

Stability Analysis and Optimal-Control Synthesis via Convex Optimization

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Pisa
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Before starting

- These slides are available at https://github.com/TobiaMarcucci/optimal_control_pisa
- In the same repo you'll also find the Drake demos I show in this presentation
- The buttons  will open the demos in Google Colab

Introduction

Quick recap

A fairly familiar problem at this point:

$$v(x_0) := \min_{u,x} \int_0^\infty l(x(t), u(t)) dt$$

subject to $x(0) = x_0$

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{for all } t \in [0, \infty)$$

$$u(t) \in U, \quad \text{for all } t \in [0, \infty)$$

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- Pontryagin's minimum principle (**analytical**):
 - necessary conditions for optimality as a two-point boundary value problem
 - closed-form solution only in very few special cases
- **Numerical** optimization:
 - in general, a nonconvex program
 - convex if system is linear, and objective and constraints are convex

Working with trajectories is not the only option

Classical example: **Lyapunov stability** is much easier in state space than “in time”

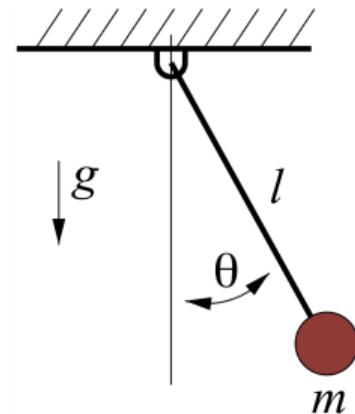
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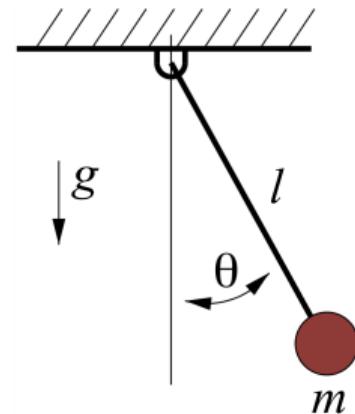
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$$v(\theta, \dot{\theta}) := \frac{1}{2}\dot{\theta}^2 - \cos(\theta)$$

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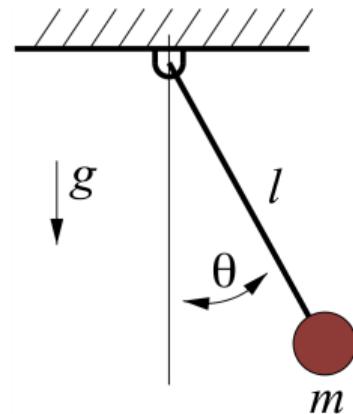
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Lyapunov theorem (sketch)

$\dot{x} = f(x)$ is stable if there exists $v(x) \geq 0$ such that $\dot{v}(x) = \frac{\partial v}{\partial x}(x)f(x) \leq 0$

Dynamic programming

- The state-space approach to optimal control is called **Dynamic Programming** (DP)
- At its core we have the **Hamilton-Jacobi-Bellman** (HJB) equation

$$\min_{u \in U} \left\{ I(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \right\} = 0, \quad \text{for all } x$$

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Dynamic programming = Lyapunov theorem ? Not quite...
 Optimal control Stability analysis

- In time, optimal control has similar issues to stability (we end up with hard ODEs)
- In state space:
 - HJB is a nasty nonlinear Partial Differential Equation (PDE)
 - Lyapunov conditions ($v(x) \geq 0$, $\dot{v}(x) \leq 0$) are simple linear differential inequalities

Quoting the authors



Bellman (1957),
“Dynamic Programming”

The problem is not to be considered solved in the mathematical sense until the structure of the optimal policy is understood.

Put another way, in place of determining the optimal sequence of decisions from some fixed state of the system, we wish to determine the optimal decision to be made at any state of the system. Only if we know the latter, do we understand the intrinsic structure of the solution.

The conceptual advantage of thinking in terms of policies is very great. It affords us a means of thinking about and treating problems which cannot be profitably discussed in any other terms. If we were to hazard a guess as to which direction of research would achieve the greatest success in the future of multi-dimensional processes, we would unhesitatingly choose this one.



Pontryagin et al. (1962),
“The Mathematical Theory of Optimal Processes”

Starting from the assumption that the synthesizing control (12) does exist, and that the corresponding functional (4), which is now a function of the point x :

$$J = J(x) = J(x^1, \dots, x^n) \quad (13)$$

is a continuously differentiable function of the variables x^1, \dots, x^n , the American mathematician R. Bellman constructed a partial differential equation for the functional (13). This equation of Bellman's gives rise to another approach to the solution of the optimal control problem (see §9). It is different from the one given in this book, but is closely related to it. It must be noted that the assumption on the continuous differentiability of the functional (13) does not hold in the simplest cases. Thus, Bellman's considerations yield a good heuristic method, rather than a mathematical solution of the problem. The maximum principle, in addition to its complete mathematical validity, also has the advantage that it results in a system of ordinary differential equations, whereas Bellman's approach requires the solution of a partial differential equation.

Our goal today

Use **convex optimization** to design functions in state space:

- SemiDefinite Programming (SDP)
- Sums-of-Squares (SOS) optimization

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Two main applications:

- Design of Lyapunov functions
 - Not really optimal control, but of fundamental importance
 - Convex optimization will be a game changer here
- Approximate dynamic programming
 - Convex optimization will be quite effective here

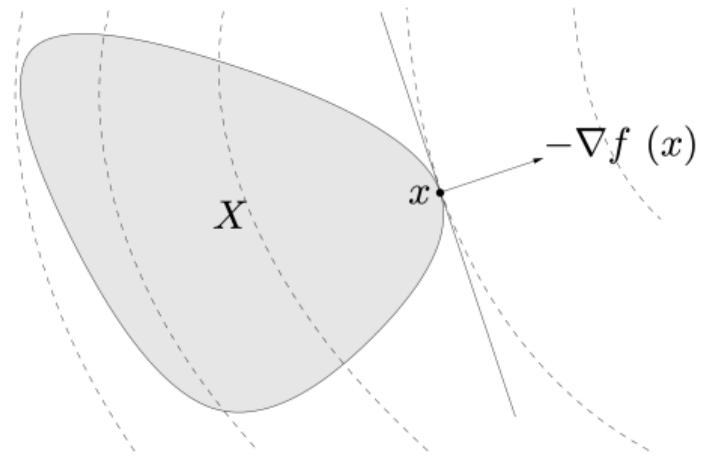
Convex-optimization background

Convex optimization recap

Standard convex program:

