Algorithms and Data Structures

Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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Structure

Binary search tree:

- With `BinarySearchTree` we could perform an `lookup` or `insert` in $O(d)$, with d being the depth of the tree
- Best case: $d \in O(\log n)$, keys are inserted randomly
- Worst case: $d \in O(n)$, keys are inserted in ascending / descending order $(20, 19, 18, \dots)$

Balanced Trees

Motivation

Gnarley trees:

- http://people.ksp.sk/~kuko/bak

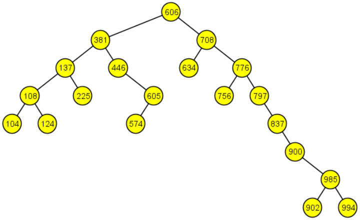


Figure: Binary search tree with random insert [Gna]

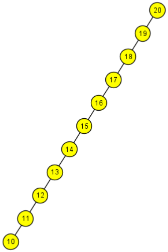


Figure: Binary search tree with descending insert [Gna]

Balanced Trees

Motivation

Balanced trees:

- We do not want to rely on certain properties of our key set
- We explicitly want a depth of $O(\log n)$
- We rebalance the tree from time to time

Balanced Trees

Motivation

How do we get a depth of $O(\log n)$?

- **AVL-Tree:**
  - Binary tree with 2 children per node
  - Balancing via "rotation"
- **(a,b)-Tree** or **B-Tree:**
  - Node has between a and b children
  - Balancing through splitting and merging nodes
  - Used in databases and file systems
- **Red-Black-Tree:**
  - Binary tree with "black" and "red" nodes
  - Balancing through "rotation" and "recoloring"
  - Can be interpreted as $(2, 4)$-tree
  - Used in C++ `std::map` and Java `SortedMap`

AVL-Tree:

- Gregory Maximovich **A**delson-**V**elskii, Yevgeniy Mikhailovlovich **L**andis (1963)
- Search tree with modified `insert` and `remove` operations while satisfying a depth condition
- Prevents degeneration of the search tree
- Height difference of left and right subtree is at maximum one
- With that the height of the search tree is always $O(\log n)$
- We can perform all basic operations in $O(\log n)$

Balanced Trees

AVL-Tree

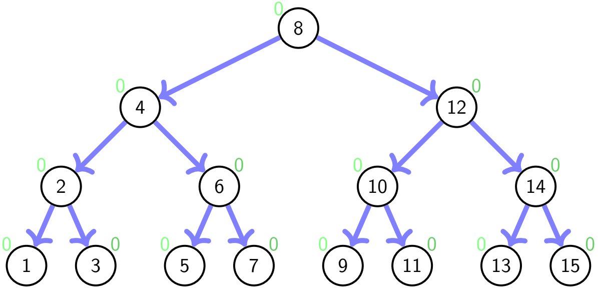


Figure: Example of an AVL-Tree

Balanced Trees

AVL-Tree

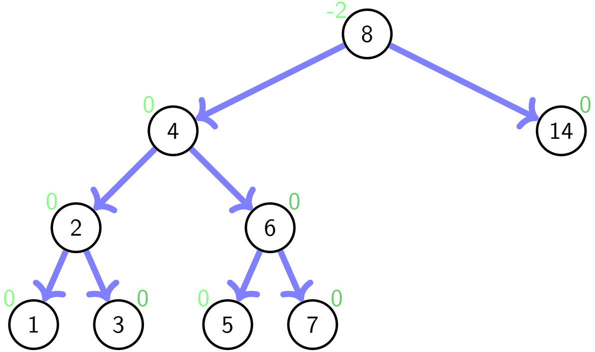


Figure: **Not** an AVL-Tree

Balanced Trees

AVL-Tree

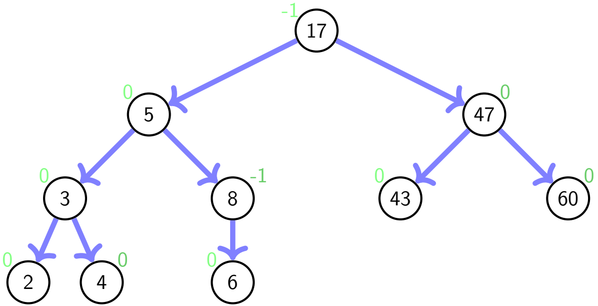


Figure: Another example of an AVL-Tree

Balanced Trees

AVL-Tree - Rebalancing

Rotation:

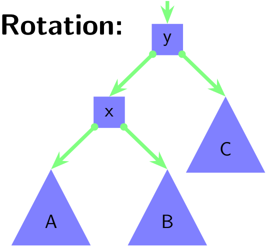


Figure: Before rotating

$\Rightarrow$

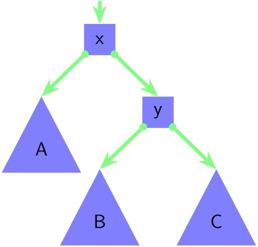


Figure: After rotating

- Central operation of rebalancing
- After rotation to the right:
  - Subtree A is a layer higher and subtree C a layer lower
  - The parent child relations between nodes x and y have been swapped

Balanced Trees

AVL-Tree - Rebalancing

AVL-Tree:

- If a height difference of ± 2 occurs on an `insert` or `remove` operation the tree is rebalanced
- Many different cases of rebalancing
- **Example:** `insert` of $1, 2, 3, \ldots$

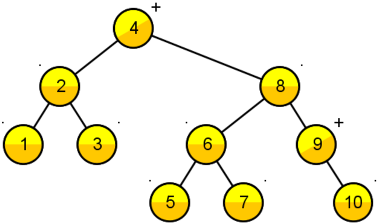


Figure: Inserting $1, \ldots, 10$ into an AVL-tree [Gna]

Balanced Trees

AVL-Tree - Summary

Summary:

- Historical the first search tree providing guaranteed `insert`, `remove` and `lookup` in $O(\log n)$
- However not amortized update costs of $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are (a,b)-trees

(a,b)-Tree:

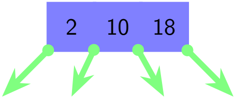
- Also known as **b-tree** (b for “balanced”)
- Used in databases and file systems

Idea:

- Save a varying number of elements per node
- So we have space for elements on an `insert` and balance operation

(a,b)-Tree:

- All leaves have the same depth
- Each inner node has $\geq a$ and $\leq b$ nodes
  (Only the root node may have less nodes)



- Each node with n children is called "node of degree n" and holds $n-1$ sorted elements
- Subtrees are located "between" the elements
- We require: $a \geq 2$ and $b \geq 2a-1$

(a,b)-Trees
Introduction

(2,4)-Tree:

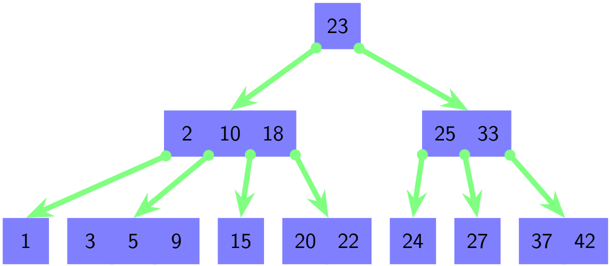


Figure: Example of an (2,4)-tree

- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

(a,b)-Trees

Introduction

Not an (2,4)-Tree:

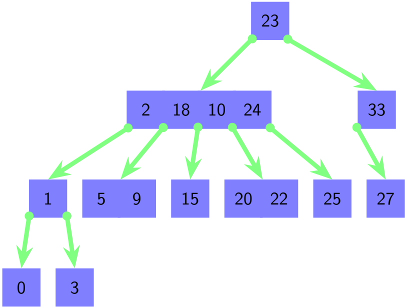


Figure: **Not** an (2,4)-tree

- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels

Searching an element: (`lookup`)

