Algorithms and Data Structures

Hash Map, Universal Hashing

Albert-Ludwigs-Universität Freiburg

UNI FREIBURG

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Algorithms and Data Structures, November 2018

Structure

Associative Arrays
- Introduction
- Hash Map

Universal Hashing
- Introduction
- Probability Calculation
- Proof
- Examples

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- An associative array is like a normal array, only that the indices are not $0, 1, 2, \ldots$, but different, e.g. telephone numbers

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- Naive solution: store pairs of key and value in a normal array
- For n keys searching requires $\Theta(n)$ time
- With a hash map this just requires $\Theta(1)$ in the best case, ... regardless of how many elements are in the map!

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- Mapping the keys onto indices with a hash function
- Store the values at the calculated indices in a normal array

Example:

- Key set: $x = \{3904433, 312692, 5148949\}$

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  A "hash table" with 5 "buckets"

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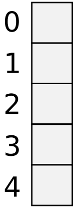
- Key set: $x = \{3904433, 312692, 5148949\}$
- Hash function: $h(x) = x \bmod 5$, in the range $[0, \ldots, 4]$
- We need an array T with 5 elements.
  A “hash table” with 5 “buckets”
- The element with the key x is stored in $T[h(x)]$

Storage:

Figure: Hash table T



Storage:

- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")

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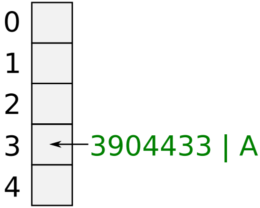

Storage:

- `insert(3904433,"A")`: $h(3904433) = 3 \Rightarrow$ T[3] = (3904433, "A")
- `insert(312692, "B")`: $h(312692) = 2 \Rightarrow$ T[2] = (312692, "B")

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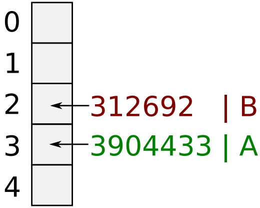

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- `insert(312692, "B")`: $h(312692) = 2 \Rightarrow$ T[2] = (312692, "B")
- `insert(5148949, "C")`: $h(5148949) = 4 \Rightarrow$ T[4] = (5148949, "C")

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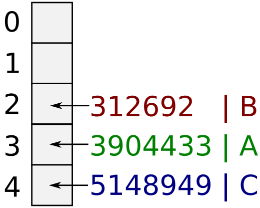

Searching:

- `search(3904433)`: $h(3904433) = 3 \Rightarrow T[3] \rightarrow$ (3904433, "A")

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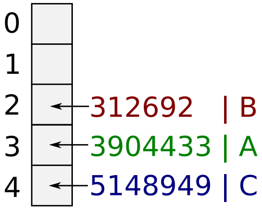

Searching:

- `search(3904433)`: $h(3904433) = 3 \;\; \Rightarrow T[3] \rightarrow$ (3904433, "A")
- `search(123459)`: $h(123459) = 4 \;\; \Rightarrow T[4]$

  $\Rightarrow$ Value with key 123459 does not exist

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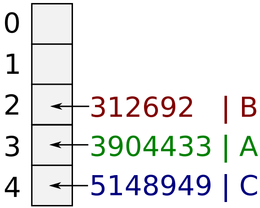

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- Search time for this example: $\mathcal{O}(1)$

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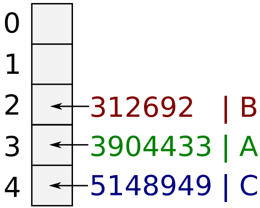
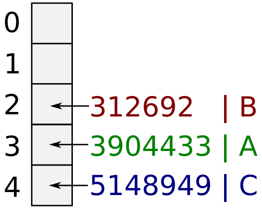


Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$

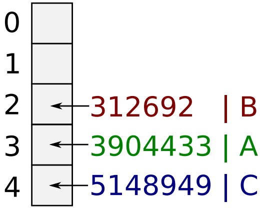
Figure: Hash table T



Further inserting:

- `insert(876543, "D"):` $h(876543) = 3$
  $\Rightarrow$ T[3] = (876543, "D") $\Rightarrow$ Collision

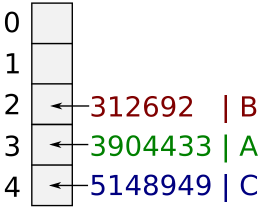
Figure: Hash table T



Further inserting:

- `insert(876543, "D")`: $h(876543) = 3$
  $\Rightarrow$ T[3] = (876543, "D") $\Rightarrow$ Collision
- This happens more often than expected
  - **Birthday problem:** with 23 people we have the probability of 50 % that 2 of them have birthday at the same day

