Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers such that $b_0 = 1$, $b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}$. Calculate

$$\sum_{n=1}^{\infty} b_n 2^n.$$