# Entwurf, Analyse und Umsetzung von Algorithmen

## Static Arrays, Dynamic Arrays, Amortized Analysis

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Entwurf, Analyse und Umsetzung von Algorithmen

iems
intelligente eingebettete
mikrosysteme

# Structure

Static Arrays

Dynamic Arrays
- Introduction
- Amortized Analysis

# Static Arrays

- Static arrays exist in nearly every programming language
- They are initialized with a fixed size $n$
- **Problem:** The needed size is not always clear at compile time

Table: Static array with size $n = 5$

| Index | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Value | "a" | "b" | "c" | "d" | "e" |

# Static Arrays

## Python

**Python:**

- We have dynamic sized lists
- Python does automatic resizing when needed

```
# Creates a list of "0"s with init. size 10
numbers = [0] * 10

# Prints number at index 7 ("0")
print("%d" % numbers[7])

# Saves number 42 at index 8
numbers[8] = 42

# Prints the number at index 8 ("42")
print("%d" % numbers[8])
```

# Static Arrays

- The name “static array” has nothing to do with the keyword `static` from Java / C++
- Nor is the array allocated before the program starts
- The size of the array is static and can not be changed after creation
- The name “fixed-size array” would be more appropiate

# Structure

Static Arrays

Dynamic Arrays
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# Dynamic Arrays

## Introduction

**Dynamic arrays:**

- The array is created with an initial size
- The size can be dynamically modified
- **Problem:** We need a dynamic structure to store the data

# Dynamic Arrays

## Python

**Python:**

```
greetings = ["Good morning", "ohai"]

greetings.append("Guten morgen")
greetings.append("bonjour")

# Prints text at index 2 ("Guten morgen")
print("%s" % greetings[2])

# Removes all elements
greetings.clear();
```

# Dynamic Arrays

## Implementation 1

- We store the data in a fixed-size array with the needed size
- **Append:**
  - Create fixed-size array with the needed size
  - Copy elements from the old to the new array
- **Remove:**
  - Create fixed-size array with the needed size
  - Copy elements from the old to the new array

# Dynamic Arrays

## Implementation 1

First implementation:

- We resize the array before each append
- We choose the size exactly as needed

# Dynamic Arrays

## Implementation 1 - Python

```
class DynamicArray:

    def __init__(self):
        self.size = 0
        self.elements = []

    def capacity(self):
        return len(self.elements)

    ...
```

# Dynamic Arrays

## Implementation 1 - Python

```
class DynamicArray:
    ...

    def append(self, item):
        newElements = [0] * (self.size + 1)

        for i in range(0, self.size):
            newElements[i] = self.elements[i]

        self.elements = newElements

        newElements[self.size] = item
        self.size += 1
```

# Dynamic Arrays

## Implementation 1

- Why is the runtime quadratic?

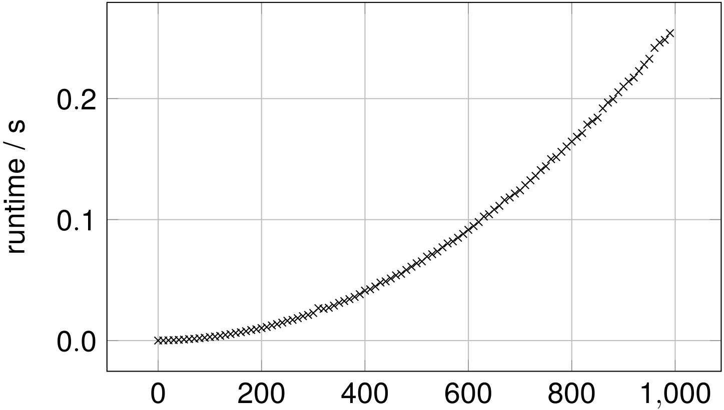
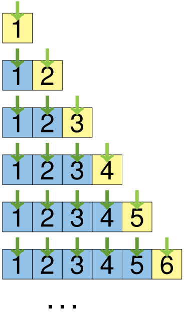


