Let

$$\mathcal{P} = \left\{ f \,\middle|\, f(x) = \sum_{k=0}^{3} a_k x^k,\ a_k \in \mathbb{R},\ |f(\pm 1)| \le 1,\ \left| f\left(\pm \frac{1}{2}\right) \right| \le 1 \right\}.$$

Evaluate

$$\sup_{f \in \mathcal{P}} \max_{-1 \le x \le 1} |f''(x)|$$

and find all polynomials $f \in \mathcal{P}$ for which the above "sup" is attained.