

# Entwurf, Analyse und Umsetzung von Algorithmen

## Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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intelligente eingebettete  
mikrosysteme

## Balanced Trees

- Motivation

- AVL-Trees

- (a,b)-Trees

  - Introduction

  - Runtime Complexity

- Red-Black Trees

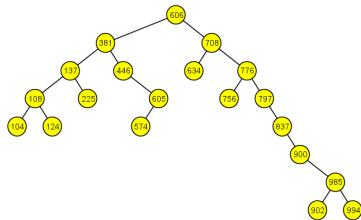
### Binary search tree:

- With `BinarySearchTree` we could perform an `lookup` or `insert` in  $O(d)$ , with  $d$  being the `depth` of the tree
- Best case:  $d \in O(\log n)$ , keys are inserted randomly
- Worst case:  $d \in O(n)$ , keys are inserted in ascending / descending order (20, 19, 18, ...)

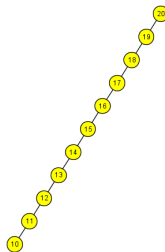


### Gnarley trees:

■ <http://people.ksp.sk/~kuko/bak>



**Figure:** Binary search tree with random insert [Gna]



**Figure:** Binary search tree with descending insert [Gna]

### Balanced trees:

- We do not want to rely on certain properties of our **key set**
- We explicitly want a depth of  $O(\log n)$
- We **rebalance** the tree from time to time

### How do we get a depth of $O(\log n)$ ?

#### ■ AVL-Tree:

- Binary tree with 2 children per node
- Balancing via “rotation”

#### ■ (a,b)-Tree or B-Tree:

- Node has between  $a$  and  $b$  children
- Balancing through **splitting** and **merging** nodes
- Used in databases and file systems

#### ■ Red-Black-Tree:

- Binary tree with “black” and “red” nodes
- Balancing through “rotation” and “recoloring”
- Can be interpreted as (2, 4)-tree
- Used in C++ `std::map` and Java `SortedMap`

### AVL-Tree:

- Gregory Maximovich **Adelson-Velskii**, Yevgeniy Mikhailovlovich **Landis** (1963)
- Search tree with modified **insert** and **remove** operations while satisfying a **depth** condition
- Prevents degeneration of the search tree
- Height difference of left and right subtree is at maximum one
- With that the height of the search tree is always  $O(\log n)$
- We can perform all basic operations in  $O(\log n)$

