Algorithms and Data Structures

Divide and Conquer, Master theorem

Albert-Ludwigs-Universität Freiburg

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Algorithms and Data Structures, December 2018

Structure

Divide and Conquer
- Concept
- Maximum Subtotal

Recursion Equations
- Substitution Method
- Recursion Tree Method
- Master theorem
  - Master theorem (Simple Form)
  - Master theorem (General Form)

Concept:

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- Recursive application of the algorithm on smaller subproblems
- Direct solving of small subproblems

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- Maximum sum of an uninterrupted subsequence of X and its index boundary

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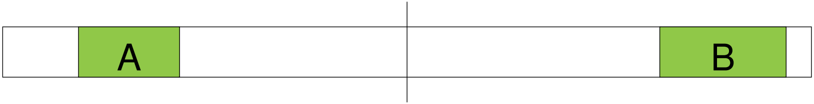
Table: input values

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

Idea:



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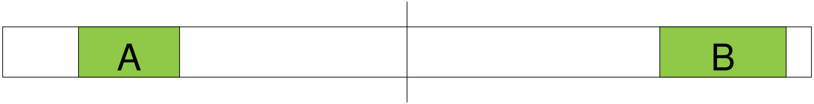

- Solve the left / right half of the problem recursively

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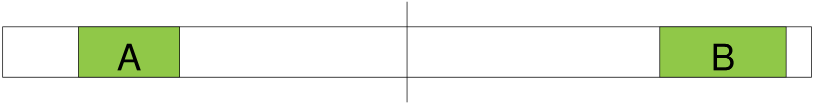

- Solve the left / right half of the problem recursively
- Combine both solutions into an overall solution

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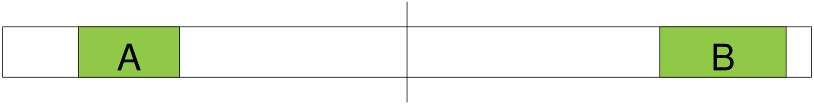

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- The maximum is located in the left half (A) or the right half (B)

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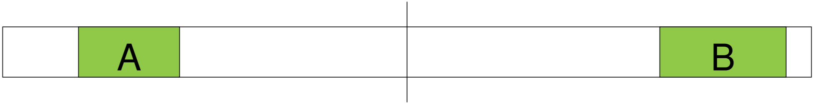
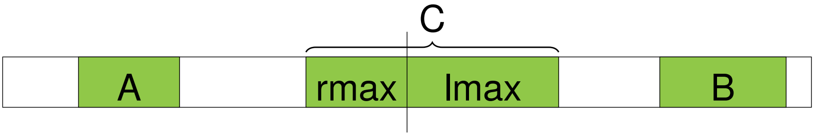


- Solve the left / right half of the problem recursively
- Combine both solutions into an overall solution
- The maximum is located in the left half (*A*) or the right half (*B*)
- The maximum interval can overlap with the border (*C*)

Principle:



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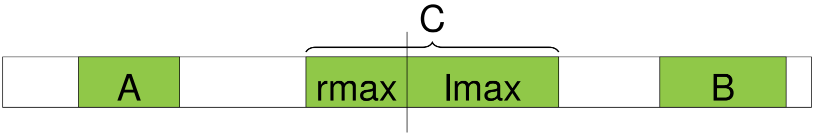

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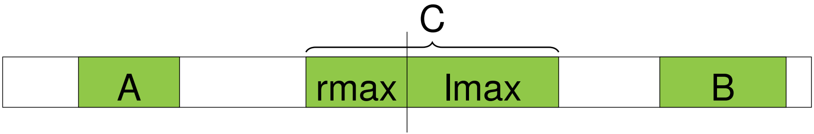

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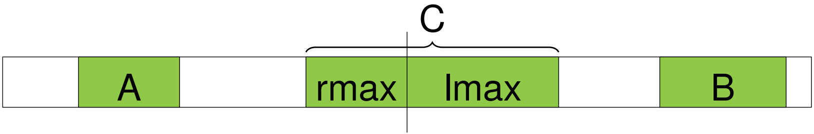

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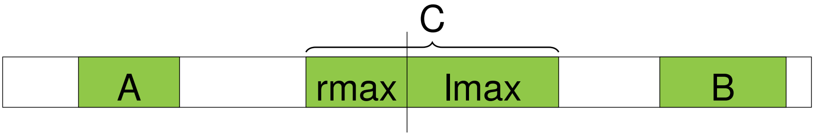

- Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$
- Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned.
- To solve C we have to calculate rmax and lmax
- The overall solution is the maximum of A, B and C

```python
def maxSubArray(X, i, j):
```

```python
def maxSubArray(X, i, j):
    if i == j: # trivial case
        return (X[i], i, i)

    # recursive subsolutions for A, B
    m = (i + j) // 2
```

Divide and Conquer

Maximum Subtotal - Python

```python
def maxSubArray(X, i, j):
    if i == j: # trivial case
        return (X[i], i, i)

    # recursive subsolutions for A, B
    m = (i + j) // 2
    A = maxSubArray(X, i, m)
    B = maxSubArray(X, m + 1, j)
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def maxSubArray(X, i, j):
    if i == j: # trivial case
        return (X[i], i, i)

    # recursive subsolutions for A, B
    m = (i + j) // 2
    A = maxSubArray(X, i, m)
    B = maxSubArray(X, m + 1, j)

    # rmax and lmax for cornercase C
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)
    C = (C1[0] + C2[0], C1[1], C2[1])
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    m = (i + j) // 2
    A = maxSubArray(X, i, m)
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    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)
    C = (C1[0] + C2[0], C1[1], C2[1])

    # compute solution from results A, B, C
    return max([A, B, C], key=lambda i: i[0])
```

```
#Alternative trivial case
def maxSubArray(X, i, j):
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def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
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Divide and Conquer

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#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```

```
#Implementation max
def max(a, b, c):
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def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
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#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
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def max(a, b):
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Divide and Conquer

Maximum Subtotal - Python

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def max(a, b):
    if a > b:
        return a
    else:
        return b