- The same algorithm as in BinarySearchTree
- Searching from the root downwards
- The keys at each node set the path

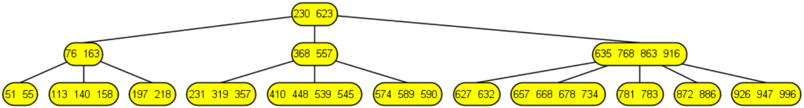


Figure: (3,5)-Tree [Gna]

Inserting an element: (`insert`)

- Search the position to insert the key into
- This position will always be an leaf
- Insert the element into the tree
- **Attention:** As a result node can overflow by one element (Degree $b+1$)
- Then we **split** the node

(a,b)-Trees
Implementation - Insert

Inserting an element: (`insert`)

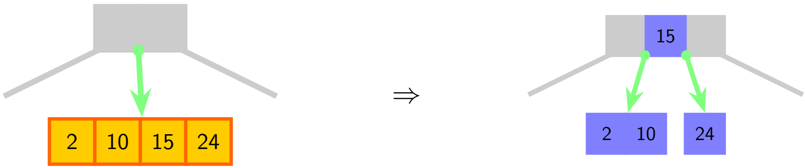


Figure: Splitting a node

- If the degree is higher than $b+1$ we split the node
- This results in a node with $\mathrm{ceil}\left(\frac{b-1}{2}\right)$ elements, a node with $\mathrm{floor}\left(\frac{b-1}{2}\right)$ elements and one element for the parent node
- Thats why we have the limit $b \geq 2a-1$

Inserting an element: (`insert`)

- If the degree is higher than $b + 1$ we split the node
- Now the parent node can be of a higher degree than $b + 1$
- We split the parent nodes the same way
- If we split the root node we create a new parent root node (The tree is now one level deeper)

(a,b)-Trees

Implementation - Remove

Removing an element: (`remove`)

- Search the element in $O(\log n)$ time
- **Case 1:** The element is contained by a leaf
  - Remove element
- **Case 2:** The element is contained by an inner node
  - Search the successor in the right subtree
  - The successor is always contained by a leaf
  - Replace the element with its successor and delete the successor from the leaf
- **Attention:** The leaf might be too small (degree of $a - 1$)
  $\Rightarrow$ We rebalance the tree

Removing an element: (`remove`)

- **Attention:** The leaf might be too small (degree of $a-1$)
  $\Rightarrow$ We rebalance the tree
  - **Case a:** If the left or right neighbour node has a degree greater than a we **borrow** one element from this node

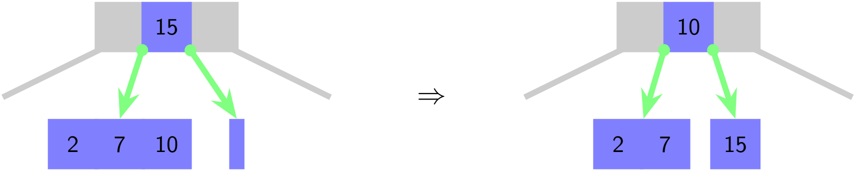


Figure: Borrow an element

Removing an element: (`remove`)

- **Attention:** The leaf might be too small (degree of $a - 1$)
  $\Rightarrow$ We rebalance the tree
  - **Case b:** We merge the node with its right or left neighbour

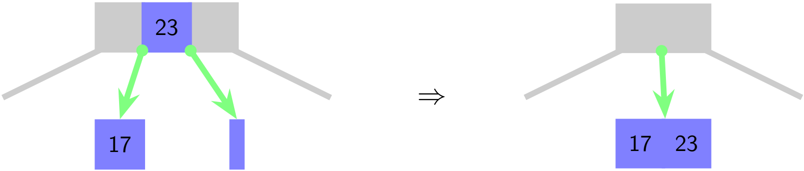


Figure: Merge two nodes

Removing an element: (`remove`)