$$\begin{aligned} \min_x f(x) \\ \text{subject to } x \in X \end{aligned}$$

- f is a convex function
- X is a convex set



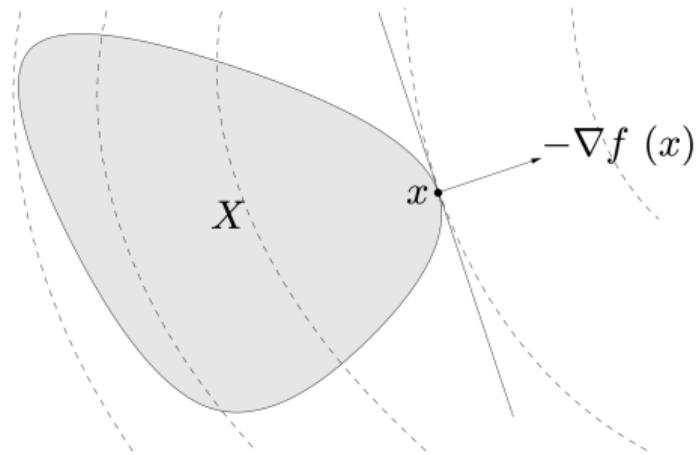
Boyd, Vandenberghe - "Convex Optimization"

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Why are we so obsessed with convex optimization?

Every local minimum is a **global minimum**

Proof of “every local minimum is global”

- If x^* is a local minimum, there exists $r > 0$ such that

$$x^* = \arg \min_{x \in X} \{f(x) : \|x - x^*\| \leq r\}$$

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- Take a point z on the line connecting x^* and y , distant $r/2$ from x^* :

$$z := x^* + \theta(y - x^*) = (1 - \theta)x^* + \theta y, \quad \theta := \frac{r}{2\|y - x^*\|}$$

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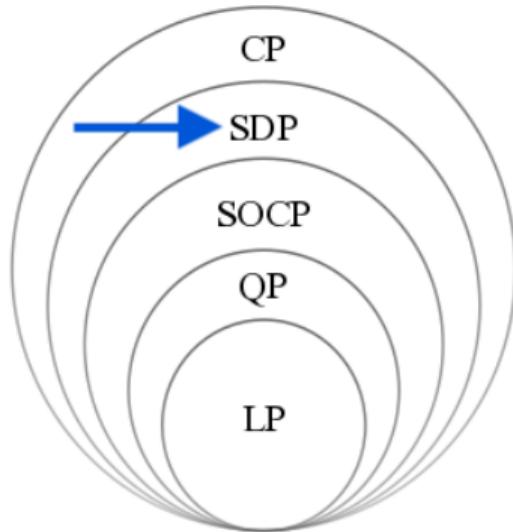
- By convexity of the feasible set X , $z \in X$
- By convexity of the objective,

$$f(z) \leq (1 - \theta)f(x^*) + \theta f(y) < (1 - \theta)f(x^*) + \theta f(x^*) = f_0(x^*)$$

which contradicts the local optimality of x^*

A hierarchy of Convex Programs (CPs)

- Linear Programs (LPs) are the easiest
- SemiDefinite Programming (SDP) is a broad class of CPs that can be solved efficiently (our main tool today)
- Some CPs outside the class of SDP can be quite hard

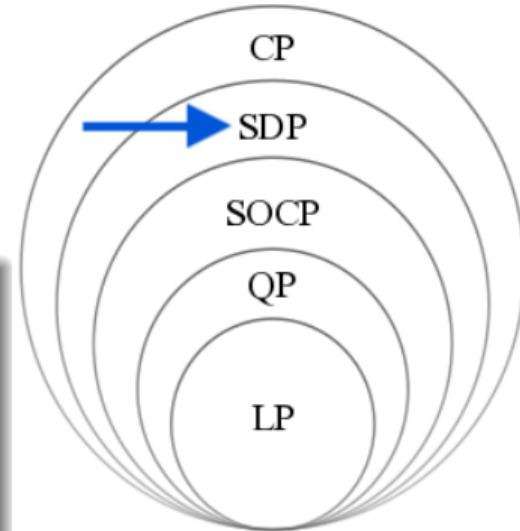


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What? Hard convex optimizations??

- Optimization algorithms require to check how far a point is from being infeasible
- Some convex sets are very hard to describe mathematically!
- We'll come back to this point...



Definite symmetric matrices

Definition

Equivalent definitions for a symmetric matrix A to be **positive semidefinite** ($A \succeq 0$):

- $x^T A x \geq 0$ for all x
- The eigenvalues $\lambda_1, \dots, \lambda_n$ of A are nonnegative
- There exists L such that $A = L^T L$

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Why do we care about positive semidefinite matrices?

- The set $\mathbb{S}_n^+ := \{(a_{11}, \dots, a_{nn}) : A \succeq 0\}$ is convex, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

Proof of “ \mathbb{S}_n^+ is convex”

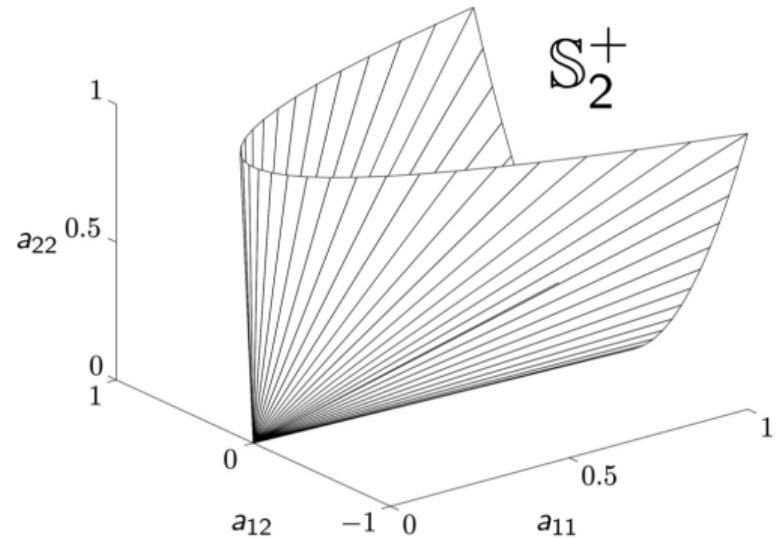
Just use the definition:

- Assume $A_1 \succeq 0$ and $A_2 \succeq 0$
- Take a convex combination of A_1 and A_2 :

$$A := \theta A_1 + (1 - \theta) A_2, \quad \theta \in [0, 1]$$

- Then, for all x ,

$$x^T A x = \theta x^T A_1 x + (1 - \theta) x^T A_2 x \geq 0$$



Boyd, Vandenberghe - “Convex Optimization”

Semidefinite program in standard form

$$\min_X \text{tr}(CX)$$

subject to $\text{tr}(A_i X) = b_i, \quad i = 1, \dots, p$
 $X \succeq 0$

- recall that $\text{tr}(CX) = \sum_{i,j=1}^n C_{ij}x_{ij}$, i.e., an arbitrary linear function of the entries of A
- linear objective function (convex)
- p linear equality constraints (convex)
- one semidefinite constraint (convex)

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Why can we solve SDPs efficiently?

- We need a function to tell how far is a point (a_{11}, \dots, a_{nn}) from being infeasible
- Interior of the feasible set is $\{(a_{11}, \dots, a_{nn}) : \lambda_1 > 0, \dots, \lambda_n > 0\}$
- Natural candidate is

$$-\sum_{i=1}^n \ln(\lambda_i) = -\ln \left(\prod_{i=1}^n \lambda_i \right) = -\ln(\det A)$$

Sums-of-squares optimization

Nonlinear parameterization of the function space

- Ultimately, we want to use convex optimization to design functions $v(x) : \mathbb{R}^n \mapsto \mathbb{R}$
- **First step:** parameterize $v(x)$ with a finite number of coefficients (optimization variables)

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Nonlinear parameterization

$$v(x) := \psi(x, \alpha)$$

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Say we want $v(0) = 0$:

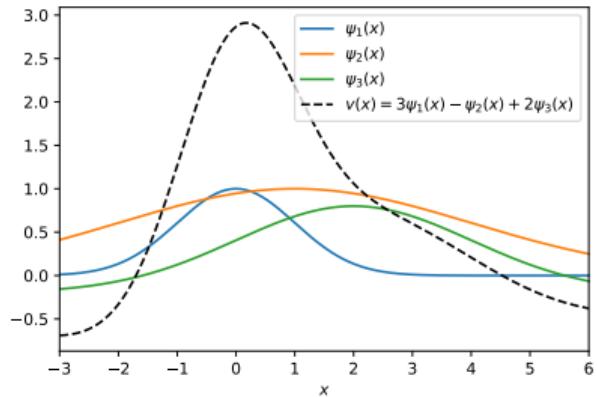
- $\psi(0, \alpha) = 0$ is a **nonlinear equality constraint** in α (not convex!)

Linear parameterization

Linear parameterization

$$v(x) := \alpha^T \psi(x)$$

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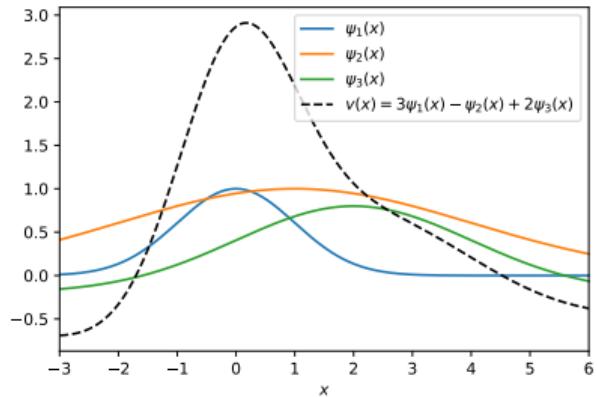


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Let's try again $v(0) = 0$:

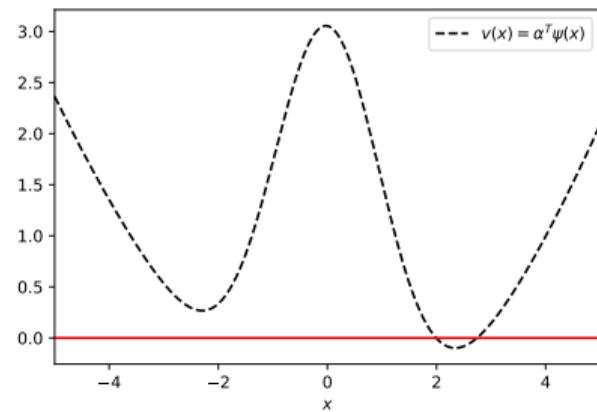
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Other ways we might want to constrain $v(x)$?

Nonnegativity constraints

For example, a Lyapunov function must be nonnegative

$$v(x) := \alpha^T \psi(x) \geq 0 \text{ for all } x$$

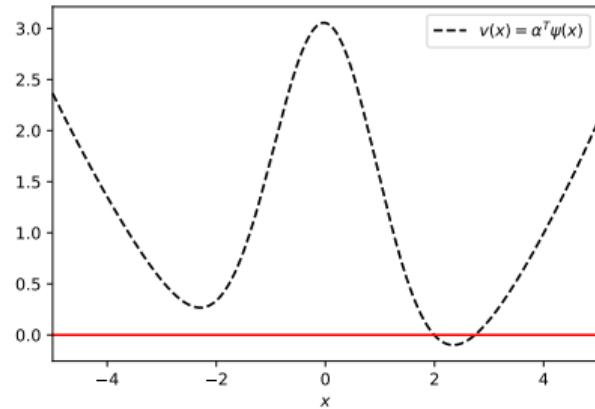


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Let's analyze the set

$$\{\alpha : \alpha^T \psi(x) \geq 0 \text{ for all } x\} \subseteq \mathbb{R}^r$$

Nonnegativity constraints and linear parameterizations

The set $\{\alpha : \alpha^T \psi(x) \geq 0\}$ is convex

- Assume α_1 is such that $v_1(x) := \alpha_1^T \psi(x) \geq 0$, and the same for α_2
- Take a convex combination of α_1 and α_2

$$\alpha := \theta \alpha_1 + (1 - \theta) \alpha_2, \quad \theta \in [0, 1]$$

- Then $v(x) := \alpha^T \psi(x) = \theta v_1(x) + (1 - \theta) v_2(x) \geq 0$

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Does this mean that optimizing over nonnegative $v(x)$ is easy?