Figure: Hash table T



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- Represent each bucket as list of key-value pairs

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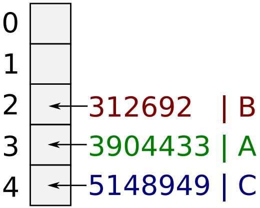

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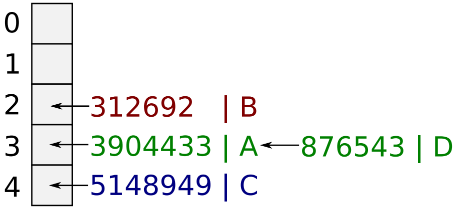
Easiest Solution:

- Represent each bucket as list of key-value pairs
- Append new values to the end of the list

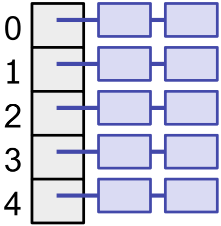
Figure: Hash table T



Best case:

- We have n keys which are equally distributed over m buckets
- We have $\approx \frac{n}{m}$ pairs per bucket
- The runtime for searching is nearly $\mathcal{O}(1)$ if **not** $n \gg m$

Best case
($m = 5, n = 10$)



Worst case:

- All n keys are mapped onto the same bucket
- The runtime is $\Theta(n)$ for searching

Worst case
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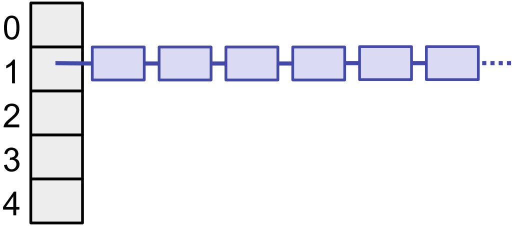

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  - *The hash function stays fixed*
  - *For table size of 100: try* $100 \times (99 + 1)$ *different numbers*
  - *Worst case: all 100 key sets map to one bucket*
- **Now:** find a solution to avoid that problem

Solution: universal hashing

- Out of a set of hash functions we randomly choose one

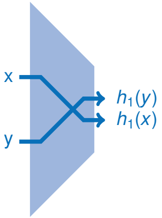


Figure: Hash func. 1

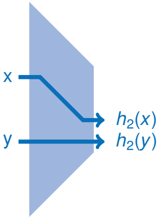
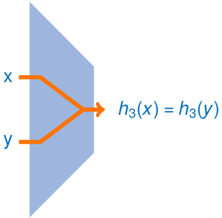


Figure: Hash func. 2



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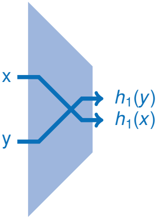
- The expected result of the hash function is an equal distribution over the buckets



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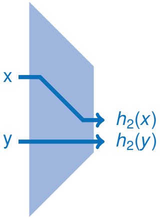

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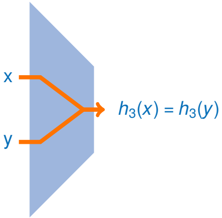

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Solution: universal hashing

- Out of a set of hash functions we randomly choose one
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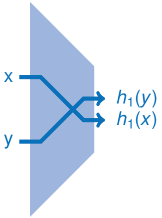
- This hash function stays fixed for the lifetime of table
  Optional: copy table with new hash when degenerated



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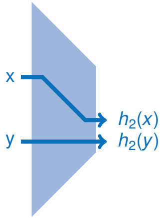

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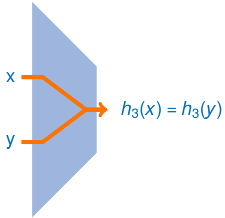

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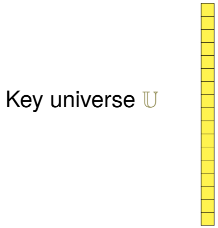
- We call $\mathbb{U}$ the set (universe) of possible keys



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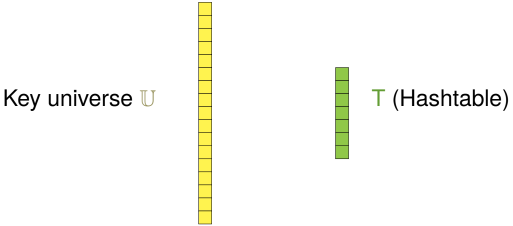

Definition:

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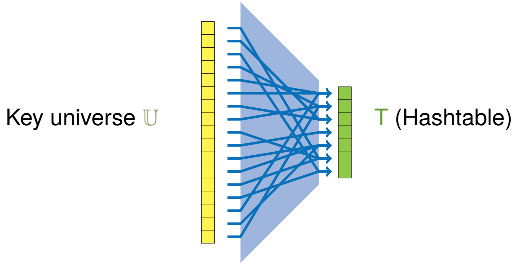