- Now the parent node can be of degree $a - 1$
- We merge parent nodes the same way
- If the root has only a single child
  - Remove the root
  - Define sole child as new root
  - The tree shrinks by one level

(a,b)-Trees

Runtime Complexity

Runtime complexity of `lookup`, `insert` and `remove`:

- All operations in $O(d)$ with d being the depth of the tree
- Each node (except the root) has more than a children
  $\Rightarrow n \geq a^{d-1}$ and $d \leq 1 + \log_a n = O(log_a n)$

In detail:

- `lookup` always takes $\Theta(d)$
- `insert` and `remove` often require only $O(1)$ time
- **Worst case:** split or merge all nodes on path up to the root
- Therefore instead of $b \geq 2\,a - 1$ we need $b \geq 2\,a$

(a,b)-Trees

Runtime Complexity - Counter-example for (2,3)-Tree

Counter example (2,3)-Tree:

- Before executing `delete(11)`

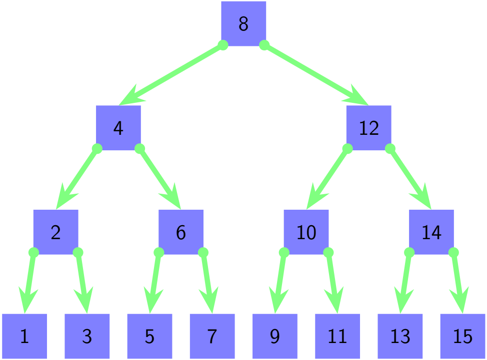


Figure: Normal (2,3)-Tree

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `delete(11)`

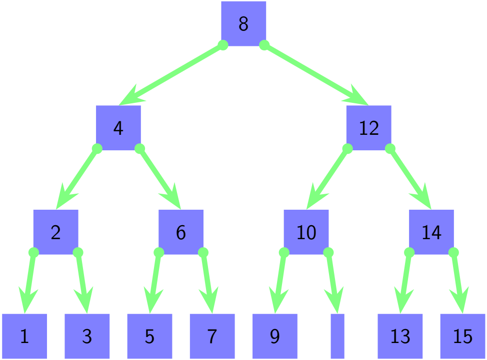


Figure: (2,3)-Tree - Delete step 1

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `delete(11)`

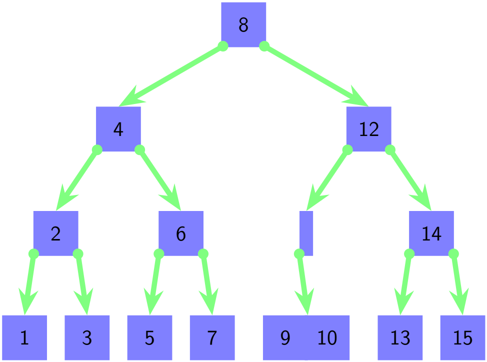


Figure: (2,3)-Tree - Delete step 2

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `delete(11)`

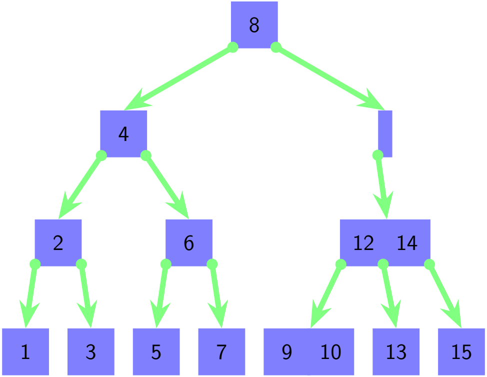


Figure: (2,3)-Tree - Delete step 3

Counter example (2,3)-Tree:

- Executed `delete(11)`

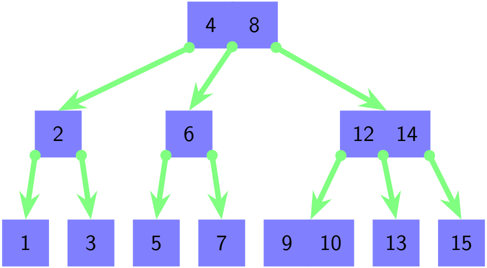


Figure: (2,3)-Tree - Delete step 4

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `insert(11)`

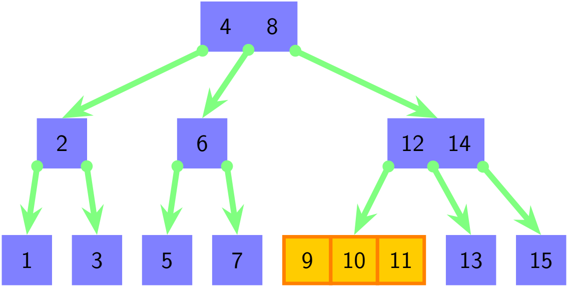


Figure: (2,3)-Tree - Insert step 1

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `insert(11)`

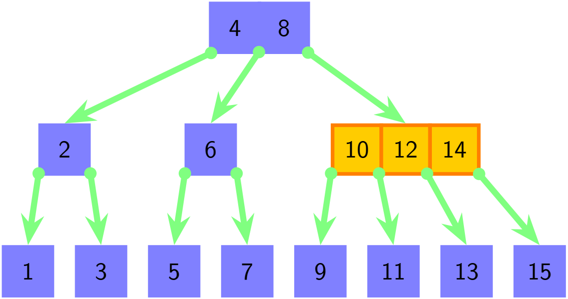


Figure: (2,3)-Tree - Insert step 2

Counter example (2,3)-Tree:

- Executing `insert(11)`

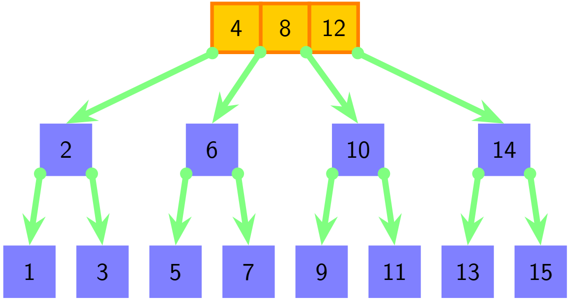


Figure: (2,3)-Tree - Insert step 3

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executed `insert(11)`

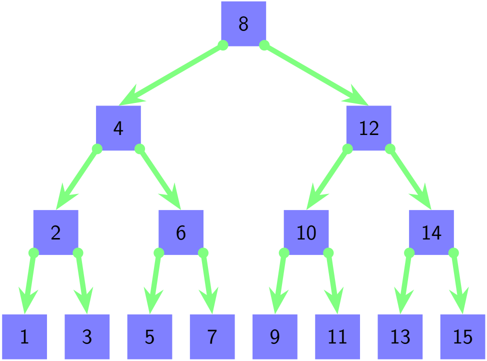


Figure: (2,3)-Tree - Insert step 4

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- We are exactly where we started
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$
- We need $b \geq 2a$ instead of $b \geq 2a - 1$

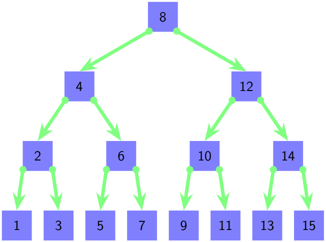


Figure: (2,3)-Tree

(2,4)-Tree:

- If all nodes have 2 children we have to merge the nodes up to the root on a `remove` operation
- If all nodes have 4 children we have to split the nodes up to the root on a `insert` operation
- If all nodes have 3 children it takes some time to reach one of the previous two states

$\Rightarrow$ **Nodes of degree 3 are stable**
Neither an `insert` nor a `remove` operation trigger rebalancing operations

(2,4)-Tree:

- **Idea:**
  - After an expensive operation the tree is in a stable state
  - It takes some time until the next expensive operation occurs
- Like with dynamic arrays:
  - Reallocation is expensive but it takes some time until the next expensive operation occurs
  - If we overallocate clever we have an amortized runtime of $O(1)$

