For $n = 1, 2, \ldots$ let

$$S_n = \log\left(\sqrt[n^2]{1^1 \cdot 2^2 \cdot \ldots \cdot n^n}\right) - \log(\sqrt{n}),$$

where log denotes the natural logarithm. Find $\lim_{n\to\infty} S_n$.