Algorithms and Data Structures

Divide and Conquer, Master theorem

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Structure

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Concept:

- Divide the problem into smaller subproblems
- Conquer the subproblems through recursive solving.
  If subproblems are small enough solve them directly
- Connect all subsolutions to solve the overall problem
- Recursive application of the algorithm on smaller subproblems
- Direct solving of small subproblems

Input:

- Sequence X of n integers

Output:

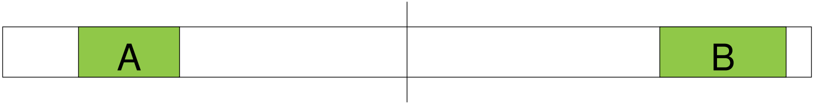
- Maximum sum of an uninterrupted subsequence of X and its index boundary

Table: input values

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

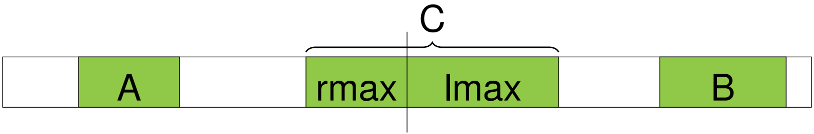
Idea:



- Solve the left / right half of the problem recursively
- Combine both solutions into an overall solution
- The maximum is located in the left half (*A*) or the right half (*B*)
- The maximum interval can overlap with the border (*C*)

Divide and Conquer
Maximum Subtotal

Principle:



- Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned.
- To solve C we have to calculate rmax and lmax
- The overall solution is the maximum of A, B and C

```python
def maxSubArray(X, i, j):
    if i == j: # trivial case
        return (X[i], i, i)

    # recursive subsolutions for A, B
    m = (i + j) // 2
    A = maxSubArray(X, i, m)
    B = maxSubArray(X, m + 1, j)

    # rmax and lmax for cornercase C
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)
    C = (C1[0] + C2[0], C1[1], C2[1])

    # compute solution from results A, B, C
    return max([A, B, C], key=lambda i: i[0])
```

```python
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```

```python
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

```python
#Alternative implementation max

def max(a, b):
    if a > b:
        return a
    else:
        return b

def maxTripel(a, b, c):
    return max(max(a,b),c)
```

```python
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Divide and Conquer

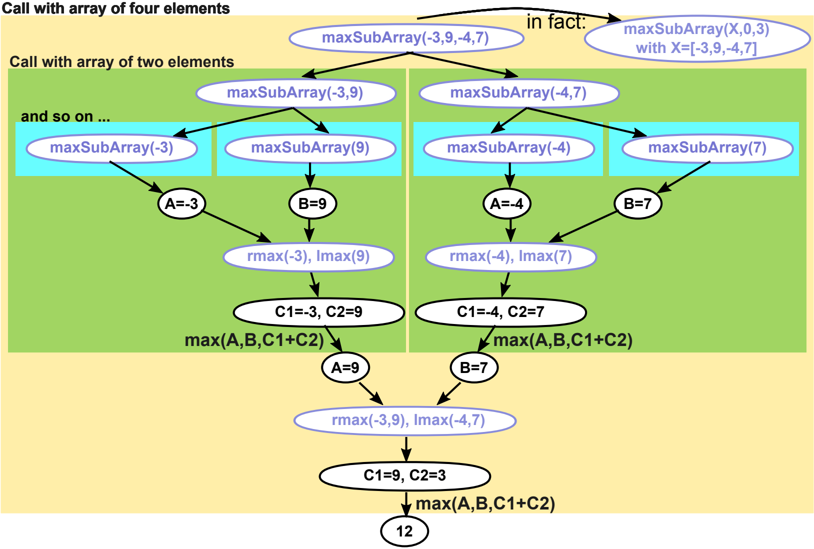
Maximum Subtotal - Python

```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[j]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

index	i	$i+1$	$\cdots$	$\cdots$	$j-1$	j
X	58	-53	26	59	-41	31
sum	58	5	31	90	49	80
lmax	58	58	58	90	90	90

- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i+1$ to j and update *sum*
- If $sum > lmax$, then *lmax* gets updated


