Algorithms and Data Structures

Runtime Complexity, Associative Arrays

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Structure

Runtime Complexity

Associative Arrays

- Introduction
- Practical Example
  - Sorting
  - Associative Array

Structure

Runtime Complexity

Runtime Complexity

- The runtime does not entirely depend on the size of the problem, but also on the type of input
- This results in:
  - Best runtime:
    Lowest possible runtime complexity for an input of size n
  - Worst runtime:
    Highest possible runtime complexity for an input of size n
  - Average / Expected runtime:
    The average of all runtime complexities for an input of size n

Runtime Complexity

Example 1 - Conditions

- Input: Array a with n elements
  $a[i] \in \mathbb{N},\ 1 \leq a[i] \leq n,\ 0 \leq i < n$
- Output: Updated a with n elements where $a[0] \neq 1$

`if a[0] == 1:`	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(?)$
`    a[0] = 2`	$\mathcal{O}(1)$		
`else :`			
`    for i in range(0, n):`	$\mathcal{O}(n)$	$\mathcal{O}(n) \cdot \mathcal{O}(1)$	
`        a[i] = 2`	$\mathcal{O}(1)$	$= \mathcal{O}(n$	

Runtime Complexity

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if $a[0] == 1$:	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(?)$
$\quad a[0] = 2$	$\mathcal{O}(1)$		
else :			
$\quad$ for i in range(0, n):	$\mathcal{O}(n)$	$\mathcal{O}(n) \cdot \mathcal{O}(1)$	
$\quad\quad a[i] = 2$	$\mathcal{O}(1)$	$= \mathcal{O}(n$	

- Best runtime: $\mathcal{O}(1) + \mathcal{O}(1) = \mathcal{O}(1)$
- Worst runtime: $\mathcal{O}(1) + \mathcal{O}(n) = \mathcal{O}(n)$

Runtime Complexity

Example 1 - Average Runtime

- The average runtime is determined by the average runtime for all input instances of size n

Runtime Complexity

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  $\Rightarrow n \cdot \ldots \cdot n = n^n$ different input instances of size n

Runtime Complexity

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- Every element of a can have n different values
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  - `a[0] == 1` in n^{n-1} instances

Runtime Complexity

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- The average runtime is determined by the average runtime for all input instances of size n
- Every element of a can have n different values
  $\Rightarrow n \cdot \ldots \cdot n = n^n$ different input instances of size n
  - a[0] $==1$ in n^{n-1} instances
  - a[0] $!= 1$ in $n^n - n^{n-1} = n^{n-1} \cdot (n-1)$ instances

Runtime Complexity

Example 1 - Average Runtime

- The average runtime is determined by the average runtime for all input instances of size n
- Every element of a can have n different values
  $\Rightarrow n \cdot \ldots \cdot n = n^n$ different input instances of size n
  - `a[0] == 1` in n^{n-1} instances
  - `a[0] != 1` in $n^n - n^{n-1} = n^{n-1} \cdot (n-1)$ instances
- Sum of all runtime complexities:

$$\underbrace{n^{n-1} \cdot \mathcal{O}(1)}_{\texttt{a[0] == 1}} + \underbrace{n^{n-1} \cdot (n-1) \cdot \mathcal{O}(n)}_{\texttt{a[0] != 1}}$$

Runtime Complexity

Example 1 - Average Runtime

- The average runtime is determined by the average runtime for all input instances of size n
- Every element of a can have n different values
  $\Rightarrow n \cdot \ldots \cdot n = n^n$ different input instances of size n
  - $\mathrm{a}[0] == 1$ in n^{n-1} instances
  - $\mathrm{a}[0]\ != 1$ in $n^n - n^{n-1} = n^{n-1} \cdot (n-1)$ instances
- Sum of all runtime complexities:

$$\underbrace{n^{n-1} \cdot \mathcal{O}(1)}_{\mathrm{a}[0]\ ==\ 1} + \underbrace{n^{n-1} \cdot (n-1) \cdot \mathcal{O}(n)}_{\mathrm{a}[0]\ !=\ 1}$$

- Average runtime: (normalize by number of instances)

$$\frac{n^{n-1} + n^{n-1} \cdot (n-1) \cdot n}{n^n} = \frac{1}{n} + n - 1 \in \mathcal{O}(n)$$

Runtime Complexity

Example 2 - Binary Addition

- Input: binary number b with n digits
- Output: binary number $b + 1$ with n digits

Table: Binary addition

Digits (n)	Input	Output	Steps
10	1011100100	1011100101	1
4	1011	1100	3
8	11111111	00000000	8

Runtime Complexity

Example 2 - Binary Addition

- Input: binary number b with n digits
- Output: binary number $b+1$ with n digits
- Runtime of the algorithm is determined by the number of bits getting changed (steps)
  1. "0" $\rightarrow$ "1"
  2. "1" $\rightarrow$ "0"

