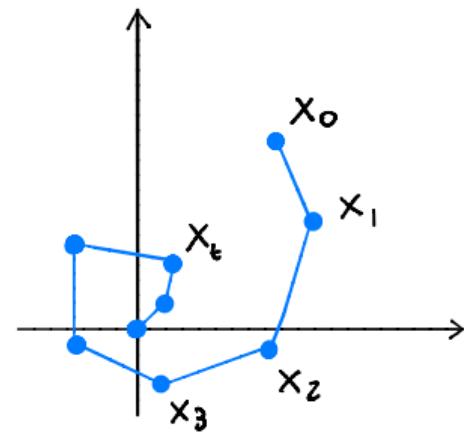
$$u_t \in U, \quad t = 0, \dots, \infty$$

HJB: informal derivation

The dynamic-programming principle:

- Assume $\mathbf{x}_0 = (x_0, x_1, x_2, x_3, \dots)$ is the optimal trajectory from x_0 to the origin
- Because of the infinite horizon, the optimal trajectory \mathbf{x}_1 from x_1 to the origin must be (x_1, x_2, x_3, \dots)
 - Otherwise (x_0, \mathbf{x}_1) would be better than \mathbf{x}_0 (contradiction)
- If we were to know the cost-to-go $v(x_1)$ from x_1 , then we could just solve a one-step problem

$$\begin{aligned} v(x_0) &= \min_{u_0 \in U} \{hl(x_0, u_0) + v(x_1)\} \\ &= \min_{u_0 \in U} \{hl(x_0, u_0) + v(x_0 + hf(x_0, u_0))\} \end{aligned}$$



HJB: informal derivation

- If h is very small,

$$v(x_0 + hf(x_0, u_0)) \approx v(x_0) + h \frac{\partial v}{\partial x}(x_0) f(x_0, u_0)$$

- Substituting

$$v(x_0) = \min_{u_0 \in U} \left\{ h l(x_0, u_0) + v(x_0) + h \frac{\partial v}{\partial x}(x_0) f(x_0, u_0) \right\}$$

- Simplifying $v(x_0)$ and dividing by h

$$0 = \min_{u_0 \in U} \left\{ l(x_0, u_0) + \frac{\partial v}{\partial x}(x_0) f(x_0, u_0) \right\}$$

Observations:

- If we know the value function $v(x)$, the optimal controller is the “argmin” of the HJB
- Why “informal”? Where does this derivation leak? (Remember the bang bang controller)

Solving the HJB

The HJB equation is a nonlinear Partial Differential Equation (PDE)

- Analytic solutions only in a few special cases (LQR)
- “Exact” numerical solution is very hard (discretization of the state and control space)
- Can we still find approximate solutions that are “good enough”?

“One side of the HJB equation is convex”

$$\min_{u \in U} \left\{ I(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \right\} \geq 0, \quad \text{for all } x$$

is equivalent to the **Bellman inequality**

$$I(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \geq 0, \quad \text{for all } x \text{ and } u \in U$$

- We can relax this condition via SOS!

Bellman inequality

What do we loose by enforcing only one side of the HJB equations?

- Let $x(t)$ and $u(t)$ be the optimal trajectory and control from $x(0)$
- Integrate the Bellman inequality along the optimal trajectory

$$\begin{aligned} 0 &\leq \int_0^\infty \left[I(x(t), u(t)) + \frac{\partial v}{\partial x}(x(t))f(x(t), u(t)) \right] dt \\ &= \int_0^\infty I(x(t), u(t))dt + \int_0^\infty \dot{v}(x(t))dt \\ &= \int_0^\infty I(x(t), u(t))dt + v(x(\infty)) - v(x(0)) \end{aligned}$$

- Let $v(0) = 0$, then $v(x)$ lower bounds the optimal value function

$$v(x(0)) \leq \int_0^\infty I(x(t), u(t))dt$$

SOS for approximate dynamic programming

$$\max_v \int_X v(x) dx$$

subject to $l(x, u) + \frac{\partial v}{\partial x}(x)f(x, u)$ is SOS for all x and $u \in U$

$$v(0) = 0$$

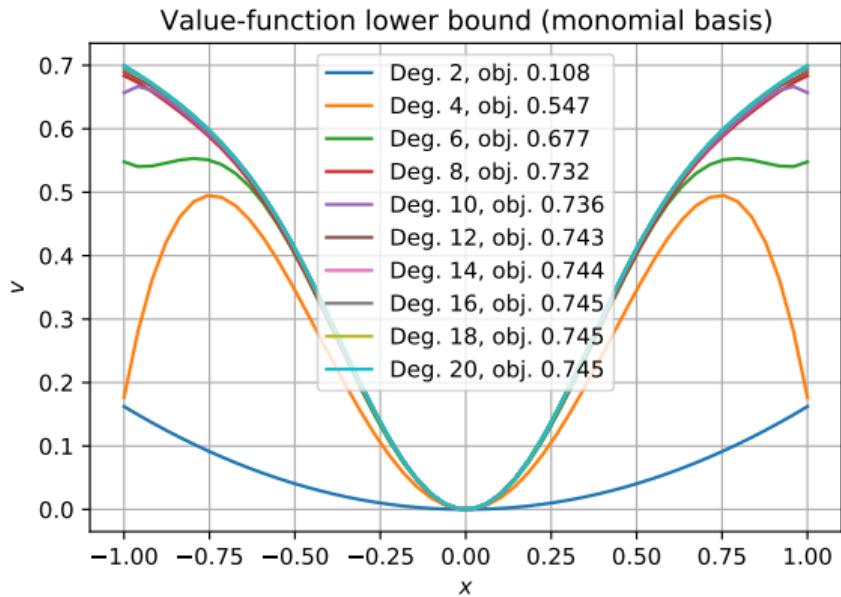
Interpretation:

- Constraints ensure that $v(x)$ lower bounds the value function
- Objective “pushes up” $v(x)$ over the set X
- As the degree of $v(x)$ increases, we get a better and better approximation of the value function over X

Toy approximate-DP problem

▶ Try this in Drake

- Scalar dynamics $f(x, u) = x - 4x^3 + u$
- Quadratic running cost $l(x, u) = x^2 + u^2$
- Control limits $u \in U = [-1, 1]$
- Approximate the value function for $x \in X = [-1, 1]$



Pendulum swing up

► Try this in Drake

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Hamilton-Jacobi-Bellman equation:

- Evans - "Partial Differential Equations"

Approximate dynamic programming:

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Software:

- [Drake](#)