Upper content of a subset E of the plane $\mathbb{R}^2$ is defined as

$$\mathcal{C}(E) = \inf\left\{\sum_{i=1}^{n} \operatorname{diam}(E_i)\right\}$$

where inf is taken over all finite families of sets $E_1, \ldots, E_n$, $n \in \mathbb{N}$, in $\mathbb{R}^2$ such that $E \subset \bigcup_{i=1}^n E_i$. Lower content of E is defined as

$$\mathcal{K}(E) = \sup\{\operatorname{length}(L) \mid \\ L \text{ is a closed line segment onto which } E \text{ can be contracted}\}.$$

Show that

(a) $\mathcal{C}(L) = \operatorname{length}(L)$ if L is a closed line segment;

(b) $\mathcal{C}(E) \geq \mathcal{K}(E)$;

(c) the equality in (b) needs not hold even if E is compact.

Remarks: All *distances* used in this Problem are Euclidian. *Diameter* of a set E is $\operatorname{diam}(E) = \sup\{\operatorname{dist}(x, y) | x, y \in E\}$. *Contraction* of a set E to a set F is a mapping $f : E \to F$ such that $\operatorname{dist}(f(x), f(y)) \leq \operatorname{dist}(x, y)$ for all $x, y \in E$. A set E can be contracted *onto* a set F if there is a contraction f of E to F which is onto, i.e., such that $f(E) = F$.