Figure: Runtime of *DynamicArray*

# Dynamic Arrays

## Implementation 1

**Runtime:**



| | |
|---|---|
| $O(1)$ | write 1 element |
| $O(1+1)$ | write 1 element, copy 1 element |
| $O(1+2)$ | write 1 element, copy 2 elements |
| $O(1+3)$ | write 1 element, copy 3 elements |
| $O(1+4)$ | write 1 element, copy 4 elements |
| $O(1+5)$ | write 1 element, copy 5 elements |
| … | … |

# Dynamic Arrays

## Implementation 1

**Analysis:**

- Let $T(n)$ be the runtime of $n$ sequential append operations
- Let $T_i$ be the runtime of each $i$-th operation
  - Then $T_i = A \cdot i$ for a constant $A$
  - We have to copy $i-1$ elements

$$T(n) = \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} (A \cdot i) = A \cdot \sum_{i=1}^{n} i = A \cdot \frac{n^2+n}{2}$$

$$= O(n^2)$$

# Dynamic Arrays

## Implementation 2

**Idea:**

- Better resize strategy
- We allocate more space than needed
- We over-allocate a constant amount of elements
  - Amount: $C = 3$ or $C = 100$

# Dynamic Arrays

## Implementation 2 - Python

```
def append(self, item):
    if self.size >= len(self.elements):
        newElements = [0] * (self.size + 100)

        for i in range(0, self.size - 1):
            newElements[i] = self.elements[i]

        self.elements = newElements

    self.elements[self.size] = item
    self.size += 1
```

# Dynamic Arrays

## Implementation 2

- Why is the runtime still quadratic?

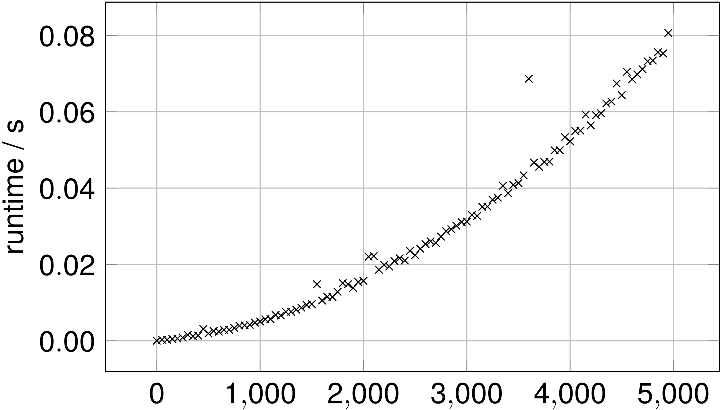
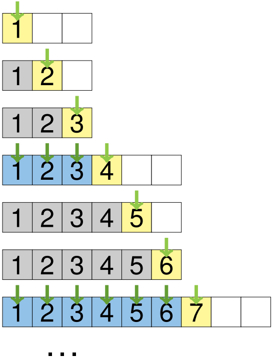


Figure: Runtime of *DynamicArray*

# Dynamic Arrays

## Implementation 2

**Runtime for $C = 3$:**



| | |
|---|---|
| $O(1)$ | write 1 element |
| $O(1)$ | write 1 element |
| $O(1)$ | write 1 element |
| $O(1+3)$ | write 1 element, copy 3 elements |
| $O(1)$ | write 1 element |
| $O(1)$ | write 1 element |
| $O(1+6)$ | write 1 element, copy 6 elements |
| … | … |

# Dynamic Arrays

## Implementation 2

**Analysis:**

- Most of the append operations now just cost $O(1)$
- Every $C$ steps the costs for copying are added: $C, 2 \cdot C, 3 \cdot C, \dots$ this means:

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{n} A \cdot 1 + \sum_{i=1}^{n/C} A \cdot i \cdot C \\
&= A \cdot n + A \cdot C \cdot \sum_{i=1}^{n/C} i \\
&= A \cdot n + A \cdot C \cdot \frac{\frac{n^2}{C^2} + \frac{n}{C}}{2} \\
&= A \cdot n + \frac{A}{2 \cdot C} \cdot n^2 + \frac{A}{2} \cdot n = O(n^2)
\end{aligned}
$$

- The factor of $n^2$ is getting smaller

# Dynamic Arrays

## Implementation 3

**Idea:**

- Double the size of the array

```
def append(self, item):
    if self.size >= len(self.elements):
        newElements = [0] \
            * max(1, 2 * self.size)

        for i in range(0, self.size):
            newElements[i] = self.elements[i]

        self.elements = newElements

    self.elements[self.size] = item
    self.size += 1
```

# Dynamic Arrays

## Implementation 3

- Now the runtime is linear with some bumps. Why?

