Let $P(x) = x^2 - 1$. How many distinct real solutions does the following equation have:

$$\underbrace{P(P(\ldots(P(x))\ldots))}_{2004} = 0?$$