Terminology:

- We analyze a sequence of n operations
- Let Φ_i be the potential of the tree after the i-*th* operation
- $\Phi_i =$ the number of stable nodes with degree 3
- Empty tree has 0 nodes: $\Phi = 0$

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Example:

- Nodes of degree 3 are highlighted

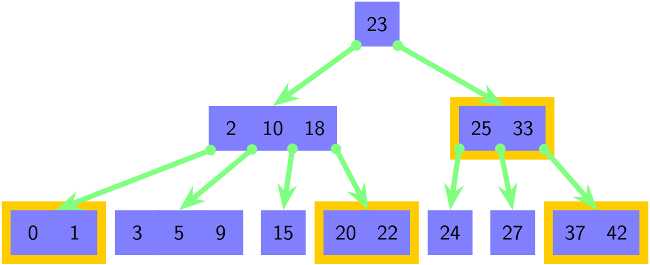


Figure: Tree with potential $\Phi = 4$

Terminology:

- Let c_i be the costs = runtime of the i-th operation
- We will show:
  - Each operation can at most destroy one stable node
  - For each cost incurring step the operation creates an additional stable node
- The costs for operation i are coupled to the difference of the potential levels

$$c_i \leq A \cdot \underbrace{(\Phi_i - \Phi_{i-1})}_{} + B, \qquad A > 0 \text{ and } B > A$$

  Number of gained stable nodes (degree 3) ≥ -1

- Each operation has an amortitzed cost of $O(1)$ summing up to $O(n)$ in total

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: *i-th* operation is an `insert` operation on a full node

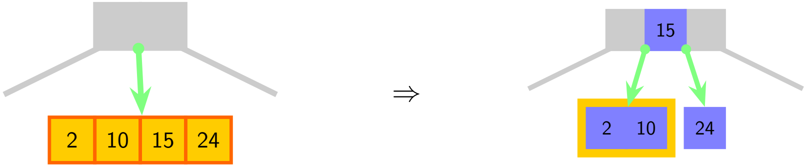


Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

Case 1: *i-th* operation is an `insert` operation on a full node

- Let m be the number of nodes split
- The potential rises by m
- If the "stop-node" is of degree 3 then the potential goes down by one

$$\Phi_i \geq \Phi_{i-1} + m - 1$$
$$\Rightarrow m \leq \Phi_i - \Phi_{i-1} + 1$$

Costs: $c_i \leq A \cdot m + B$

$$\Rightarrow c_i \leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B$$
$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}$$

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Inner node
  - Searching the successor in a tree is $O(d) = O(\log n)$
  - Normally the tree is coupled with a doubly linked list
    $\Rightarrow$ We can find the succcessor in $O(1)$

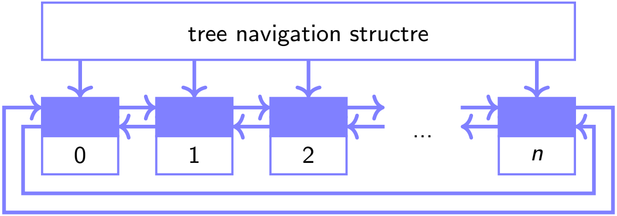


Figure: Tree with doubly linked list

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
  - Case 2.1.1: Potential rises by one

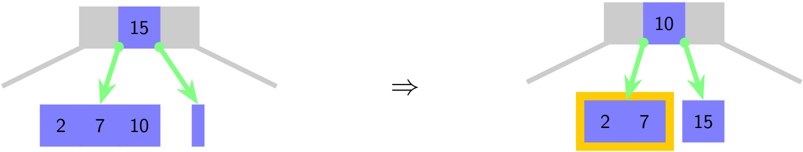


Figure: Case 2.1.1: Borrow an element

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
  - Case 2.1.2: Potential is lowered by one

