Entwurf, Analyse und Umsetzung von Algorithmen

Hash Map, Universal Hashing

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Entwurf, Analyse und Umsetzung von Algorithmen

iems
intelligente eingebettete
mikrosysteme

Structure

Associative Arrays
- Introduction
- Hash Map

Universal Hashing
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- Probability Calculation
- Proof
- Examples

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Reminder:

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Associative Arrays

How do we build a Map?

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Problem:

- Quickly find an element with a specific key
- Naive solution: store pairs of key and value in a normal array
- For n keys searching requires $\Theta(n)$ time
- With a hash map this just requires $\Theta(1)$ in the best case, ... regardless of how many elements are in the map!

Structure

Idea:

- Mapping the keys onto indices with a hash function
- Store the values at the calculated indices in a normal array

Example:

- Key set: $x = \{3904433, 312692, 5148949\}$

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  A "hash table" with 5 "buckets"

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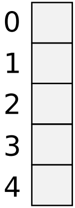
- Hash function: $h(x) = x \bmod 5$, in the range $[0, \ldots, 4]$
- We need an array T with 5 elements.
  A "hash table" with 5 "buckets"
- The element with the key x is stored in $T[h(x)]$

Storage:

Figure: Hash table T



Storage:

- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$

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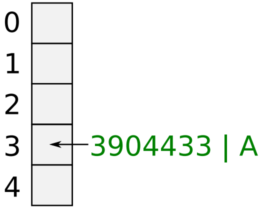

Storage:

- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")
- `insert(312692, "B")`: $h(312692) = 2 \Rightarrow$ T[2] = (312692, "B")

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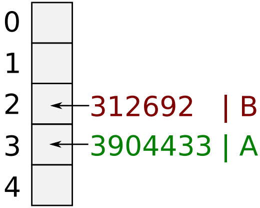

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- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")
- `insert(312692, "B")`: $h(312692) = 2 \Rightarrow$ T[2] = (312692, "B")
- `insert(5148949, "C")`: $h(5148949) = 4 \Rightarrow$ T[4] = (5148949, "C")

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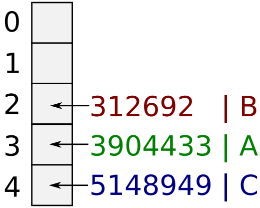

Searching:

- `search(3904433)`: $h(3904433) = 3 \Rightarrow T[3] \rightarrow$ (3904433, "A")

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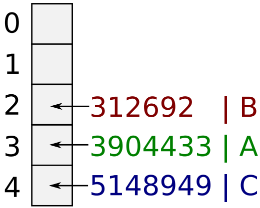

Searching:

- `search(3904433)`: $h(3904433) = 3 \;\; \Rightarrow$ T[3] $\rightarrow$ (3904433, "A")
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  $\Rightarrow$ Value with key 123459 does not exist

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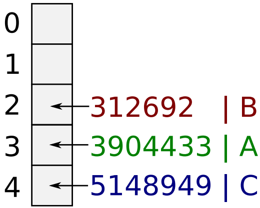

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- `search(3904433)`: $h(3904433) = 3 \Rightarrow T[3] \rightarrow$ (3904433, "A")
- `search(123459)`: $h(123459) = 4 \Rightarrow T[4]$

  $\Rightarrow$ Value with key 123459 does not exist
- Search time for this example: $\mathcal{O}(1)$

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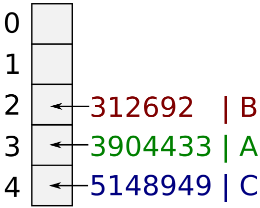
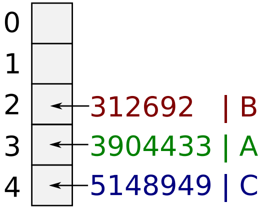


Further inserting:

- `insert(876543, "D"):` $h(876543) = 3$

Figure: Hash table T



Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$
  $\Rightarrow$ T[3] = (876543, "D") $\Rightarrow$ Collision

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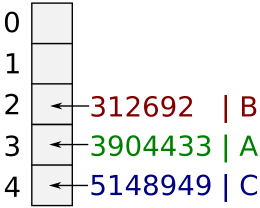
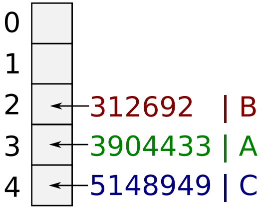


