For a natural number n consider the hyperplane

$$R_0^n = \left\{ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{n=1}^{n} x_i = 0 \right\}$$

and the lattice $Z_0^n = \{y \in R_0^n : \text{ all } y_i \text{ are integers}\}$. Define the (quasi-)norm in $\mathbb{R}^n$ by $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ if $0 < p < \infty$, and $\|x\|_\infty = \max_i |x_i|$.

a) Let $x \in R_0^n$ be such that

$$\max_i x_i - \min_i x_i \leq 1.$$

For every $p \in [1, \infty]$ and for every $y \in Z_0^n$ prove that

$$\|x\|_p \leq \|x + y\|_p.$$

b) For every $p \in (0, 1)$, show that there is an n and an $x \in R_0^n$ with $\max_i x_i - \min_i x_i \leq 1$ and $y \in Z_0^n$ such that

$$\|x\|_p > \|x + y\|_p.$$