# Entwurf, Analyse und Umsetzung von Algorithmen

## Linked Lists, Binary Search Trees

Albert-Ludwigs-Universität Freiburg

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science
Entwurf, Analyse und Umsetzung von Algorithmen

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mikrosysteme

# Structure

Sorted Sequences

Linked Lists

Binary Search Trees

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  - `next()/previous()`: returns the element with the next bigger/smaller key. This enables iteration over all elements

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- How could we implement this?

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- Let's have a closer look

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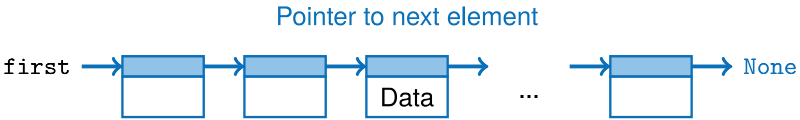


Figure: Linked list

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- No direct access of elements
  $\Rightarrow$ We have to iterate over the list

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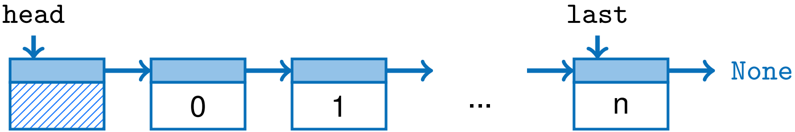


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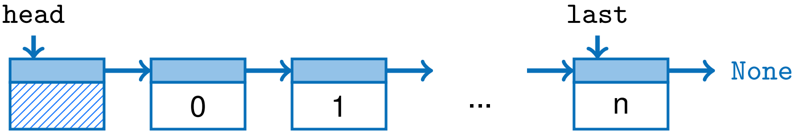


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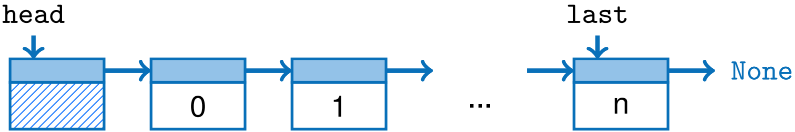


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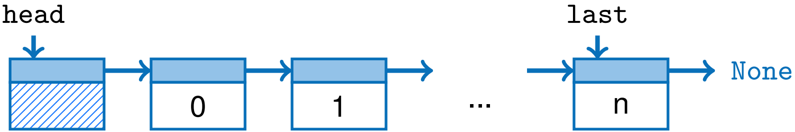


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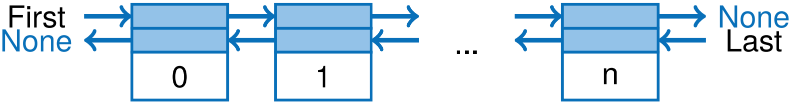


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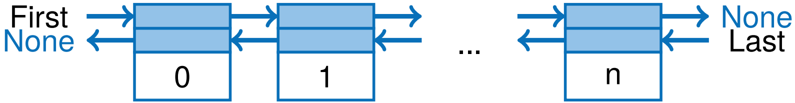


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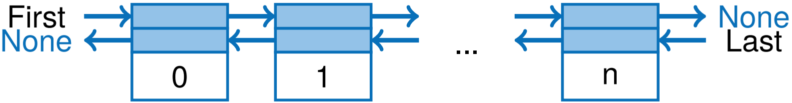


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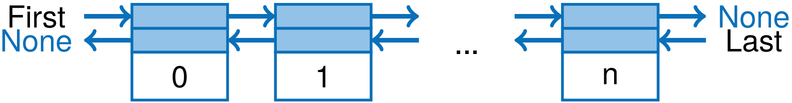


Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward

# Linked Lists

## Implementation - Node/Element - Python

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode=None):
        self.value = value
        self.nextNode = nextNode
```

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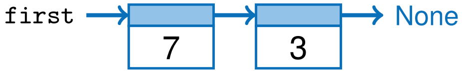

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## Usage examples

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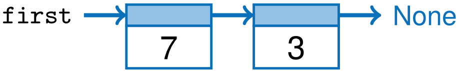
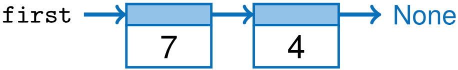



- `first.nextNode.value = 4`



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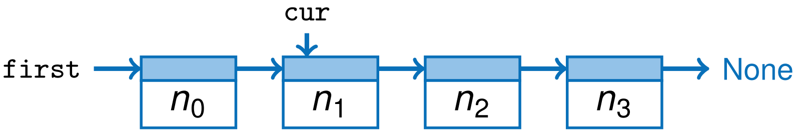

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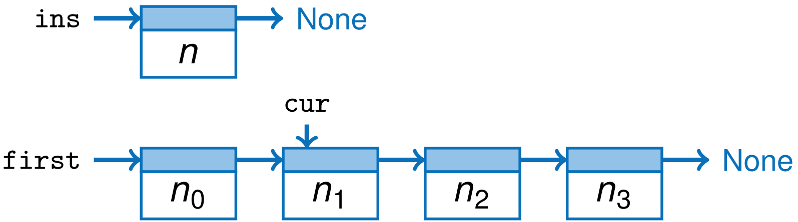

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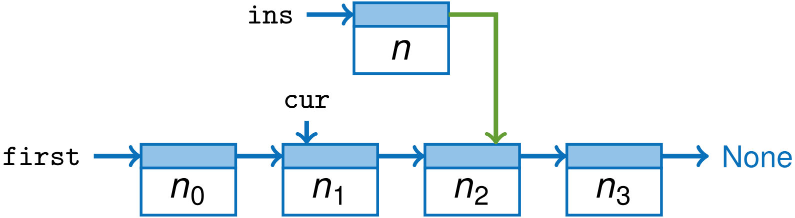

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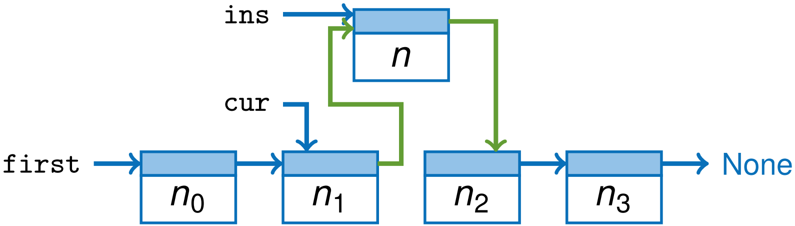

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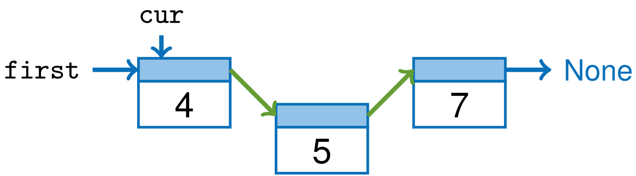

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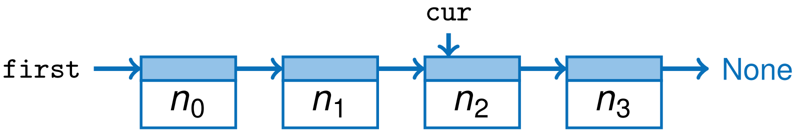

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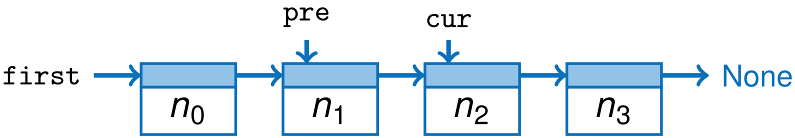

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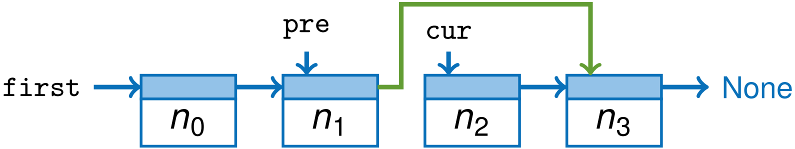

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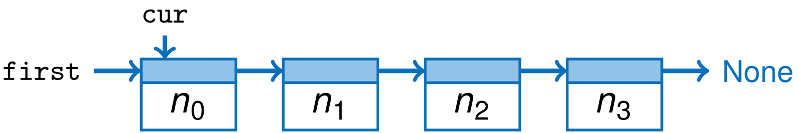

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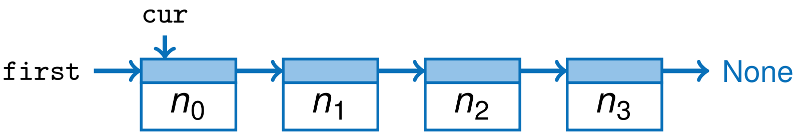


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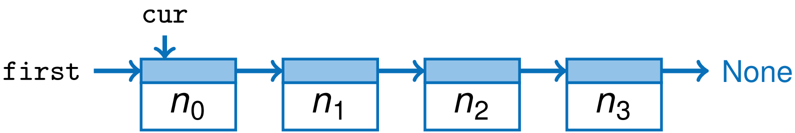


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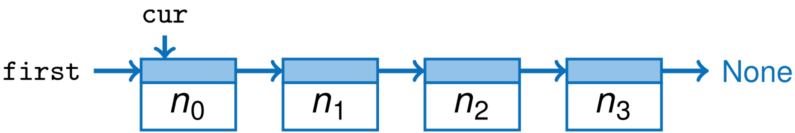


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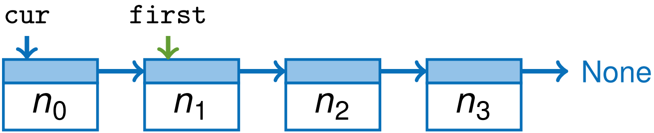

**Removing a node `cur`:** (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```

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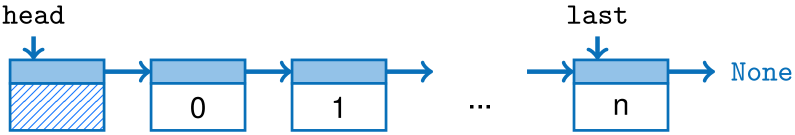