Associative Arrays
Hash Collisions

Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$
  $\Rightarrow$ T[3] = (876543, "D") $\Rightarrow$ Collision
- This happens more often than expected
  - **Birthday problem:** with 23 people we have the probability of 50 % that 2 of them have birthday at the same day

Figure: Hash table T



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Easiest Solution:

- Represent each bucket as list of key-value pairs

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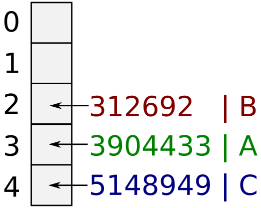


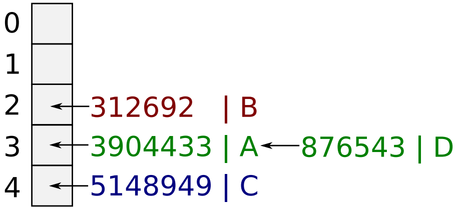
Problem:

- Two keys are equal $h(x) = h(y)$ but not the values $x \neq y$

Easiest Solution:

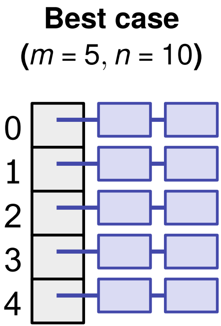
- Represent each bucket as list of key-value pairs
- Append new values to the end of the list

Figure: Hash table T



Best case:

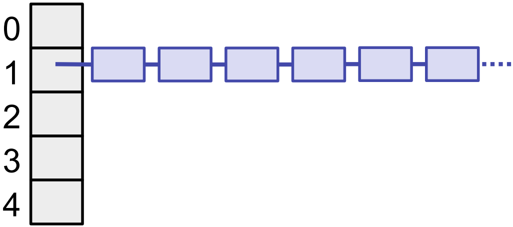
- We have n keys which are equally distributed over m buckets
- We have $\approx \frac{n}{m}$ pairs per bucket
- The runtime for searching is nearly $\mathcal{O}(1)$ if **not** $n \gg m$



Worst case:

- All n keys are mapped onto the same bucket
- The runtime is $\Theta(n)$ for searching

Worst case
($m = 5$, $n = 10$)



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- A hash function is defined for a given key set

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- A hash function is defined for a given key set
- Find a set of keys resulting in a degenerated hash table
  - *The hash function stays fixed*
  - *For table size of 100: try* $100 \times (99+1)$ *different numbers*
  - *Worst case: all 100 key sets map to one bucket*
- **Now:** find a solution to avoid that problem

Solution: universal hashing

- Out of a set of hash functions we randomly choose one

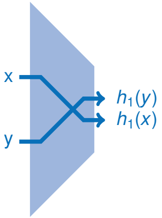


Figure: Hash func. 1

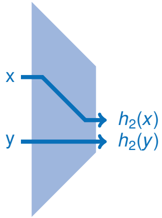
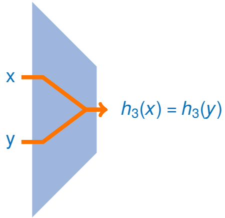


Figure: Hash func. 2



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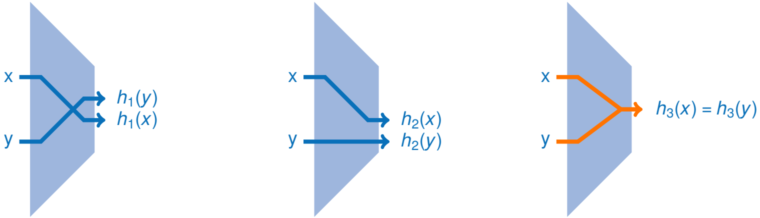
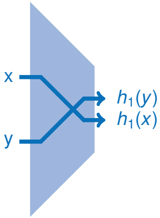


Figure: Hash func. 1 Figure: Hash func. 2 Figure: Hash func. coll.

