Entwurf, Analyse und Umsetzung von Algorithmen

Open Addressing, Priority Queue

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Entwurf, Analyse und Umsetzung von Algorithmen

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Structure

Hashing

- Recapitulation
- Treatment of hash collisions
- Open Addressing
- Summary

Priority Queue

- Introduction

Hashing

Recapitulation

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  - Then the random keys make sure that it is distributed evenly
- To find a good hash function for every key set, universal hashing is needed
  - Then however, for a fixed set of keys not every hash function is suitable, but only some

Hashing

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    - Expensive but does not happen often
    - Therefore the average cost is low
    - Look at amortized analysis in the next lecture

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Buckets as linked list:

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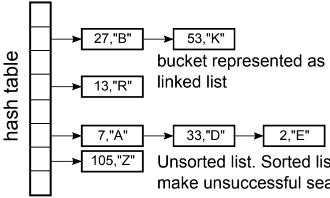

bucket represented as linked list

Unsorted list. Sorted list would make unsuccessful search faster

Hashing

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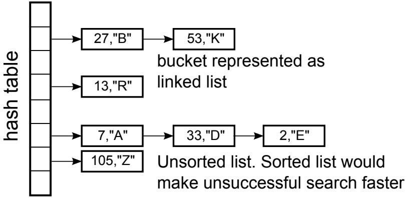

- Operations in $O(1)$ are possible if a suitable table size and hash function is selected

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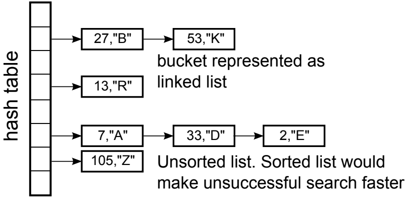

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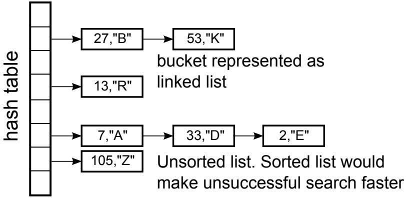

bucket represented as linked list

Unsorted list. Sorted list would make unsuccessful search faster

- Operations in $O(1)$ are possible if a suitable table size and hash function is selected
- Worst case $O(n)$, e.g. table size of 1
- Dynamic number of elements is possible

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  - If an entry is already occupied, then iteratively the following entry is checked. If a free entry is found the element is inserted

Hashing

Open Addressing

- For colliding keys we choose a new free entry
- Static, fixed number of elements
- The **probe sequence** determines for each key, in which sequence all hash table entries are searched for a free bucket
    - If an entry is already occupied, then iteratively the following entry is checked. If a free entry is found the element is inserted
    - If element is not found at the corresponding table entry, even if the entry is occupied, then probing has to be performed until the element or a free entry has been found

Hashing

Open Addressing

Definitions:

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$h(s)$ Hash function for key s

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$g(s,j)$	Probing function for key s with overflow positions $j \in \{0,\ldots,m-1\}$ e.g. g(s,j)=j

Hashing

Open Addressing

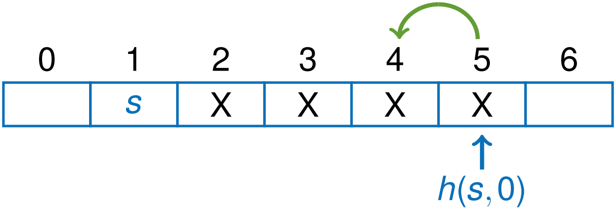
Definitions:

$h(s)$ Hash function for key s

$g(s,j)$ Probing function for key s with overflow positions $j \in \{0,\ldots,m-1\}$ e.g. g(s,j)=j

- The **probe sequence** is calculated by

$$h(s,j) = (h(s) - g(s,j)) \mod m \in \{0,\ldots,m-1\}$$



Hashing

Open Addressing - Python

```
def insert(s, value):
    j = 0

    while t[(h(s) - g(s, j)) mod m] \
            is not None:
        j += 1

    t[(h(s) - g(s, j)) mod m] \
        = (s, value)
```

Hashing

Open Addressing - Python

```
def lookup(s):
    j = 0

    while t[(h(s) - g(s, j)) mod m] \
            is not None:
        if t[(h(s) - g(s, j)) mod m][0] != s:
                        j += 1
        if t[(h(s) - g(s, j)) mod m][0] == s:
            return t[(h(s) - g(s, j)) mod m]
    return None
```

Hashing

Open Addressing - Linear Probing

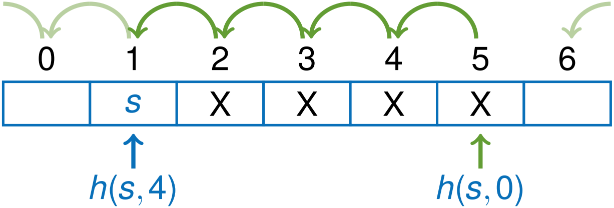


Figure: Linear probe sequence

Hashing

Open Addressing - Linear Probing

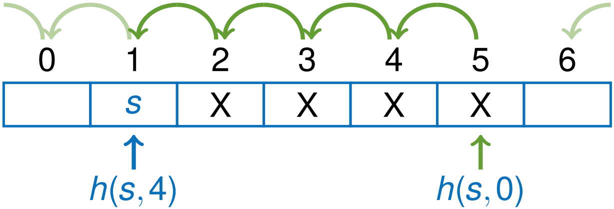


Figure: Linear probe sequence

- Check the element with lower index: $g(s,j) := j$
  $\Rightarrow$ Hash function: $h(s,j) = (h(s) - j) \mod m$

Hashing

Open Addressing - Linear Probing

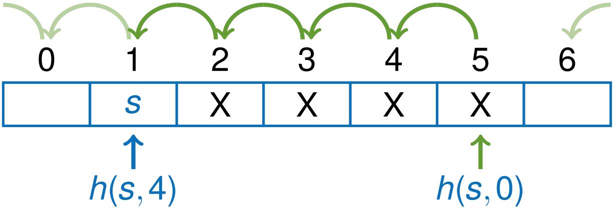


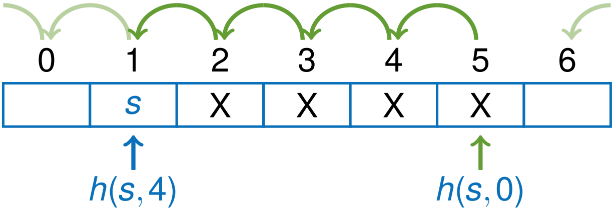
Figure: Linear probe sequence

- Check the element with lower index: $g(s,j) := j$
  $\Rightarrow$ Hash function: $h(s,j) = (h(s) - j) \mod m$
- This leads to the following probe sequence

$$h(s), h(s)-1, h(s)-2, \ldots, \underbrace{0, m-1}_{\text{clipping}}, m-2, \ldots, h(s)+1$$

Hashing

Open Addressing - Linear Probing



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Hashing

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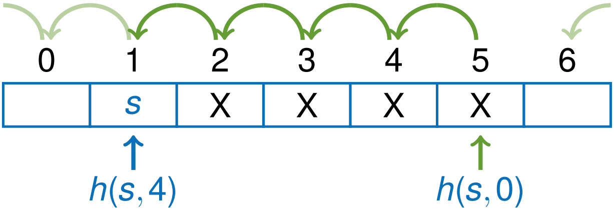

Figure: Linear probe sequence

- Can result in primary clustering

Hashing

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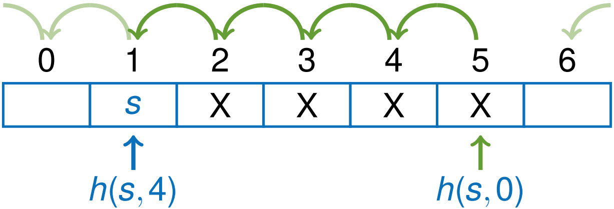