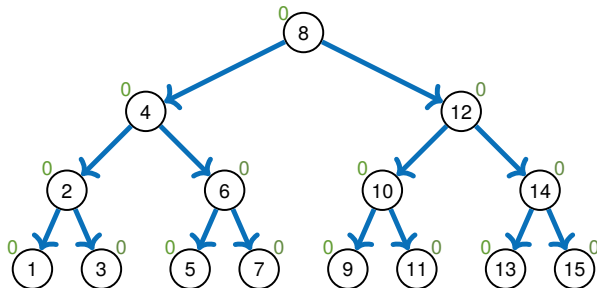


Figure: Example of an AVL-Tree



# Balanced Trees

## AVL-Tree

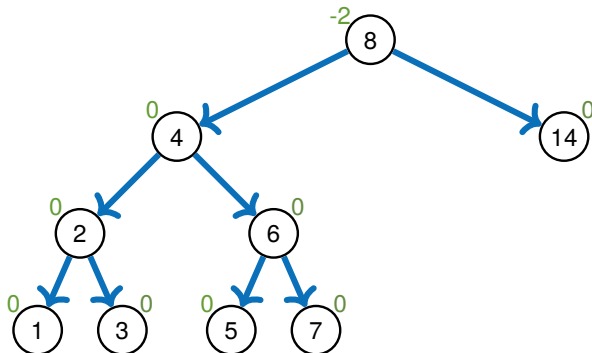


Figure: **Not** an AVL-Tree

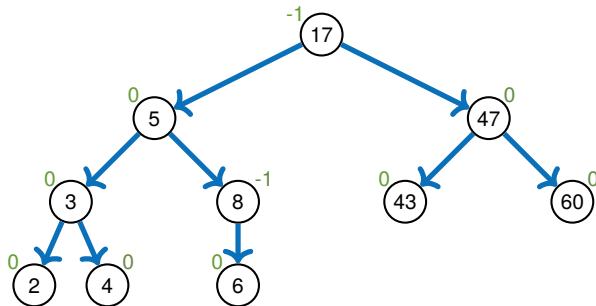


Figure: Another example of an AVL-Tree

### Rotation:

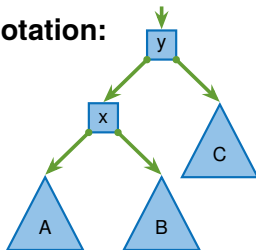


Figure: Before rotating

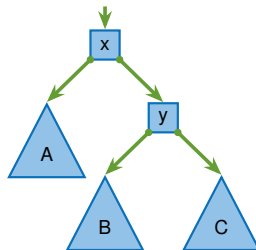


Figure: After rotating

- Central operation of **rebalancing**
- After rotation to the right:
  - Subtree **A** is a layer higher and subtree **C** a layer lower
  - The parent child relations between nodes **x** and **y** have been swapped

### AVL-Tree:

- If a height difference of  $\pm 2$  occurs on an **insert** or **remove** operation the tree is rebalanced
- Many different cases of rebalancing
- **Example:** **insert** of 1,2,3,...

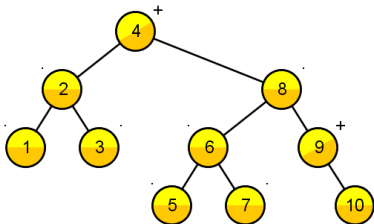


Figure: Inserting 1,...,10 into an AVL-tree [Gna]

### Summary:

- Historical the first search tree providing guaranteed `insert`, `remove` and `lookup` in  $O(\log n)$
- However not amortized update costs of  $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are  $(a,b)$ -trees

### (a,b)-Tree:

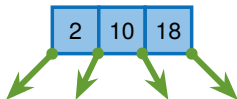
- Also known as **b-tree** (b for “balanced”)
- Used in databases and file systems

### Idea:

- Save a varying number of elements per node
- So we have space for elements on an **insert** and balance operation

### (a,b)-Tree:

- All leaves have the same depth
- Each inner node has  $\geq a$  and  $\leq b$  nodes  
(Only the root node may have less nodes)



- Each node with  $n$  children is called “node of degree  $n$ ” and holds  $n - 1$  sorted elements
- Subtrees are located “between” the elements
- We require:  $a \geq 2$  and  $b \geq 2a - 1$

### (2,4)-Tree:

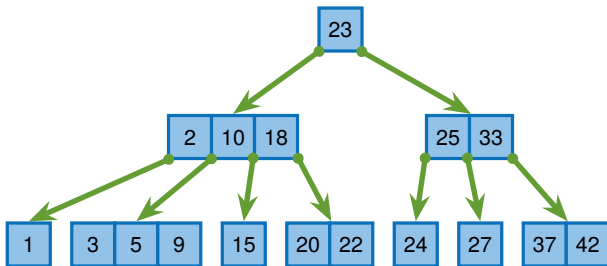


Figure: Example of an (2,4)-tree

- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)



### Not an (2,4)-Tree:

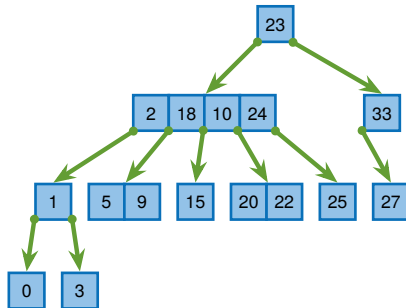


Figure: **Not** an (2,4)-tree

- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels

### Searching an element: (lookup)

- The same algorithm as in [BinarySearchTree](#)
- Searching from the root downwards
- The keys at each node set the path