def maxTripel(a, b, c):
    return max(max(a,b),c)
```

```python
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

```python
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[j]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Table: *lmax* example

index	i	$i+1$	$\cdots$	$\cdots$	$j-1$	j
X	58	-53	26	59	-41	31
sum	58	5	31	90	49	80
lmax	58	58	58	90	90	90

Table: *lmax* example

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X	58	-53	26	59	-41	31
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- The *sum* and *lmax* are initialized with $X[i]$

Table: *lmax* example

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- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i+1$ to j and update *sum*

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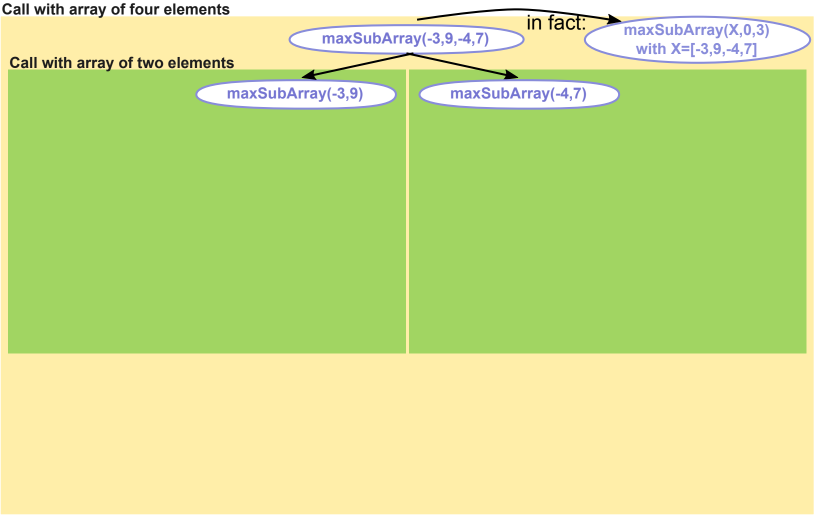
- The *sum* and *lmax* are initialized with $X[i]$
- We iterate over X from $i+1$ to j and update *sum*
- If $sum > lmax$, then *lmax* gets updated

Call with array of four elements

maxSubArray(-3,9,-4,7)

in fact:

maxSubArray(X,0,3)
with X=[-3,9,-4,7]



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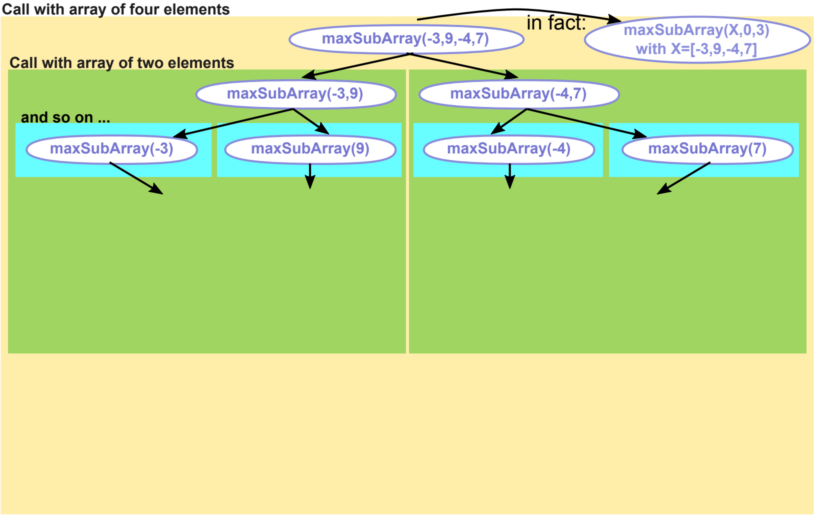

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and so on ...
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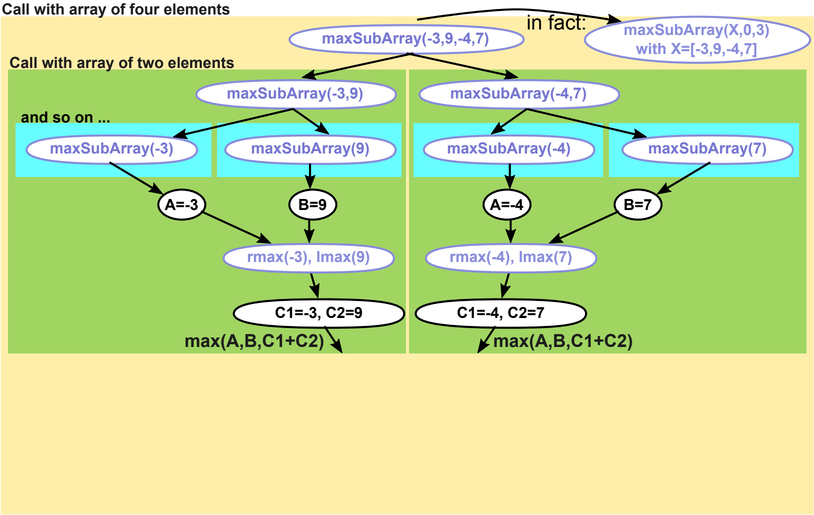
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B=9
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B=7
rmax(-3), lmax(9)
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C1=-3, C2=9
C1=-4, C2=7
max(A,B,C1+C2)
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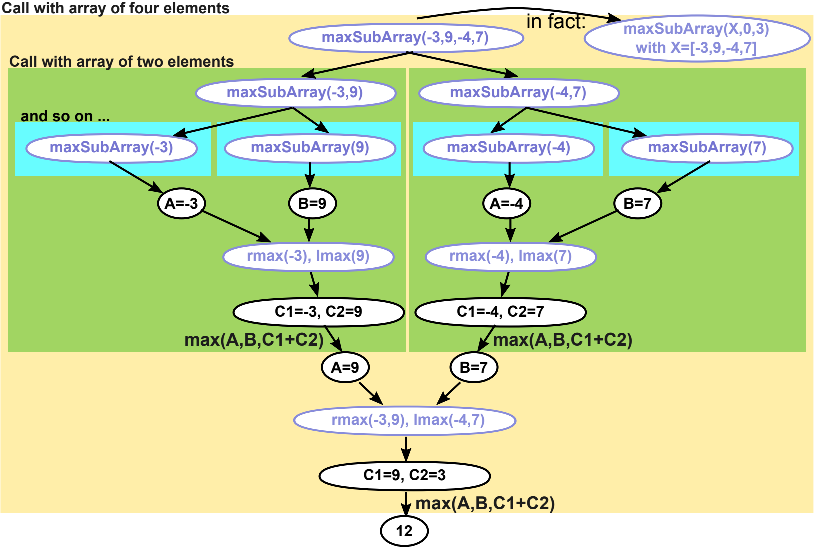

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C1=9, C2=3
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12

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Divide and Conquer

Maximum Subtotal - Python

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    A = maxSubArray(X, i, m)                    # T(n/2)
    B = maxSubArray(X, m + 1, j)

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Divide and Conquer

Maximum Subtotal - Python