- No, in general, no
- Except for a few cases, for a given α , determining if $\alpha^T \psi(x) \geq 0$ is NP-hard!

Back to the black board

What about a parameterization that is nonnegative by construction?

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Sums Of Squares (SOS)

$$v(x) := \psi^T(x)Q\psi(x), \quad Q \succeq 0$$

- Q is an r -by- r matrix of coefficients (entries of Q are our optimization variables)
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Remarks:

- $v(x) = \sum_{i,j=1}^r Q_{ij}\psi_i(x)\psi_j(x)$ is still linear in Q
 - E.g., $v(0) = 0$ is a **linear constraint** on the entries of Q
- $Q \succeq 0$ is a **convex constraint**

Use SOS as a constraints

What if we want to certify that a given function $v(x)$ is nonnegative?

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- The issue is how to pick $\psi(x)$...

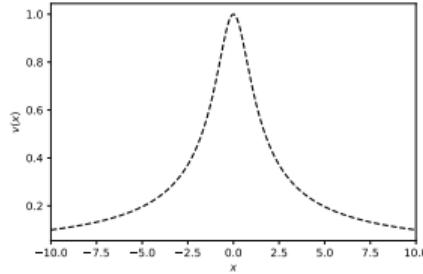
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Example

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- Nonnegative for all x
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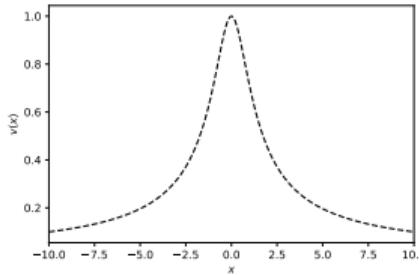
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Can you think of any class of functions for which this approach might work?

SOS polynomials

Example

- Given $v(x) = 2 - 2x + 3x^2 + 2x^3 + x^4$, we want $\psi(x)$ such that $v(x) = \psi^T(x)Q\psi(x)$
- We can just pick $\psi(x) := (1, x, x^2)$
- Then, we look for $Q \succeq 0$ such that

$$2 - 2x + 3x^2 + 2x^3 + x^4 = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

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- For $v(x)$ **polynomial** of degree $2d$, we fill $\psi(x)$ with all the **monomials** up to degree d
- This works even if the coefficients of v are (linear functions of) optimization variables

SOS polynomials

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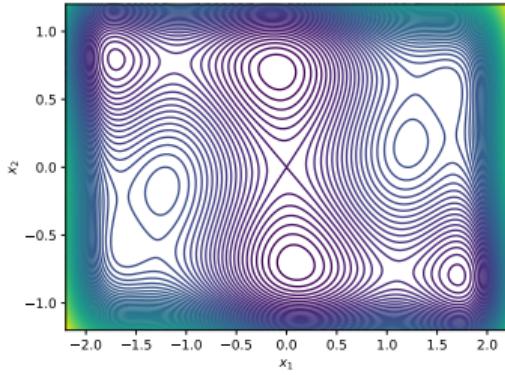
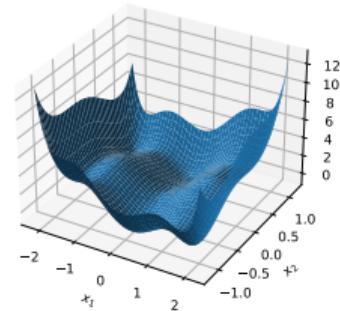
- Comparing the coefficients we get **linear constraints** on Q , e.g., $Q_{11} = 2$
- For $v(x)$ **polynomial** of degree $2d$, we fill $\psi(x)$ with all the **monomials** up to degree d
- This works even if the coefficients of v are (linear functions of) optimization variables
- Other classes of functions could work, polynomials have many properties

The power of SOS optimization

▶ Try this in Drake

Minimize the six-hump-camel function

$$\min_x p(x) = 4x_1^2 + x_1 x_2 - 4x_2^2 - 2.1x_1^4 + 4x_2^4 + x_1^6/3$$



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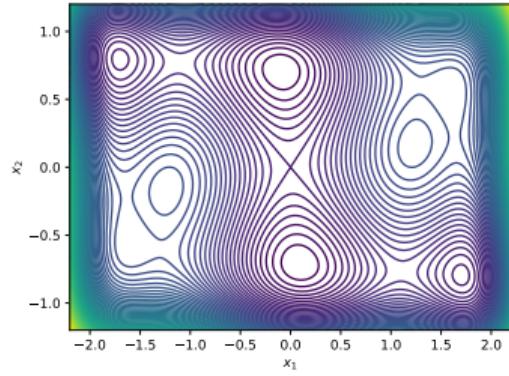
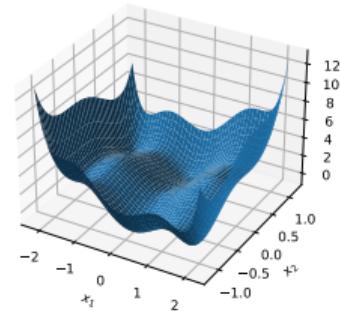
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- Write it as

$$\max_{\lambda, Q} \lambda$$

subject to $p(x) - \lambda$ is SOS

where “is SOS” means “equal to a SOS polynomial in x parameterized by Q ”



Did we just solve global optimization over polynomials?

Not all the nonnegative polynomials are SOS!

- Hilbert in 1888¹ showed that “SOS = nonnegative” only in 3 cases²:
 - Univariate polynomials (1 variable)
 - Quadratic polynomials (degree 2)
 - Bivariate quartics (2 variables, degree 4)



¹Hilbert - “Ueber die Darstellung definiter Formen als Summe von Formenquadraten”

²See also [Hilbert's 17th problem](#)

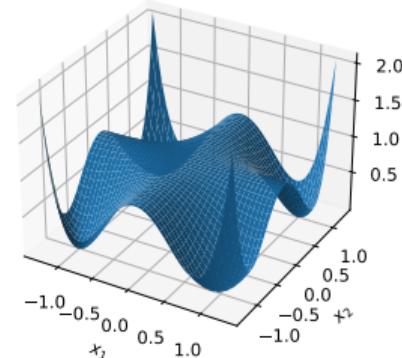
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 - Univariate polynomials (1 variable)
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 - Bivariate quartics (2 variables, degree 4)
- Famous counterexample by Motzkin

$$x_1^4 x_2^2 + x_1^2 x_2^4 + 1 - 3x_1^2 x_2^2$$

- Counterexamples are “rare enough that people give names to them”



