Find all $r > 0$ such that whenever $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $|\mathrm{grad} f(0,0)| = 1$ and $|\mathrm{grad} f(u) - \mathrm{grad} f(v)| \le |u - v|$ for all $u, v \in \mathbb{R}^2$, then the maximum of f on the disk $\{u \in \mathbb{R}^2 : |u| \le r\}$ is attained at exactly one point.
($\mathrm{grad} f(u) = (\partial_1 f(u), \partial_2 f(u))$ is the gradient vector of f at the point u. For a vector $u = (a, b)$, $|u| = \sqrt{a^2 + b^2}$.)