```python
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```

# Linked Lists

## Implementation - LinkedList - Python

```python
def append(self, value):
    ...

def insertAfter(self, cur, value):
    ...

def remove(self, cur):
    ...

def get(self, position):
    ...

def contains(self, value):
    ...
```

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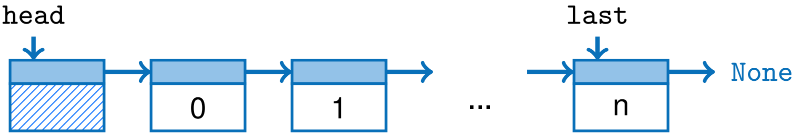

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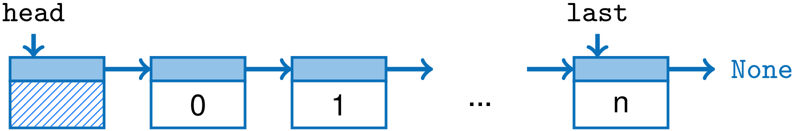


- `Head` points to the first node, `last` to the last node

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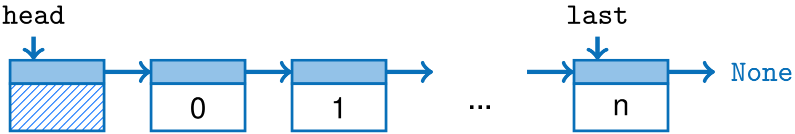


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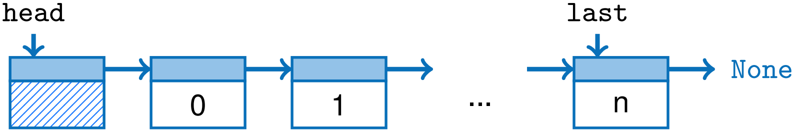


- `Head` points to the first node, `last` to the last node
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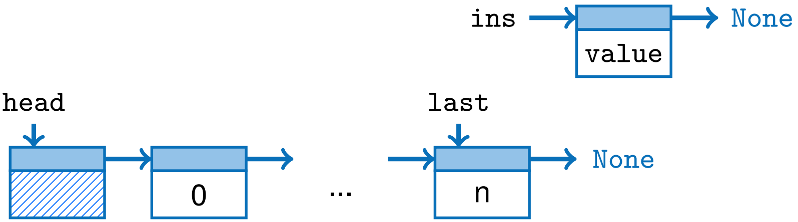