Figure: Linear probe sequence

- Can result in primary clustering
- Dealing with a hash collision will result in a higher probability of hash collisions in close entries

Hashing

Open Addressing - Linear Probing

Example:

Hashing

Open Addressing - Linear Probing

Example:

- Keys: $\{12, 53, 5, 15, 2, 19\}$

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Open Addressing - Linear Probing

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- Keys: $\{12, 53, 5, 15, 2, 19\}$
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Example:

- Keys: $\{12, 53, 5, 15, 2, 19\}$
- Hash function: $h(s,j) = (s \mod 7 - j) \mod 7$

- t.insert(12, "A"), $h(12,0) = 5$

0	1	2	3	4	5	6
					12, A	

Hashing

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Example:

- Keys: $\{12, 53, 5, 15, 2, 19\}$
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0	1	2	3	4	5	6
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- t.insert(53, "B"), $h(53,0) = 4$

				53, B	12, A	

Figure: Probe/Insertion sequence on a hash map

Hashing

Open Addressing - Linear Probing

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Hashing

Open Addressing - Linear Probing

Example:

- Hash function: $h(s,j) = (s \mod 7 - j) \mod 7$
- t.insert(5, "C"), $h(5,0) = 5$, $h(5,1) = 4$, $h(5,2) = 3$

0	1	2	3	4	5	6
			5, C	53, B	12, A	

Hashing

Open Addressing - Linear Probing

Example:

- Hash function: $h(s,j) = (s \mod 7 - j) \mod 7$

- t.insert(5, "C"), $h(5,0) = 5$, $h(5,1) = 4$, $h(5,2) = 3$

0	1	2	3	4	5	6
			5, C	53, B	12, A	

- t.insert(15, "D"), $h(15,0) = 1$

	15, D		5, C	53, B	12, A	

Figure: Probe/Insertion sequence on a hash map

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Hashing

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Example:

- Hash function: $h(s,j) = (s \mod 7 - j) \mod 7$
- t . insert (2, "E"), $h(2,0) = 2$

0	1	2	3	4	5	6
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- Hash function: $h(s,j) = (s \mod 7 - j) \mod 7$
- t.insert(2, "E"), $h(2,0) = 2$

0	1	2	3	4	5	6
	15, D	2, E	5, C	53, B	12, A	

- t.insert(19, "F"), $h(19,0) = 5$, $h(19,1) = 4$, $h(19,2) = 3$, $h(19,3) = 2$, $h(19,4) = 1$, $h(19,5) = 0$

19, F	15, D	2, E	5, C	53, B	12, A	

Figure: Probe/Insertion sequence on a hash map

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Open Addressing - Squared Probing

Squared probing:

Hashing

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Squared probing:

- Motivation: avoid local clustering

$$g(s,j) := (-1)^j \left\lceil \frac{j}{2} \right\rceil^2$$

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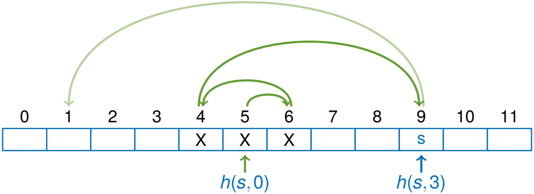

Figure: Squared probe sequence

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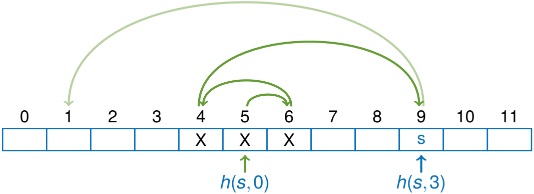

Figure: Squared probe sequence

- This leads to the following probe sequence

$h(s),\ h(s)+1,\ h(s)-1,\ h(s)+4,\ h(s)-4,\ h(s)+9,\ h(s)-9,\ \ldots$

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- Alternatively: $h(s,j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \mod m$

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- If m is a prime number for which $m = 4 \cdot k + 3$ then the probe sequence is a permutation of the indices of the hash tables
- Alternatively: $h(s,j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \mod m$
- Problem of secondary clustering:
  No local clustering anymore, but keys with same hash value have similar probe sequence

Hashing

Open Addressing - Uniform Probing

Uniform Probing:

Hashing

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Uniform Probing:

- Motivation: so far function $g(s,j)$ uses only the step counter j for linear and squared probing
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- Uniform probing computes the sequence $g(s,j)$ of permutations of all possible indices dependent on key s
- **Advantage:** prevents clustering because different keys with the same hash value do not produce the same probe sequence
- **Disadvantage:** hard to implement

Double Hashing:

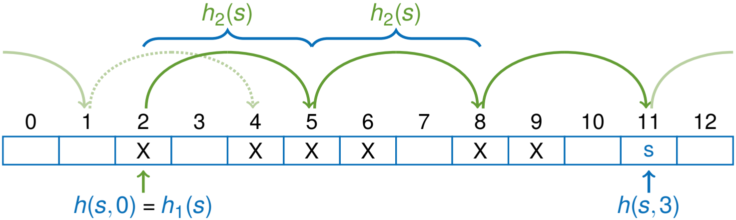
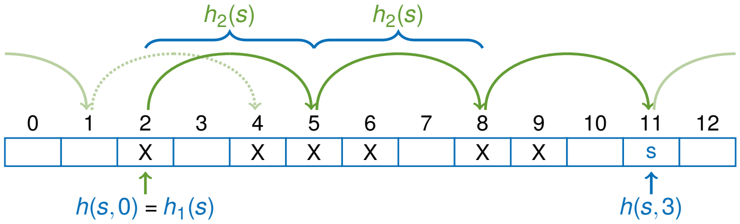


Figure: double hashing probe sequence

Hashing

Open Addressing - Double Hashing

Double Hashing:



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Hashing

Open Addressing - Double Hashing

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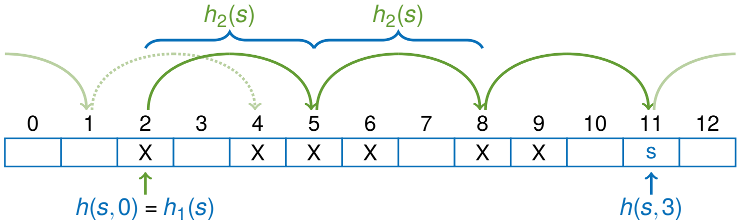

Figure: double hashing probe sequence

- Motivation: consider key s in probe sequence
- Use two independent hash functions $h_1(s), h_2(s)$

Hashing

Open Addressing - Double Hashing

Double Hashing:

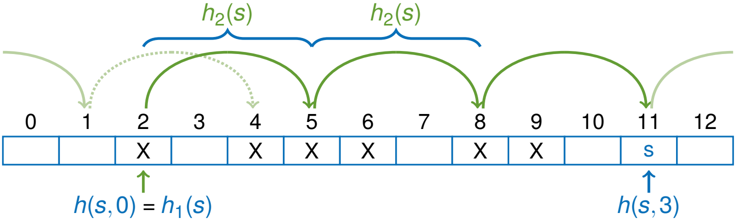

Figure: double hashing probe sequence

- Motivation: consider key s in probe sequence
- Use two independent hash functions $h_1(s), h_2(s)$
- Hash function: $h(s,j) = (h_1(s) + j \cdot h_2(s)) \mod m$

Hashing

Open Addressing - Double Hashing

Double Hashing:

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Double Hashing:

- Hash function: $h(s,j) = (h_1(s) + j \cdot h_2(s)) \mod m$
- Probe sequence:

$$h_1(s),\ h_1(s) + h_2(s),\ h_1(s) + 2 \cdot h_2(s),\ h_1(s) + 3 \cdot h_2(s),\ \ldots$$

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- Works well in practical use
- This method is an approximation of uniform probing

Hashing

Open Addressing - Double Hashing - Example

Example:

Hashing

Open Addressing - Double Hashing - Example

Example:

$$h_1(s) = s \mod 7$$

$$h_2(s) = (s \mod 5) + 1$$

$$h(s,j) = h_1(s) + j \cdot h_2(s) \mod 7$$

Hashing

Open Addressing - Double Hashing - Example

Example:

$$h_1(s) = s \mod 7$$
$$h_2(s) = (s \mod 5) + 1$$
$$h(s,j) = h_1(s) + j \cdot h_2(s) \mod 7$$

Table: comparing both hash functions

s	10	19	31	22	14	16
$h_1(s)$	3	5	3	1	0	2
$h_2(s)$	1	5	2	3	5	2

- The efficiency of double hashing is dependent on $h_1(s) \neq h_2(s)$

Hashing

Open Addressing - Double Hashing - Optimization

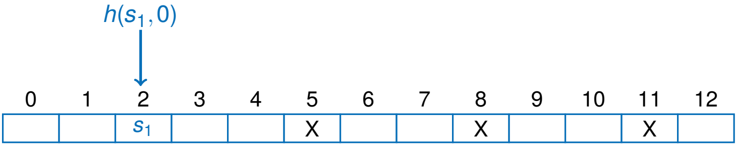

Figure: double hashing

Double hashing by Brent:

Hashing

Open Addressing - Double Hashing - Optimization

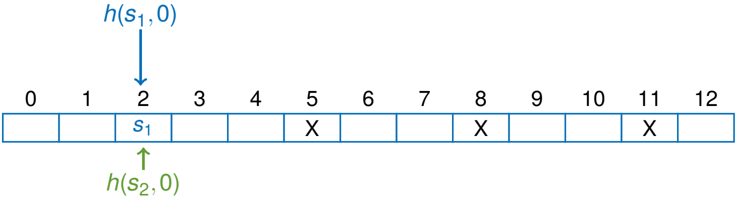

Figure: double hashing

Double hashing by Brent:

- Motivation:
  Because different keys have different probe sequences, the sequence of the insertions has impact on efficiency of a sucessful search

Hashing

Open Addressing - Double Hashing - Optimization

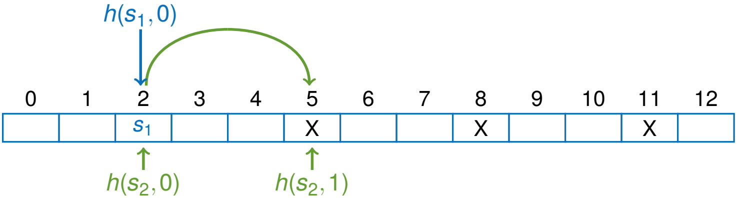

Figure: double hashing

Double hashing by Brent:

- Motivation:
  Because different keys have different probe sequences, the sequence of the insertions has impact on efficiency of a sucessful search

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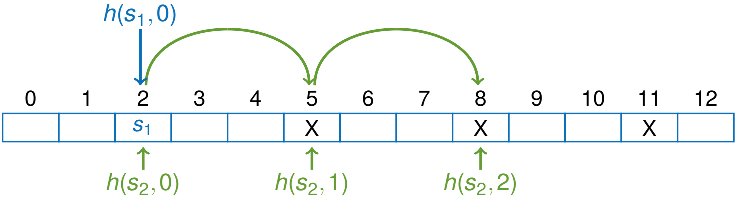

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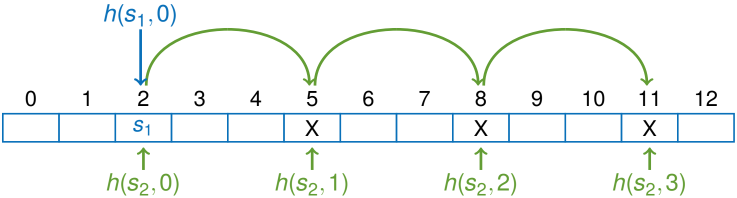

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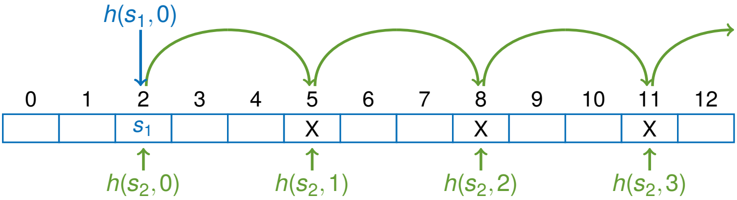

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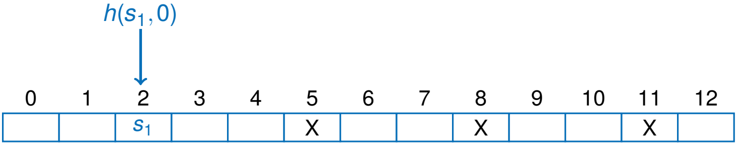

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Example:

- The key s_1 is inserted at position $p_1 = h(s_1, 0)$

Hashing

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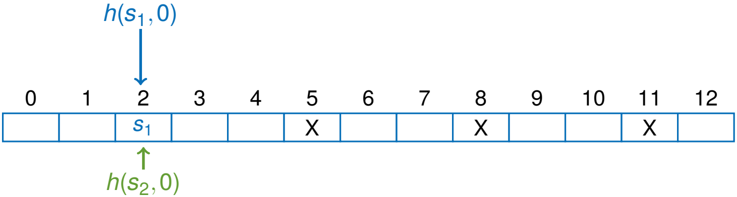

Figure: double hashing

Example:

- The key s_1 is inserted at position $p_1 = h(s_1, 0)$
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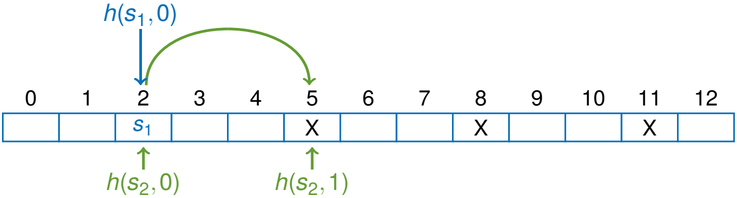
Hashing

Open Addressing - Double Hashing - Optimization



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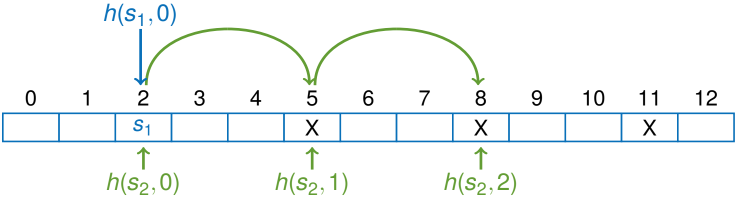
Example:

- The key s_1 is inserted at position $p_1 = h(s_1, 0)$
- The hash function for s_2 also results in $p_2 = h(s_2, 0) = p_1$
- The locations $h(s_2, j)$, $j \in \{1, \ldots, n\}$ are also occupied

Hashing

Open Addressing - Double Hashing - Optimization



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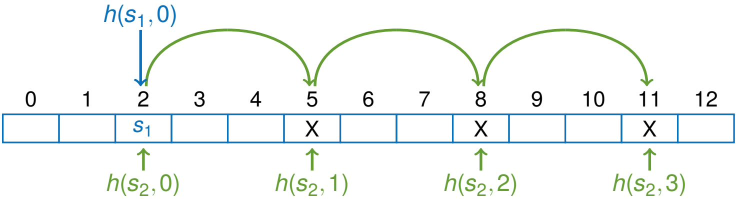

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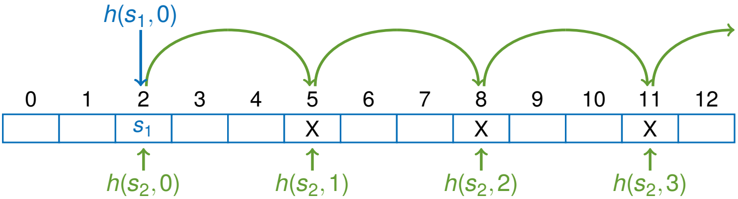
Open Addressing - Double Hashing - Optimization



