For $R > 1$ let $\mathcal{D}_R = \{(a, b) \in \mathbb{Z}^2 : 0 < a^2 + b^2 < R\}$. Compute

$$\lim_{R\to\infty} \sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2 + b^2}.$$