Call with array of four elements
in fact:
maxSubArray(-3,9,-4,7)
maxSubArray(X,0,3)
with X=[-3,9,-4,7]
Call with array of two elements
maxSubArray(-3,9)
maxSubArray(-4,7)
and so on ...
maxSubArray(-3)
maxSubArray(9)
maxSubArray(-4)
maxSubArray(7)
A=-3
B=9
A=-4
B=7
rmax(-3), lmax(9)
rmax(-4), lmax(7)
C1=-3, C2=9
C1=-4, C2=7
max(A,B,C1+C2)
max(A,B,C1+C2)
A=9
B=7
rmax(-3,9), lmax(-4,7)
C1=9, C2=3
max(A,B,C1+C2)
12

Divide and Conquer

Maximum Subtotal - Python

```python
def maxSubArray(X, i, j):
    if i == j:                                          # O(1)
        return (X[i], i, i)                             # O(1)

    m = (i + j) // 2                                    # O(1)
    A = maxSubArray(X, i, m)                            # T(n/2)
    B = maxSubArray(X, m + 1, j)                        # T(n/2)

    C1 = rmax(X, i, m)                                  # O(n)
    C2 = lmax(X, m + 1, j)                              # O(n)
    C = (C1[0] + C2[0], C1[1], C2[1])                   # O(1)

    return max([A, B, C], \                             # O(1)
        key=lambda item: item[0])
```

Recursion equation:

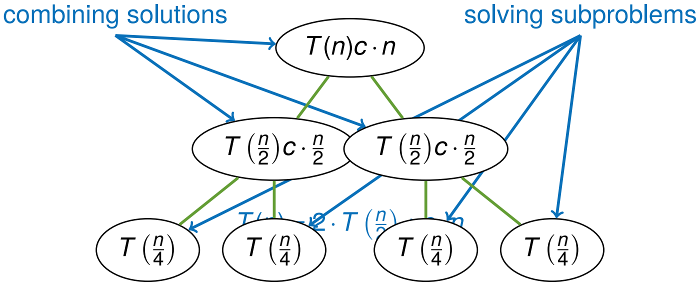
$$T(n) = \begin{cases} \underbrace{\Theta(1)}_{\text{trivial case}} & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

- There exist two constants a and b with:

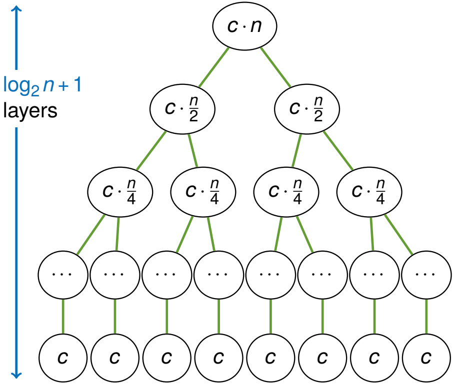
$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$



1 node processing n elements
$\Rightarrow c \cdot n$

2 nodes processing $\frac{n}{2}$ elements
$\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$

4 nodes processing $\frac{n}{4}$ elements
$\Rightarrow 4c \cdot \frac{n}{4} = c \cdot n$

2^i nodes processing $\frac{n}{2^i}$ elements
$\Rightarrow 2^i c \cdot \frac{n}{2^i} = c \cdot n$

n nodes processing 1 element
$\Rightarrow c \cdot n$

Figure: recursion tree method

Depth:

- Top level with depth $i = 0$
- Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- A total of $\log_2 n + 1$ levels costing $c \cdot n$ each
  The costs of merging the solutions and solving the trivial problems are the same in this case

$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

Summary:

- Direct solution is slow with $\mathcal{O}(n^3)$
- Better solution with incremental update of sum was $\mathcal{O}(n^2)$
- Divide and conquer approach results in $\mathcal{O}(n \log n)$
- There is an approach running in $\mathcal{O}(n)$, under the assumption that all subtotals are positive

