Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties:

(i) $\mathcal{F}$ contains at least $\binom{n}{k}+1$ distinct sets containing exactly k elements;

(ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$.

Prove that $\mathcal{F}$ contains at least three sets with at least n elements.