a) Let a, b be real numbers such that $b \leq 0$ and $1+ax+bx^2 \geq 0$ for every x in $[0,1]$. Prove that

$$\lim_{n\to+\infty} n \int_0^1 (1+ax+bx^2)^n \, \mathrm{d}x = \begin{cases} -\frac{1}{a} & \text{if } a<0, \\ +\infty & \text{if } a \geq 0. \end{cases}$$

b) Let $f : [0,1] \to [0,\infty)$ be a function with a continuous second derivative and let $f''(x) \leq 0$ for every x in $[0,1]$. Suppose that $L = \lim_{n\to\infty} n \int_0^1 (f(x))^n \, \mathrm{d}x$ exists and $0 < L < +\infty$. Prove that f' has a constant sign and $\min_{x\in[0,1]}|f'(x)| = L^{-1}$.