

# Entwurf, Analyse und Umsetzung von Algorithmen

Cache Efficiency, Divide and Conquer

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science

Entwurf, Analyse und Umsetzung von Algorithmen



**iems**

intelligente eingebettete  
mikrosysteme

# Structure

## Cache Efficiency

- Introduction

- Cache Organization

## Divide and Conquer

- Introduction

# Cache Efficiency

## Introduction

### **Background:**

- ▶ Up to now we always counted the **number of operations**
- ▶ Assuming this is a good measure for the runtime of an algorithm/tool
- ▶ Today we will see examples where this is not suitable

# Cache Efficiency

## Introduction

### Example:

- ▶ We sum up all elements of an array  $a$  of size  $n$  in ...

- ▶ natural order:

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

- ▶ random order:

$$\text{sum}(a) = a[21] + a[5] + \cdots + a[8]$$

# Cache Efficiency

## Linear Order - Python

### Python:

```
def init(size):  
    """Creates the dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

# Cache Efficiency

## Linear Order - Python

### Python:

```
def run(param):  
    """Processes the dataset."""  
  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```

# Cache Efficiency

## Linear Order

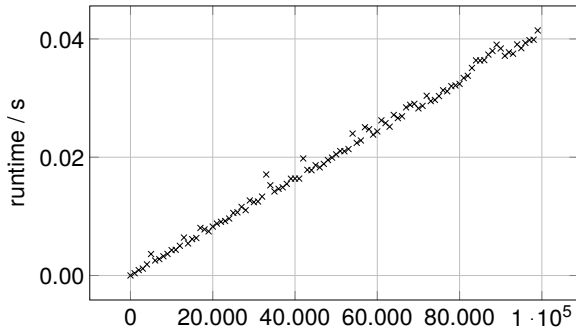


Figure: summing elements in linear order

# Cache Efficiency

## Random Order - Python

```
def init(size):  
    """Creates a randomly ordered dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```



# Cache Efficiency

## Random Order

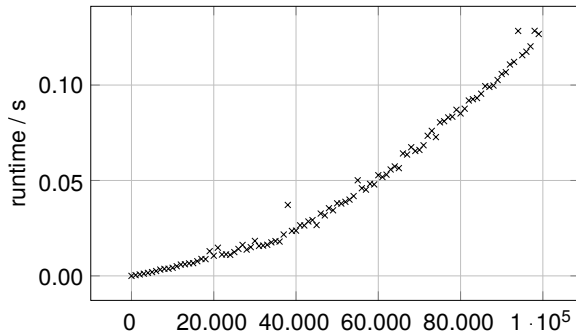


Figure: summing elements in random order

# Cache Efficiency

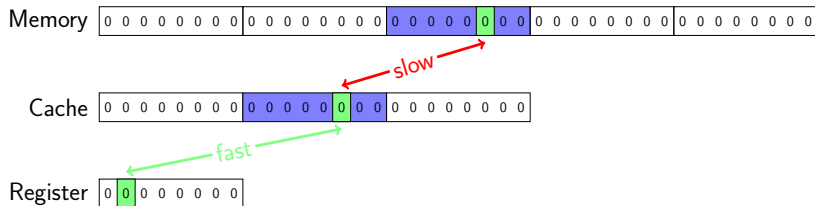
## Algorithm Comparision

### **Conclusion:**

- ▶ The number of operations is identical for both algorithms
- ▶ Accessing elements in random order takes a lot longer (factor 10)
- ▶ The costs in terms of memory access are very different

# Cache Efficiency

## CPU Cache

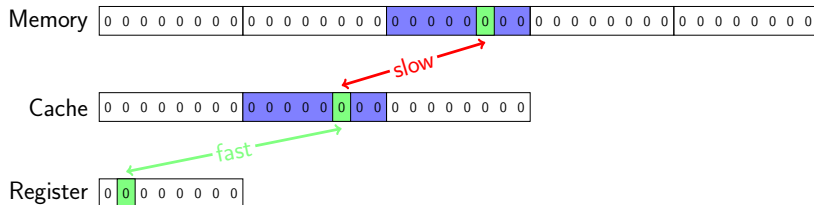


## Principle / organization:

- ▶ Accessing one byte of the main memory takes  $\approx 100$  ns
- ▶ Accessing one byte of (L1-)cache takes  $\approx 1$  ns
- ▶ Accessing one or more byte/s of main memory loads a whole block  $\approx 100$  B into the cache
- ▶ As long as this block is in the cache, it is not necessary to access the memory for bytes of this block

# Cache Efficiency

## CPU Cache

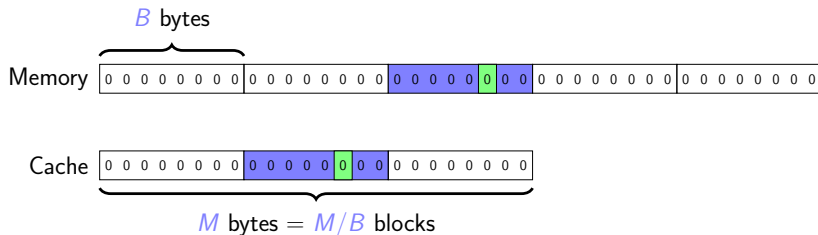


## Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks
  - ▶ Cache lines  $\approx$  100 kB
- ▶ If the capacity is reached unused blocks are discarded
  - ▶ Least recently used (LRU)
  - ▶ Least frequently used (LFU)
  - ▶ First in first out (FIFO)
- ▶ Details of discarding not discussed today

# Cache Efficiency

## Block Operations

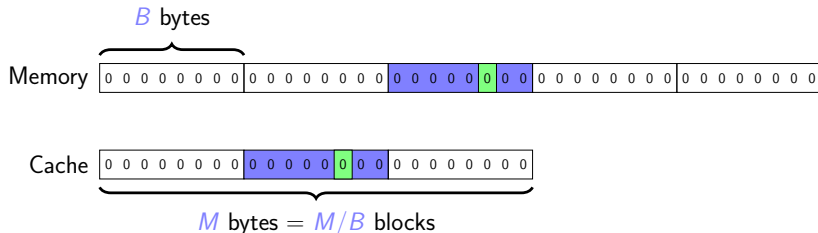


### Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size  $B$**
- ▶ The **fast cache** has size  $M$  and can store  $M/B$  blocks
- ▶ If data is not in fast memory, the corresponding block is loaded into the **cache**

# Cache Efficiency

## Block Operations



## Terminology:

- ▶ The program defines which blocks are held in the **cache**
- ▶ We use the number of **block operations** as runtime estimation