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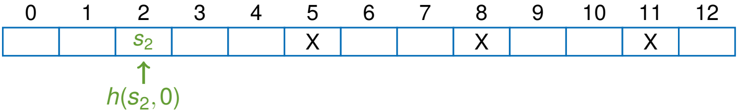
Example:

- The key s_1 is inserted at position $p_1 = h(s_1, 0)$
- The hash function for s_2 also results in $p_2 = h(s_2, 0) = p_1$
- The locations $h(s_2, j)$, $j \in \{1, \ldots, n\}$ are also occupied
- If we insert s_2 at position $h(s_2, n + 1)$ the search will be inefficient

Hashing

Open Addressing - Double Hashing - Optimization



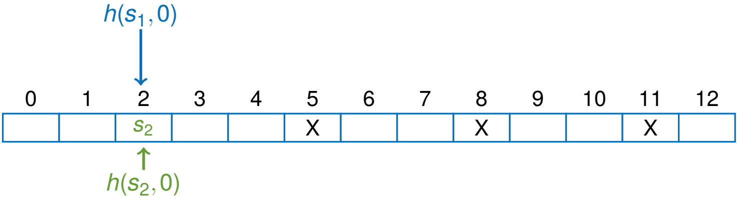
0	1	2	3	4	5	6	7	8	9	10	11	12
		s_2			X			X			X	

$h(s_2, 0)$

Figure: double hashing by Brent

Hashing

Open Addressing - Double Hashing - Optimization



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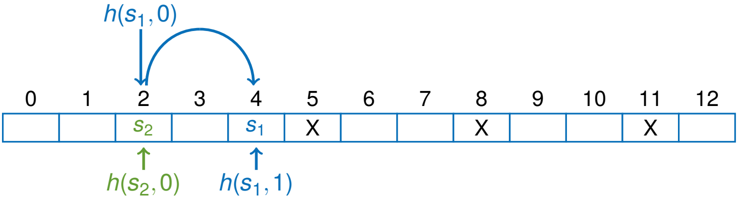
Hashing

Open Addressing - Double Hashing - Optimization



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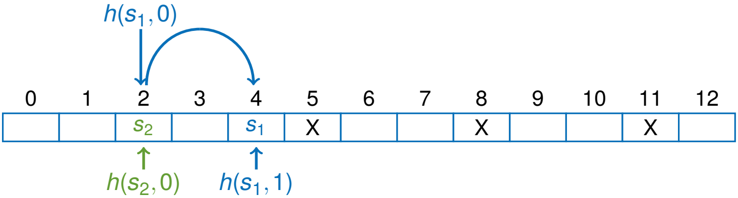

Figure: double hashing by Brent

- Reversed sequence of keys would have been better

Hashing

Open Addressing - Double Hashing - Optimization

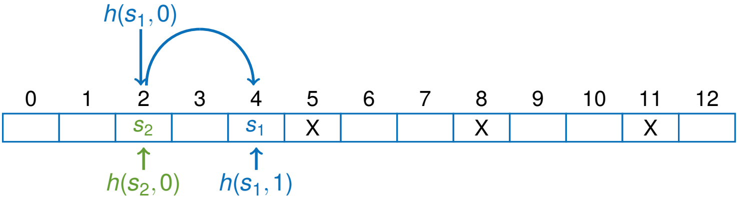

Figure: double hashing by Brent

- Reversed sequence of keys would have been better
- **Brent's idea:**

Hashing

Open Addressing - Double Hashing - Optimization

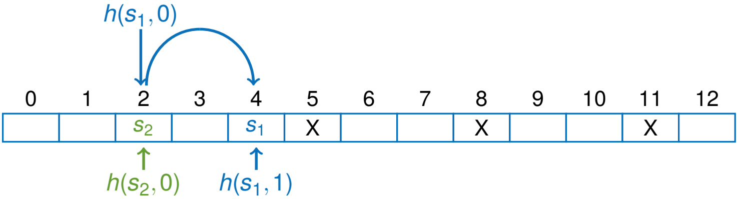

Figure: double hashing by Brent

- Reversed sequence of keys would have been better
- **Brent's idea:**
  - Test if location $h(s_1, 1)$ is free

Hashing

Open Addressing - Double Hashing - Optimization

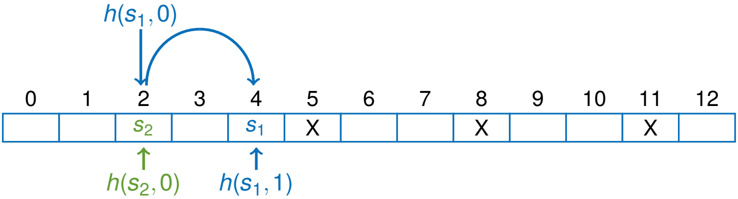

Figure: double hashing by Brent

- Reversed sequence of keys would have been better
- **Brent's idea:**
    - Test if location $h(s_1, 1)$ is free
    - If yes, move s_1 from $h(s_1, 0)$ to $h(s_1, 1)$ and insert s_2 at $h(s_2, 0)$

Hashing

Open Addressing - Ordered Hashing

Idea:

- Motivation: colliding elements are inserted in the hash table sorted.
- Therefore, in case of an unsuccessful search of elements in combination with linear probing or double hashing, aborting is possible earlier because single probing steps have a fixed length

Implementation:

- Compare both keys if a collision occurs
- Insert the smaller key at p_1
- Search a position based on the diversion order for the bigger key

Hashing

Open Addressing - Ordered Hashing

Example:

- The key 12 is saved at position $p_1 = h(12, 0)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $5 < 12$ we insert the key 5 at position p_1
- For the key 12 we iterate through the sequence

$$h(12, 1), h(12, 2), h(12, 3), \dots$$

Hashing

Open Addressing - Robin-Hood Hashing

Motivation:

Hashing

Open Addressing - Robin-Hood Hashing

Motivation:

- Having similiar length of probe sequences for all elements. Total costs stay the same, but they are distributed evenly. Results in approximately similar search times for all elements

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Open Addressing - Robin-Hood Hashing

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- Having similiar length of probe sequences for all elements. Total costs stay the same, but they are distributed evenly. Results in approximately similar search times for all elements

Implementation:

- If two keys s_1, s_2 collide ($p_1 = h(s_1, j_1) = h(s_2, j_2)$) we compare the length of the sequence (j_1 or j_2)

Hashing

Open Addressing - Robin-Hood Hashing

Motivation:

- Having similiar length of probe sequences for all elements. Total costs stay the same, but they are distributed evenly. Results in approximately similar search times for all elements

Implementation:

- If two keys s_1, s_2 collide ($p_1 = h(s_1, j_1) = h(s_2, j_2)$) we compare the length of the sequence (j_1 or j_2)
- The key with the bigger search sequence is inserted at p_1. The other key is assigned to a new location based on the sequence

Example:

- The key 12 is saved at position $p_1 = h(12, 7)$
- We insert the key 5 into the hash map
- We assume $h(5, 0)$ results in location p_1
- Because $j_1 < j_2$ $(0 < 7)$ key 12 stays at position p_1
- For key 5 we iterate through the sequence

$$h(5, 1),\ h(5, 2),\ h(5, 3),\ \dots$$

Problem:

- The key s_1 is inserted at position p_1
- The key s_2 returns the same hash value, but is inserted at position p_2 because of the probing order
- If s_1 is removed, it is impossible to find s_2

Hashing

Open Addressing - Implement Insert / Remove

Problem:

- The key s_1 is inserted at position p_1
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Solution:

Hashing

Open Addressing - Implement Insert / Remove

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Solution:

- **Remove:** elements are marked as removed, but not deleted

Hashing

Open Addressing - Implement Insert / Remove

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- If s_1 is removed, it is impossible to find s_2

Solution:

- **Remove:** elements are marked as removed, but not deleted
- **Inserting:** elements marked as removed will we overwritten