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Divide and Conquer

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```

Recursion equation:

$$T(n) = \begin{cases} \underbrace{\Theta(1)}_{\text{trivial case}} & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

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- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

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- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

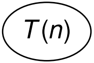
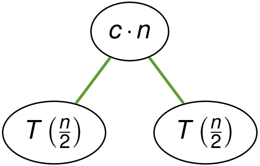


Figure: illustration of the runtime



$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

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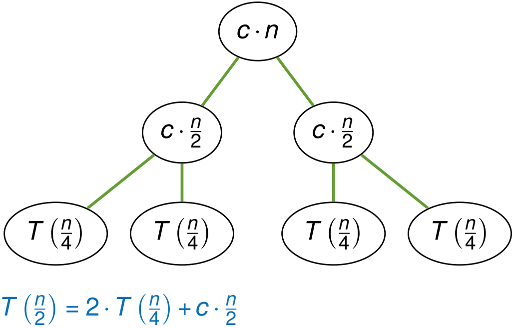


Figure: illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

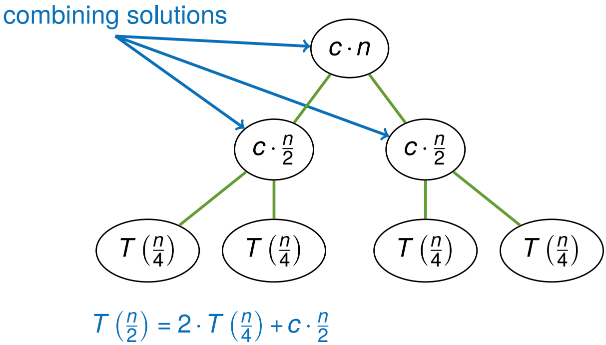
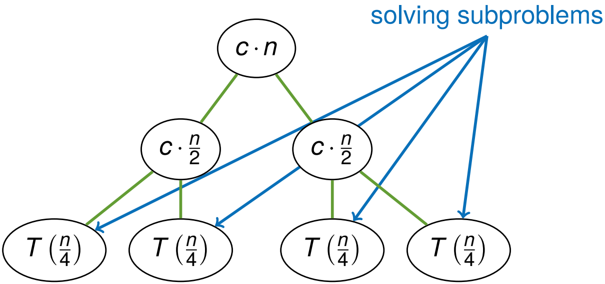


Figure: illustration of the runtime



$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$

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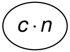
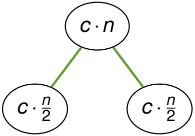


1 node processing n elements
$\Rightarrow c \cdot n$

Figure: recursion tree method



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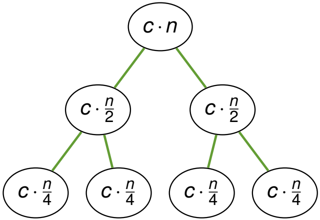

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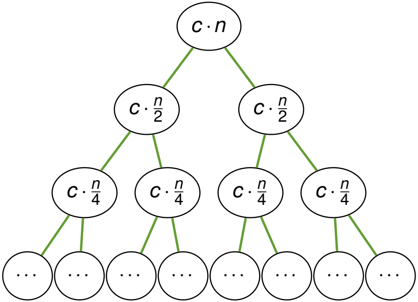

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Divide and Conquer

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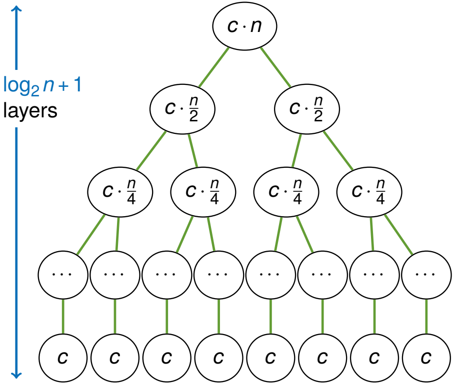

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$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

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- Better solution with incremental update of sum was $\mathcal{O}(n^2)$
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- There is an approach running in $\mathcal{O}(n)$, under the assumption that all subtotals are positive

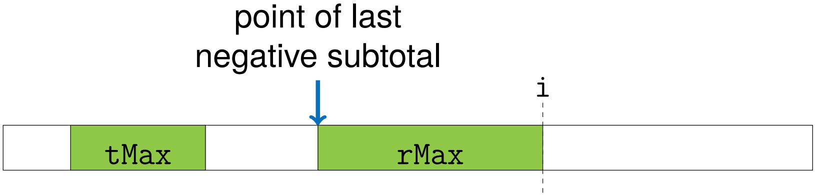

Figure: scanning the array in linear time

Divide and Conquer

Maximum Subtotal - Python