- ▶ We ignore runtime costs of cache access / management



# Cache Efficiency

## Block Operations

### Additional factors:

- ▶ The following settings change only a small constant factor in number of block operations
  - ▶ Partitioning of the slow memory into blocks
  - ▶ Regardless of the block size: 1 bytes or 4 bytes or 8 bytes

### Note:

- ▶ If the input size is smaller than  $M$  we load the complete data chunk directly into the cache
- ▶ Cache handling is only interesting when the input size is greater than  $M$



# Cache Efficiency

## Block Operations

**Typical values:** (Intel® i7-4770 Haswell, WD® Blue 2 TB)

- ▶ CPU L1 Cache:  $B = 64 \text{ B}$ ,  $M = 4 \times (32 \text{ kB} + 32 \text{ kB})$
- ▶ CPU L2 Cache:  $B = 64 \text{ B}$ ,  $M = 4 \times 256 \text{ kB}$
- ▶ CPU L3 Cache:  $B = 64 \text{ B}$ ,  $M = 8 \text{ MB}$
- ▶ Disk Cache:  $B = 64 \text{ kB}$ ,  $M = 64 \text{ MB}$ 
  - ▶ Many operating systems use free system memory as disk cache

# Cache Efficiency

## Block Operations

### **Terminology:**

- ▶ Block loads on CPU cache are called **cache misses**
- ▶ Block operations on disk cache are called **IOs**  
(input / output operations)
- ▶ These also fall under the term **cache efficiency** or **IO efficiency**

# Cache Efficiency

## Block Operations - Linear Order

### Example 1 - Linear order:

- We sum up all elements in **natural order**

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

- The number of block operations is  $\text{ceil}\left(\frac{n}{B}\right)$

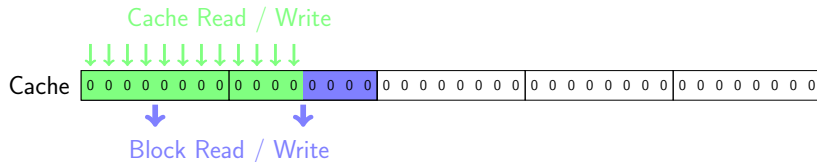


Figure: good locality of sum operation

# Cache Efficiency

## Block Operations - Random Order

### Example 2 - Random order:

- ▶ We sum up all elements in **random order**

$$\text{sum}(a) = a[21] + a[5] + \dots + a[8]$$

- ▶ The number of block operations is  $n$  in the **worst case**
- ▶ This leads to a runtime factor difference of  $B$

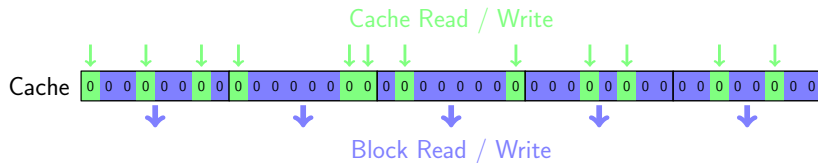


Figure: bad locality of sum operation

# Cache Efficiency

## Block Operations

**Generally the factor is substantially  $< B$**

- ▶ Even with a **random order** we access 4 neighboring bytes at once per `int` (`int32_t`)
- ▶ The next element might already be loaded into the cache
- ▶ If **not  $n \gg M$**  this might occur with a high probability

# Cache Efficiency

## Block Operations - Quicksort

### Quicksort:

- ▶ **Strategy:** Divide and Conquer
- ▶ Divide the data into two parts where the “left” part contains all values  $\leq$  the values in the right part
- ▶ Choose one element (e.g the first one) as “pivot” element
- ▶ Ideally both parts are the same size
- ▶ Both parts are sorted recursively

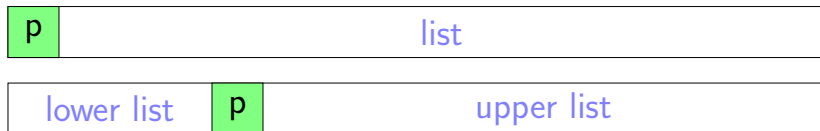
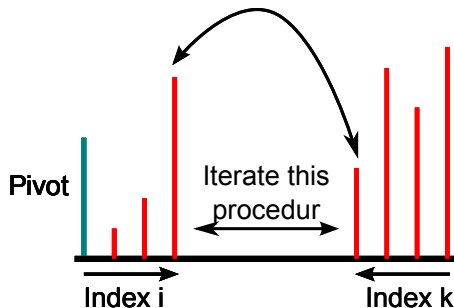


Figure: *Quicksort* with pivot element

# Idea of Quicksort

- ▶ **At start:** pivot in first position, first re-arrange list such that left part contains smaller and right part larger elements
- ▶ Do required changes *in place*



- ▶ **End point:**  $k$  is left to left-most element greater than pivot  
*swap position 0 (pivot) with  $k$  (smaller than pivot)*

# Cache Efficiency

## Block Operations - Quicksort - Python

### Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[start]  
  
    ...
```