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Constrained SOS: the S-procedure

- To constrain the polynomial $v(x)$ to be nonnegative over the set

$$\{x : g(x) = 0\}$$

with $g(x)$ polynomial, we can add a new polynomial $\lambda(x)$ and write

$$v(x) + \lambda(x)g(x) \text{ is SOS}$$

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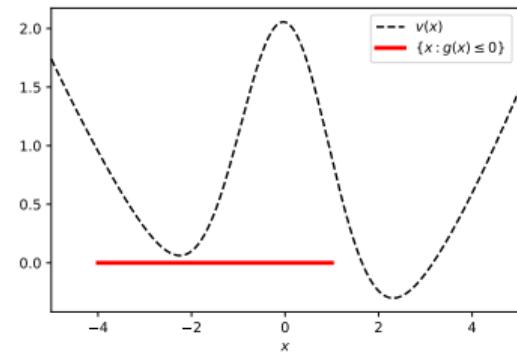
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- Similarly, for the set

$$\{x : g(x) \leq 0\}$$

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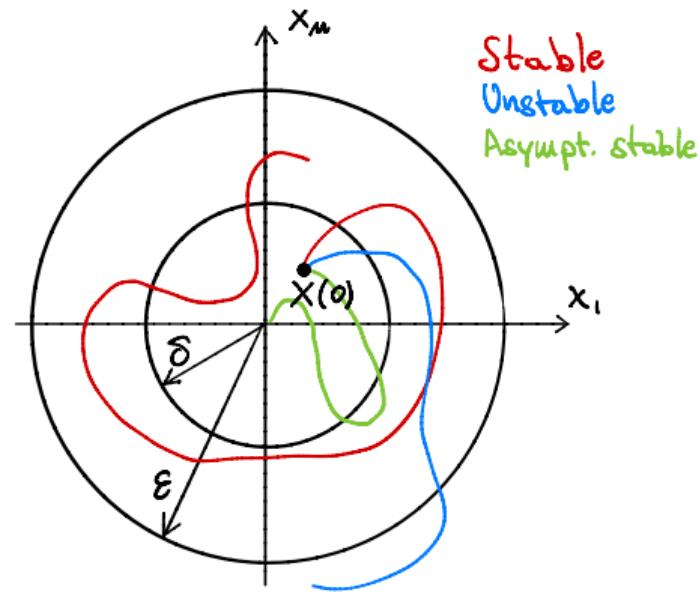
Design of Lyapunov functions

Recap on Lyapunov stability

The equilibrium $x = 0$ for the system $\dot{x} = f(x)$ is:

- **stable** if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \text{ for all } t \geq 0$$



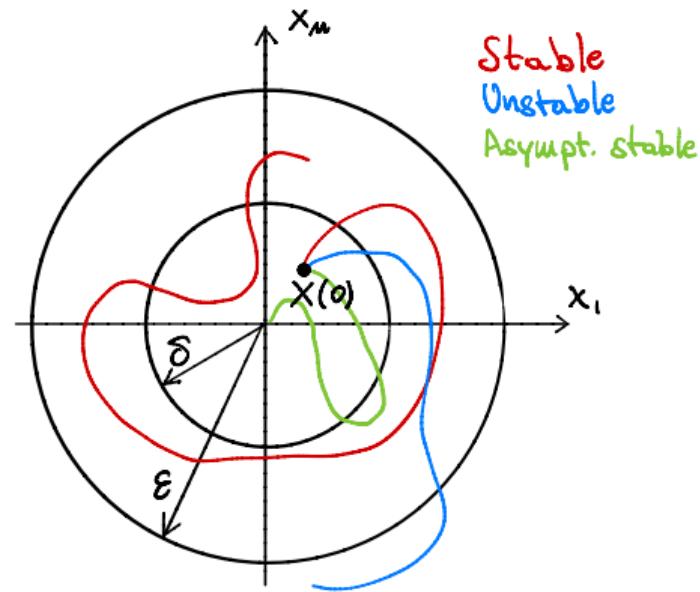
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- **asymptotically stable** if it is stable, and $\lim_{t \rightarrow \infty} \|x(t)\| = 0$



Recap on Lyapunov direct method

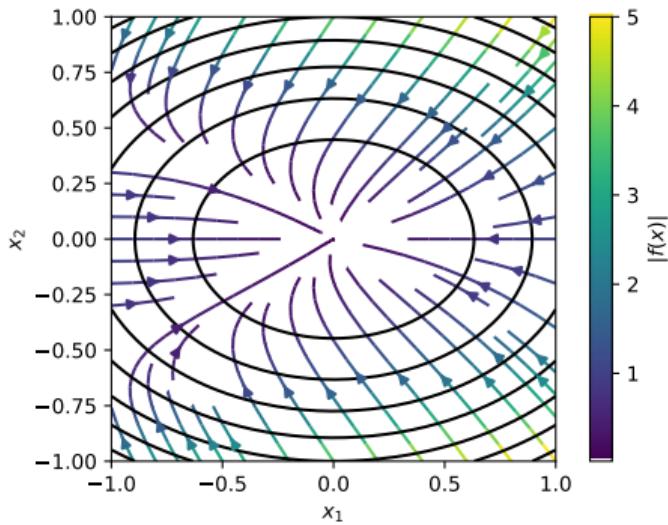
The equilibrium $x = 0$ for the system $\dot{x} = f(x)$ is:

- **stable** iff there exists $v(x)$ such that

$$v(0) = 0, \quad v(x) > 0 \text{ for all } x \neq 0,$$

and

$$\dot{v}(x) = \frac{\partial v}{\partial x}(x)f(x) \leq 0 \text{ for all } x \neq 0$$



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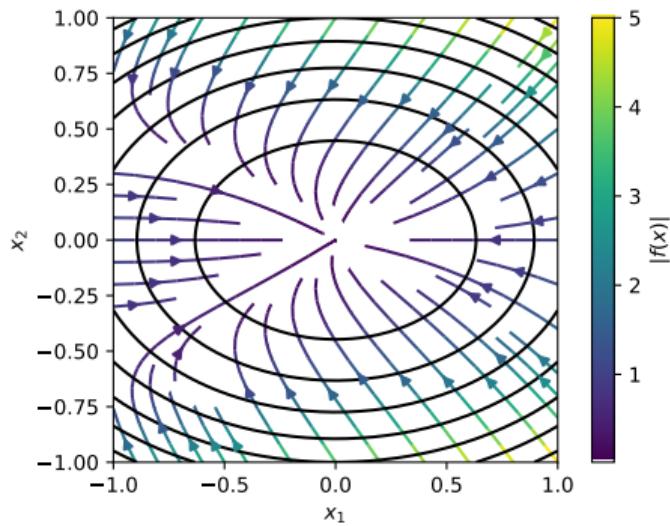
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Design of Lyapunov functions via SOS