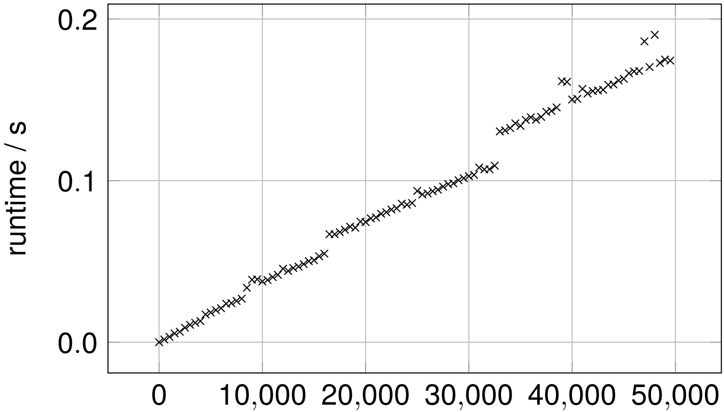
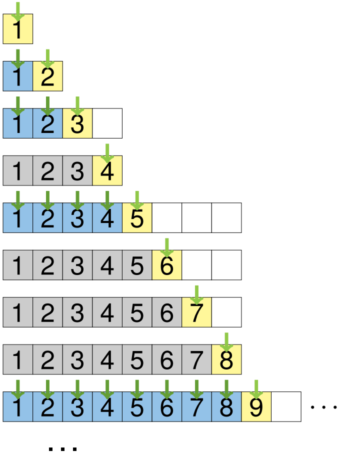


Figure: Runtime of *DynamicArray*

# Dynamic Arrays

## Implementation 2

**Runtime for $C = 2$ (Double the size):**



| | |
|---|---|
| $O(1)$ | write 1 |
| $O(1 + 1)$ | write 1, copy 1 element |
| $O(1 + 2)$ | write 1, copy 2 elements |
| $O(1)$ | write 1 |
| $O(1 + 4)$ | write 1, copy 4 elements |
| $O(1)$ | write 1 |
| $O(1)$ | write 1 |
| $O(1)$ | write 1 |
| $O(1 + 8)$ | write 1, copy 8 elements |
| … | … |

# Dynamic Arrays

## Implementation 3

**Analysis:**

- Now all appends cost $O(1)$
- Every $2^i$ steps we have to add the cost $A \cdot 2^i$ (for $i = 0, 1, 2, \ldots, k$ with $k = floor(log_2(n-1))$
- In total that accounts to:

$$\begin{aligned} T(n) &= n \cdot A + A \cdot \sum_{i=0}^{k} 2^i = n \cdot A + A(2^{k+1} - 1) \\ &\leq n \cdot A + A \cdot 2^{(k+1)} \\ &= n \cdot A + 2 \cdot A \cdot 2^{(k)} \\ &\leq n \cdot A + 2 \cdot A \cdot n \\ &= 3 \cdot A \cdot n \\ &= O(n) \end{aligned}$$

# Dynamic Arrays

## Shrinking

**How do we shrink the array?**

- If the array is half-full, we can shrink it by half, like for the append operation
- If we *append* directly after *shrinking* we have to extend the array again
  - We leave some space for following append operations

  ⇒ We only shrink the array to 75%

# Dynamic Arrays

## Shrinking

**Analysis:**

- **Difficult:** We have a random number of *append* / *remove* operations
- We can not exactly predict when resizing is happening