Figure: (3,5)-Tree [Gna]

### Inserting an element: (`insert`)

- Search the position to insert the key into
- This position will always be an leaf
- Insert the element into the tree
- **Attention:** As a result node can overflow by one element (Degree  $b+1$ )
- Then we **split** the node

### Inserting an element: (`insert`)



Figure: Splitting a node

- If the degree is higher than  $b + 1$  we split the node
- This results in a node with  $\text{ceil}\left(\frac{b-1}{2}\right)$  elements, a node with  $\text{floor}\left(\frac{b-1}{2}\right)$  elements and one element for the parent node
- That's why we have the limit  $b \geq 2a - 1$

### Inserting an element: (`insert`)

- If the degree is higher than  $b+1$  we split the node
- Now the parent node can be of a higher degree than  $b+1$
- We `split` the parent nodes the same way
- If we split the root node we create a new parent root node  
(The tree is now one level deeper)

### Removing an element: (remove)

- Search the element in  $O(\log n)$  time
- **Case 1:** The element is contained by a leaf
  - Remove element
- **Case 2:** The element is contained by an inner node
  - Search the **successor** in the right subtree
  - The **successor** is always contained by a leaf
  - Replace the element with its **successor** and delete the **successor** from the leaf
- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree

### Removing an element: (`remove`)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree
- **Case a:** If the left or right neighbour node has a degree greater than  $a$  we **borrow** one element from this node

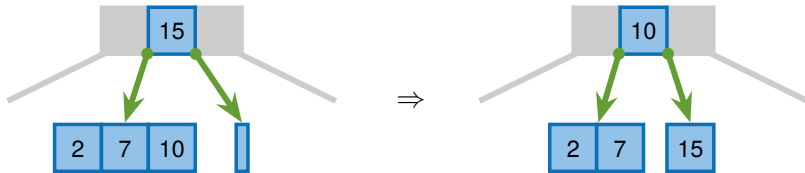


Figure: Borrow an element

### Removing an element: (remove)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree

- **Case b:** We **merge** the node with its right or left neighbour

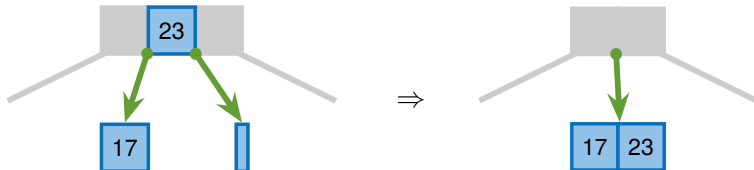


Figure: Merge two nodes



### Removing an element: (remove)

- Now the parent node can be of degree  $a - 1$
- We merge parent nodes the same way
- If the root has only a single child
  - Remove the root
  - Define sole child as new root
  - The tree shrinks by one level

### Runtime complexity of **lookup**, **insert** and **remove**:

- All operations in  $O(d)$  with  $d$  being the depth of the tree
- Each node (except the root) has **more than  $a$  children**  
 $\Rightarrow n \geq a^{d-1}$  and  $d \leq 1 + \log_a n = O(\log_a n)$

### In detail:

- **lookup** always takes  $\Theta(d)$
- **insert** and **remove** often require only  $O(1)$  time
- **Worst case:** **split** or **merge** all nodes on path up to the root
- Therefore instead of  $b \geq 2a - 1$  we need  $b \geq 2a$