# Cache Efficiency

## Block Operations - Quicksort - Python

```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```

# Cache Efficiency

## Block Operations - Quicksort

### Number of operations for Quicksort:

- ▶ Let  $T(n)$  be the runtime for the input size  $n$

### Assumptions:

- ▶ Arrays are always separated perfectly in the middle
- ▶  $n$  is a power-of-two and recursion depth is  $k = \log_2 n$

# Cache Efficiency

## Block Operations - Quicksort

$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left( A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$

# Cache Efficiency

## Block Operations - Quicksort

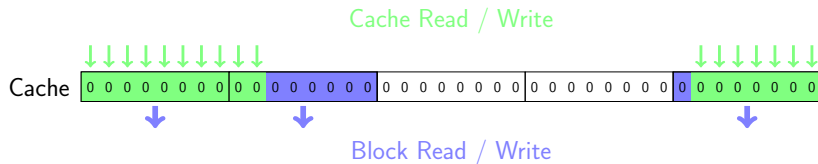


Figure: locality of Quicksort

- ▶ Let  $IO(n)$  be the number of **block operations** for input size  $n$
- ▶ Assumptions as before but recursion depth is  $k = \log_2 \frac{n}{B}$

# Cache Efficiency

## Block Operations - Quicksort

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in \mathcal{O}\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

# Divide and Conquer

## Introduction

### Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.  
If subproblems are small enough solve them directly
- ▶ **Connect** all solutions of the subproblems to the solution of the full problem
- ▶ **Recursive** application of the algorithm to ever smaller subproblems
- ▶ **Direct** solving of sufficiently small subproblems

# Divide and Conquer

## Introduction - Python

- ▶ Function `solve` for solving a `problem` of size `n`

