Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair (x, y) of real numbers, if

$$|x - y| < \min \bigl\{g(x), g(y)\bigr\},$$

then

$$\bigl|f(x) - f(y)\bigr| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \ldots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.