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Runtime Complexity

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- Output: binary number $b+1$ with n digits
- Runtime of the algorithm is determined by the number of bits getting changed (steps)
  1. "0" $\rightarrow$ "1"
  2. "1" $\rightarrow$ "0"
- Best runtime: $1\,\text{step} = \mathcal{O}(1)$
- Worst runtime: $n\,\text{steps} = \mathcal{O}(n)$

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10	1011100100	1011100101	1
4	1011	1100	3
8	11111111	00000000	8

Runtime Complexity

Example 2 - Average Steps

Table: Binary addition with $n = 1$

Input	Output	Steps
0	1	1
1	0	1

$$\overline{\text{steps}} = \frac{1+1}{2} = 1$$

Runtime Complexity

Example 2 - Average Steps

Table: Binary addition with $n = 1$

Input	Output	Steps
0	1	1
1	0	1

Table: Binary addition with $n = 2$

Input	Output	Steps
00	01	1
01	10	2
10	11	1
11	00	2

$$\overline{\text{steps}} = \frac{1+1}{2} = 1$$

$$\overline{\text{steps}} = \frac{1+2+1+2}{4} = \frac{3}{2}$$

Runtime Complexity

Example 2 - Average Steps

Table: Binary addition with $n = 3$

Input	Output	Steps
000	001	1
001	010	2
010	011	1
011	100	3
100	101	1
101	110	2
110	111	1
111	000	3

$$\overline{\text{steps}} = \frac{1+2+1+3+1+2+1+3}{8} = \frac{7}{4}$$

Runtime Complexity
Example 2 - Average Steps

Table: Binary addition with $n = 3$

Input	Output	Steps
000	001	1
001	010	2
010	011	1
011	100	3
100	101	1
101	110	2
110	111	1
111	000	3

$$\overline{\text{steps}} = \frac{1+2+1+3+1+2+1+3}{8} = \frac{7}{4}$$

$$= 2 - \frac{1}{4} = 2 - \frac{1}{2^{n-1}}$$

$\Rightarrow$ Average runtime:

$$2 - \frac{1}{2^{n-1}} \in \mathcal{O}(1)$$

Runtime Complexity

Example 2 - Average Steps

Table: Case analysis for instances of size n

Input	Output	Instances	Steps
$\_\_\_\ldots\_\_\_0$	$\_\_\_\ldots\_\_\_1$	2^{n-1}	1
$\_\_\_\ldots\_\_01$	$\_\_\_\ldots\_\_10$	2^{n-2}	2
$\_\_\_\ldots\_011$	$\_\_\_\ldots\_100$	2^{n-3}	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\_01\ldots1111$	$\_10\ldots0000$	2^1	n-1
$011\ldots1111$	$100\ldots0000$	2^0	n
$111\ldots1111$	$000\ldots0000$	1	n

Runtime Complexity

Example 2 - Average Steps

Table: Case analysis for instances of size n

Input	Output	Instances	Steps
___ . . . ___0	___ . . . ___1	2^{n-1}	1
___ . . . __01	___ . . . __10	2^{n-2}	2
___ . . . _011	___ . . . _100	2^{n-3}	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
_01 . . . 1111	_10 . . . 0000	2^1	n-1
011 . . . 1111	100 . . . 0000	2^0	n
111 . . . 1111	000 . . . 0000	1	n

Average steps:

$$\frac{1 \cdot 2^{n-1} + 2 \cdot 2^{n-2} + \cdots + (n-1) \cdot 2^1 + n \cdot 2^0 + n \cdot 1}{2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 + 1} =$$

Runtime Complexity

Example 2 - Average Steps

Table: Case analysis for instances of size n

Input	Output	Instances	Steps
___ . . . ___0	___ . . . ___1	2^{n-1}	1
___ . . . __01	___ . . . __10	2^{n-2}	2
___ . . . _011	___ . . . _100	2^{n-3}	3
⋮	⋮	⋮	⋮
_01 . . . 1111	_10 . . . 0000	2^1	n-1
011 . . . 1111	100 . . . 0000	2^0	n
111 . . . 1111	000 . . . 0000	1	n

Average steps:

$$\frac{1 \cdot 2^{n-1} + 2 \cdot 2^{n-2} + \cdots + (n-1) \cdot 2^1 + n \cdot 2^0 + n \cdot 1}{2^{n-1} + 2^{n-2} + \cdots + 2^1 + 2^0 + 1} = \frac{\left(\sum_{i=1}^{n} i \cdot 2^{n-i}\right) +}{\left(\sum_{i=0}^{n-1} 2^i\right) + 1}$$