```python
#Implementation - linear runtime
def maxSubArray(X):
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def maxSubArray(X):
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        if rMax == 0:
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def maxSubArray(X):
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    for i in range(len(X)):
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        rMax = max(0, rMax + X[i])

        if rMax > tMax:
            tMax, itMax = rMax, irMax

    return (tMax, itMax)
```

Structure

Recursion equation:

- Runtime description for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$

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- $T(n)$ is only defined for integers of $\frac{n}{b}$, which is often ignored in benefit of a simpler solution

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&\leq c \cdot n \log_2 n, \quad c \geq 1
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$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

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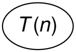

Figure: recursion tree of example

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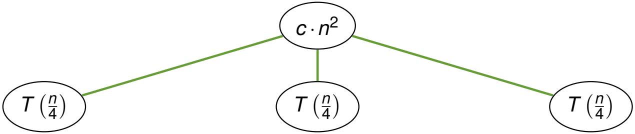

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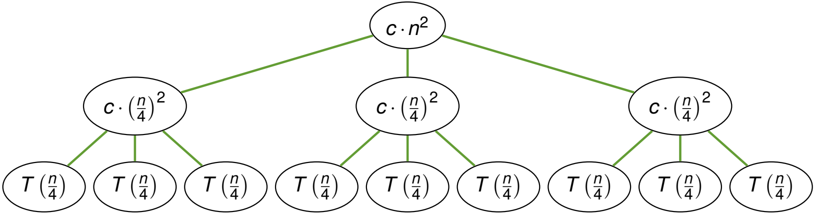

$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

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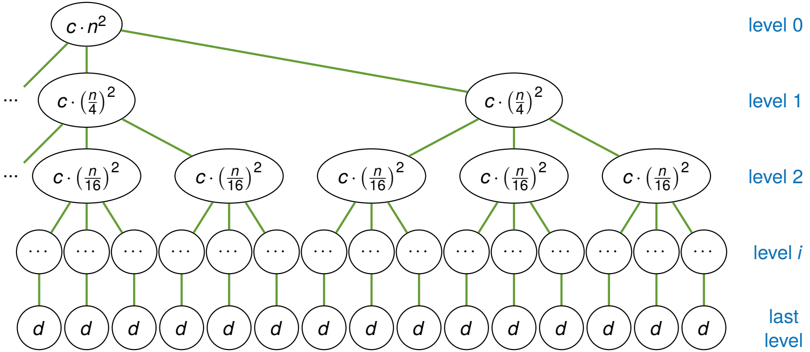

Figure: levels of the recursion tree

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$$T_i(n) = 3^i \cdot c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n\right)^2 = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$$

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$$= n^{\log_4 3}$$

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$$
\begin{aligned}
\log_4 n &= \log_4\left(3^{\log_3 n}\right) && \text{using } n = 3^{\log_3 n} \\
&= \log_3 n \cdot \log_4 3 && \text{using } \log a^b = b \cdot \log a
\end{aligned}
$$

- This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- Now the whole expression:

$$
\begin{aligned}
3^{\log_4 n} &= 3^{\log_3 n \cdot \log_4 3} && \text{using reformulation above} \\
&= \left(3^{\log_3 n}\right)^{\log_4 3} && \text{using } x^{a \cdot b} = (x^a)^b \\
&= n^{\log_4 3}
\end{aligned}
$$

- This term will recur in the master theorem

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$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\text{geometric series, constant}\ \left(\text{even with infinite elements}\right)} + \underbrace{d \cdot n^{\log_4 3}}_{\log_4 3 < 1,\ \text{grows a lot slower than } n^2} \in \mathcal{O}(n^2)$$

