a) Let the mapping $f : M_n \to \mathbb{R}$ from the space $M_n = \mathbb{R}^{n^2}$ of $n \times n$ matrices with real entries to reals be linear, i.e.:

$$f(A+B) = f(A) + f(B), \; f(cA) = cf(A) \quad (1)$$

for any $A, B \in M_n, c \in \mathbb{R}$. Prove that there exists a unique matrix $C \in M_n$ such that $f(A) = \operatorname{tr}(AC)$ for any $A \in M_n$. (If $A = \{a_{ij}\}_{i,j=1}^n$ then $\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$).

b) Suppose in addition to (1) that

$$f(AB) = f(BA) \quad (2)$$

for any $A, B \in M_n$. Prove that there exists $\lambda \in \mathbb{R}$ such that $f(A) = \lambda \operatorname{tr}(A)$.