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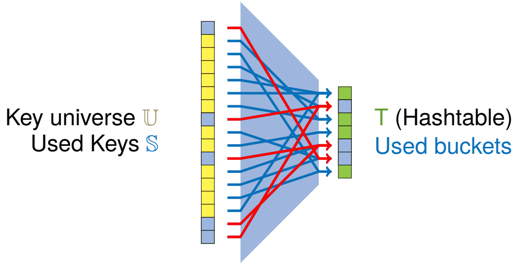

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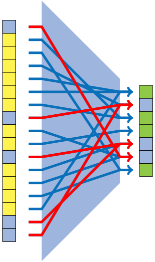
- Idea: runtime should be $O(1 + \frac{|\mathbb{S}|}{m})$, where $\frac{|\mathbb{S}|}{m}$ is the table load

Key universe $\mathbb{U}$
Used Keys $\mathbb{S}$



T (Hashtable)
Used buckets

Figure: Hash function h_1

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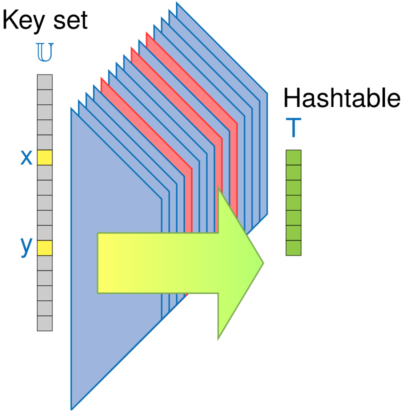

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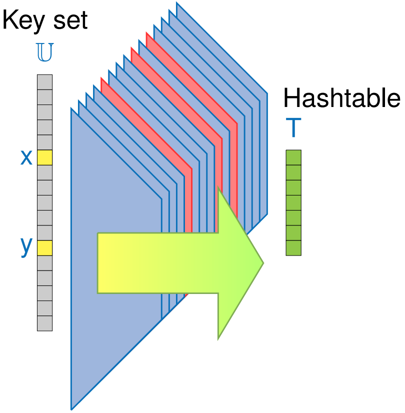
- We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$
- An average of 3 out of 15 functions produce collisions



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$$\frac{\overbrace{|\{h \in \mathbb{H} : h(x) = h(y)\}|}^{\text{Number of hash functions that create collisions}}}{\underbrace{|\mathbb{H}|}_{\text{Number of hash functions}}} \leq c \cdot \frac{1}{m}, \qquad c \in \mathbb{R}$$

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Note: If the hash function assigns each key x and y randomly to buckets then:

$$Prob(Collision) = \frac{1}{m} \Leftrightarrow c = 1$$

- $\mathbb{U}$: key universe
- $\mathbb{S}$: used Keys
- $\mathbb{S}_i \subseteq \mathbb{S}$: keys mapping to Bucket i (“synonyms”)
- Ideal would be $|\mathbb{S}_i| = \frac{|\mathbb{S}|}{m}$

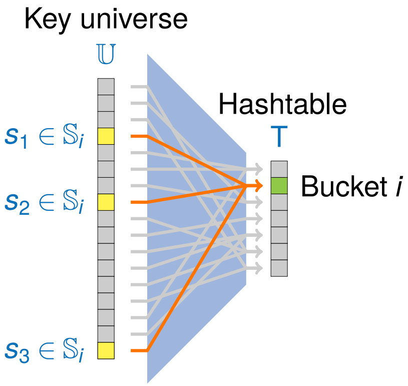

Figure: Hash function $h \in \mathbb{H}$

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- Let $\mathbb{S}$ be a set of keys and $h \in \mathbb{H}$ selected randomly
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- The expected average number of elements to search through per bucket is

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- Particulary: if $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

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Table: throwing a dice

e	$P(e)$
1	$^1/_6$
2	$^1/_6$
3	$^1/_6$
4	$^1/_6$
5	$^1/_6$
6	$^1/_6$

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- Each event $e \in \Omega$ has the probability $P(e) = {}^1/_{36}$

Table: throwing a dice twice

e	$P(e)$
$(1,1)$	${}^1/_{36}$
$(1,2)$	${}^1/_{36}$
$(1,3)$	${}^1/_{36}$
$\ldots$	$\ldots$
$(6,5)$	${}^1/_{36}$
$(6,6)$	${}^1/_{36}$

Example:

- Rolling a dice twice ($\Omega = \{1, \ldots, 6\}^2$)
- Each event $e \in \Omega$ has the probability $P(e) = {}^1/_{36}$
- E = if both results are even, then $P(E) =$

