# Algorithmns and Datastructures

## O-Notation, L'Hopital

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Algorithmns and Datastructures, November 2017

# Structure

# $\mathcal{O}$-Notation

## Motivation

**We are interested in:**

- Example: `sorting`
  - Runtime of `Minsort` "is growing as" $n^2$
  - Runtime of `HeapSort` "is growing as" $n \log n$
- Growth of a function in runtime $T(n)$
  - The role of constants (e.g. $1ns$) is minor
  - it is enough if relation holds for some $n \geq \ldots$
- Describe the growth of the function **more formally**
  - By the means of Landau-Symbols [Wik]):
    - $\mathcal{O}(n)$ (Big O of $n$),
    - $\Omega(n)$ (Omega of $n$),
    - $\Theta(n)$ (Theta of $n$)

# $\mathcal{O}$-Notation

## Definition

**Big $\mathcal{O}$-Notation:**

- Consider the function: $f : \mathbb{N} \to \mathbb{R},\ n \mapsto f(n)$
  - $\mathbb{N}$: Natural numbers $\to$ input size
  - $\mathbb{R}$: Real numbers $\to$ runtime
- **Example:**
  - $f(n) = 3\,n$
  - $f(n) = 2\,n \log n$
  - $f(n) = \frac{1}{10} n^2$
  - $f(n) = n^2 + 3\,n \log n - 4\,n$

# $\mathcal{O}$-Notation

## Definition

**Big $\mathcal{O}$-Notation:**

- Given two functions $f$ and $g$:
  $f, g : \mathbb{N} \to \mathbb{R}$
- **Intuitive:** $f$ is Big-O of $g$ ($f$ is $\mathcal{O}(g)$)
  - ... if $f$ relative to $g$ does not grow faster than $g$
  - the growth rate matters, not the absolute values

# $\mathcal{O}$-Notation

## Definition

**Big $\mathcal{O}$-Notation:**

- **Informal:** $f = \mathcal{O}(g)$
  - "$=$" corresponds to *is* not *isequal*
  - ... if for some value $n_0$ for all $n \geq n_0$
  - $f(n) \leq C \cdot g(n)$ for a constant C
  - ($f = \mathcal{O}(g)$: From a value $n_0$ for all $n \geq n_0 \rightarrow f(n) \leq C \cdot g(n)$)
- **Formal:** $f \in \mathcal{O}(g)$

**Formal:** $f \in \mathcal{O}(g)$

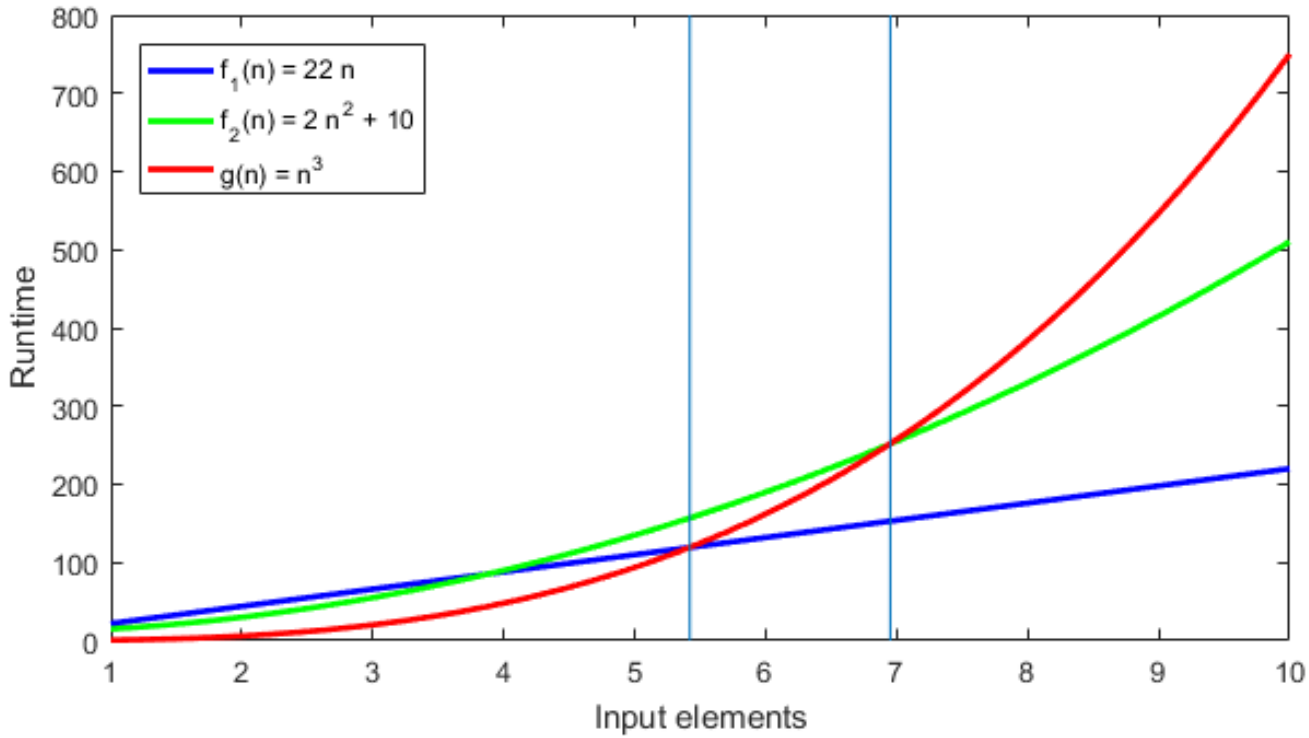
$$\mathcal{O}(g) = \{ \text{f} : \mathbb{N} \rightarrow \mathbb{R} \ | \ \exists n_0 \in \mathbb{N}, \ \exists C > 0, \ \forall n > n_0 \colon f(n) \leq C \cdot g(n) \}$$

