

Let \mathcal{H} be an infinite-dimensional real Hilbert space, let $d > 0$, and suppose that S is a set of points (not necessarily countable) in \mathcal{H} such that the distance between any two distinct points in S is equal to d . Show that there is a point $y \in \mathcal{H}$ such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : x \in S \right\}$$

is an orthonormal system of vectors in \mathcal{H} .