

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function with the property that for any x and y in the interval,

$$xf(y) + yf(x) \leq 1.$$

a) Show that

$$\int_0^1 f(x) \, dx \leq \frac{\pi}{4}.$$

b) Find $x_1, y_1 \in [0, 1]$ and a function f with the same property for which

$$x_1 f(y_1) + y_1 f(x_1) = 1.$$