# Structure

Static Arrays

Dynamic Arrays

- Introduction
- Amortized Analysis

# Dynamic Arrays

## Amortized Analysis

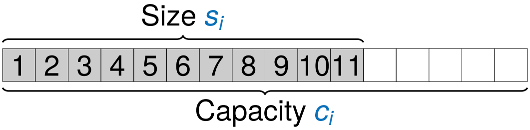


Figure: Static array with capacity $c_i$

**Notation:**

- We have $n$ instructions $O = \{O_1, \ldots, O_n\}$
- The size after operation $i$ is $s_i$, with $s_0 := 0$
- The capacity after operation $i$ is $c_i$, with $c_0 := 0$
- The cost of operation $i$ is $\mathrm{cost}(O_i)$ (previously named $T_i$)

Reallocation: $\mathrm{cost}(O_i) \leq A \cdot s_i$,
Insert / Delete (Update): $\mathrm{cost}(O_i) \leq A$,

# Dynamic Arrays

## Amortized Analysis - Example

| Operation | | | Size $s_i$ | Capactity $c_i$ | Costs $\mathrm{cost}(O_i)$ |
|---|---|---|---|---|---|
| $O_1$ | append | realloc. | $s_1$ | $c_1$ | $A \cdot s_1$ |
| $O_2$ | append | | $s_2$ | $c_2 = c_1$ | $A \cdot 1$ |
| $O_3$ | append | | $s_3$ | $c_3 = c_1$ | $A \cdot 1$ |
| $O_4$ | remove | | $s_4$ | $c_4 = c_1$ | $A \cdot 1$ |
| $O_5$ | remove | realloc. | $s_5$ | $c_5$ | $A \cdot s_5$ |
| $O_6$ | append | | $s_6$ | $c_6 = c_5$ | $A \cdot 1$ |
| $O_7$ | remove | | $s_7$ | $c_7 = c_5$ | $A \cdot 1$ |
| $O_8$ | append | | $s_8$ | $c_8 = c_5$ | $A \cdot 1$ |
| $O_9$ | append | realloc. | $s_9$ | $c_9$ | $A \cdot s_9$ |
| ... | ... | ... | ... | ... | ... |
| $O_n$ | append | | $s_n$ | $c_n$ | $A \cdot 1$ |

# Dynamic Arrays

## Amortized Analysis - Example

**Implementation:**

- If $O_i$ is an *append* operation and $s_{i-1} = c_{i-1}$:
  $\Rightarrow$ Resize array to $c_i = \left\lfloor \frac{3}{2} s_i \right\rfloor = \text{floor}\left(\frac{3}{2} s_i\right)$
  $\Rightarrow$ $cost(O_i) = A \cdot s_i$

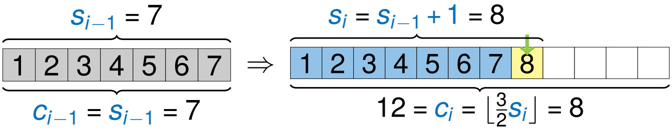


Figure: *Append* operation with reallocation

**Result:** after operation we have $c_i = \frac{3}{2} \cdot s_i$

# Dynamic Arrays

## Amortized Analysis - Example

**Implementation:**

- If $O_i$ is an *remove* operation and $s_{i-1} \leq \frac{1}{3} c_{i-1}$:
  $\Rightarrow$ Resize array to $c_i = \left\lfloor \frac{3}{2} s_i \right\rfloor = \text{floor}\left(\frac{3}{2} s_i\right)$
  $\Rightarrow cost(O_i) = A \cdot s_i$

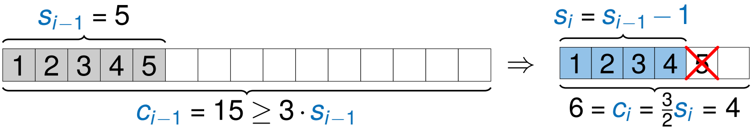


Figure: *Remove* operation with reallocation

**Result:** after operation we have again $c_i = \frac{3}{2} \cdot s_i$

# Dynamic Arrays

## Amortized Analysis - Proof

**Idea for proof:**

- Expensive are only operations where reallocations are necessary
- If we just reallocated, it takes some time until the next reallocation is required.
- **Assumption:** After a costly *reallocation* of size $X$ we have at least $X$ operations of runtime $O(1)$
- **Then:** Total cost of $n$ operations is maximally $2 \cdot n$

