(i) Prove that for every sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} < \mathrm{e} \sum_{n=1}^{\infty} a_n,$$

where e is the natural log base.

(ii) Prove that for every $\varepsilon > 0$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} > (\mathrm{e} - \varepsilon) \sum_{n=1}^{\infty} a_n.$$