### Counter example (2,3)-Tree:

- Before executing `delete(11)`

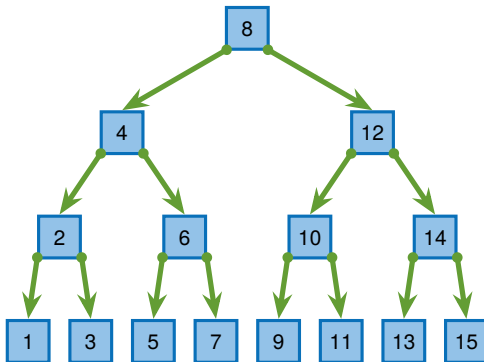


Figure: Normal (2,3)-Tree

### Counter example (2,3)-Tree:

- Executing `delete(11)`

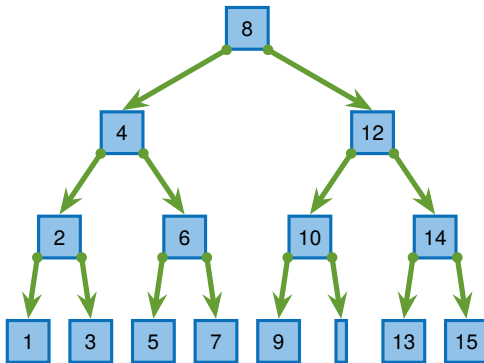


Figure: (2,3)-Tree - Delete step 1

### Counter example (2,3)-Tree:

- Executing `delete(11)`

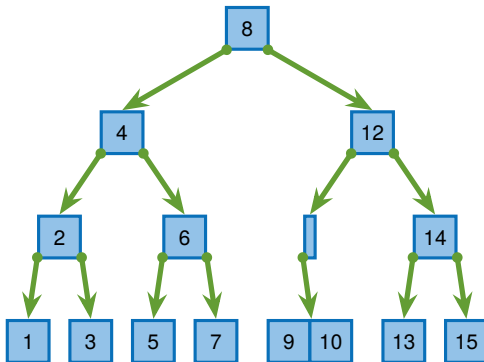


Figure: (2,3)-Tree - Delete step 2

### Counter example (2,3)-Tree:

- Executing `delete(11)`

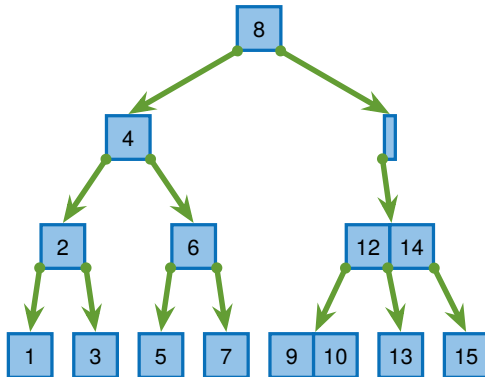


Figure: (2,3)-Tree - Delete step 3

### Counter example (2,3)-Tree:

- Executed `delete(11)`

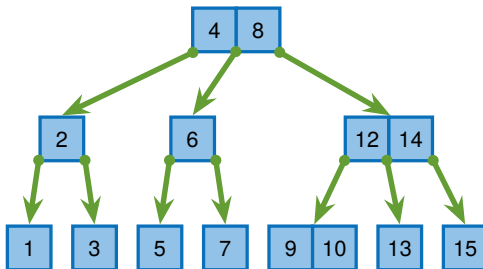


Figure: (2,3)-Tree - Delete step 4

### Counter example (2,3)-Tree:

- Executing `insert(11)`

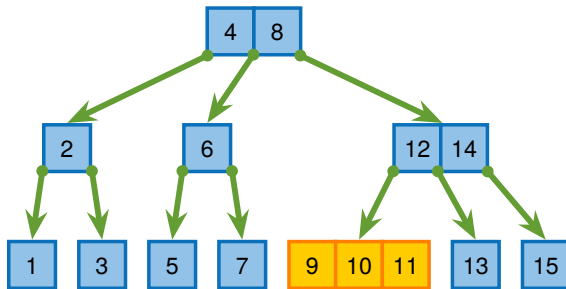


Figure: (2,3)-Tree - Insert step 1



### Counter example (2,3)-Tree:

- Executing `insert(11)`

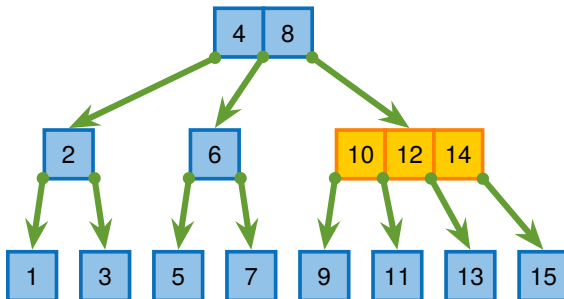


Figure: (2,3)-Tree - Insert step 2

### Counter example (2,3)-Tree:

- Executing `insert(11)`

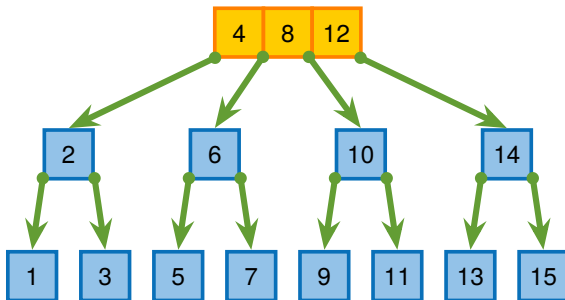


Figure: (2,3)-Tree - Insert step 3

### Counter example (2,3)-Tree:

- Executed `insert(11)`

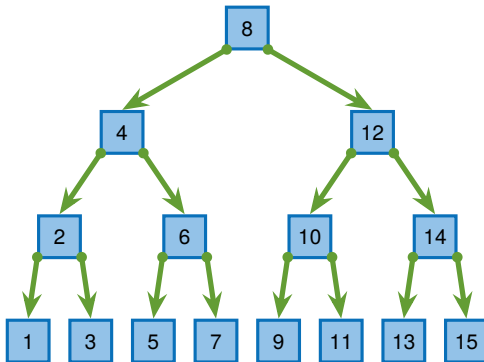


Figure: (2,3)-Tree - Insert step 4

### Counter example (2,3)-Tree:

- We are exactly where we started
