

Algorithms and Datastructures

Linked Lists, Binary Search Trees

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Algorithms and Datastructures, January 2018

Structure

Sorted Sequences

Linked Lists

Binary Search Trees

Sorted Sequences

Introduction

Structure:

Sorted Sequences

Introduction

Structure:

- ▶ We have a set of **keys** mapped to **values**

Sorted Sequences

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- ▶ We have a set of **keys** mapped to **values**
- ▶ We have a ordering **i** applied to the keys

Sorted Sequences

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- ▶ We need the following operations:

Sorted Sequences

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 - ▶ **insert(key, value)**: Insert the given pair

Sorted Sequences

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Sorted Sequences

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 - ▶ **insert(key, value)**: Insert the given pair
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 - ▶ **lookup(key)**: Find the element with the given **key**, if it is not available find the element with the next smallest key

Sorted Sequences

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 - ▶ **insert(key, value)**: Insert the given pair
 - ▶ **remove(key)**: Remove the pair with the given **key**
 - ▶ **lookup(key)**: Find the element with the given **key**, if it is not available find the element with the next smallest key
 - ▶ **next()/previous()**: Returns the element with the next bigger/smaller **key**. This enables iteration over all elements

Sorted Sequences

Introduction

Application examples:

Sorted Sequences

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- ▶ Example: Database for books, products or apartments

Sorted Sequences

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- ▶ Large number of records (data sets / tuples)

Sorted Sequences

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Application examples:

- ▶ Example: Database for books, products or apartments
- ▶ Large number of records (data sets / tuples)
- ▶ Typical query: Return all apartments with a monthly rent between 400€ and 600€

Sorted Sequences

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 - ▶ This is called a **range query**

Sorted Sequences

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 - ▶ We can implement this with a combination of **lookup(key)** and **next()**

Sorted Sequences

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Sorted Sequences

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Sorted Sequences

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- ▶ Typical query: Return all apartments with a monthly rent between 400€ and 600€
 - ▶ This is called a **range query**
 - ▶ We can implement this with a combination of **lookup(key)** and **next()**
 - ▶ It's not essential if an apartments exists with **exactly** 400€ monthly rent
- ▶ We do not want to sort all elements every time on an **insert** operation
- ▶ How could we implement this?

Sorted Sequences

Implementation 1 (not good) - Static Array

Static array:

3	5	9	14	18	21	26	40	41	42	43	46
---	---	---	----	----	----	----	----	----	----	----	----

Sorted Sequences

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- **lookup** in time $O(\log n)$

Sorted Sequences

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 - ▶ With **binary search**

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 - ▶ Example: **lookup(41)**

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- ▶ **next** / **previous** in time $O(1)$
 - ▶ They are next to each other

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- ▶ **insert** and **remove** up to $\Theta(n)$

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 - ▶ With **binary search**
 - ▶ Example: **lookup(41)**
- ▶ **next** / **previous** in time $O(1)$
 - ▶ They are next to each other
- ▶ **insert** and **remove** up to $\Theta(n)$
 - ▶ We have to copy up to n elements

Sorted Sequences

Implementation 2 (bad) - Hash Table

Hash map:

Sorted Sequences

Implementation 2 (bad) - Hash Table

Hash map:

- ▶ `insert` and `remove` in $O(1)$

Sorted Sequences

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If the hash table is big enough and we use a good hash function

Sorted Sequences

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Sorted Sequences

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Hash map:

- ▶ `insert` and `remove` in $O(1)$

If the hash table is big enough and we use a good hash function

- ▶ `lookup` in time $O(1)$

If element with **exactly** this key exists, otherwise we get `None` as result

- ▶ `next` / `previous` in time up to $\Theta(n)$

Sorted Sequences

Implementation 2 (bad) - Hash Table

Hash map:

- ▶ `insert` and `remove` in $O(1)$

If the hash table is big enough and we use a good hash function

- ▶ `lookup` in time $O(1)$

If element with **exactly** this key exists, otherwise we get `None` as result

- ▶ `next` / `previous` in time up to $\Theta(n)$

Order of the elements is independent of the order of the keys

Sorted Sequences

Implementation 3 (good?) - Linked List

Linked list:

Sorted Sequences

Implementation 3 (good?) - Linked List

Linked list:

- ▶ Runtimes for doubly linked lists:

Sorted Sequences

Implementation 3 (good?) - Linked List

Linked list:

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Sorted Sequences

Implementation 3 (good?) - Linked List

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- ▶ Runtimes for doubly linked lists:
 - ▶ `next` / `previous` in time $O(1)$
 - ▶ `insert` and `remove` in $O(1)$