Table: throwing a dice twice

e	$P(e)$
$(1,1)$	$^1/_{36}$
$(1,2)$	$^1/_{36}$
$(1,3)$	$^1/_{36}$
$\ldots$	$\ldots$
$(6,5)$	$^1/_{36}$
$(6,6)$	$^1/_{36}$

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$(1,1)$	$1/36$	2
$(1,2)$	$1/36$	3
$(1,3)$	$1/36$	4
$\dots$	$\dots$	$\dots$
$(6,5)$	$1/36$	11
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  - Example 2: $P(X = 4) =$

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Table: throwing a dice once

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- Example "rolling once": $\mathbb{E}(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5$

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- Example "rolling twice": $\mathbb{E}(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 12 \cdot \frac{1}{36} = 7$

Sum of expected values: for arbitrary discrete random variables $X_1, \ldots, X_n$ we can write:

$$\mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n)$$

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Example: throwing two dice

- X_1: result of dice 1: $\mathbb{E}(X_1) = 3.5$
- X_2: result of dice 2: $\mathbb{E}(X_2) = 3.5$
- $X = X_1 + X_2$: total number
- Expected number when rolling two dices:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

Corollary:

The probability of the event E is $p = P(E)$. Let X be the occurences of the event E and n be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

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Example (Rolling the dice 60 times)

$$\mathbb{E}(\text{occurences of } 6) = \frac{1}{6} \cdot 60 = 10$$

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Def. $\mathbb{E}$-value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

Structure

Associative Arrays
Introduction
Hash Map

Universal Hashing
Introduction
Probability Calculation
Proof
Examples

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Given:

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To proof:

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \forall i$$

We know:

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We define an indicator variable:

$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \qquad y \in \mathbb{S} \setminus \{x\}$$

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$$\Rightarrow \qquad |\mathbb{S}_i| = 1 + \sum_{y \in \mathbb{S} \setminus x} I_y$$

$$\Rightarrow \qquad \mathbb{E}\left(|\mathbb{S}_i|\right) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

Auxiliary calculation:

$$
\begin{aligned}
\mathbb{E}[I_y] &= P(I_y = 1) \\
&= P(h(y) = i) \\
&= P(h(y) = h(x)) \\
&\leq c \cdot \frac{1}{m}
\end{aligned}
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Hence:

$$\begin{aligned}\mathbb{E}[|\mathbb{S}_i|] = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}\left[I_y\right] &\leq 1 + \sum_{y \in \mathbb{S} \setminus x} c \cdot \frac{1}{m}\\ &= 1 + (|\mathbb{S}| - 1) \cdot c \cdot \frac{1}{m}\\ &\leq 1 + |\mathbb{S}| \cdot c \cdot \frac{1}{m}\\ &= 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \square\end{aligned}$$

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Auxiliary calculation:

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$$\leq 1 + |\mathbb{S}| \cdot c \cdot \frac{1}{m}$$

$$= 1 + c \cdot \frac{|\mathbb{S}|}{m} \qquad \square$$

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&= 1 + (|\mathbb{S}| - 1) \cdot c \cdot \frac{1}{m} \\
&\leq 1 + |\mathbb{S}| \cdot c \cdot \frac{1}{m} \\
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- The set of all h for which $h_a(x) = (a \cdot x) \mod m$, for a $a \in \mathbb{U}$

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$$\forall x, y \quad x \neq y : \frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}$$

- Which x, y lead to a relative collision count bigger than $\frac{c}{m}$?

Positive example:

- Let p be a big prime number, $p > m$ and $p \geq |\mathbb{U}|$

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- Let $\mathbb{H}$ be the set of all h for which:

$$h_{a,b}(x) = ((a \cdot x + b) \mod p) \mod m,$$
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- Easy to implement but hard to proof
- Exercise: show empirically that it is 2-universal

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- **Intuitive**: scalar product with base m

$$a \bullet x = \sum_{0,\dots,k-1} a_i \cdot x_i$$

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Example ($\mathbb{U} = \{0, \ldots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned} h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\ &= (3x_2 + 4x_1 + 8x_0) \mod 10 \end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned} h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\ &= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\ &= 7 \end{aligned}$$

Further Literature

- **Course literature**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Hash Map - Theory**

  [Wik] Hash table
  `https://en.wikipedia.org/wiki/Hash_table`

- **Hash Map - Implementations / API**

  [Cpp] C++ - hash_map
  `http://www.sgi.com/tech/stl/hash_map.html`

  [Jav] Java - HashMap
  `https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html`

  [Pyt] Python - Dictionaries (Hash table)
  `https://en.wikipedia.org/wiki/Hash_table`