- If  $b = 2a - 1$  then we can create a sequence of **insert** and **remove** operations where each operation costs  $O(\log n)$
- We need  $b \geq 2a$  instead of  $b \geq 2a - 1$

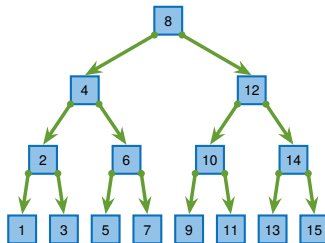


Figure: (2,3)-Tree

### (2,4)-Tree:

- If all nodes have **2 children** we have to **merge** the nodes up to the root on a **remove** operation
- If all nodes have **4 children** we have to **split** the nodes up to the root on a **insert** operation
- If all nodes have **3 children** it takes some time to reach one of the previous two states

⇒ **Nodes of degree 3 are stable**

Neither an **insert** nor a **remove** operation trigger rebalancing operations

### (2,4)-Tree:

#### ■ Idea:

- After an expensive operation the tree is in a stable state
- It takes some time until the next expensive operation occurs
- Like with dynamic arrays:
  - **Reallocation** is expensive but it takes some time until the next expensive operation occurs
  - If we **overallocate** clever we have an amortized runtime of  $O(1)$

### Terminology:

- We analyze a sequence of  $n$  operations
- Let  $\Phi_i$  be the potential of the tree after the  $i$ -th operation
- $\Phi_i$  = the number of stable nodes with degree 3
- Empty tree has 0 nodes:  $\Phi = 0$