Solution: universal hashing

- Out of a set of hash functions we randomly choose one
- The expected result of the hash function is an equal distribution over the buckets
- This hash function stays fixed for the lifetime of table
  Optional: copy table with new hash when degenerated



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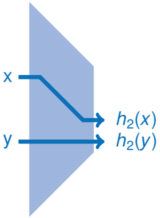

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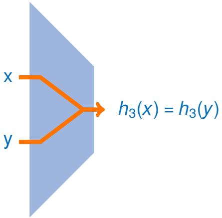

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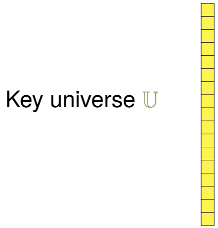
Definition:

- We call $\mathbb{U}$ the set (universe) of possible keys



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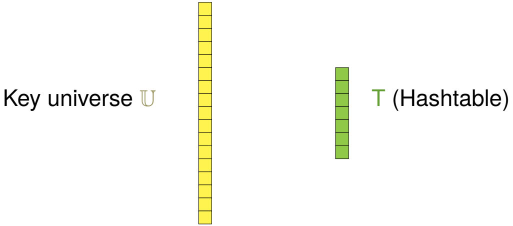

Definition:

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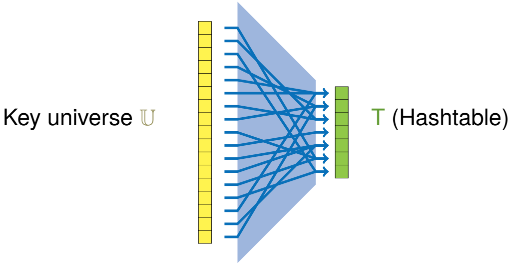

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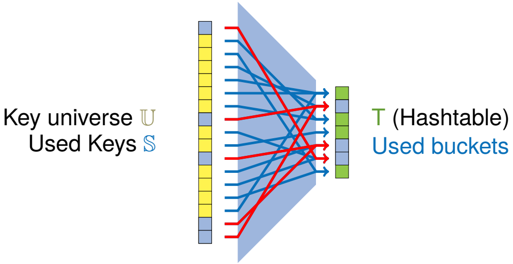

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- Idea: runtime should be $O(1 + \frac{|\mathbb{S}|}{m})$, where $\frac{|\mathbb{S}|}{m}$ is the table load

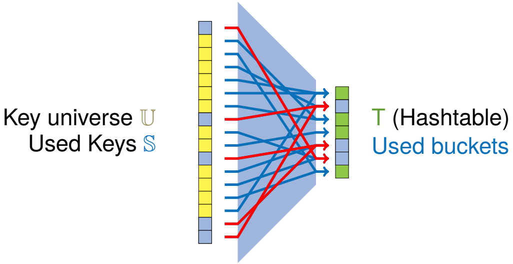


Figure: Hash function h_1

- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$

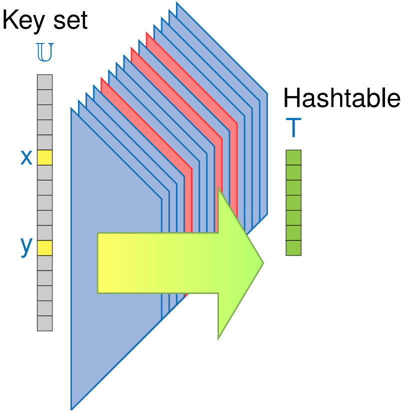
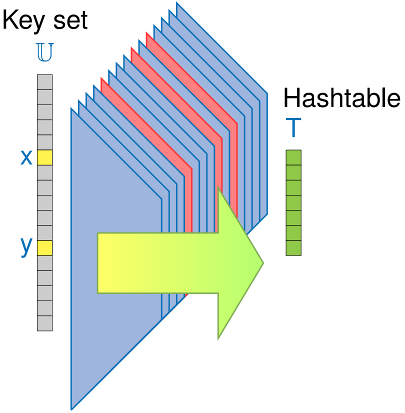


Figure: Set of hash functions $\mathbb{H}$

- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$
- An average of 3 out of 15 functions produce collisions