# Dynamic Arrays

## Amortized Analysis - Proof

Table: Dynamic Array with $C_{\text{ext}} = \frac{3}{2}$

| Operation | | | Size | Capacity | Costs |
|---|---|---|---|---|---|
| | | | $s_i$ | $c_i$ | $\mathrm{cost}(O_i)$ |
| $O_1$ | app. | realloc. | $s_1$ | $c_1 = 4$ | $C_1 \cdot s_1$ |
| $O_2$ | app. | | $s_2$ | $c_2 = c_1$ | $C_2$ |
| $O_3$ | app. | | $s_3$ | $c_3 = c_1$ | $C_2$ |
| $O_4$ | app. | | $s_4$ | $c_4 = c_1$ | $C_2$ |
| $O_5$ | app. | realloc. | $s_5$ | $c_5 = \lfloor \frac{3}{2} s_5 \rfloor = 7$ | $C_1 \cdot s_5$ |
| $O_6$ | app. | | $s_6$ | $c_6 = c_5$ | $C_2$ |
| $O_7$ | app. | | $s_7$ | $c_7 = c_5$ | $C_2$ |
| $O_8$ | app. | realloc. | $s_8$ | $c_8 = \frac{3}{2} s_8 = 12$ | $C_1 \cdot s_8$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |

distance $4 \geq \left\lfloor \frac{s_1}{2} \right\rfloor$

distance $3 \geq \left\lfloor \frac{s_5}{2} \right\rfloor$

# Dynamic Arrays

## Amortized Analysis - Proof

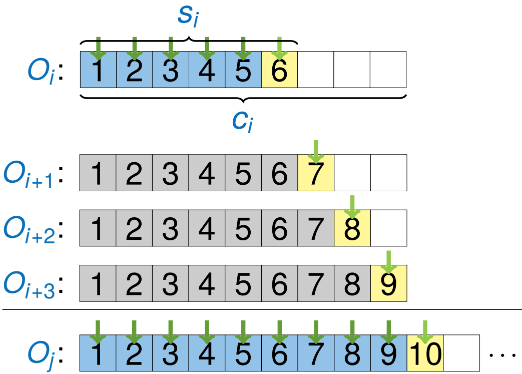
**To show:**

- **Lemma:** If a *reallocation* occurs at $O_i$ the nearest *reallocation* is at $O_j$ with $j - i > \frac{s_i}{2}$
- **Corollary:** $\mathrm{cost}(O_1) + \cdots + \mathrm{cost}(O_n) \leq 4A \cdot n$

# Dynamic Arrays

## Proof: Worst Case Same Operation

Table: Case 1: $\frac{1}{2}s_i$ *appends*



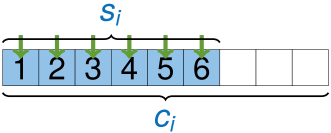
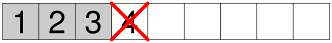
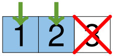
| Array | Costs |
|---|---|
| $O_i$: 1 2 3 4 5 6 | reallocation<br>$A \cdot s_i$ (linear) |
| $O_{i+1}$: 1 2 3 4 5 6 7 | $A$ (constant) |
| $O_{i+2}$: 1 2 3 4 5 6 7 8 | $A$ (constant) |
| $O_{i+3}$: 1 2 3 4 5 6 7 8 9 | $A$ (constant) |
| $O_j$: 1 2 3 4 5 6 7 8 9 10 … | reallocation<br>$A \cdot s_j$ (earliest realloc) |

$\frac{s_i}{2}$ times

# Dynamic Arrays

## Amortized Analysis - Proof

Table: Case 2: $\frac{1}{2}s_i$ *removes*

| | Array | Costs |
|---|---|---|
| $O_i$: |  | reallocation<br>$A \cdot s_i$ (linear) |
| $O_{i+1}$: |  | $A$ (constant) |
| $O_{i+2}$: |  | $A$ (constant) |
| $O_{i+3}$: |  | $A$ (constant) |
| $O_j$: |  | reallocation<br>$A \cdot s_j$ (earliest reallocation) |