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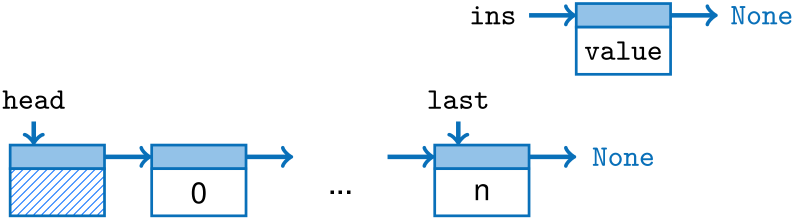


```
def append(self, value):
    last.nextNode = Node(value)
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    itemCount += 1
```

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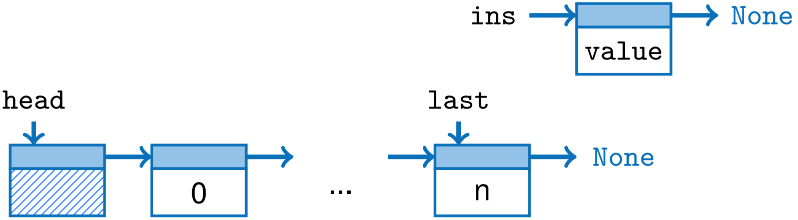


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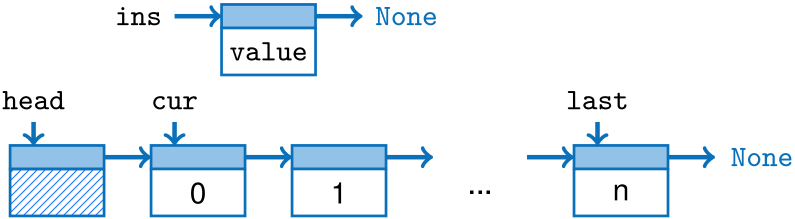
- The pointer to `last` avoids the iteration of the whole list

**Inserting after node `cur`:**



**Inserting after node `cur`:**

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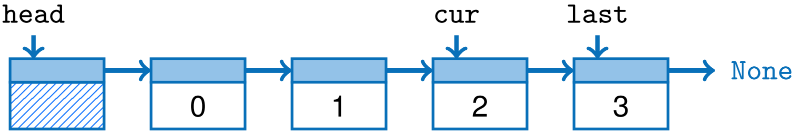
```
def insertAfter(self, cur, value):
    if cur == last:
        # also update last node
        append(value)
    else:
        # last node is not modified
        cur.nextNode = Node(value, \
            cur.nextNode)
        itemCount += 1
```

**Remove node `cur`:**



**Remove node `cur`:**

- Searching the predecessor in $O(n)$

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```
def remove(self, cur):
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
    itemCount -= 1

    if pre.nextNode == None:
        last = pre
```

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**Getting a reference to node at `pos`:**

- Iterate the entries of the list until position in $O(n)$

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```
def get(self, pos):
    if pos < 0 or pos >= itemCount:
        return None

    cur = head
    for i in range(0, pos):
        cur = cur.nextNode

    return cur
```

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**Searching a `value`:**

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```
def contains(self, value):
    cur = head

    for i in range(0, itemCount):
        cur = cur.nextNode
        if cur.value == value:
            return True