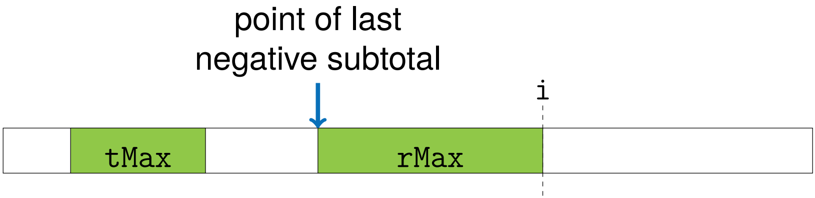


Figure: scanning the array in linear time

```python
#Implementation - linear runtime
def maxSubArray(X):
    # sum, start index
    rMax, irMax = 0, 0 # current maximum
    tMax, itMax = 0, 0 # total maximum

    for i in range(len(X)):
        if rMax == 0:
            irMax = i
        rMax = max(0, rMax + X[i])

        if rMax > tMax:
            tMax, itMax = rMax, irMax

    return (tMax, itMax)
```

Recursion equation:

- Runtime description for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$

Recursion equation:

- Runtime descripion for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

- n_0 is usually small, $f_0(n_0) \in \Theta(1)$
- Usually, $a > 1$ and $b > 1$
- Dependent on the strategy of solving $T(n)$ f_0 is ignored
- $T(n)$ is only defined for integers of $\frac{n}{b}$, which is often ignored in benefit of a simpler solution

Substitution Method:

- Guess the solution and prove it with induction
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

- Assumption: $T(n) = n + n \cdot \log_2 n$

Induction:

- Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- Induction step (from $\frac{n}{2}$ to n):

$$
\begin{aligned}
T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\
&\overset{IA}{=} 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot \log_2 \frac{n}{2}\right) + n \\
&= 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot (\log_2 n - 1)\right) + n \\
&= n + n \log_2 n - n + n \\
&= n + n \log_2 n
\end{aligned}
$$

Substitution Method:

- Alternative assumption
- Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 0 \end{cases}$$

- Assumption: $T(n) \in O(n \log n)$
- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$

Induction:

- Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$
- Induction step (from $\frac{n}{2}$ to n):

$$\begin{aligned}
T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\
&\leq 2 \cdot \left(c \cdot \frac{n}{2} \log_2 \frac{n}{2}\right) + n \\
&= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\
&= c \cdot n \log_2 n - c \cdot n + n \\
&\leq c \cdot n \log_2 n, \quad c \geq 1
\end{aligned}$$

Recursion tree method:

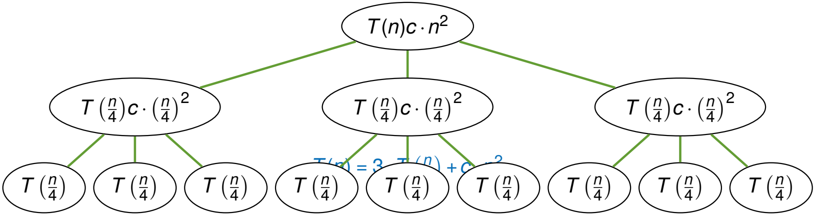
- Can be used to make assumptions about the runtime
- Example:

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$

$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$

$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$



$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$

Figure: recursion tree of example

Recursion Equations
Recursion Tree Method

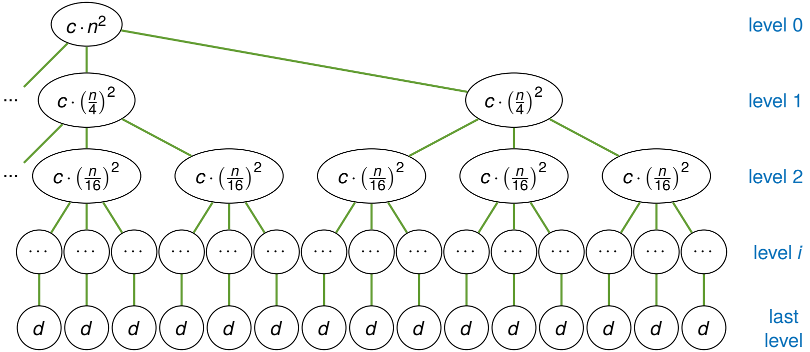


Figure: levels of the recursion tree

Costs of connecting the partial solutions:
(excludes the last layer)