Sorted Sequences

Implementation 3 (good?) - Linked List

Linked list:

- ▶ Runtimes for doubly linked lists:
 - ▶ `next` / `previous` in time $O(1)$
 - ▶ `insert` and `remove` in $O(1)$
 - ▶ `lookup` in time $\Theta(n)$

Sorted Sequences

Implementation 3 (good?) - Linked List

Linked list:

- ▶ Runtimes for doubly linked lists:
 - ▶ `next` / `previous` in time $O(1)$
 - ▶ `insert` and `remove` in $O(1)$
 - ▶ `lookup` in time $\Theta(n)$
- ▶ Not yet what we want, but structure is related to binary search trees

Sorted Sequences

Implementation 3 (good?) - Linked List

Linked list:

- ▶ Runtimes for doubly linked lists:
 - ▶ `next` / `previous` in time $O(1)$
 - ▶ `insert` and `remove` in $O(1)$
 - ▶ `lookup` in time $\Theta(n)$
- ▶ Not yet what we want, but structure is related to binary search trees
- ▶ Let's have a closer look

Structure

Sorted Sequences

Linked Lists

Binary Search Trees

Linked Lists

Introduction

Linked list:

Linked Lists

Introduction

Linked list:

- ▶ Dynamic datastructure

Linked Lists

Introduction

Linked list:

- ▶ Dynamic datastructure
- ▶ Number of elements changeable

Linked Lists

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- ▶ Number of elements changeable
- ▶ Data elements can be simple types or composed datastructures

Linked Lists

Introduction

Linked list:

- ▶ Dynamic datastructure
- ▶ Number of elements changeable
- ▶ Data elements can be simple types or composed datastructures
- ▶ **Elements are linked** through references / pointer to the predecessor / successor

Linked Lists

Introduction

Linked list:

- ▶ Dynamic datastructure
- ▶ Number of elements changeable
- ▶ Data elements can be simple types or composed datastructures
- ▶ **Elements are linked** through references / pointer to the predecessor / successor
- ▶ Single / doubly linked lists possible

Linked Lists

Introduction

Linked list:

- ▶ Dynamic datastructure
- ▶ Number of elements changeable
- ▶ Data elements can be simple types or composed datastructures
- ▶ Elements are linked through references / pointer to the predecessor / successor
- ▶ Single / doubly linked lists possible

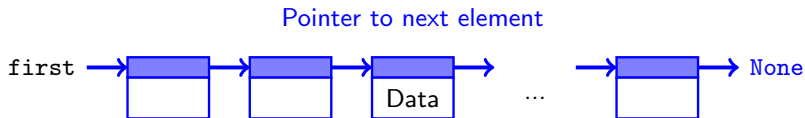


Figure: Linked list

Linked Lists

Introduction

Properties in comparison to an array:

Linked Lists

Introduction

Properties in comparison to an array:

- ▶ Minimal extra space for storing pointer

Linked Lists

Introduction

Properties in comparison to an array:

- ▶ Minimal extra space for storing pointer
- ▶ We do not need to copy elements on `insert` or `remove`

Linked Lists

Introduction

Properties in comparison to an array:

- ▶ Minimal extra space for storing pointer
- ▶ We do not need to copy elements on `insert` or `remove`
- ▶ The number of elements can be simply modified

Linked Lists

Introduction

Properties in comparison to an array:

- ▶ Minimal extra space for storing pointer
- ▶ We do not need to copy elements on `insert` or `remove`
- ▶ The number of elements can be simply modified
- ▶ No direct access of elements
⇒ We have to iterate over the list

Linked Lists

Variants

List with head / last element pointer:

Linked Lists

Variants

List with head / last element pointer:

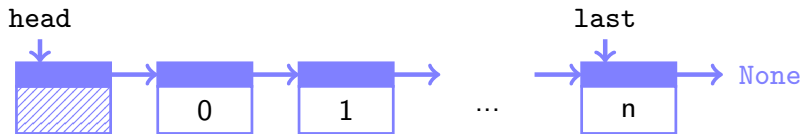


Figure: Singly linked list

Linked Lists

Variants

List with head / last element pointer:



Figure: Singly linked list

- Head element has pointer to first list element

Linked Lists

Variants

List with head / last element pointer:



Figure: Singly linked list

- ▶ Head element has pointer to first list element
- ▶ May also hold additional information:

Linked Lists

Variants

List with head / last element pointer:



Figure: Singly linked list

- ▶ Head element has pointer to first list element
- ▶ May also hold additional information:
 - ▶ Number of elements

Linked Lists

Variants

Doubly linked list:

Linked Lists

Variants

Doubly linked list:



Figure: Doubly linked list

Linked Lists

Variants

Doubly linked list:



Figure: Doubly linked list

- Pointer to successor element

Linked Lists

Variants

Doubly linked list:



Figure: Doubly linked list

- ▶ Pointer to successor element
- ▶ Pointer to predecessor element

Linked Lists

Variants

Doubly linked list:



Figure: Doubly linked list

- ▶ Pointer to successor element
- ▶ Pointer to predecessor element
- ▶ Iterate forward and backward

Linked Lists

Implementation - Node/Element - Python

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```


Linked Lists

Usage examples

Creating linked lists - Python:

Linked Lists

Usage examples

Creating linked lists - Python:

► `first = Node(7)`



Linked Lists

Usage examples

Creating linked lists - Python:

- ▶ `first = Node(7)`



- ▶ `first.nextNode = Node(3)`



Linked Lists

Usage examples

Creating linked lists - Python:

- ▶ `first = Node(7)`



- ▶ `first.nextNode = Node(3)`



- ▶ `first.nextNode.value = 4`



Linked Lists

Implementation - Insert

Inserting a node after node cur:



Linked Lists

Implementation - Insert

Inserting a node after node `cur`:

Linked Lists

Implementation - Insert

Inserting a node after node cur:

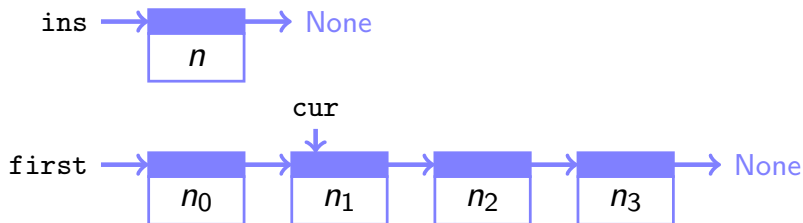
- ▶ `ins = Node(n)`

Linked Lists

Implementation - Insert

Inserting a node after node cur:

► `ins = Node(n)`



Linked Lists

Implementation - Insert

Inserting a node after node cur:

- ▶ `ins.nextNode = cur.nextNode`

Linked Lists

Implementation - Insert

Inserting a node after node cur:

► `ins.nextNode = cur.nextNode`



Linked Lists

Implementation - Insert

Inserting a node after node cur:

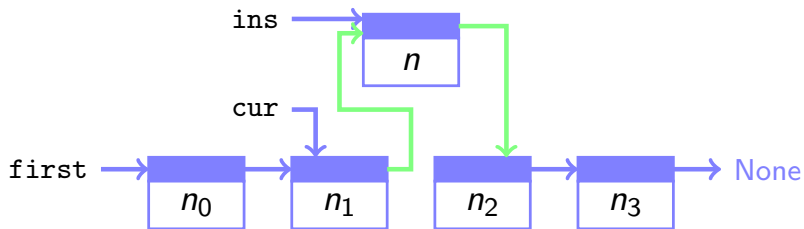
- ▶ `cur.nextNode = ins`

Linked Lists

Implementation - Insert

Inserting a node after node `cur`:

► `cur.nextNode = ins`



Linked Lists

Implementation - Insert

Inserting a node after node `cur` - single line of code:

Linked Lists

Implementation - Insert

Inserting a node after node `cur` - single line of code:



Linked Lists

Implementation - Insert

Inserting a node after node `cur` - single line of code:



► `cur.nextNode = Node(value, cur.nextNode)`

Linked Lists

Implementation - Insert

Inserting a node after node `cur` - single line of code:



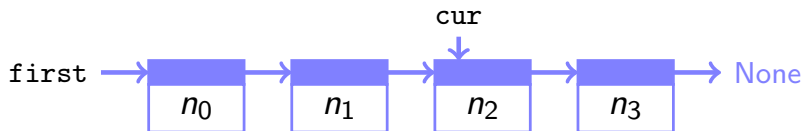
► `cur.nextNode = Node(value, cur.nextNode)`



Linked Lists

Implementation - Remove

Removing a node `cur`:



Linked Lists

Implementation - Remove

Removing a node `cur:`

Linked Lists

Implementation - Remove

Removing a node cur:

- Find the predecessor of cur:

```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```

Linked Lists

Implementation - Remove

Removing a node `cur`:

- ▶ Find the predecessor of `cur`:

```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```

- ▶ Runtime of $O(n)$

Linked Lists

Implementation - Remove

Removing a node `cur`:

- ▶ Find the predecessor of `cur`:

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pre = first
while pre.nextNode != cur:
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- ▶ Runtime of $O(n)$
- ▶ Does not work for first node!