    return False
```

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- Better with doubly linked lists

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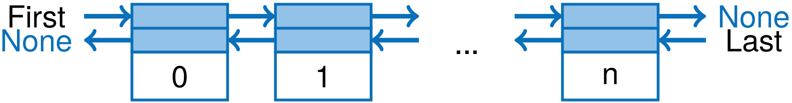

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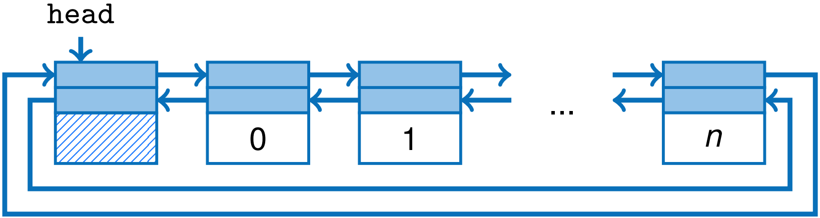

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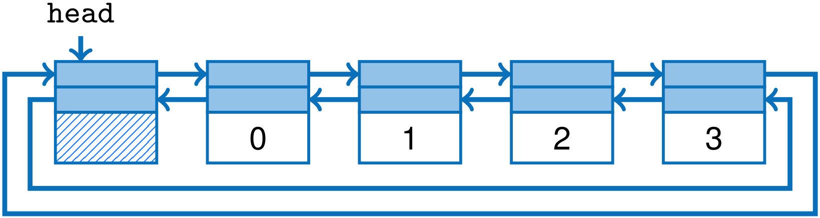
**Linked list in book:**



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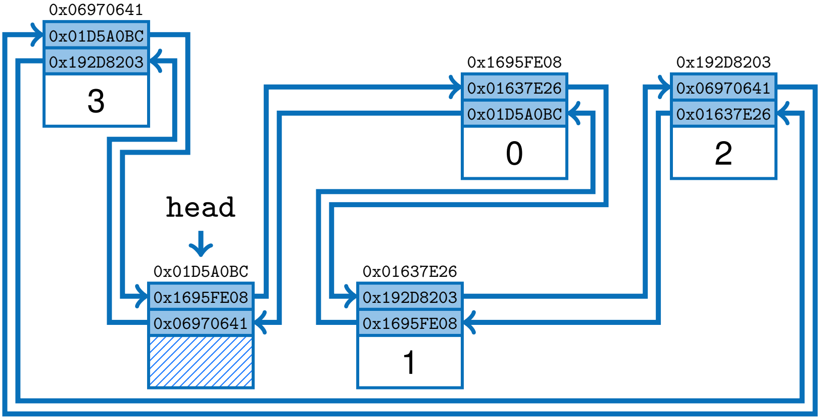
## Linked list in memory:



# Structure

Sorted Sequences

Linked Lists

Binary Search Trees

**Runtime of a search tree:**

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  The structure helps searching efficiently

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- Edge direction indicates ordering

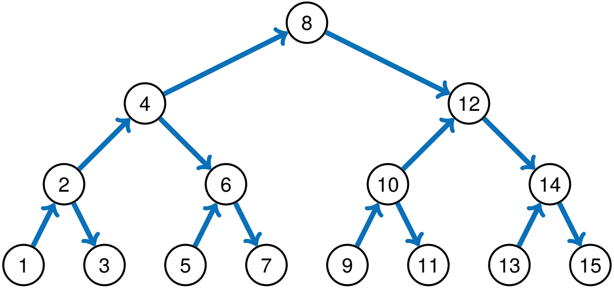


Figure: a binary search tree

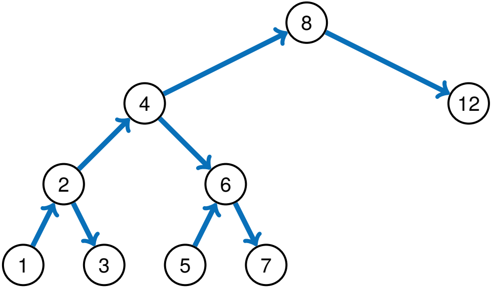


Figure: another binary search tree

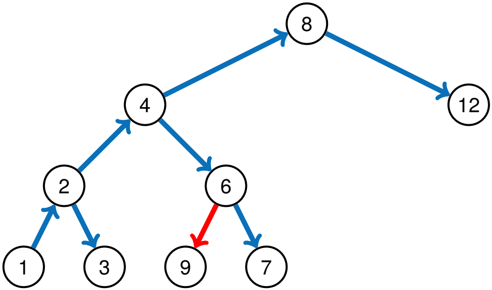


Figure: **not** a binary search tree

## Implementation:

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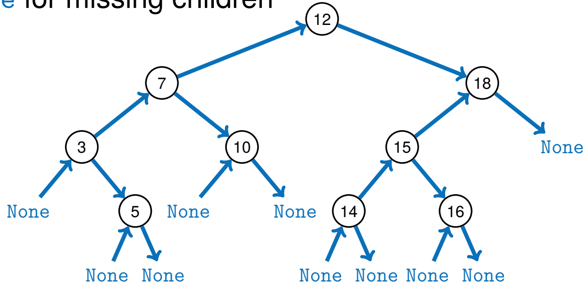

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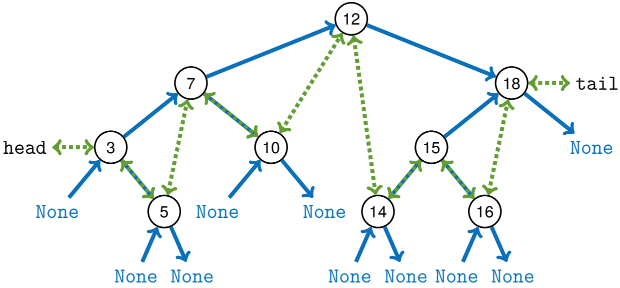