SOS parameterization of the Lyapunov function

$$v(x) = \psi^T(x)Q\psi(x), \quad Q \succeq 0$$

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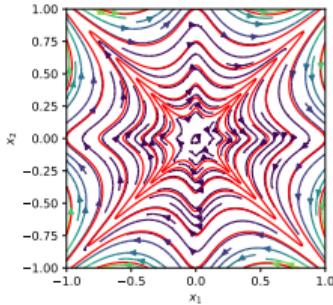
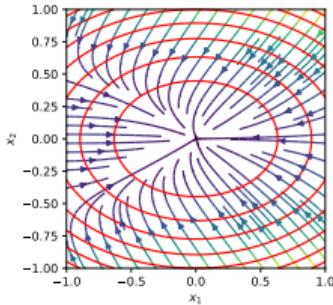
- If $f(x)$ is not polynomial we can use, e.g., Taylor approximation

Design of Lyapunov functions via SOS

▶ Try this in Drake

The overall SOS program

$$\text{find } v(x) \text{ SOS : } -\frac{\partial v}{\partial x}(x)f(x) \text{ is SOS}$$



¹Ahmadi, Krstic, Parrilo - “A Globally Asymptotically Stable Polynomial Vector Field with no Polynomial Lyapunov Function”

Design of Lyapunov functions via SOS

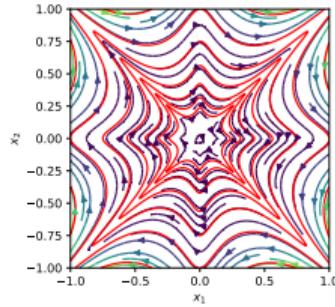
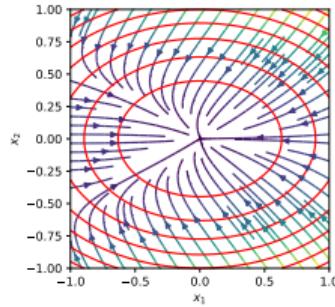
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Miscellaneous:

- To rule out the trivial solution $v(x) = 0$ we can require $v(1) = 1$
- Complexity is polynomial in the number of states x , but exponential in the degree of v
- Another “leak”: not all the stable polynomial systems admit a polynomial Lyapunov function³



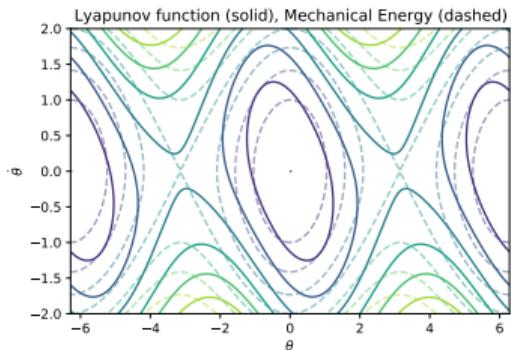
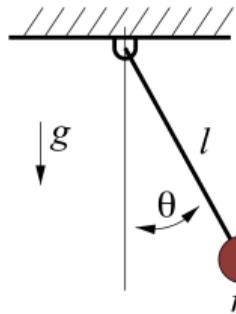
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Verification of non-polynomial systems via SOS

▶ Try this in Drake

- Some non-polynomial systems can be verified exactly using SOS⁴

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - \sin(x_1)$$



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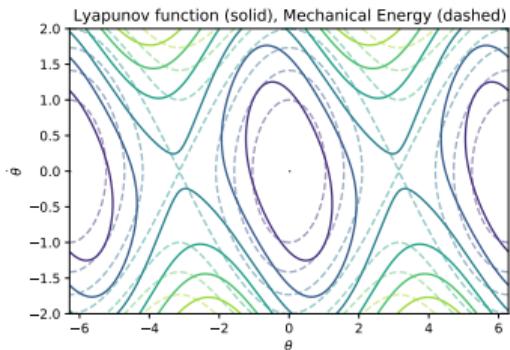
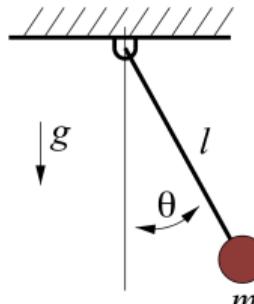
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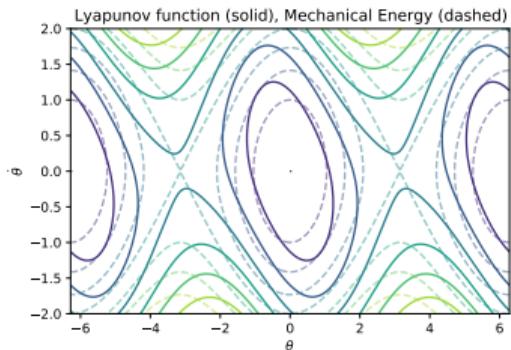
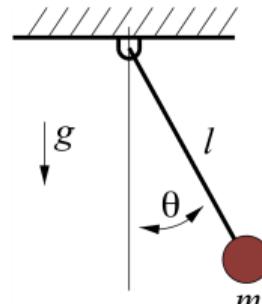
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- Introduce auxiliary variables $s = \sin(x_1)$, $c = \cos(x_1)$
- Substitute to get the polynomial system

$$\dot{s} = cx_2, \quad \dot{c} = -sx_2, \quad \dot{x}_2 = -x_2 - s$$

subject to the polynomial constraint $s^2 + c^2 = 1$



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Approximation of region of attraction via SOS