Runtime Complexity

Example 2 - Average Steps

- Denominator:

$$\left(\sum_{i=0}^{n-1} 2^i\right) + 1 \overset{\substack{\text{geometric}\\\text{series}}}{=} (2^n - 1) + 1 = 2^n$$

Runtime Complexity

Example 2 - Average Steps

- Denominator:

$$\left(\sum_{i=0}^{n-1} 2^i\right) + 1 \overset{\text{geometric series}}{=} (2^n - 1) + 1 = 2^n$$

- Counter:

$$\left(\sum_{i=1}^{n} i \cdot 2^{n-i}\right) + n$$

Runtime Complexity

Example 2 - Average Steps

- Denominator:

$$\left(\sum_{i=0}^{n-1} 2^i\right) + 1 \overset{\substack{\text{geometric}\\ \text{series}}}{=} (2^n - 1) + 1 = 2^n$$

- Counter:

$$\left(\sum_{i=1}^{n} i \cdot 2^{n-i}\right) + n \overset{[x=2x-x]}{=} \left(2\sum_{i=1}^{n} i \cdot 2^{n-i}\right) - \left(\sum_{i=1}^{n} i \cdot 2^{n-i}\right) + n$$

Runtime Complexity

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- Denominator:

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$$= 1 \cdot 2^n + 2 \cdot 2^{n-1} + 3 \cdot n^{n-2} + \cdots + (n-1) \cdot 2^2 + n \cdot 2^1$$

$$-1 \cdot 2^{n-1} - 2 \cdot 2^{n-2} - \cdots - (n-2) \cdot 2^2 - (n-1) \cdot 2^1 - n \cdot 2^0$$

Runtime Complexity

Example 2 - Average Steps

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$$\left(\sum_{i=1}^{n} i \cdot 2^{n-i}\right) + n \overset{[x=2x-x]}{=} \left(2\sum_{i=1}^{n} i \cdot 2^{n-i}\right) - \left(\sum_{i=1}^{n} i \cdot 2^{n-i}\right) + n$$

$$= 1 \cdot 2^n + 2 \cdot 2^{n-1} + 3 \cdot n^{n-2} + \cdots + (n-1) \cdot 2^2 + n \cdot 2^1$$

$$-1 \cdot 2^{n-1} - 2 \cdot 2^{n-2} - \cdots - (n-2) \cdot 2^2 - (n-1) \cdot 2^1 - n \cdot 2^0$$

$$= \underbrace{2^n + 2^{n-1} + \cdots + 2^1 + 2^0}_{2^{n+1}-1} - 2^0 = 2^{n+1} - 2$$

Runtime Complexity

Example 2 - Average Steps

Average steps:

$$\overline{steps} = \frac{\left(\sum_{i=1}^{n} i \cdot 2^{n-i}\right) + n}{\left(\sum_{i=0}^{n-1} 2^{i}\right) + 1} = \frac{2^{n+1} - 2}{2^{n}} = 2 - \frac{1}{2^{n-1}}$$

$$\lim_{n \to \infty} \left(2 - \frac{1}{2^{n-1}}\right) = 2 \in \mathcal{O}(1)$$

Structure

Associative Arrays
Introduction

Normal array:

$$A = [0 \Rightarrow A_0,\ 1 \Rightarrow A_1,\ 2 \Rightarrow A_2,\ 3 \Rightarrow A_3,\ \ldots]$$

- Searching elements by index
- Lookup of element with index "3":
  $\Rightarrow A[3] = A_3$

Associative Arrays

Introduction

- In practice: all major programming project require associative arrays
- In our lecture: example of countries with associated information

Associative array:

$$A = \begin{bmatrix} \text{"}Europa\text{"} \Rightarrow A_0, \text{"}Amerika\text{"} \Rightarrow A_1, \\ \text{"}Asien\text{"} \Rightarrow A_2, \text{"}Afrika\text{"} \Rightarrow A_3, \\ \dots \end{bmatrix}$$

- Searching elements by key
- The keys can be of any type with unique elements
- Lookup of element with key "*Afrika*":
  $\Rightarrow A[\text{"}Afrika\text{"}] = A_3$

Structure

Associative Arrays
Practical Example

Table: Country data query from `http://geonames.org`

ISO	ISO3	Country	Continent	...
AD	AND	Andorra	EU	...
AE	ARE	United Arab Emirates	AS	...
AF	AFG	Afghanistan	AS	...
AG	ATG	Antigua and Barbuda	NA	...
AI	AIA	Anguilla	NA	...
AL	ALB	Albania	EU	...
AM	ARM	Armenia	AS	...
AO	AGO	Angola	AF	...
AQ	ATA	Antarctica	AN	...
⋮	⋮	⋮	⋮	⋱

Associative Arrays

Practical Example

Task: How many countries belong to each continent?