Figure: binary search tree with links

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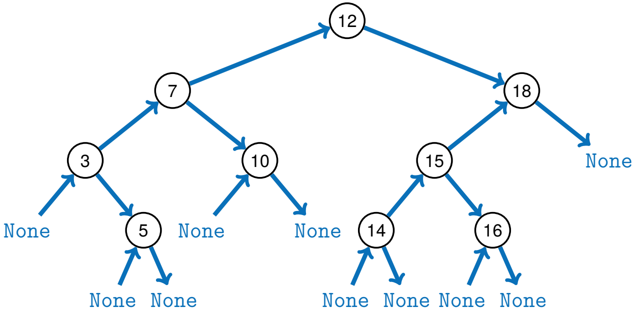


**Examples:**

Figure: binary search tree with total order "<"

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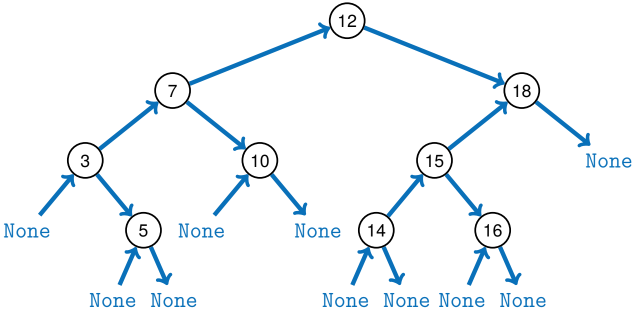


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**Examples:**

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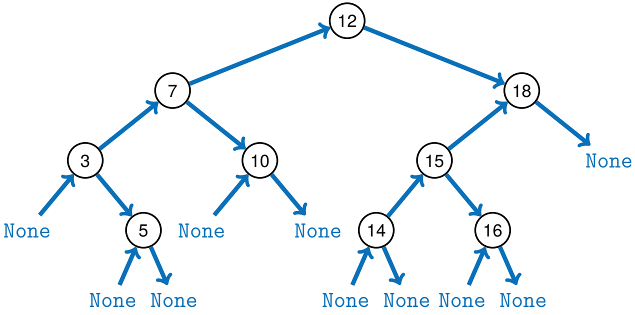


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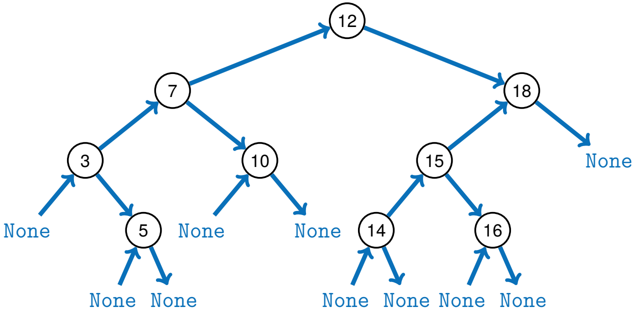


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# Binary Search Trees

## Implementation - Insert

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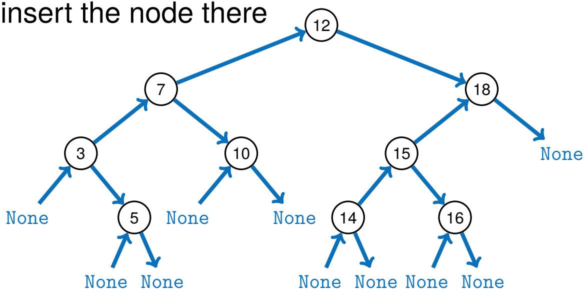


Figure: Binary search tree with total order “<”

**Remove:** case 1: the node “5” has no children

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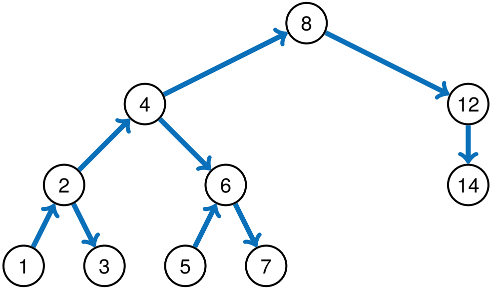


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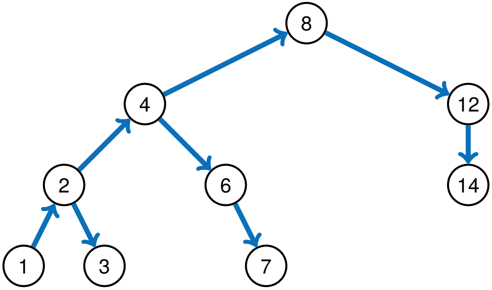


Figure: binary search tree after deleting node “5”

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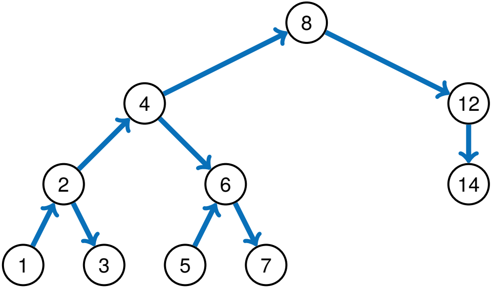