Linked Lists

Implementation - Remove

Removing a node cur:

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pre = first
while pre.nextNode != cur:
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- ▶ Runtime of $O(n)$
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Linked Lists

Implementation - Remove

Removing a node `cur:`

Linked Lists

Implementation - Remove

Removing a node `cur`:

- ▶ Update the pointer to the next element:
`pre.nextNode = cur.nextNode`

Linked Lists

Implementation - Remove

Removing a node `cur`:

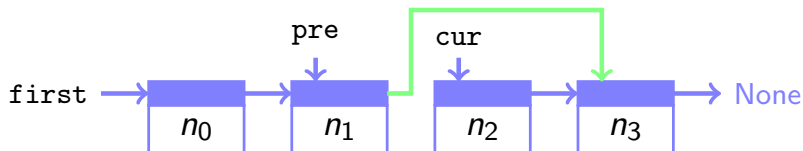
- ▶ Update the pointer to the next element:
`pre.nextNode = cur.nextNode`
- ▶ `cur` will get automatically destroyed if no more references exist (`cur=None`)

Linked Lists

Implementation - Remove

Removing a node `cur`:

- ▶ Update the pointer to the next element:
`pre.nextNode = cur.nextNode`
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Linked Lists

Implementation - Remove

Removing the first node:

Linked Lists

Implementation - Remove

Removing the first node:



Linked Lists

Implementation - Remove

Removing the first node:



- Update the pointer to the next element:
`first = first.nextNode`

Linked Lists

Implementation - Remove

Removing the first node:



- ▶ Update the pointer to the next element:
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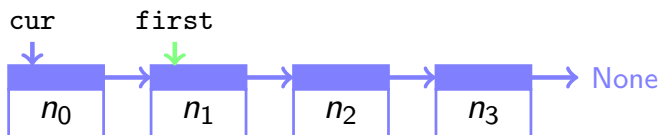
Linked Lists

Implementation - Remove

Removing the first node:



- Update the pointer to the next element:
`first = first.nextNode`
- `cur` will get automatically destroyed if no more references exist
(`cur=None`)



Linked Lists

Implementation - Remove

Removing a node cur: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```


Linked Lists

Implementation - Head Node

Using a head node:

Linked Lists

Implementation - Head Node

Using a head node:

- ▶ Advantage:

Linked Lists

Implementation - Head Node

Using a head node:

- ▶ Advantage:
 - ▶ Deleting the first node is no special case

Linked Lists

Implementation - Head Node

Using a head node:

- ▶ Advantage:
 - ▶ Deleting the first node is no special case
- ▶ Disadvantage
 - ▶ We have to consider the first node at other operations

Linked Lists

Implementation - Head Node

Using a head node:

- ▶ Advantage:
 - ▶ Deleting the first node is no special case
- ▶ Disadvantage
 - ▶ We have to consider the first node at other operations
 - ▶ Iterating all nodes
 - ▶ Counting of all nodes

Linked Lists

Implementation - Head Node

Using a head node:

- ▶ Advantage:
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 - ▶ We have to consider the first node at other operations
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 - ▶ ...

Linked Lists

Implementation - Head Node

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- ▶ Disadvantage
 - ▶ We have to consider the first node at other operations
 - ▶ Iterating all nodes
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 - ▶ ...



Linked Lists

Implementation - LinkedList - Python

```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```


Linked Lists

Implementation - LinkedList - Python

```
def append(self, value):
```

```
...
```

```
def insertAfter(self, cur, value):
```

```
...
```

```
def remove(self, cur):
```

```
...
```

```
def get(self, position):
```

```
...
```

```
def contains(self, value):
```

```
...
```

Linked Lists

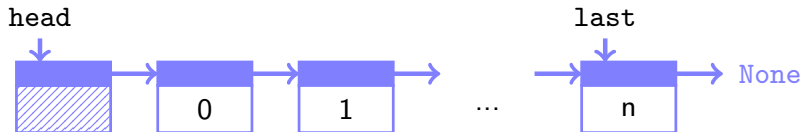
Implementation

Head, last:

Linked Lists

Implementation

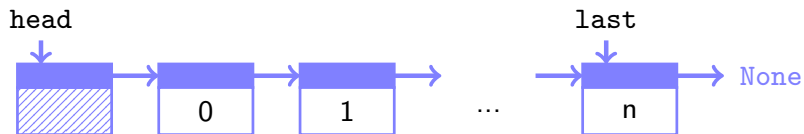
Head, last:



Linked Lists

Implementation

Head, last:



- Head points to the first node, last to the last node

Linked Lists

Implementation

Head, last:



- ▶ Head points to the first node, last to the last node
- ▶ We can append elements to the end of the list in $O(1)$ through the last node

Linked Lists

Implementation

Head, last:



- ▶ Head points to the first node, last to the last node
- ▶ We can append elements to the end of the list in $O(1)$ through the last node
- ▶ We have to keep the pointer to last updated after all operations

Linked Lists

Implementation - Append

Appending an element:

Linked Lists

Implementation - Append

Appending an element:



Linked Lists

Implementation - Append

Appending an element:



```
def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```

Linked Lists

Implementation - Append

Appending an element:



```
def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```

- The pointer to last avoids the iteration of the whole list

Linked Lists

Implementation - Insert After

Inserting after node cur:



Linked Lists

Implementation - Insert After

Inserting after node `cur`:

- ▶ The pointer to head is not modified

Linked Lists

Implementation - Insert After

Inserting after node cur:

- The pointer to head is not modified

```
def insertAfter(self, cur, value):  
    if cur == last:  
        # also update last node  
        append(value)  
    else:  
        # last node is not modified  
        cur.nextNode = Node(value, \  
                             cur.nextNode)  
        itemCount += 1
```

Linked Lists

Implementation - Remove

Remove node `cur`:



Linked Lists

Implementation - Remove

Remove node `cur`:

- ▶ Searching the predecessor in $O(n)$

Linked Lists

Implementation - Remove

Remove node cur:

- ▶ Searching the predecessor in $O(n)$

```
def remove(self, cur):  
    pre = first  
    while pre.nextNode != cur:  
        pre = pre.nextNode  
  
    pre.nextNode = cur.nextNode  
    itemCount -= 1  
  
    if pre.nextNode == None:  
        last = pre
```


Linked Lists

Implementation - Get

Getting a reference to node at pos:

- ▶ Iterate the entries of the list until at position in $O(n)$

Linked Lists

Implementation - Get

Getting a reference to node at pos:

- Iterate the entries of the list until at position in $O(n)$

```
def get(self, pos):  
    if pos < 0 or pos >= itemCount:  
        return None  
  
    cur = head  
    for i in range(0, pos):  
        cur = cur.nextNode  
  
    return cur
```

Linked Lists

Implementation - Contains

Searching a value:

Linked Lists

Implementation - Contains

Searching a value:

- ▶ First element is head without an assigned value

Linked Lists

Implementation - Contains

Searching a value:

- ▶ First element is head without an assigned value
- ▶ Iterate the entries of the list until value found in $O(n)$

Linked Lists

Implementation - Contains

Searching a value:

- ▶ First element is head without an assigned value
- ▶ Iterate the entries of the list until value found in $O(n)$

```
def contains(self, value):  
    cur = head  
  
    for i in range(0, itemCount):  
        cur = cur.nextNode  
        if cur.value == value:  
            return True  
  