Structure

Hashing

- Recapitulation
- Treatment of hash collisions
- Open Addressing
- Summary

Priority Queue

- Introduction

Hashing

Open Addressing - Summary Collision Handling

Bucket as linked list: (dynamic, number of elements variable)

- Save colliding elements as linked list

Open hashing: (static, number of elements fixed)

- Determine a probe sequence, permutation of all hash values
- Linear, quadratic probing:
  - Easy to implement
  - Raises the probability of collisions because probing order does not depend on the key

Hashing

Open Addressing - Summary Collision Handling

Open hashing: (static, number of elements fixed)

Hashing

Open Addressing - Summary Collision Handling

Open hashing: (static, number of elements fixed)

- Uniform probing, double hashing:
    - Different probing orders for different keys
    - Avoids clustering of elements

Hashing

Open Addressing - Summary Collision Handling

Open hashing: (static, number of elements fixed)

- Uniform probing, double hashing:
  - Different probing orders for different keys
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Improving efficiency: (Brent, Ordered Hashing)

- Improve search efficiency by sorting colliding insertions
  - Abortion of unsuccessfull search
  - Search sequence length balancing

Hashing

Open Addressing - Summary Hashing

Hashing:

Hashing

Open Addressing - Summary Hashing

Hashing:

- Efficiency of dictionary operations:

  `Insert`: $O(1)\ldots O(n)$

  `Search`: $O(1)\ldots O(n)$

  `Remove`: $O(1)\ldots O(n)$

Hashing

Open Addressing - Summary Hashing

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- Efficiency of dictionary operations:
  `Insert`: $O(1) \dots O(n)$
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  `Remove:` $O(1) \dots O(n)$
- Direct access oto all elements in a hash table
- Using a hash function to find the position (hash value) in the hash table
- Hash function, size of the hash table and strategy to avoid hash collisions all influence the efficiency of the data structure

Structure

Hashing
    Recapitulation
    Treatment of hash collisions
    Open Addressing
    Summary

Priority Queue
    Introduction

Priority Queue

Introduction

Definition:

Priority Queue

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- A priority queue saves a set of elements

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- Each element contains a key and a value like a map

Priority Queue

Introduction

Definition:

- A priority queue saves a set of elements
- Each element contains a key and a value like a map
- There is a total order (like $\leq$) defined on the keys

Priority Queue

Introduction

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Priority Queue

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Priority Queue

Introduction

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- The priority queue supports the following operations:

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`deleteMin():` removes the element with the smallest key

Priority Queue

Introduction

Definition:

- The priority queue supports the following operations:

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- Sometimes additional operations are defined:

Priority Queue

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Priority Queue

Introduction

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- Sometimes additional operations are defined:

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`remove(item):`	removes the element from the queue

Priority Queue

Introduction

Special features:

Priority Queue

Introduction

Special features:

- Multiple elements with the same key

Priority Queue

Introduction

Special features:

- Multiple elements with the same key
    - No problem and for many applications necessary
    - If there is more than one element with the smallest key

      `getMin()`: returns just one of the possible elements
      `deleteMin()`: deletes the element returned by `getMin`

Priority Queue

Introduction

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- Argument of `changeKey` and `remove` operations

Priority Queue

Introduction

Special features:

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  - If there is more than one element with the smallest key

    `getMin():` returns just one of the possible elements

    `deleteMin():` deletes the element returned by `getMin`

- Argument of `changeKey` and `remove` operations
  - There is no **quick access** to an element in the queue
  - That is why `insert` and `getMin` return a reference (handle, accessor object)
  - `changeKey` and `remove` take this reference as argument
  - Therefore each element has to store its current position in the heap.

Priority Queue

Python

```
from queue import PriorityQueue

q = PriorityQueue()

e1 = (5, "A") # element with priority 5
q.put(e1); # insert element e1

# remove and return the lowest item
e2 = q.get()
```

Priority Queue

Application Example

Example 1:

- Calculation of the sorted union of *k* sorted lists (multi-way merge or *k*-way merge)

L_1 :	3	5	8	12	...
L_3 :	1	10	11	24	...
L_2 :	4	5	6	7	...

Priority Queue

Application Example

Example 1:

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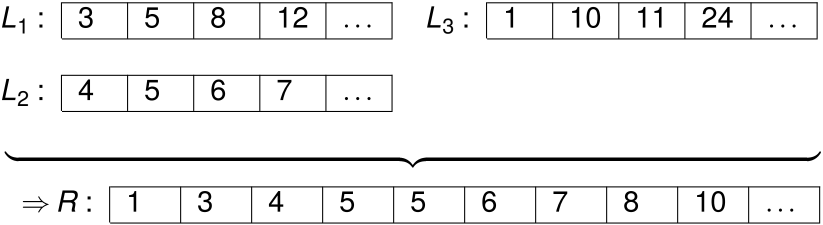

L_1 :	3	5	8	12	...
L_2 :	4	5	6	7	...
L_3 :	1	10	11	24	...

$\Rightarrow R$:	1	3	4	5	5	6	7	8	10	...

Figure: 3-way merge

Priority Queue

Application Example

Example 1:

Priority Queue

Application Example

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Priority Queue

Application Example

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Priority Queue

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  - Trivial: $\Theta(N \cdot k)$, minimum calculation $\Theta(k)$

Priority Queue

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  - Priority queue: $\Theta(N \cdot \log k)$, minimum calculation $\Theta(\log k)$

Priority Queue

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Example 2:

Priority Queue

Application Example

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Example 2:

- For example Dijkstra's algorithm for computing the shortest path (following lecture)

Priority Queue

Application Example

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Example 2:

- For example Dijkstra's algorithm for computing the shortest path (following lecture)
- Among other applications it can be used for sorting

Priority Queue

Implementation

Idea:

Priority Queue

Implementation

Idea:

- Save elements as tuples in a binary heap

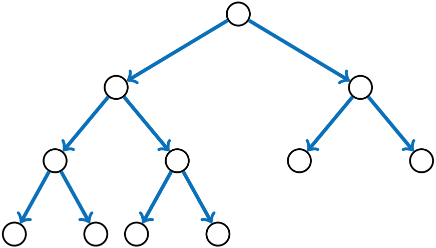


Figure: heap with 11 nodes

Priority Queue

Implementation

Idea:

- Save elements as tuples in a binary heap
- Summary from lecture 1 (*HeapSort*):
  - Nearly complete binary tree
  - Heap condition:
    The key of each node $\leq$ the keys of the children

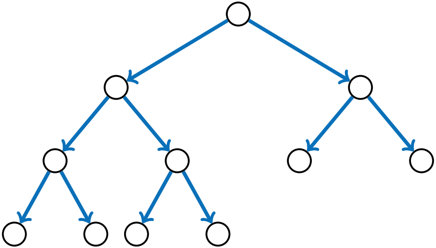
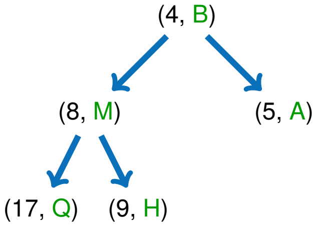


Figure: heap with 11 nodes

Priority Queue

Implementation



0	1	2	3	4
4, B	8, M	5, A	17, Q	9, H

Figure: min heap stored in array

Priority Queue

Implementation

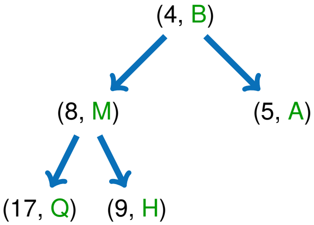


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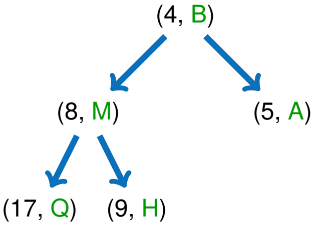
Figure: min heap stored in array

Storing a binary heap:

Priority Queue

Implementation



0	1	2	3	4
4, B	8, M	5, A	17, Q	9, H

Figure: min heap stored in array

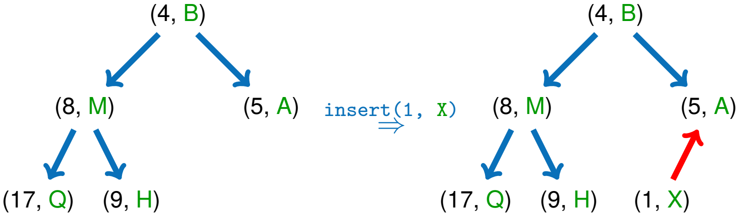
Storing a binary heap:

- Number nodes from top to bottom and left to right starting with 0 and store entries in array
- Children of node i are the nodes $2i+1$ and $2i+2$
- Parent node of node i is $\mathrm{floor}((i-1)/2)$

Priority Queue

Implementation - Insertion

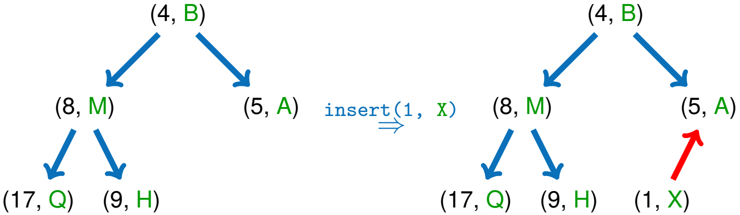
Inserting an element: `insert(key, item)`



Priority Queue

Implementation - Insertion

Inserting an element: `insert(key, item)`

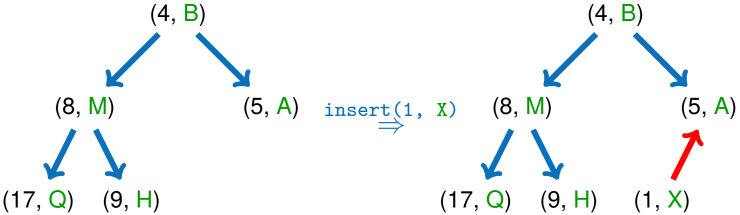


- Append the element at the end of the array

Priority Queue

Implementation - Insertion

Inserting an element: `insert(key, item)`



- Append the element at the end of the array
- The heap condition may be violated, but only at the last index

Priority Queue

Implementation - Insertion

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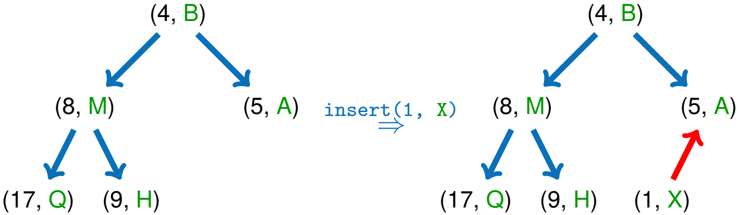
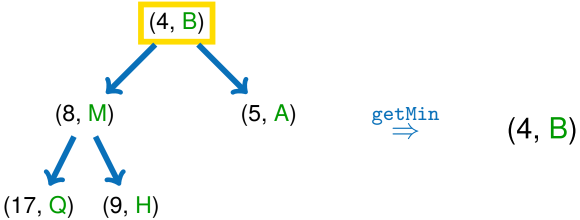


- Append the element at the end of the array
- The heap condition may be violated, but only at the last index
- Repair heap condition ⇒ We will see later how to do this

Priority Queue

Implementation

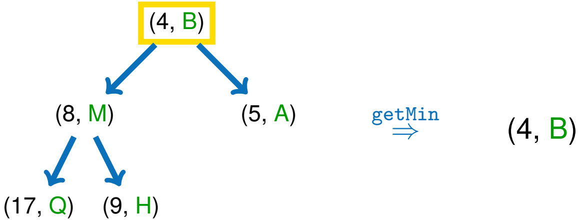
Returning the minimum: `getMin()`



Priority Queue

Implementation

Returning the minimum: `getMin()`



(4, B)

(8, M)

(5, A)

(17, Q)

(9, H)

$\overset{\texttt{getMin}}{\Rightarrow}$

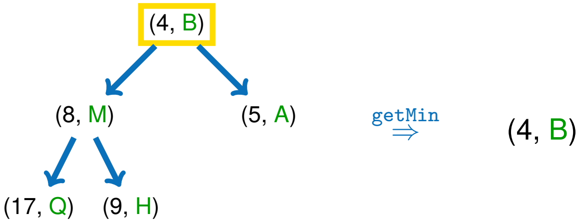
(4, B)

- Else return the first element

Priority Queue

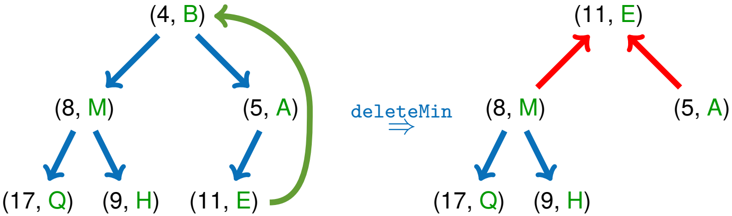
Implementation

Returning the minimum: `getMin()`



- Else return the first element
- If the heap is empty return None

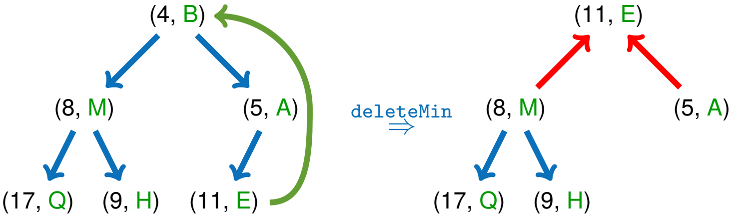
Removing the minimum: `deleteMin()`



Priority Queue

Implementation

Removing the minimum: `deleteMin()`

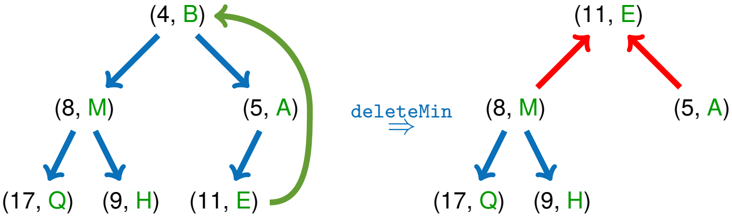


- Deleting the element with the lowest key

Priority Queue

Implementation

Removing the minimum: `deleteMin()`

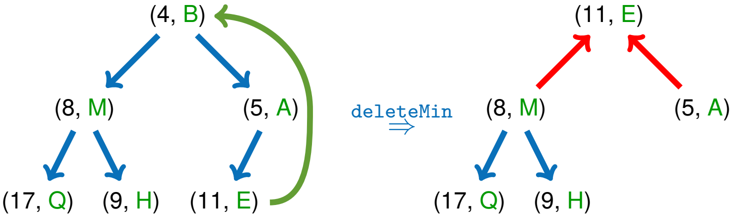


- Deleting the element with the lowest key
- Swap the last element with the first element and shrink the heap by one

Priority Queue

Implementation

Removing the minimum: `deleteMin()`



- Deleting the element with the lowest key
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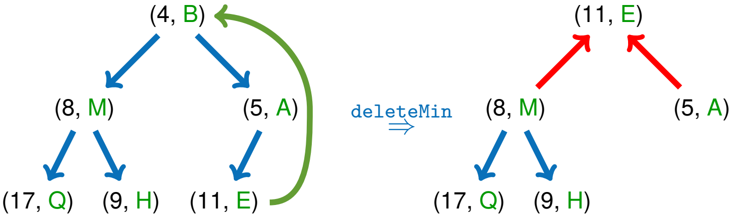
Priority Queue