```
def solve(problem):  
    if n < threshold:  
        return solution # solve directly  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
    return S1 + S2 + ... + Sk
```

# Divide and Conquer

## Features

### **Divide and Conquer:**

- ▶ Can help with conceptual hard problems
- ▶ **Solution** of the trivial problems has to be known
- ▶ **Dividing** into subproblems has to be possible
- ▶ **Combination** of solutions has to be possible



# Divide and Conquer

## Features

### Features:

- ▶ Realization of **efficient solutions**
  - ▶ If trivial solution is  $\in O(1)$
  - ▶ And separation / combination of subproblems is  $\in O(n)$
  - ▶ And the number of subproblems is limited
  - ▶ The runtime is  $\in O(n \cdot \log n)$
- ▶ Suitable for parallel processing
  - ▶ Parallel processing of subproblems possible since subproblems are **independent** of each other

# Divide and Conquer

## Implementation

### **Definition of the trivial case:**

- ▶ Smaller subproblems are elegant and simple
- ▶ On the other hand the efficiency will be improved if relatively big subproblems can be solved directly
- ▶ Recursion depth should not get too big (stack / memory overhead)

# Divide and Conquer

## Implementation

### **Division in subproblems:**

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

### **Combination of solutions:**

- ▶ Typically conceptionally demanding

# Divide and Conquer

## Example - Maximum Subtotal

### Example - Maximum Subtotal Input:

- ▶ Sequence  $X$  of  $n$  integers

### Output:

- ▶ Maximum sum of related subsequence and its index boundary

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

**Output:** sum: 187, start: 2, end: 6

# Divide and Conquer

## Example - Maximum Subtotal

### Application:

- ▶ Maximum profit of buying and selling shares

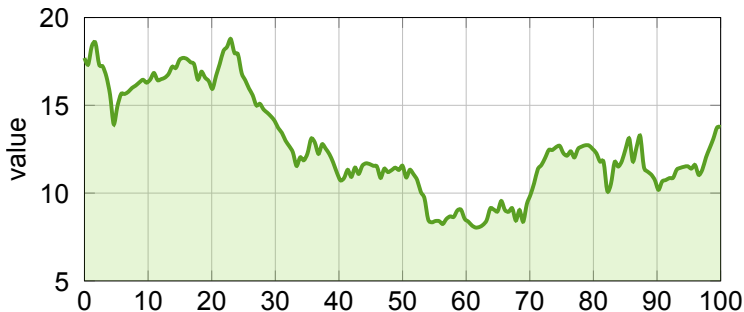


Figure: stock value over time

# Divide and Conquer

## Example - Maximum Subtotal - Python

### Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

## Example - Maximum Subtotal - Python

### Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

## Example - Maximum Subtotal

### Upper bound:

- ▶ Three nested loops
- ▶ Each loop with runtime  $O(n)$
- ▶ Algorithm runtime of  $O(n^3)$



# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Lower bound:

Table: Operations

$i$	Additions	$j$
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least  $\frac{n}{3}$  values for  $i$
- ▶ For each  $i$  we iterate at least  $\frac{n}{3}$  values for  $j$
- ▶ For each  $j$  we have at least  $\frac{n}{3}$  additions
- ▶ We need at least  $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$  steps

# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Runtime:

- ▶ With  $T(n) \in O(n^3)$  and  $T(n) \in \Omega(n^3)$  we know:

$$T(n) \in \Theta(n^3)$$

- ▶ It is hard to solve the problem in a worse way ...

# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Current approach:

- ▶ Calculating the sum for range from  $i$  to  $j$  with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

### Better approach:

- ▶ Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i + 1] + \cdots + X[j] + X[j + 1]$$

$$S_{i,j+1} = S_{i,j} + X[j + 1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

# Divide and Conquer

## Example - Maximum Subtotal - Python

### Better solution:

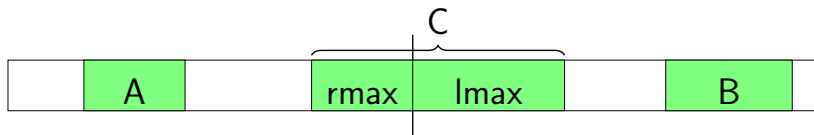
```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

► Runtime  $\in O(n^2)$

# Divide and Conquer

## Example - Maximum Subtotal

### Divide and Conquer:



### Divide and Conquer idea to solve:

- ▶ Split the sequence in the middle
- ▶ Solve left half of the problem
- ▶ Solve right half and combine both solutions into one
- ▶ Maximum might be located in left half ( $A$ ) or right half ( $B$ )
- ▶ Problem: Maximum can overlap the split
- ▶ To solve this case we have to calculate  $rmax$  and  $lmax$
- ▶ The overall solution is the maximum of  $A$ ,  $B$  and  $C$

# Divide and Conquer

## Example - Maximum Subtotal

### Principle - Divide and Conquer:

- ▶ Small problems are solved directly:  $n = 1 \Rightarrow \text{max} = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and solved recursively. Subsolutions  $A$  and  $B$  are returned
- ▶ To determine subsolution  $C$ ,  $\text{rmax}$  and  $\text{lmax}$  for the subproblems are computed
- ▶ The overall solution is the maximum of  $A$ ,  $B$  and  $C$

# Divide and Conquer

## Example - Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) // 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

# Further Literature

## ► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

*Introduction to Algorithms.*

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.



# Further Literature

## ► Caching

[Wik] [Cache](https://en.wikipedia.org/wiki/Cache)

`https://en.wikipedia.org/wiki/Cache`