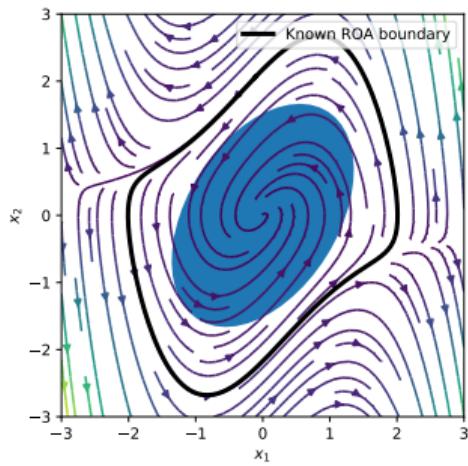
▶ Try this in Drake

Time-reversed Van der Pol oscillator

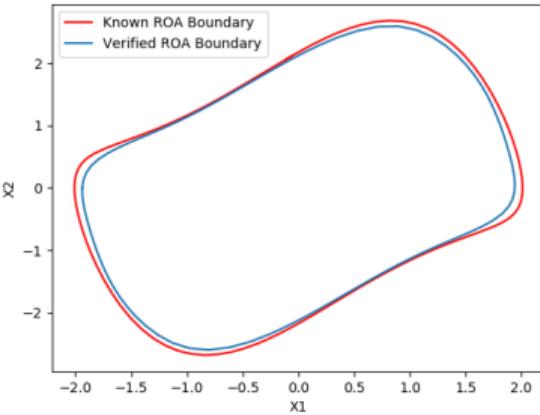
$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

Maximize volume of level set



Alternate maximization of level set and reshape Lyapunov function



SOS for control-Lyapunov function?

We can't do control synthesis with Lyapunov and SOS:

- SOS Lyapunov function $v(x)$
- Linear parameterization for $u(x)$
- For simplicity, let the system be control-affine

$$\dot{x} = f(x) + G(x)u$$

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- Lyapunov condition is

$$-\frac{\partial v}{\partial x}(x)[f(x) + G(x)u(x)] \text{ is SOS}$$

- Multiplication of $\frac{\partial v}{\partial x}$ and u leads to **nonlinear equality constraints**

Approximate dynamic programming

Hamilton-Jacobi-Bellman (HJB) equation

How to arrive to the HJB equation

$$\min_{u \in U} \left\{ l(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \right\} = 0, \quad \text{for all } x$$

from the Optimal Control Problem (OCP)

$$v(x_0) := \min_{u,x} \int_0^\infty l(x(t), u(t)) dt$$

subject to $x(0) = x_0$

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{for all } t \in [0, \infty)$$

$$u(t) \in U, \quad \text{for all } t \in [0, \infty)$$

HJB: informal derivation

First order approximation with time step h :

- Cost function

$$\int_0^\infty I(x(t), u(t))dt \approx h \sum_{k=0}^{\infty} I(x(hk), u(hk))$$

- Dynamics

$$x(t + h) \approx x(t) + hf(x(t), u(t))$$

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- Calling $x_k := x(hk)$ and $u_k := u(hk)$, we get

$$v(x_0) := \min_{u,x} h \sum_{k=0}^{\infty} I(x_k, u_k)$$

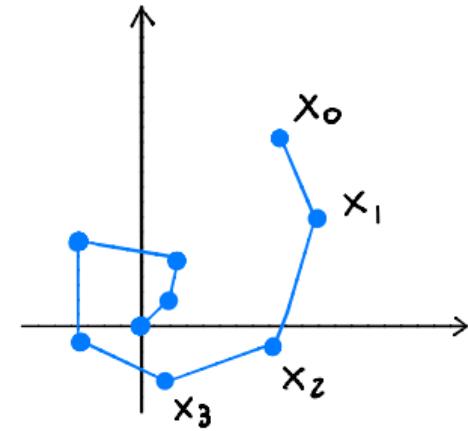
subject to x_0 given

$$\begin{aligned} x_{k+1} &= x_k + hf(x_k, u_k), & k = 0, \dots, \infty \\ u_k &\in U, & k = 0, \dots, \infty \end{aligned}$$

HJB: informal derivation

The dynamic-programming principle:

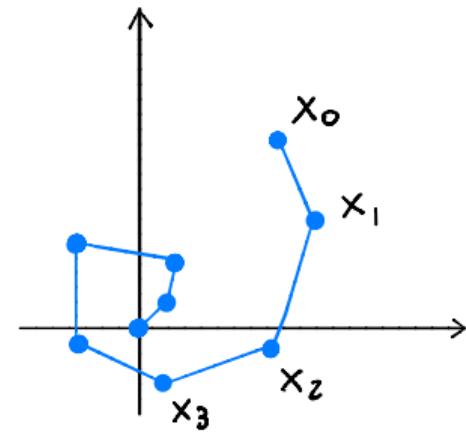
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- Assume $\mathbf{x}_0 = (x_0, x_1, x_2, x_3, \dots)$ is the optimal trajectory from x_0 to the origin
- Because of the infinite horizon, the optimal trajectory \mathbf{x}_1 from x_1 to the origin must be (x_1, x_2, x_3, \dots)
 - Otherwise (x_0, \mathbf{x}_1) would be better than \mathbf{x}_0 (contradiction)

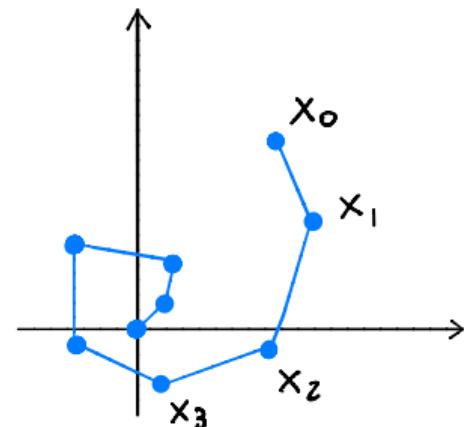


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 - Otherwise (x_0, \mathbf{x}_1) would be better than \mathbf{x}_0 (contradiction)
- If we were to know the cost-to-go $v(x_1)$ from x_1 , then we could just solve a one-step problem

$$\begin{aligned} v(x_0) &= \min_{u_0 \in U} \{ h_l(x_0, u_0) + v(x_1) \} \\ &= \min_{u_0 \in U} \{ h_l(x_0, u_0) + v(x_0 + h_f(x_0, u_0)) \} \end{aligned}$$



HJB: informal derivation

- If h is very small,

$$v(x_0 + hf(x_0, u_0)) \approx v(x_0) + h \frac{\partial v}{\partial x}(x_0) f(x_0, u_0)$$

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- Simplifying $v(x_0)$ and dividing by h

$$0 = \min_{u_0 \in U} \left\{ l(x_0, u_0) + \frac{\partial v}{\partial x}(x_0) f(x_0, u_0) \right\}$$