$\left.\begin{matrix} O_{i+1} \\ O_{i+2} \\ O_{i+3} \end{matrix}\right\} \frac{s_i}{2}$ times

**Proof of lemma:**

- If a reallocation happens at $O_i$ and then again at $O_j$, then $j - i \geq s_i/2$
- After operation $O_i$ the capacity is

$$c_i = \left\lfloor \frac{3}{2} \cdot s_i \right\rfloor$$

- Lets consider a operation $O_i$ to $O_k$ with $k - i \leq \frac{s_i}{2}$:
    - Case 1: Since the *reallocation* we have inserted at maximum $\text{floor}\left(\frac{1}{2} \cdot s_i\right)$ elementsation

$$s_k \leq s_i + \left\lfloor \frac{s_i}{2} \right\rfloor = \left\lfloor \frac{3}{2} s_i \right\rfloor = c_i$$

no reallocation needed

**Proof of lemma - continued:**

- Case 2: Since the *reallocation* we have removed at maximum $\left\lfloor \frac{1}{2} s_i \right\rfloor$ elements

$$s_k \geq s_i - \left\lfloor \frac{s_i}{2} \right\rfloor = \left\lceil \frac{1}{2} s_i \right\rceil$$

no reallocation needed

$$\Rightarrow 3 \cdot s_k \geq \left\lceil \frac{3}{2} s_i \right\rceil \geq \left\lfloor \frac{3}{2} s_i \right\rfloor = c_i$$

# Dynamic Arrays

## Amortized Analysis - Proof of Corollary

### Corollary:

$$\text{cost}(O_1) + \cdots + \text{cost}(O_n) \leq 4A \cdot n$$

- Let the *reallocations* be at operations $\text{cost}(O_{i_1}), \ldots, \text{cost}(O_{i_m})$
- The cost of all *reallocations* are $A \cdot \left(s_{i_1} + \cdots + s_{i_m}\right)$
- With the lemma we know:

$$i_2 - i_1 > \frac{s_{i_1}}{2}, \quad i_3 - i_2 > \frac{s_{i_2}}{2}, \quad \ldots, \quad i_m - i_{m-1} > \frac{s_{i_{m-1}}}{2}$$

# Dynamic Arrays

## Amortized Analysis - Proof of Corollary

- We can conclude that:

$$
\begin{aligned}
i_2 - i_1 &> \frac{s_{i_1}}{2} &\Rightarrow&& s_{i_1} &< 2(i_2 - i_1) \\
i_3 - i_2 &> \frac{s_{i_2}}{2} &\Rightarrow&& s_{i-2} &< 2(i_3 - i_2) \\
&& \vdots && & \\
i_m - i_{m-1} &> \frac{s_{i_{m-1}}}{2} &\Rightarrow&& s_{i_{m-1}} &< 2(i_m - i_{m-1}) \\
&&&& s_{i_m} &\leq n \qquad \text{(trivial)}
\end{aligned}
$$

# Dynamic Arrays

## Amortized Analysis - Proof of Corollary

- The costs of all reallocations are:

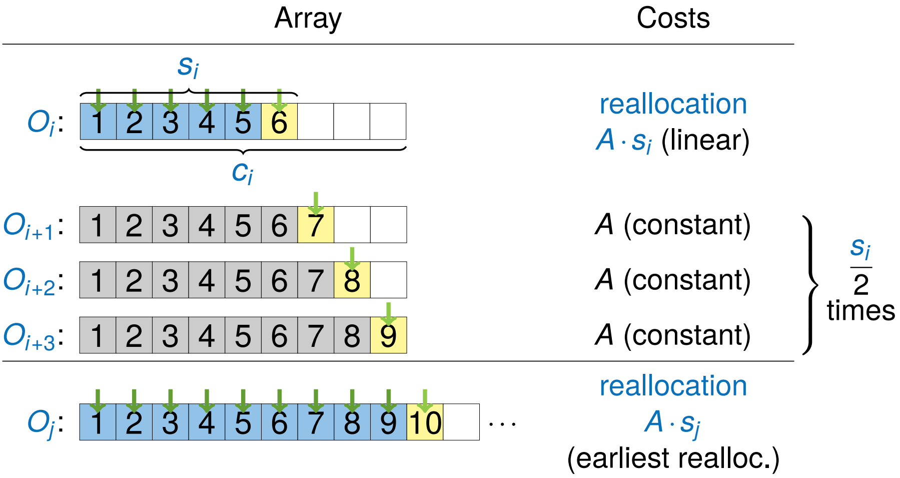
$$\begin{aligned}
cost(realloc.) &= A \cdot \left(s_{i_1} + \cdots + s_{i_m}\right) \\
&< A \cdot (2(i_2 - i_1) + 2(i_3 - i_2) + \cdots + 2(i_m - i_{m-1}) + n) \\
&= A \cdot (2(i_m - i_1) + n) \\
&\leq A \cdot (2n + n) = 3A \cdot n
\end{aligned}$$

- Additionally we have to consider the respective constant costs for a normal append or remove ($\leq A \cdot n$) therefore in total $\leq 4 \cdot A \cdot n$

# Dynamic Arrays

## Amortized Analysis - Alternate Proof of Corollary

Table: Case 1: $\frac{1}{2}s_i$ *appends*



| Array | Costs |
|---|---|
| $O_i$: $\underbrace{\overbrace{1\ 2\ 3\ 4\ 5\ 6}^{s_i}\ \square\ \square\ \square}_{c_i}$ | reallocation<br>$A \cdot s_i$ (linear) |
| $O_{i+1}$: 1 2 3 4 5 6 7 | $A$ (constant) |
| $O_{i+2}$: 1 2 3 4 5 6 7 8 | $A$ (constant) |
| $O_{i+3}$: 1 2 3 4 5 6 7 8 9 | $A$ (constant) |
| $O_j$: 1 2 3 4 5 6 7 8 9 10 $\cdots$ | reallocation<br>$A \cdot s_j$<br>(earliest realloc.) |

$\frac{s_i}{2}$ times

# Dynamic Arrays

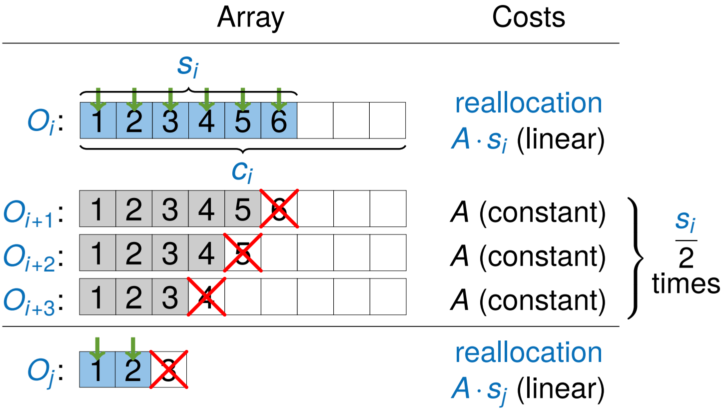
## Amortized Analysis - Alternate Proof of Corollary

- Total costs of $A \cdot \frac{3}{2} \cdot s_i$ for $\frac{s_i}{2} + 1$ operations
- Cost per operation:

$$\frac{\frac{3}{2}A \cdot s_i}{\frac{1}{2}s_i + 1} \leq \frac{\frac{3}{2}A \cdot s_i}{\frac{1}{2}s_i} = 3 \cdot A = \text{const.}$$

# Dynamic Arrays

## Amortized Analysis - Alternate Proof of Corollary



| Array | Costs | |
|---|---|---|
| $s_i$ <br> $O_i$: 1 2 3 4 5 6 <br> $c_i$ | reallocation <br> $A \cdot s_i$ (linear) | |
| $O_{i+1}$: 1 2 3 4 5 ~~6~~ | $A$ (constant) | $\frac{s_i}{2}$ times |
| $O_{i+2}$: 1 2 3 4 ~~5~~ | $A$ (constant) | |
| $O_{i+3}$: 1 2 3 ~~4~~ | $A$ (constant) | |
| $O_j$: 1 2 ~~3~~ | reallocation <br> $A \cdot s_j$ (linear) | |