Implementation

Removing the minimum: `deleteMin()`



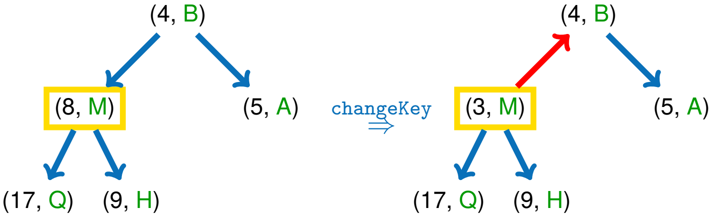
- Deleting the element with the lowest key
- Swap the last element with the first element and shrink the heap by one
- The heap condition may be violated, but only at the first index
- Repair heap condition

Priority Queue

Implementation

Changing the key (priority): `changeKey(item, key)`

- The element (queue item) is given as argument
- Replace the key of the element
- The heap condition may be violated, but only at the element index and only in one direction (up / down)
- Repair heap condition



Priority Queue

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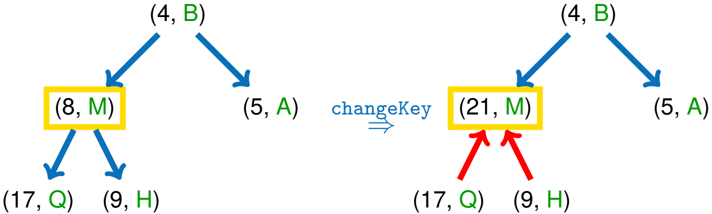

Priority Queue

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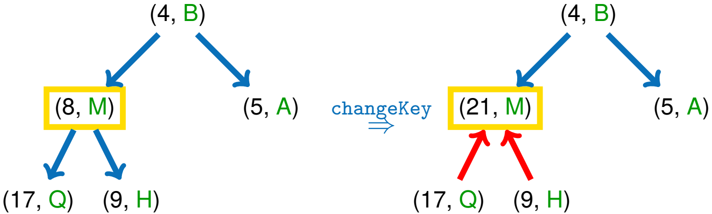

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Priority Queue

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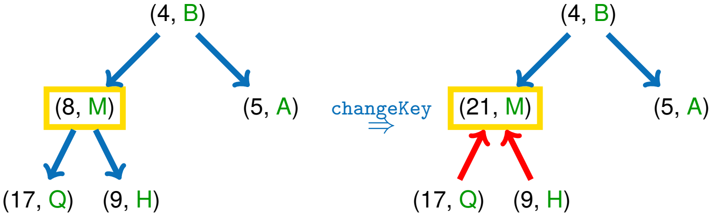

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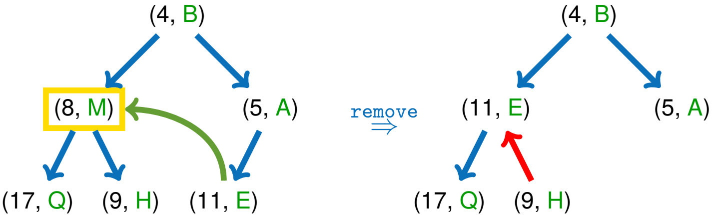
Priority Queue

Implementation

Removing an element: `remove(item)`



Priority Queue

Implementation

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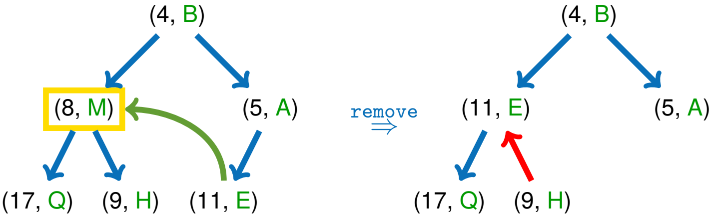

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Priority Queue

Implementation

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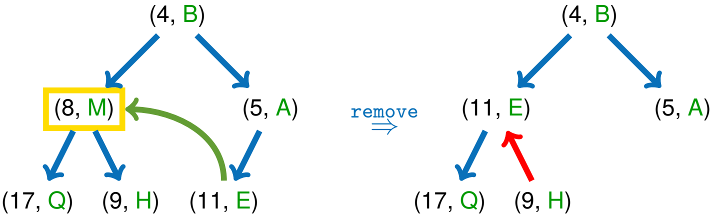
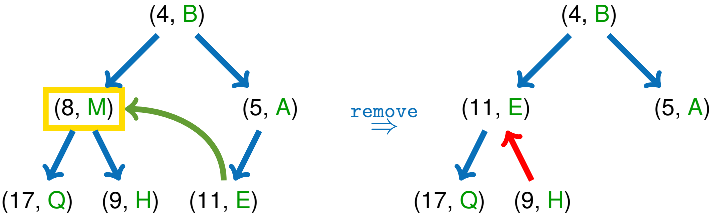


- The element (queue item) is given as argument
- Replace the element with the last element and shrink the heap by one

Priority Queue

Implementation

Removing an element: `remove(item)`



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Priority Queue

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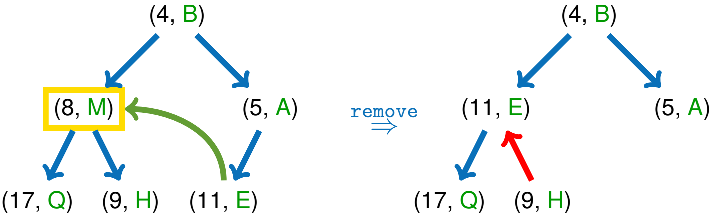

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Priority Queue

Implementation - Reparing the Heap

Repairing after modifying operations:

Priority Queue

Implementation - Reparing the Heap

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- The heap condition can be violated after using `insert`, `deleteMin`, `changeKey`, `remove`, but only at one known position with index i

Priority Queue

Implementation - Reparing the Heap

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Priority Queue

Implementation - Reparing the Heap

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- Heap conditions can be violated in two directions:
    - Downwards: the key at index *i* is not $\leq$ than the value of its children

Priority Queue

Implementation - Reparing the Heap

Repairing after modifying operations:

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- Heap conditions can be violated in two directions:
  - Downwards: the key at index i is not $\leq$ than the value of its children
  - Upwards: the key at index i is not $\geq$ than the value of its parent

Priority Queue

Implementation - Reparing the Heap

Repairing after modifying operations:

- The heap condition can be violated after using `insert`, `deleteMin`, `changeKey`, `remove`, but only at one known position with index *i*
- Heap conditions can be violated in two directions:
    - Downwards: the key at index *i* is not $\leq$ than the value of its children
    - Upwards: the key at index *i* is not $\geq$ than the value of its parent
- We need two repair methods: `repairHeapUp`, `repairHeapDown`

Priority Queue

Implementation - Reparing the Heap

`repairHeapDown:`

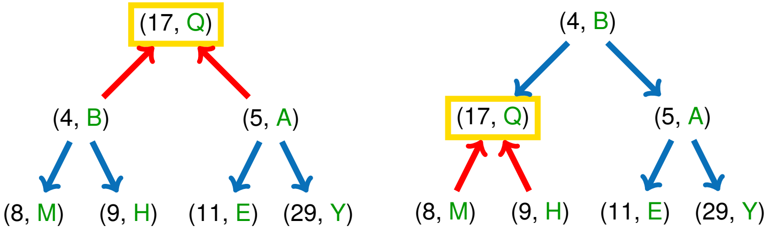


Figure: repairing the heap downwards

Priority Queue

Implementation - Reparing the Heap

`repairHeapDown:`

- Sift the element until the heap condition is valid

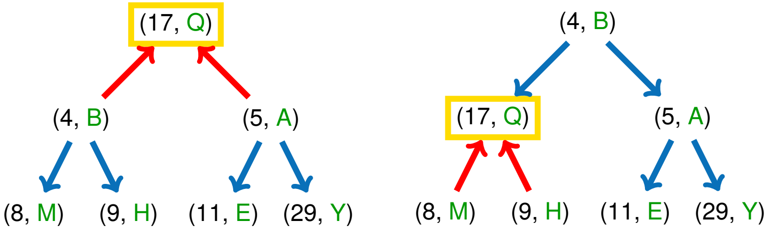


Figure: repairing the heap downwards

Priority Queue

Implementation - Reparing the Heap

`repairHeapDown:`

- Sift the element until the heap condition is valid
  - Change node with child, which has the lower key of both children

