Let $\mathbb{M}$ be the vector space of $m \times p$ real matrices. For a vector subspace $S \subseteq \mathbb{M}$, denote by $\delta(S)$ the dimension of the vector space generated by all columns of all matrices in S.

Say that a vector subspace $T \subseteq \mathbb{M}$ is a *covering matrix space* if

$$\bigcup_{A \in T, A \neq 0} \ker A = \mathbb{R}^p.$$

Such a T is *minimal* if it does not contain a proper vector subspace $S \subset T$ which is also a covering matrix space.

(a) Let T be a minimal covering matrix space and let $n = \dim T$. Prove that $\delta(T) \leq \binom{n}{2}$.

(b) Prove that for every positive integer n we can find m and p, and a minimal covering matrix space T as above such that $\dim T = n$ and $\delta(T) = \binom{n}{2}$.