For R > 1 let  $\mathcal{D}_R = \{(a, b) \in \mathbb{Z}^2 : 0 < a^2 + b^2 < R\}$ . Compute

pute 
$$\lim_{R \to \infty} \sum_{\substack{(a,b) \in \mathcal{D}_R \\ a^2 + b^2}} \frac{(-1)^{a+b}}{a^2 + b^2}.$$