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- Here: The costs of connecting the partial problems dominate

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  Quotient of two neighboring sequence parts is constant

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- **Geometric series:**
  The series (cumulative sum) of a geometric sequence
- For $|\, q \,| < 1$:

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1-q} \qquad \Rightarrow \text{constant}$$

Proof of $\mathcal{O}(n^2)$:

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- We know:

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{4}\right) + \Theta(n^2) \\
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&= \frac{3}{16} k \cdot n^2 + c \cdot n^2 \\
&\leq k \cdot n^2 \qquad\qquad \text{for } k \geq \frac{16}{13} c
\end{aligned}$$

Structure

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- **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

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$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

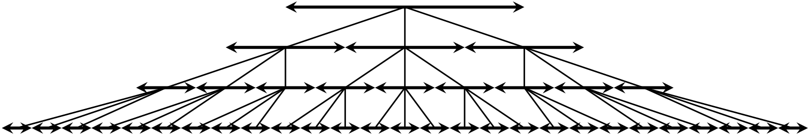
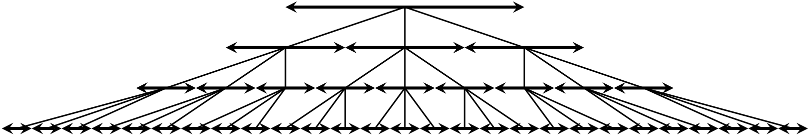


Figure: simple recursion equation with $a = 3, b = 2$



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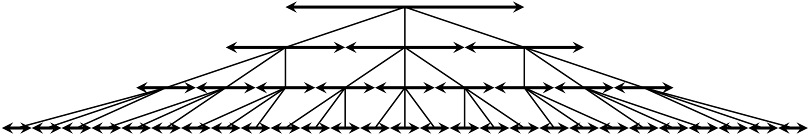

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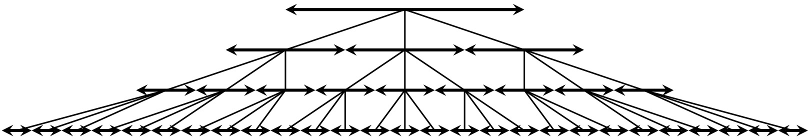

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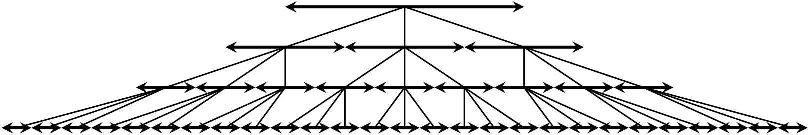

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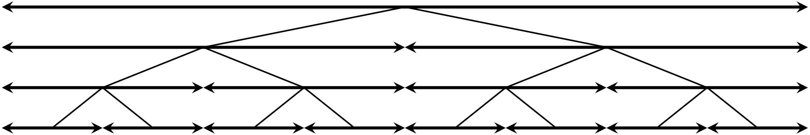
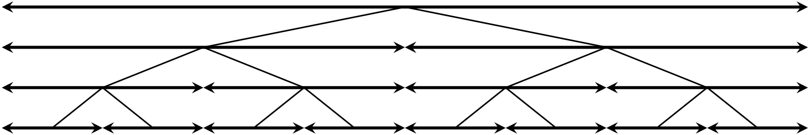


Figure: simple recursion equation with $a = 2, b = 2$



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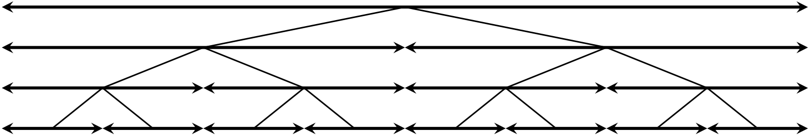

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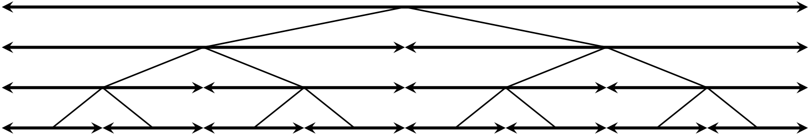

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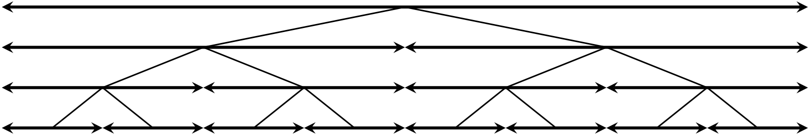

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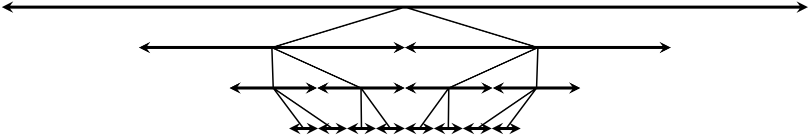

Figure: simple recursion equation with $a = 2, b = 3$

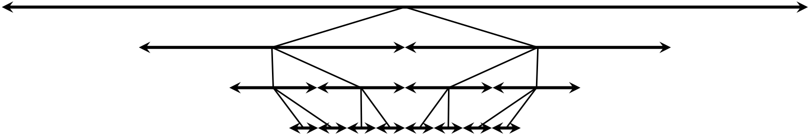

Figure: simple recursion equation with $a = 2, b = 3$

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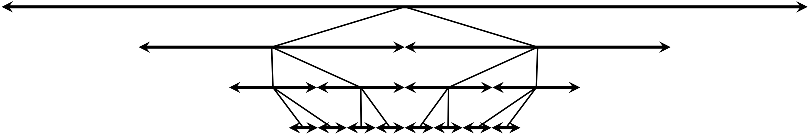

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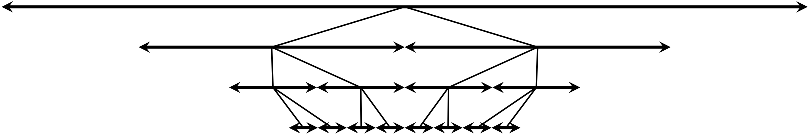

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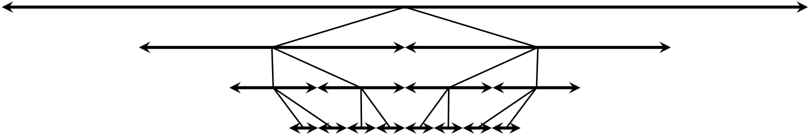

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- Runtime of $\Theta(n)$