### Example:

- Nodes of degree 3 are highlighted

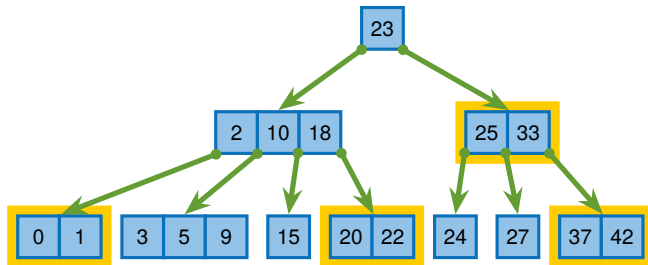


Figure: Tree with potential  $\Phi = 4$



### Terminology:

- Let  $c_i$  be the costs = runtime of the  $i$ -th operation
- We will show:
  - Each operation can at most destroy one stable node
  - For each cost incurring step the operation creates an additional stable node
- The costs for operation  $i$  are coupled to the difference of the potential levels

$$c_i \leq A \cdot (\underbrace{\phi_i - \phi_{i-1}}_{\text{difference of potential levels}}) + B, \quad A > 0 \text{ and } B > A$$

Number of gained stable nodes (degree 3)  $\geq -1$

- Each operation has an amortized cost of  $O(1)$  summing up to  $O(n)$  in total

**Case 1:**  $i$ -th operation is an **insert** operation on a full node



Figure: Splitting a node on **insert**

- Each splitted node creates a node of **degree 3**
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

**Case 1:**  $i$ -th operation is an **insert** operation on a full node

- Let  $m$  be the number of nodes split
- The potential rises by  $m$
- If the “stop-node” is of **degree 3** then the potential goes down by one

$$\begin{aligned}\phi_i &\geq \phi_{i-1} + m - 1 \\ \Rightarrow m &\leq \phi_i - \phi_{i-1} + 1\end{aligned}$$

Costs:  $c_i \leq A \cdot m + B$

$$\begin{aligned}\Rightarrow c_i &\leq A \cdot (\phi_i - \phi_{i-1} + 1) + B \\ c_i &\leq A \cdot (\phi_i - \phi_{i-1}) + \underbrace{A + B}_{B'}\end{aligned}$$

**Case 2:** *i*-th operation is an **remove** operation

■ **Case 2.1:** Inner node

- Searching the successor in a tree is  $O(d) = O(\log n)$
- Normally the tree is coupled with a doubly linked list  
⇒ We can find the successor in  $O(1)$

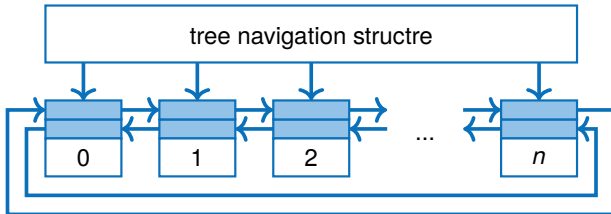


Figure: Tree with doubly linked list

**Case 2:**  $i$ -th operation is an **remove** operation

■ **Case 2.1:** Borrow a node

- Creates no additional operations
- Case 2.1.1: Potential rises by one



Figure: Case 2.1.1: Borrow an element

**Case 2:**  $i$ -th operation is an **remove** operation

■ **Case 2.1:** Borrow a node

- Creates no additional operations
- Case 2.1.2: Potential is lowered by one

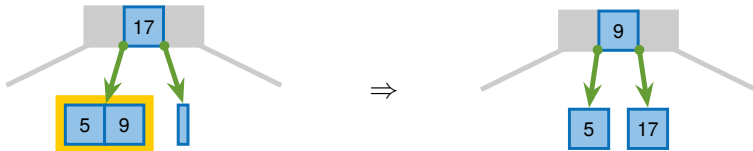


Figure: Case 2.1.2: Borrow an element

**Case 2:** *i*-th operation is an **remove** operation

■ **Case 2.2:** Merging two node

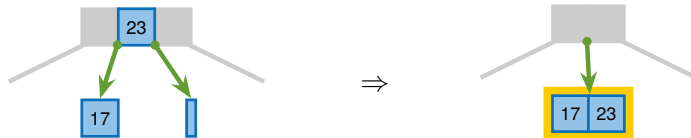


Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree  $> 2$  or a node of degree 2, which can borrow from a neighbour