    return False
```

Linked Lists

Runtime

Runtime:

Linked Lists

Runtime

Runtime:

- ▶ Singly linked list:

Linked Lists

Runtime

Runtime:

- ▶ Singly linked list:
 - ▶ `next` in $O(1)$

Linked Lists

Runtime

Runtime:

- ▶ Singly linked list:
 - ▶ `next` in $O(1)$
 - ▶ `previous` in $\Theta(n)$

Linked Lists

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 - ▶ `insert` in $O(1)$

Linked Lists

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Linked Lists

Runtime

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- ▶ Singly linked list:
 - ▶ `next` in $O(1)$
 - ▶ `previous` in $\Theta(n)$
 - ▶ `insert` in $O(1)$
 - ▶ `remove` in $\Theta(n)$
 - ▶ `lookup` in $\Theta(n)$

Linked Lists

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Runtime:

- ▶ Singly linked list:
 - ▶ `next` in $O(1)$
 - ▶ `previous` in $\Theta(n)$
 - ▶ `insert` in $O(1)$
 - ▶ `remove` in $\Theta(n)$
 - ▶ `lookup` in $\Theta(n)$
- ▶ Better with `doubly linked lists`

Linked Lists

Doubly Linked List

Doubly linked list:

Linked Lists

Doubly Linked List

Doubly linked list:

- ▶ Each node has a reference to its successor and its predecessor

Linked Lists

Doubly Linked List

Doubly linked list:

- ▶ Each node has a reference to its successor and its predecessor
- ▶ We can iterate the list forward and backward

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Linked Lists

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Doubly linked list:

- ▶ It is helpful to have a **head** node

Linked Lists

Doubly Linked List

Doubly linked list:

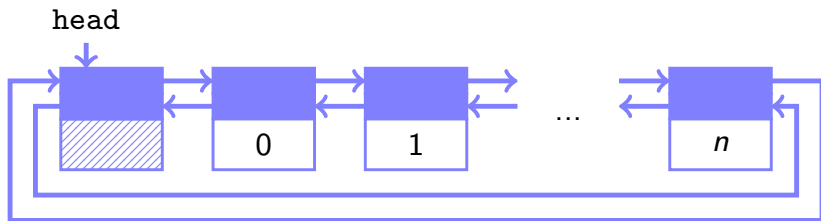
- ▶ It is helpful to have a **head** node
- ▶ We only need **one head** node if we connect the list cyclic

Linked Lists

Doubly Linked List

Doubly linked list:

- ▶ It is helpful to have a **head** node
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Linked Lists

Runtime

Runtime of doubly linked list:

Linked Lists

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Each element has a pointer to pred-/sucessor

Linked Lists

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A constant number of pointers needs to be modified

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Even if the elements are sorted we can only retrieve them in $\Theta(n)$

Linked Lists

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Even if the elements are sorted we can only retrieve them in $\Theta(n)$ Why?

Linked Lists

List in real program

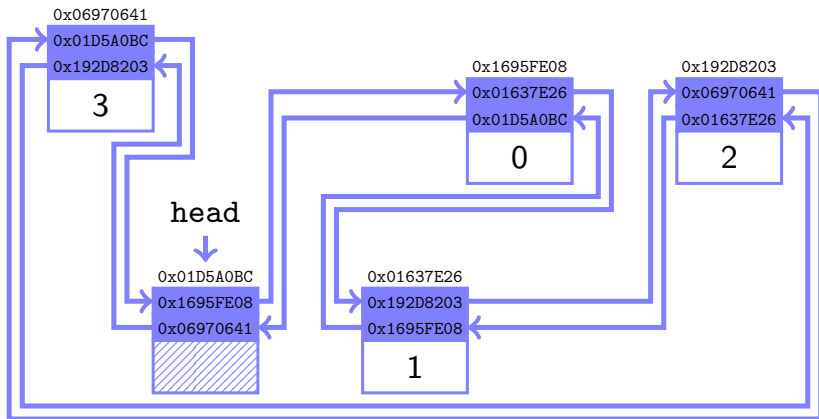
Linked list in book:



Linked Lists

List in real program

Linked list in memory:



Structure

Sorted Sequences

Linked Lists

Binary Search Trees

Binary Search Trees

Introduction

Runtime of a search tree:

Binary Search Trees

Introduction

Runtime of a search tree:

- ▶ `next` and `previous` in $O(1)$

Binary Search Trees

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Pointers corresponding to linked list

Binary Search Trees

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- ▶ `insert` and `remove` in $O(\log n)$

Binary Search Trees

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Binary Search Trees

Introduction

Runtime of a search tree:

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Pointers corresponding to linked list

- ▶ `insert` and `remove` in $O(\log n)$

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The structure helps searching efficiently

Binary Search Trees

Introduction

Idea:

Binary Search Trees

Introduction

Idea:

- ▶ We define a total order for the search tree

Binary Search Trees

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- ▶ All nodes of the left subtree have **smaller keys** than the current node

Binary Search Trees

Introduction

Idea:

- ▶ We define a total order for the search tree