- Runtime analysis for local worst-case sequence
- Same total cost as previous slide

# Dynamic Arrays

## Amortized Analysis - Yet Another Proof of Corollary

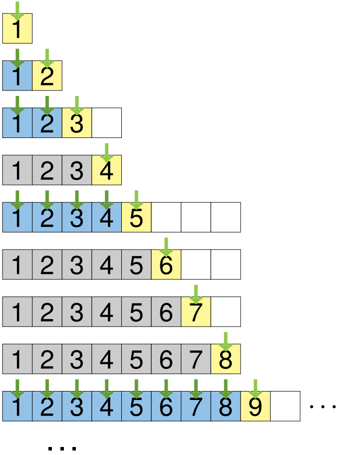
**Bank account paradigm:**

- **Idea:** “Save first, spend later”
- For each operation we deposit some coins on an “bank account”
  $\Rightarrow$ We still have constant costs
- When we have a linear operation (reallocation) we pay with the coins from our “bank account”
- For the “double the size” strategy we have to pay two coins per operation

# Dynamic Arrays

## Amortized Analysis - Yet Another Proof of Corollary

**Double the size:**



| $\mathrm{cost}(O_i)$ | deposit / withdraw | account value |
|---|---|---|
| $O(1)$ | +2 | 2 |
| $O(1+1)$ | +2 -1 | 3 |
| $O(1+2)$ | +2 -2 | 3 |
| $O(1)$ | +2 | 5 |
| $O(1+4)$ | +2 -4 | 3 |
| $O(1)$ | +2 | 5 |
| $O(1)$ | +2 | 7 |
| $O(1)$ | +2 | 9 |
| $O(1+8)$ | +2 -8 | 3 |
| ... | ... | ... |

# Dynamic Arrays

## Amortized Analysis - Yet Another Proof of Corollary

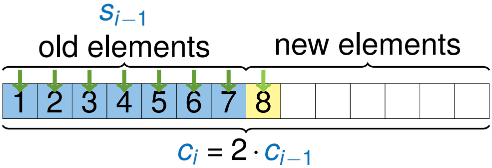


Figure: Array after realloc. (insert) operation

**Why do we need to deposit 2 coints per operation?**

1. Each newly inserted element has to be copied later (first coin)
2. Due to the factor of two there is for each new element also an old one in the array that also has to be copied (second coin)

# Dynamic Arrays

## Amortized Analysis - Yet Another Proof of Corollary

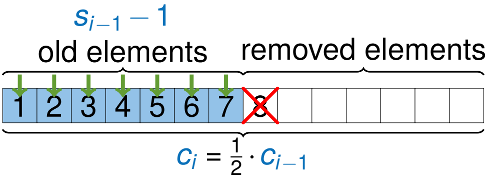


Figure: Array after realloc. (remove) operation

**Shrinking strategy: If array 1/4 full shrink by half**

- How many coins do we need per *remove* operation?
- **Worst case:** The previous remove operation triggered a *reallocation*

⇒ Array is half full

# Dynamic Arrays

## Amortized Analysis - Yet Another Proof of Corollary

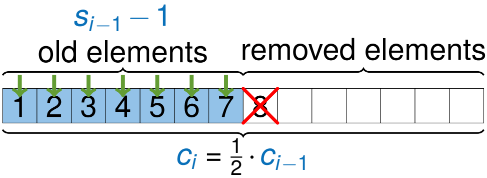


Figure: Array after realloc. (remove) operation

**Shrinking strategy: If array 1/4 full shrink by half**

- Array is half full
- The nearest *reallocation* is after removing $\frac{1}{4}c_i$ elements
- We have to copy $\frac{1}{4}c_i$ elements

⇒ 1 coin per operation is enough

# Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
**Introduction to Algorithms.**
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
`https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf`.

# Further Literature

- **Amortized Analysis**

[Wik] Amortized analysis
https://en.wikipedia.org/wiki/Amortized_analysis