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Observations

- If we know the value function $v(x)$, the optimal controller $u(x)$ is the argmin of the HJB
- Why “informal”? (Remember the bang bang controller)

Solving the HJB

▶ Try this in Drake

The HJB equation is a nonlinear PDE

- Analytic solutions only in a few special cases (LQR)
- “Exact” numerical solution is very hard:
 - mainly variants of the **Value-Iteration (VI) algorithm**
 - require discretization of the state and control space

Solving the HJB

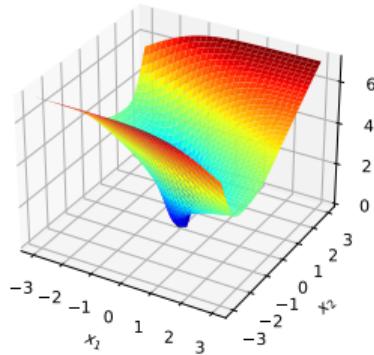
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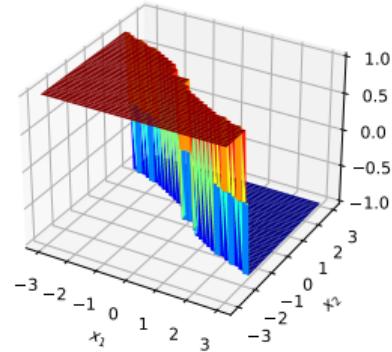
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Bang-bang control of the double integrator via VI

Value function $v(x)$



Controller $u(x)$



Solving the HJB approximately

Can we still find approximate solutions of the HJB that are “good enough”?

“One side of the HJB equation is convex”

$$\min_{u \in U} \left\{ I(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \right\} \geq 0, \quad \text{for all } x$$

is equivalent to the so-called **Bellman inequality**

$$I(x, u) + \frac{\partial v}{\partial x}(x) f(x, u) \geq 0, \quad \text{for all } x \text{ and } u \in U$$

- We can relax this condition via SOS!

Bellman inequality

What do we loose by enforcing only one side of the HJB?

Bellman inequality

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Bellman inequality

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- Let $x(t)$ and $u(t)$ be the optimal trajectory and control from $x(0)$
- Integrate the Bellman inequality along this optimal trajectory

$$\begin{aligned} 0 &\leq \int_0^\infty \left[I(x(t), u(t)) + \frac{\partial v}{\partial x}(x(t))f(x(t), u(t)) \right] dt \\ &= \int_0^\infty I(x(t), u(t))dt + \int_0^\infty \dot{v}(x(t))dt \\ &= \int_0^\infty I(x(t), u(t))dt + v(x(\infty)) - v(x(0)) \end{aligned}$$

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- Assume $v(0) = 0$, then $v(x)$ lower bounds the optimal value function

$$v(x(0)) \leq \int_0^\infty I(x(t), u(t))dt$$

SOS for approximate dynamic programming

$$\max_v \int_X v(x) dx$$

subject to $I(x, u) + \frac{\partial v}{\partial x}(x)f(x, u)$ is SOS for all x and $u \in U$

$$v(0) = 0$$

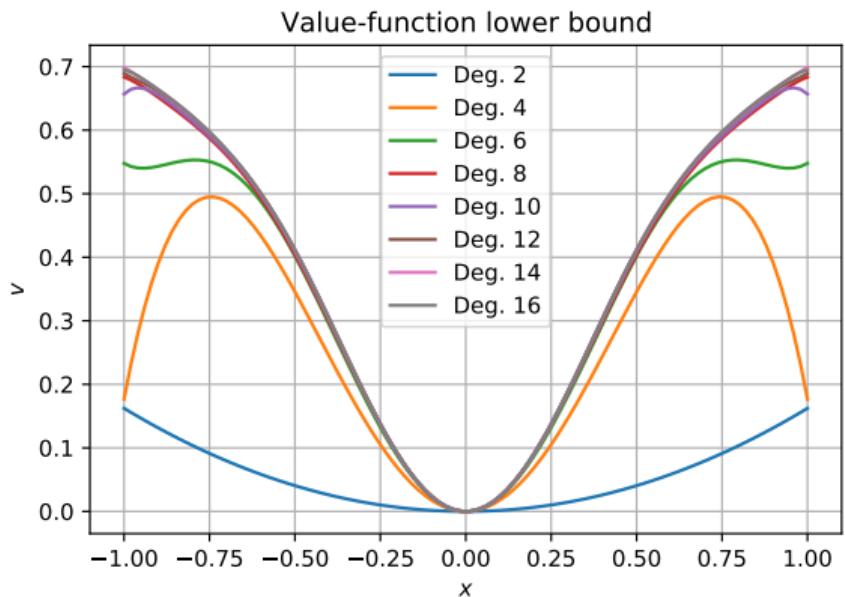
Interpretation

- Constraints ensure that $v(x)$ lower bounds the value function
- Objective “pushes up” $v(x)$ over the set X (note that the objective is linear in Q)
- As the degree of $v(x)$ increases, we get a better and better approximation of the value function over the set X

Toy approximate-DP problem

▶ Try this in Drake

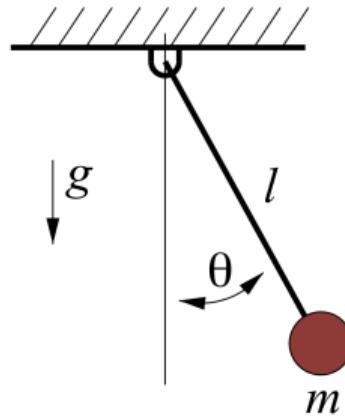
- Scalar dynamics $f(x, u) = x - 4x^3 + u$
- Quadratic running cost $l(x, u) = x^2 + u^2$
- Control limits $u \in U = [-1, 1]$
- Approximate the value function for $x \in X = [-1, 1]$



Pendulum swing up

► Try this in Drake

- Find a controller that stabilizes the pendulum in the upright configuration
- Use S-procedure to handle sines and cosines
- Approximate value function of degree 10



Conclusions

Take-home messages

- Convex optimization is a mature and extremely powerful tool
- Not all the convex optimizations are easy
- Semidefinite programs are easy and they can tackle surprisingly many problems
- Sums-of-squares optimization is a game changer in stability analysis
- SOS is quite effective also in optimal control

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