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Definition: $\mathbb{H}$ is *c-universal* if $\forall x, y \in \mathbb{U} \mid x \neq y$:

$$\frac{\overbrace{|\{h \in \mathbb{H} : h(x) = h(y)\}|}^{\text{Number of hash functions that create collisions}}}{\underbrace{|\mathbb{H}|}_{\text{Number of hash functions}}} \leq c \cdot \frac{1}{m}, \qquad c \in \mathbb{R}$$

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$$Prob(Collision) = \frac{1}{m} \Leftrightarrow c = 1$$

- $\mathbb{U}$: key universe
- $\mathbb{S}$: used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$: keys mapping to Bucket i ("synonyms")
- Ideal would be $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$

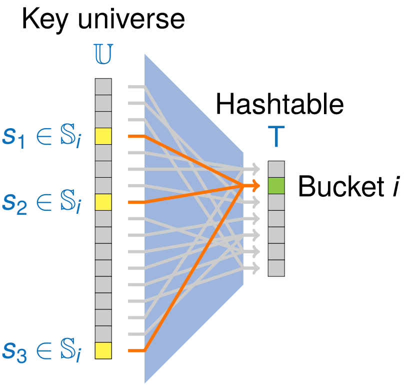

Figure: Hash function $h \in \mathbb{H}$

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- Particulary: if $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

Structure

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Table: throwing a dice

e	$P(e)$
1	$^1/_6$
2	$^1/_6$
3	$^1/_6$
4	$^1/_6$
5	$^1/_6$
6	$^1/_6$

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$(1,1)$	${}^1\!/_{36}$
$(1,2)$	${}^1\!/_{36}$
$(1,3)$	${}^1\!/_{36}$
$\ldots$	$\ldots$
$(6,5)$	${}^1\!/_{36}$
$(6,6)$	${}^1\!/_{36}$

Example:

- Rolling a dice twice ($\Omega = \{1,\ldots,6\}^2$)
- Each event $e \in \Omega$ has the probability $P(e) = {}^1\!/_{36}$
- E = if both results are even, then $P(E) =$

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  - Assigns a number to the result of an experiment
  - For example: X = Sum of results for rolling twice
  - $X = 12$ and $X \geq 7$ are regarded as events

Table: throwing a dice twice

e	$P(e)$	X
$(1,1)$	$^1/_{36}$	2
$(1,2)$	$^1/_{36}$	3
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$\dots$	$\dots$	$\dots$
$(6,5)$	$^1/_{36}$	11
$(6,6)$	$^1/_{36}$	12

Example:

- Random variable
  - Assigns a number to the result of an experiment
  - For example: X = Sum of results for rolling twice
  - $X = 12$ and $X \geq 7$ are regarded as events
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Sum of expected values: for arbitrary discrete random variables $X_1, \ldots, X_n$ we can write:

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n)$$

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- X_2: result of dice 2: $\mathbb{E}(X_2) = 3.5$
- $X = X_1 + X_2$: total number
- Expected number when rolling two dices:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

Corollary:

The probability of the event E is $p = P(E)$. Let X be the occurences of the event E and n be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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Example (Rolling the dice 60 times)

$$\mathbb{E}(\text{occurences of 6}) = \frac{1}{6} \cdot 60 = 10$$

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UNI FREIBURG

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Def. $\mathbb{E}$-value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

Structure

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To proof:

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \forall i$$

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Universal Hashing
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We define an indicator variable:

$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \qquad y \in \mathbb{S} \setminus \{x\}$$

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$$\Rightarrow \qquad \mathbb{E}(|\mathbb{S}_i|) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

Auxiliary calculation:

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- Which x, y lead to a relative collision count bigger than $\frac{c}{m}$?

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- Exercise: show empirically that it is 2-universal

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- **Intuitive**: scalar product with base m

$$a \bullet x = \sum_{0,\ldots,k-1} a_i \cdot x_i$$

Example ($\mathbb{U} = \{0, \ldots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned} h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10 \end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned} h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7 \end{aligned}$$

Further Literature

- **Course literature**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Hash Map - Theory**

  [Wik] Hash table
  https://en.wikipedia.org/wiki/Hash_table

- **Hash Map - Implementations / API**

  [Cpp] C++ - hash_map
  http://www.sgi.com/tech/stl/hash_map.html

  [Jav] Java - HashMap
  https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html

  [Pyt] Python - Dictionaries (Hash table)
  https://en.wikipedia.org/wiki/Hash_table