- Size of partial problems on level i: $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$
- Costs of partial problems on level i:

$$T_{i_p}(n) = c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n\right)^2$$

- Number of partial problems on level i: $n_i = 3^i$
- Costs on level i:

$$T_i(n) = 3^i \cdot c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n\right)^2 = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$$

Costs of solving partial solutions: (only the last layer)

- Size of partial problems on the last level: $s_{i+1}(n) = 1$
- Costs of partial problem on the last level: $T_{i+1_p}(n) = d$
- With this the depth of the tree is:

$$\left(\frac{1}{4}\right)^i \cdot n = 1 \qquad \Rightarrow n = 4^i \qquad \Rightarrow i = \log_4 n$$

- Number of partial problems on the last level:

$$n_{i+1} = 3^{\log_4 n} \quad = n^{\log_4 3} \quad \leftarrow \text{next slide}$$

- Costs on the last level: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

- Transforming $3^{\log_4 n}$ using general log rules

$$\log_4 n = \log_4\left(3^{\log_3 n}\right) \qquad \text{using } n = 3^{\log_3 n}$$
$$= \log_3 n \cdot \log_4 3 \qquad \text{using } \log a^b = b \cdot \log a$$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- Now the whole expression:

$$3^{\log_4 n} = 3^{\log_3 n \cdot \log_4 3} \qquad \text{using reformulation above}$$
$$= \left(3^{\log_3 n}\right)^{\log_4 3} \qquad \text{using } x^{a \cdot b} = (x^a)^b$$
$$= n^{\log_4 3}$$

- This term will recur in the master theorem

Total costs:

- Costs of level i: $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
- Costs of last level: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \left(\text{even with infinite elements}\right)}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

- Here: The costs of connecting the partial problems dominate

- **Geometric progression:**
  Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \ldots, 2^k$$

- **Geometric series:**
  The series (cumulative sum) of a geometric sequence
- For $|\, q \,| < 1$:

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \qquad \Rightarrow \text{constant}$$

Proof of $\mathcal{O}(n^2)$:

- We know:

$$T(n) = 3T\left(\frac{n}{4}\right) + \Theta(n^2)$$
$$\leq 3T\left(\frac{n}{4}\right) + c \cdot n^2$$

- Assumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) \leq k \cdot n^2$$

Proof of $\mathcal{O}(n^2)$:

- Presumption: $T(n) \in \mathcal{O}(n^2)$, so there exists a $k > 0$ with

$$T(n) < k \cdot n^2$$

- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \\ &= \frac{3}{16} k \cdot n^2 + c \cdot n^2 \\ &\leq k \cdot n^2 \qquad\qquad \text{for } k \geq \frac{16}{13} c \end{aligned}$$

Master theorem:

- Solution approach for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \qquad a \geq 1, b > 1$$

- $T(n)$ is the runtime of an algorithm ...
  - ... which divides a problem of size n in a partial problems
  - ... which solves each partial problem recursively with a runtime of $T\left(\frac{n}{b}\right)$
  - ... which takes $f(n)$ steps to merge all partial solutions

Master theorem:

- In the examples we have seen that ...
  - Either the runtime of connecting the solutions dominates
  - Or the runtime of solving the problems dominates
  - Or both have equal influence on runtime
- **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\substack{\text{Is any } f(n) \\ \text{in general form}}}, \qquad a \geq 1, b > 1, c > 0$$

- This yields a runtime of:

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

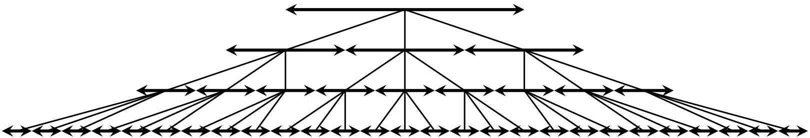


Figure: simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- Three partial problems with $\frac{1}{2}$ the size
- Solving the partial problems dominates (last layer, leaves)
- Runtime of $\Theta(n^{\log_b a})$