**Case 2:**  $i$ -th operation is an **remove** operation

■ **Case 2.2:** Merging two nodes

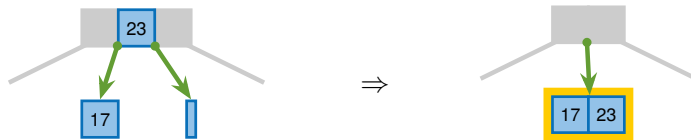


Figure: Merging two nodes

- The potential rises by  $m$
- If the “stop-node” is of **degree 2** then the potential eventually goes down by one
- Same costs as **insert**



### Lemma:

- We know:

$$c_i \leq A \cdot (\phi_i - \phi_{i-1}) + B, \quad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^n c_i \in O(n)$$

### Proof:

$$\begin{aligned}\sum_{i=0}^n c_i &\leq \underbrace{A \cdot (\phi_1 - \phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\phi_2 - \phi_1) + B}_{\leq c_2} + \dots + \underbrace{A \cdot (\phi_n - \phi_{n-1}) + B}_{\leq c_n} \\ &= A \cdot (\phi_n - \phi_0) + B \cdot n \quad | \text{ telescope sum} \\ &= A \cdot \phi_n + B \cdot n \quad | \text{ we start with an empty tree} \\ &< A \cdot n + B \cdot n \in O(n) \quad | \text{ number of degree 3 nodes} \\ &\quad < \text{ number of nodes}\end{aligned}$$

### Red-Black Tree:

- Binary tree with **red** and **black** nodes
- Number of **black** nodes on path to leaves is equal
- Can be interpreted as **(2,4)-tree** (also named 2-3-4-tree)
- Each **(2,4)-tree**-node is a small red-black-tree with a **black** root node

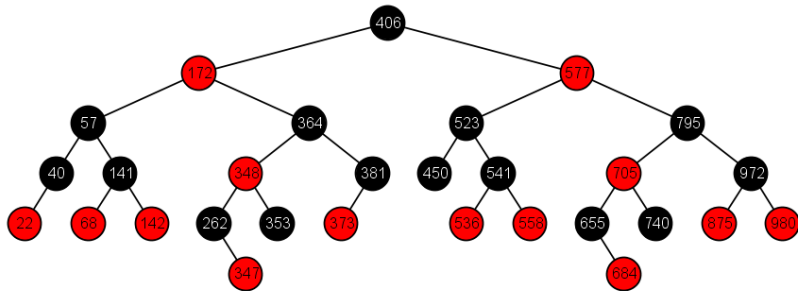


Figure: Example of an red-black-tree [Gna]

## ■ General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

### **Introduction to Algorithms.**

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ■ Gnarley Trees

[Gna] **Gnarley Trees**

<https://people.ksp.sk/~kuko/gnarley-trees/>

## ■ AVL-Tree

[Wik] [AVL tree](#)

[https://en.wikipedia.org/wiki/AVL\\_tree](https://en.wikipedia.org/wiki/AVL_tree)

## ■ (a,b)-Tree

[Wika] [2-3-4 tree](#)

[https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934\\_tree](https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%944_tree)

[Wikb] [\(a,b\)-tree](#)

[https://en.wikipedia.org/wiki/\(a,b\)-tree](https://en.wikipedia.org/wiki/(a,b)-tree)

## ■ Red-Black-Tree

[Wik] [Red-black tree](https://en.wikipedia.org/wiki/Red%E2%80%93black_tree)

`https://en.wikipedia.org/wiki/Red%E2%80%93black\_tree`