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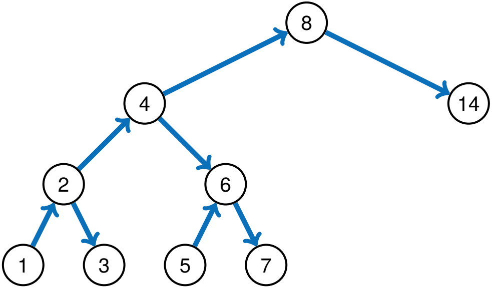


Figure: binary search tree after delting node “12”

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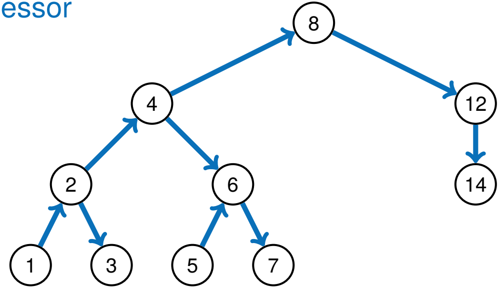

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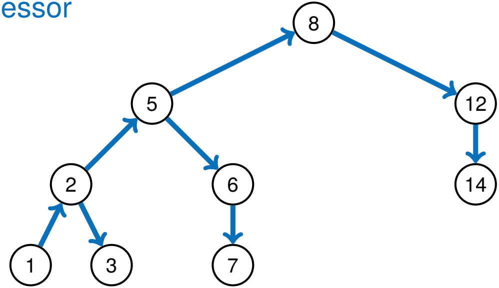

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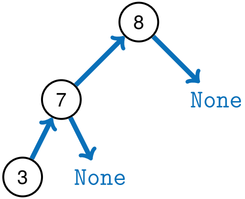


Figure: degenerated binary tree $d = n$

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- Worst case with $d = n$ the runtime is $\Theta(n)$
- If we **always** want to have a runtime of $\Theta(\log n)$ then we have to rebalance the tree

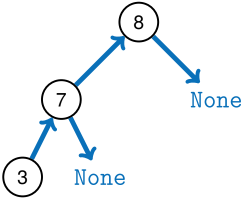


Figure: degenerated binary tree $d = n$

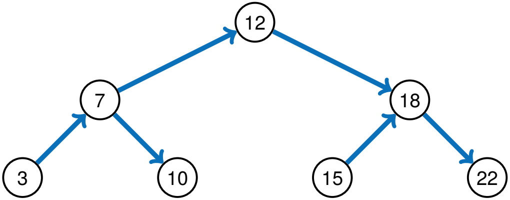


Figure: complete binary tree $d = \log n$

## Course literature


[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
**Introduction to Algorithms.**
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
`https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf`.

- **Linked List**

  [Wik] Linked list
  `https://en.wikipedia.org/wiki/Linked_list`

- **Binary Search Tree**

  [Wik] Binary search tree
  `https://en.wikipedia.org/wiki/Binary_search_tree`