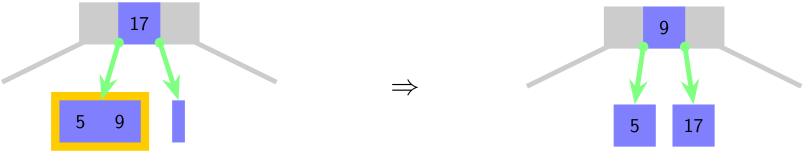


Figure: Case 2.1.2: Borrow an element

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

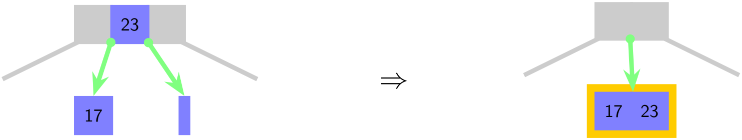


Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree > 2 or a node of degree 2, which can borrow from a neighbour

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

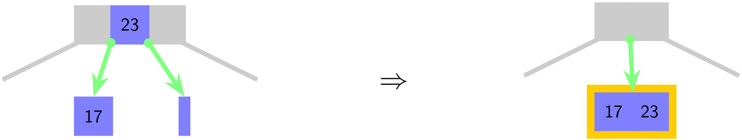


Figure: Merging two nodes

- The potential rises by m
- If the "stop-node" is of degree 2 then the potential eventually goes down by one
- Same costs as `insert`

Lemma:

- We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^{n} c_i \in O(n)$$

(a,b)-Trees

Runtime Complexity - (2,4)-Tree - Lemma - Proof

Proof:

$$\sum_{i=0}^{n} c_i \leq \underbrace{A \cdot (\Phi_1 - \Phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\Phi_2 - \Phi_1) + B}_{\leq c_2} + \cdots + \underbrace{A \cdot (\Phi_n - \Phi_{n-1})}_{\leq c_n}$$

$$= A \cdot (\Phi_n - \Phi_0) + B \cdot n \qquad | \text{ telescope sum}$$

$$= A \cdot \Phi_n + B \cdot n \qquad | \text{ we start with an empty tree}$$

$$< A \cdot n + B \cdot n \in O(n) \qquad | \text{ number of degree 3 nodes}$$

¡ number of nodes

Red-Black-Trees

Introduction

Red-Black Tree:

- Binary tree with red and black nodes
- Number of black nodes on path to leaves is equal
- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- Each (2,4)-tree-node is a small red-black-tree with a black root node

Red-Black-Trees

Introduction

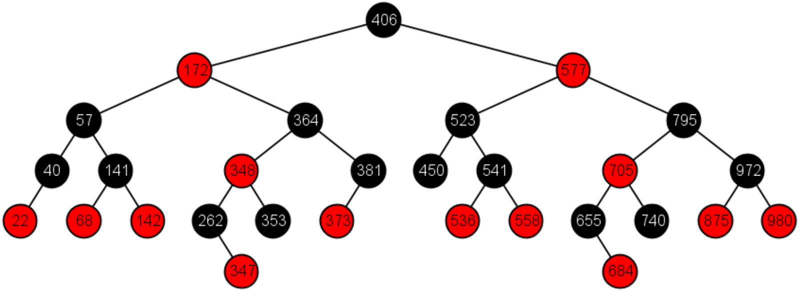


Figure: Example of an red-black-tree [Gna]

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

- **Gnarley Trees**

[Gna] Gnarley Trees
https://people.ksp.sk/~kuko/gnarley-trees/

- **AVL-Tree**

  [Wik] AVL tree
  https://en.wikipedia.org/wiki/AVL_tree

- **(a,b)-Tree**

  [Wika] 2-3-4 tree
  https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934_tree

  [Wikb] (a,b)-tree
  https://en.wikipedia.org/wiki/(a,b)-tree

- **Red-Black-Tree**

[Wik] Red-black tree
https://en.wikipedia.org/wiki/Red%E2%80%93black_tree