Define the sequence $x_1, x_2, \ldots$ by the initial terms $x_1 = 2$, $x_2 = 4$, and the recurrence relation

$$x_{n+2} = 3x_{n+1} - 2x_n + \frac{2^n}{x_n} \quad \text{for } n \geq 1.$$

Prove that $\lim_{n\to\infty} \frac{x_n}{2^n}$ exists and satisfies

$$\frac{1+\sqrt{3}}{2} \leq \lim_{n\to\infty} \frac{x_n}{2^n} \leq \frac{3}{2}.$$