Define the sequence $f_1, f_2, \ldots : [0,1) \to \mathbb{R}$ of continuously differentiable functions by the following recurrence:

$$f_1 = 1;\ f'_{n+1} = f_n f_{n+1} \text{ on } (0,1), \text{ and } f_{n+1}(0) = 1.$$

Show that $\lim_{n\to\infty} f_n(x)$ exists for every $x \in [0,1)$ and determine the limit function.