- ▶ All nodes of the left subtree have **smaller keys** than the current node
- ▶ All nodes of the right subtree have **bigger keys** than the current node

Binary Search Trees

Introduction

- ▶ Edge direction indicates ordering

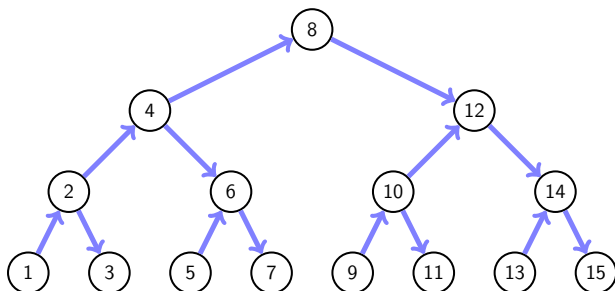


Figure: A binary search tree

Binary Search Trees

Introduction

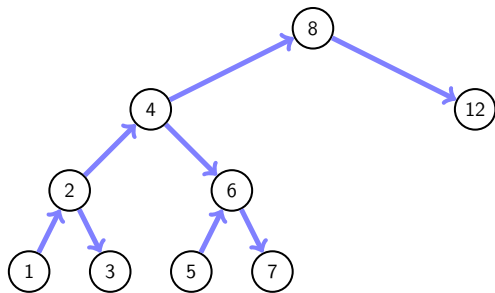


Figure: Another binary search tree

Binary Search Trees

Introduction

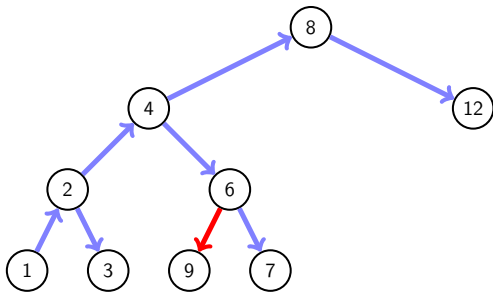


Figure: **Not** a binary search tree

Binary Search Trees

Implementation

Implementation:

Binary Search Trees

Implementation

Implementation:

- ▶ For the heap we had all elements stored in an array
- ▶ Here we link all nodes through pointer / references, like linked lists

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Binary Search Trees

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- ▶ Here we link all nodes through pointer / references, like linked lists
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- ▶ `None` for missing children

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Binary Search Trees

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- ▶ This enables an efficient implementation of (`next` / `previous`)

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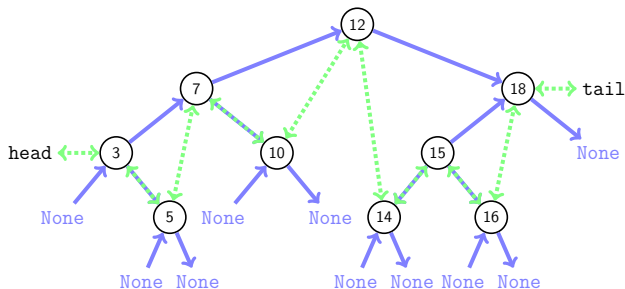


Figure: Binary search tree with links

Binary Search Trees

Implementation - Lookup

Lookup:

Binary Search Trees

Implementation - Lookup

Lookup:

- ▶ Definition:
“ Search the element with the given key. If no element is found return the element with the next (bigger) key. ”

Binary Search Trees

Implementation - Lookup

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Binary Search Trees

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 - ▶ Compare the searched key with the key of the node

Binary Search Trees

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Binary Search Trees

Implementation - Lookup

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 - ▶ Compare the searched key with the key of the node
 - ▶ Go to the left / right until the child is **None** or the key is found
 - ▶ If the key is not found return the next bigger one

Binary Search Trees

Implementation - Lookup

For each node applies the total order:

Binary Search Trees

Implementation - Lookup

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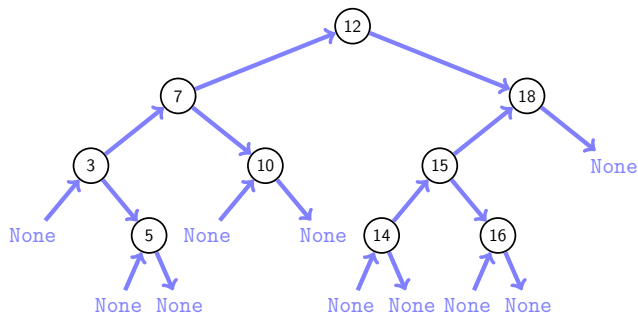
keys of left subtree | `node.key` | keys of right subtree

Binary Search Trees

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Examples:

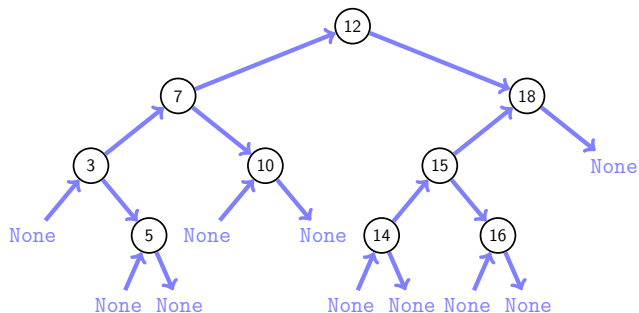
Figure: Binary search tree with total order “i”

Binary Search Trees

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Examples:

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Binary Search Trees

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Examples:

`lookup(14)`

`lookup(6)`

Figure: Binary search tree with total order “`i`”

Binary Search Trees

Implementation - Lookup

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`lookup(6)`

`lookup(19)`

Figure: Binary search tree with total order “i”

Binary Search Trees

Implementation - Insert

Insert:

Binary Search Trees

Implementation - Insert

Insert:

- ▶ We search for the key in our search tree

Binary Search Trees

Implementation - Insert

Insert:

- ▶ We search for the key in our search tree
- ▶ If a node is found we replace the value with the new one

Binary Search Trees

Implementation - Insert

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- ▶ If a node is found we replace the value with the new one
- ▶ Else we insert a new node

Binary Search Trees

Implementation - Insert

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Binary Search Trees

