Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions on the interval $[0,1]$ such that

$$\int_0^1 f_m(x) f_n(x)\,\mathrm{d}x = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\sup\{|f_n(x)| \mid x \in [0,1] \text{ and } n = 1, 2, \dots\} < +\infty.$$

Show that there exists no subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{k\to\infty} f_{n_k}(x)$ exists for all $x \in [0,1]$.