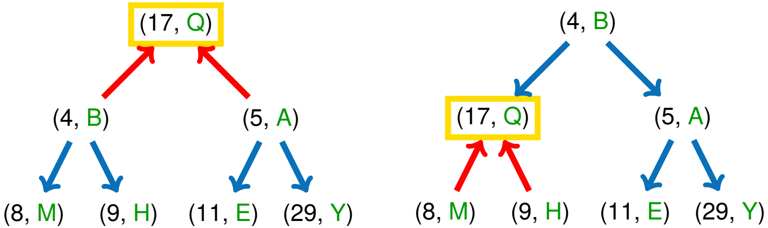


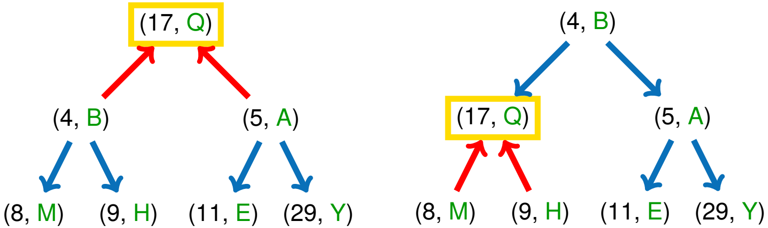
Figure: repairing the heap downwards

Priority Queue

Implementation - Reparing the Heap

`repairHeapDown:`

- Sift the element until the heap condition is valid
  - Change node with child, which has the lower key of both children
  - If the heap condition is violated repeat for the child node



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Priority Queue

Implementation - Reparing the Heap

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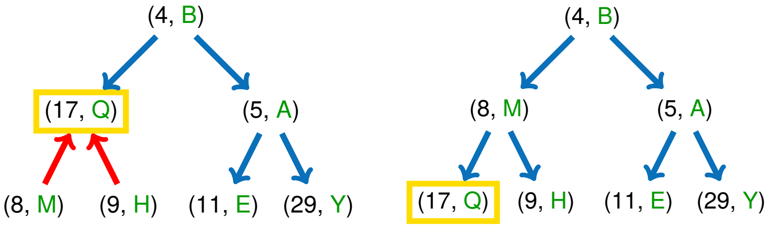

Figure: repairing the heap downwards

Priority Queue

Implementation - Reparing the Heap

`repairHeapUp:`

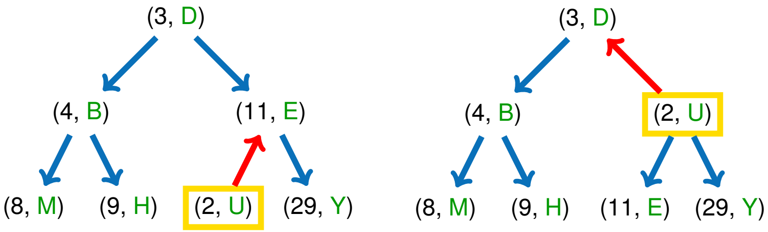

Figure: repairing the heap upwards

Priority Queue

Implementation - Reparing the Heap

`repairHeapUp:`

- Change node with parent

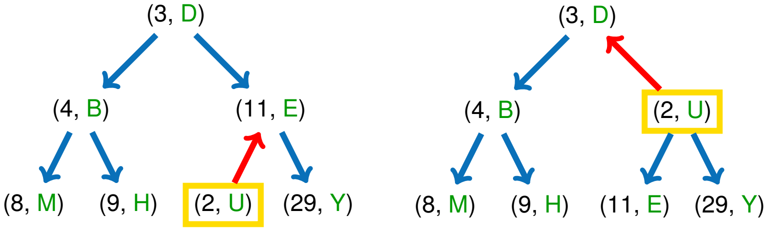

Figure: repairing the heap upwards

Priority Queue

Implementation - Reparing the Heap

`repairHeapUp:`

- Change node with parent
- If the heap condition is violated repeat for parent node

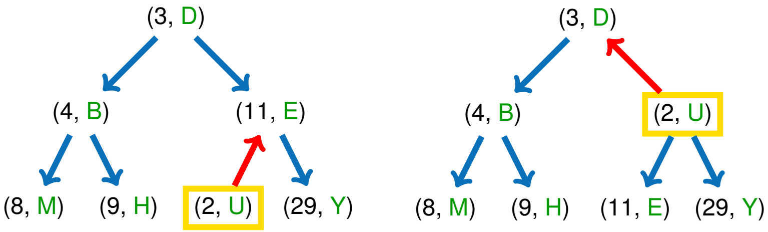

Figure: repairing the heap upwards

Priority Queue

Implementation - Reparing the Heap

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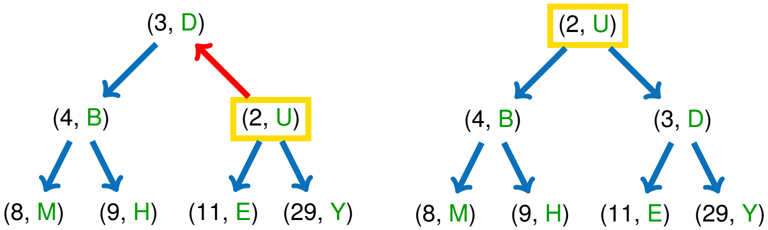

Figure: repairing the heap upwards

Priority Queue

Implementation - Priority Queue Item

Index of a priority queue item:

Priority Queue

Implementation - Priority Queue Item

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- Attention: for `changeKey` and `remove` the item has to "know" where it is located in the heap

Priority Queue

Implementation - Priority Queue Item

Index of a priority queue item:

- Attention: for `changeKey` and `remove` the item has to "know" where it is located in the heap
- Remember for `repairHeapUp` and `repairHeapDown`: update the index if moving an heap element

Priority Queue

Implementation - Priority Queue Item - Python

```
class PriorityQueueItem:

    """Provides a handle for a queue item.

    This handle can be used to remove or
    update the queue item.
    """

    def __init__(self, key, value, index):
        self.key = key
        self.value = value
        self.index = index
```

Priority Queue

Complexity

Summary lecture 1:

Priority Queue

Complexity

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Priority Queue

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Priority Queue

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  We have only one path to traverse: $O(\log n)$

Priority Queue

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Runtime for methods

Priority Queue

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Priority Queue

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  We have only one path to traverse: $O(\log n)$

Runtime for methods

- `insert`, `deleteMin`, `changeKey`, `remove`:
  we have to repair the heap: $O(\log n)$
- `getMin`: return the element at index 0: $O(1)$

Priority Queue

Complexity

Improvements (Fibonacci heaps):

Priority Queue

Complexity

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- `getMin`, `insert` and `decreaseKey` in amortized time of $O(1)$

Priority Queue

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Priority Queue
Complexity

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Practical experience:

Priority Queue

Complexity

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Practical experience:

- The binary heap is simpler: costs for managing the structure are low

Priority Queue

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Priority Queue

Complexity

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- `deleteMin` in amortized time $O(\log n)$

Practical experience:

- The binary heap is simpler: costs for managing the structure are low
- The difference is negligible if the number of elements is relatively small
- Example:
  - For $n = 2^{10} \approx 1,000$, the depth $\log_2 n$ is only 10
  - For $n = 2^{20} \approx 1,000,000$, the depth $\log_2 n$ is only 20

Further Literature

- **Course literature**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
`https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf`.

Further Literature

- **Priority Queue - Implementations / API**

  [Cpp] C++ - priority_queue
  `http://www.sgi.com/tech/stl/priority_queue.html`

  [Jav] Java - PriorityQueue
  `https://docs.oracle.com/javase/7/docs/api/java/util/PriorityQueue.html`

  [Pyt] Python - PriorityQueue
  `https://docs.python.org/3/library/queue.html#queue.PriorityQueue`