Implementation - Remove

Remove: Case 1: The node “5” has no children

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node “5” has no children

- ▶ Find **parent** of node “5” (“6”)

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node “5” has no children

- ▶ Find **parent** of node “5” (“6”)
- ▶ Set left / right child of node “6” to **None** depending on position of node “5”

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node "5" has no children

- ▶ Find **parent** of node "5" ("6")
- ▶ Set left / right child of node "6" to **None** depending on position of node "5"

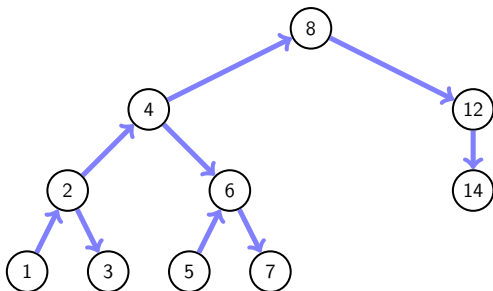


Figure: Binary search tree with total order "i"

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node "5" has no children

- ▶ Find **parent** of node "5" ("6")
- ▶ Set left / right child of node "6" to **None** depending on position of node "5"



Figure: Binary search tree after deleting node "5"

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node “12” has one child

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node “12” has one child

- Find the **child** of node “12” (“14”)

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node “12” has one child

- ▶ Find the **child** of node “12” (“14”)
- ▶ Find the **parent** of node “12” (“8”)

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node "12" has one child

- ▶ Find the **child** of node "12" ("14")
- ▶ Find the **parent** of node "12" ("8")
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Binary Search Trees

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Binary Search Trees

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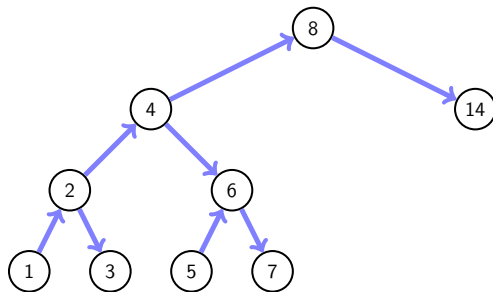


Figure: Binary search tree after deleting node "12"

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

- ▶ Find the **successor** of node “4” (“5”)

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

- ▶ Find the **successor** of node “4” (“5”)
- ▶ Replace the value of node “4” with the value of node “5”

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

- ▶ Find the **successor** of node “4” (“5”)
- ▶ Replace the value of node “4” with the value of node “5”
- ▶ Delete node “5” (the **successor** of node “4”) with remove-case 1 or 2

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

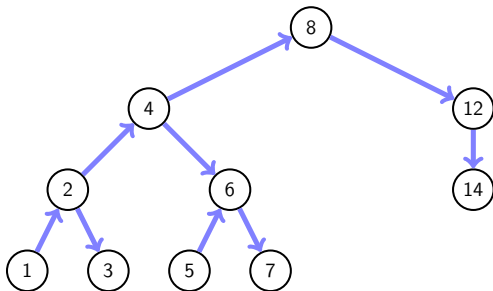
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Binary Search Trees

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Binary Search Trees

Runtime Complexity

How long takes `insert` and `lookup`?

Binary Search Trees

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- ▶ Up to $\Theta(d)$, with d being the `depth of the tree`
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Binary Search Trees

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- ▶ **Best case** with $d = \log n$ the runtime is $\Theta(\log n)$
- ▶ **Worst case** with $d = n$ the runtime is $\Theta(n)$
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Binary Search Trees

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Figure: Degenerated binary tree $d = n$

Binary Search Trees

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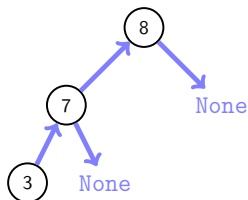


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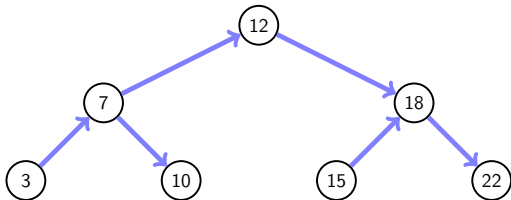


Figure: Complete binary tree $d = \log n$

► **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

► **Linked List**

[Wik] [Linked list](https://en.wikipedia.org/wiki/Linked_list)

`https://en.wikipedia.org/wiki/Linked_list`

► **Binary Search Tree**

[Wik] [Binary search tree](https://en.wikipedia.org/wiki/Binary_search_tree)

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`//en.wikipedia.org/wiki/Binary_search_tree`