- We are interested in column 3 (Country) and 4 (Continent)
- There are two typical ways to solve this:
  - Using sorting
  - Using an associative array

Structure

Associative Arrays
Practical Example

Idea using sorting:

- We sort the table by Continent, so that all countries from the same continent are grouped in one block
- We count the size of the blocks

  Disadvantage:
  - Runtime of $\Theta(n \log n)$
  - We have to iterate the array twice (sort and then count)

  Advantage:
  - Easy to implement (even with simple linux / unix commands)

Input:

- The data is saved as tab seperated text (countryInfo.txt)
- Comments begin with a hash sign #

Commands:

- **grep**: Selects a specific set of lines (filter by ...)
  `grep -v '^#' countryInfo.txt`
    - `-v`: not
    - `^#`: # at start of line
- **cut**: Selects specific columns of each line (tab separated)
  `cut -f5,9`
    - `-f5,9`: columns 5 and 9 (= columns 3+4 of shown Table 17)

Associative Arrays

Practical Example - Sorting With Linux / Unix Commands

Commands:

- **sort**: Sorts lines by a key
  `sort -t '	' -k2,2`
  - `-t '	'`: Separator: Tab (Insert with `CTRL-V TAB`)
  - `-k2,2`: Key from column 2 to 2
- **uniq**: Finds or counts unique keys
  `uniq -c`
  - `-c`: count occurences of keys
- **head**: Displays a provided number of lines
  `head -n30`
  - `-n30`: Displays the first 30 lines
- **less**: Displays the file page wise

Associative Arrays

Practical Example - Sorting With Linux / Unix Commands

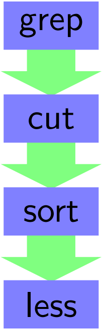
Sort countries by continent:

```
grep -v '^#' countryInfo.txt | cut -f5,9 \
   | sort -t '   ' -k2,2 | less
```

Table: Resulting data

Algeria	AF
Angola	AF
Benin	AF
Botswana	AF
Burkina Faso	AF
Burundi	AF
Cameroon	AF
Cape Verde	AF
⋮	⋮

Figure: Data pipeline



Associative Arrays

Practical Example - Sorting With Linux / Unix Commands

Count countries per continent:

```
grep -v '^#' countryInfo.txt | cut -f9 \
   | sort | uniq -c | sort -nr
```

Table: Resulting data

58	AF
54	EU
52	AS
42	NA
27	OC
14	SA
5	AN

Figure: Data pipeline

Structure

Associative Arrays

Practical Example - Associative Array

Idea using associative arrays:

- Take the continent as key
- Use a counter (occurences) or a list with all countries associated with this continent as value

Advantage:

- Runtime $\mathcal{O}(n)$, implied we can find an element in $\mathcal{O}(1)$ as in normal arrays

Associative Arrays
Python

Python:

```
# creates a new map (called dictionary)
countries = {"DE" : "Deutschland", \
    "EN" : "England"}

# check if element exists
if "EN" in countries:
    print("Found %s!" % countries["EN"])

# map key "DE" to value "Germany"
countries["DE"] = "Germany"

# delete key "DE"
del countries["DE"]
```

Efficiency:

- Depends on implementation
- Two typical implementations:
  - **Hashing:** Calculates a checksum of the key used as key of a normal array
    `search`: $\mathcal{O}(1) \ldots \mathcal{O}(n)$
    `insert`: $\mathcal{O}(1) \ldots \mathcal{O}(n)$
    `delete`: $\mathcal{O}(1) \ldots \mathcal{O}(n)$
  - **(Binary-)Tree:** Creates a sorted (binary) tree
    `search`: $\mathcal{O}(\log n) \ldots \mathcal{O}(n)$
    `insert`: $\mathcal{O}(\log n) \ldots \mathcal{O}(n)$
    `delete`: $\mathcal{O}(\log n) \ldots \mathcal{O}(n)$

Table: Map implementions of programming languages

	Hashing	(Binary-)Tree
Python	all dictionaries	
Java	`java.util.HashMap`	`java.util.TreeMap`
C++11/14	`std::unordered_map`	`std::map`
C++98	`__gnu_cxx::hash_map`	`std::map`

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Map - Implementations / API**

[Java] Java - HashMap
https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html

[Javb] Java - TreeMap
https://docs.oracle.com/javase/7/docs/api/java/util/TreeMap.html

[Pyt] Python - Dictionaries (Hash table)
https://docs.python.org/3/tutorial/datastructures.html#dictionaries

Further Literature

- **Map - Implementations / API**

[Cppa] C++ - hash_map
http://www.sgi.com/tech/stl/hash_map.html

[Cppb] C++ - map
http://www.sgi.com/tech/stl/Map.html