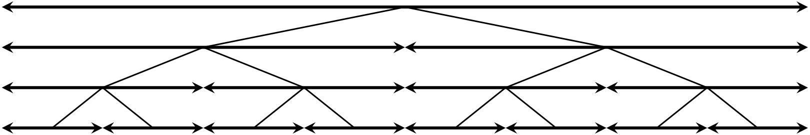


Figure: simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

- Two partial problems with $\frac{1}{2}$ the size
- Each layer has equal costs, $\log n$ layers
- Runtime of $\Theta(n \log n)$

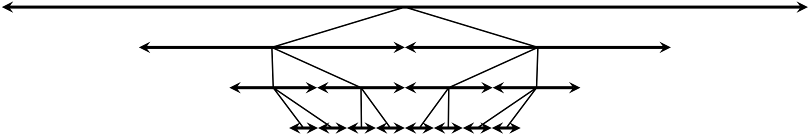


Figure: simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

- Two partial problems with $\frac{1}{3}$ the size
- Connecting all partial solutions dominates (first layer, root)
- Runtime of $\Theta(n)$

For a recursion equation like

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \qquad a \geq 1, b > 1, c > 0$$

- … yields a runtime of:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b \\ \Theta(n \log_b n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

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Master theorem (general form):

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \qquad a \geq 1, b > 1$$

- **Case 1:** $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon}),\ \varepsilon > 0$
  Solving the partial problems dominates
  (last layer, leaves)

- **Case 2:** $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$
  Each layer has equal costs, $\log_b n$ layers

Master theorem (general form):

- **Case 3:** $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a+\varepsilon}),\ \varepsilon > 0$
  Connecting all partial solutions in first layer (root) dominates

  Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1,$$
$$n > n_0$$

Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in O(n^{\log_b a - \varepsilon}),\ \varepsilon > 0$

Solving the partial problems dominates (last layer, leaves)

- $T(n) = 8 \cdot T(\frac{n}{2}) + 1000 \cdot n^2$

  $$a = 8,\ b = 2,\ f(n) = 1000 \cdot n^2,\ \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

  $$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

- $T(n) = 9 \cdot T(\frac{n}{3}) + 17 \cdot n$

  $$a = 9,\ b = 3,\ f(n) = 17 \cdot n,\ \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

  $$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$

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Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

- $T(n) = 2 \cdot T(\frac{n}{2}) + 10 \cdot n$

$$a = 2,\ b = 2,\ f(n) = 10 \cdot n,\ \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

- $T(n) = T(\frac{2n}{3}) + 1$

$$a = 1,\ b = \frac{3}{2},\ f(n) = 1,\ \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a+\varepsilon}),\ \varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T(\frac{n}{2}) + n^2$

$$a = 2,\ b = 2,\ f(n) = n^2,\ \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$f(n) \in \Omega(n^{1+\varepsilon})$

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a+\varepsilon}),\ \varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T(\frac{n}{2}) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$
- Check if regularity condition also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow c \geq \frac{1}{2}$$

$\Rightarrow T(n) \in \Theta(n^2)$

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Master theorem:

- Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2,\ b = 2,\ f(n) = n \log n,\ \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$
- **Case 2:** $f(n) \notin \Theta(n^1)$
- **Case 3:** $f(n) \notin \Omega(n^{1+\varepsilon})$

$n \log n$ is *asymptotically* larger than n,
but not *polynominal* larger

Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- Three cases depending on the dominance of the terms
- **Case 1:** Solving the partial problems is *polynominal* bigger than merging all solutions
  $T(n) \in \Theta(n^{\log_b a})$, $T(n) \in \Theta(\text{number of leaves})$
- **Case 2:** Each layer has equal costs
  $T(n) \in \Theta(n^{\log_b a} \log n)$, $\log n$ layers
- **Case 3:** Connecting all partial solutions is *polynominal* bigger than solving all partial problems
  $T(n) \in \Theta(f(n))$

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Master theorem**

  [Wik] Master theorem
  https://en.wikipedia.org/wiki/Master_theorem