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$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \qquad a \geq 1, b > 1, c > 0$$

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- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

Structure

Master theorem (general form):

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- **Case 3:** $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon}),\ \varepsilon > 0$
  Connecting all partial solutions in first layer (root) dominates

  Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1,$$
$$n > n_0$$

Case 1 - Example: if $f(n) \in O(n^{\log_b a - \varepsilon}),\ \varepsilon > 0$

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Solving the partial problems dominates (last layer, leaves)

- $T(n) = 8 \cdot T(\frac{n}{2}) + 1000 \cdot n^2$

$$a = 8,\ b = 2,\ f(n) = 1000 \cdot n^2,\ \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$

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$$f(n) \in \mathcal{O}(n^{3-\varepsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

- $T(n) = 9 \cdot T(\frac{n}{3}) + 17 \cdot n$

$$a = 9,\ b = 3,\ f(n) = 17 \cdot n,\ \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\varepsilon}) \Rightarrow T(n) \in \Theta(n^2)$$

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$$a = 2,\ b = 2,\ f(n) = 10 \cdot n,\ \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

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$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

- $T(n) = T(\frac{2n}{3}) + 1$

$$a = 1,\ b = \frac{3}{2},\ f(n) = 1,\ \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

Case 3: if $f(n) \in \Omega(n^{\log_b a+\varepsilon}),\ \varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

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Connecting all partial solutions in first layer (root) dominates

- $T(n) = 2 \cdot T(\frac{n}{2}) + n^2$
- $f(n) \in \Omega(n^{1+\varepsilon})$
- Check if regularity condition also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow c \geq \frac{1}{2}$$

$\Rightarrow T(n) \in \Theta(n^2)$

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- **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$

Master theorem:

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$n \log n$ is *asymptotically* larger than n,
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- **Case 3:** Connecting all partial solutions is *polynominal* bigger than solving all partial problems
  $T(n) \in \Theta(f(n))$

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Master theorem**

  [Wik] Master theorem
  https://en.wikipedia.org/wiki/Master_theorem