"set of all functions" — "for which" — "it exists" — "for all" — "such that"

# $\mathcal{O}$-Notation

## Examples

**Illustration of the Big O-Notation:**




$g(n)$

$f_2 \in \mathcal{O}(g)$

$f_1 \in \mathcal{O}(g)$

Figure: Runtime of two algorithms $f_1, f_2$

# $\mathcal{O}$-Notation

## Examples

**Example:**

- $f(n) = 5\,n + 7,\ g(n) = n$
  $\Rightarrow 5\,n + 7 \in \mathcal{O}(g)$
  $\Rightarrow f \in \mathcal{O}(g)$
- **Intuitive:**
  $f(n) = 5\,n + 7 \rightarrow$ linear growth

Attention

$f(n) \leq g(n)$ is not guaranteed, better is $f(n) \leq C \cdot g(n)\ \ \forall n > n_0$.

# $\mathcal{O}$-Notation

## Proof

We have to proof: $\exists n_0, \exists C, \forall n \geq n_0 \colon\ 5\,n + 7 \leq C \cdot n$ .

$$\begin{aligned} 5\,n + 7 &\leq 5\,n + n \quad (\text{for } n \geq 7) \\ &= 6\,n \end{aligned}$$

$\Rightarrow n_0 = 7,\ C = 6$ $\square$

# $\mathcal{O}$-Notation

## Proof

Alternate proof:

$$\begin{aligned} 5\,n + 7 &\leq 5\,n + 7\,n \quad (\text{for } n \geq 1) \\ &= 12\,n \end{aligned}$$

$\Rightarrow n_0 = 1,\ C = 12$ □

# $\mathcal{O}$-Notation

## Examples

**Big O-Notation:**

- We are only interested in the term with the highest-order, the fastest growing summand, the others will be ignored
- $f(n)$ is limited from above by $C \cdot g(n)$

**Examples:**

$$2\,n^2 + 7\,n - 20 \in \mathcal{O}(n^2)$$

$$2\,n^2 + 7\,n \log n - 20 \in$$

$$7\,n \log n - 20 \in$$

$$5 \in$$

$$2\,n^2 + 7\,n \log n + n^3 \in$$

**Harder Example:**

- Polynomes are simple
- More problematic: combination of complex functions

$$2\sqrt{x} + 3 \ln x \in \mathcal{O}(??)$$

# Ω-Notation

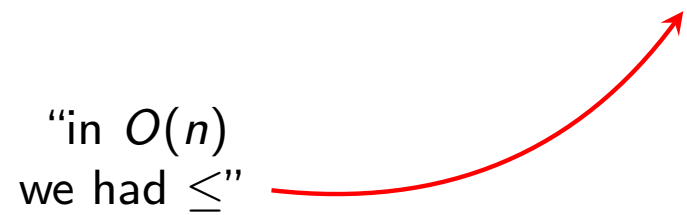
## Definition

**Omega-Notation:**

- **Intuitive**:
  - $f \in \Omega(g)$, $f$ is growing at least as fast as $g$
  - So the same as Big-O but with *at-least* and not *at-most*

**Formal:** $f \in \Omega(g)$

$$\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} \mid \exists n_0 \in \mathbb{N}, \exists C > 0, \forall n > n_0 : \ f(n) \geq C \cdot g(n)\}$$



# Ω-Notation

## Proof

**Example:**

Proof of $f(n) = 5n + 7 \in \Omega(n)$:

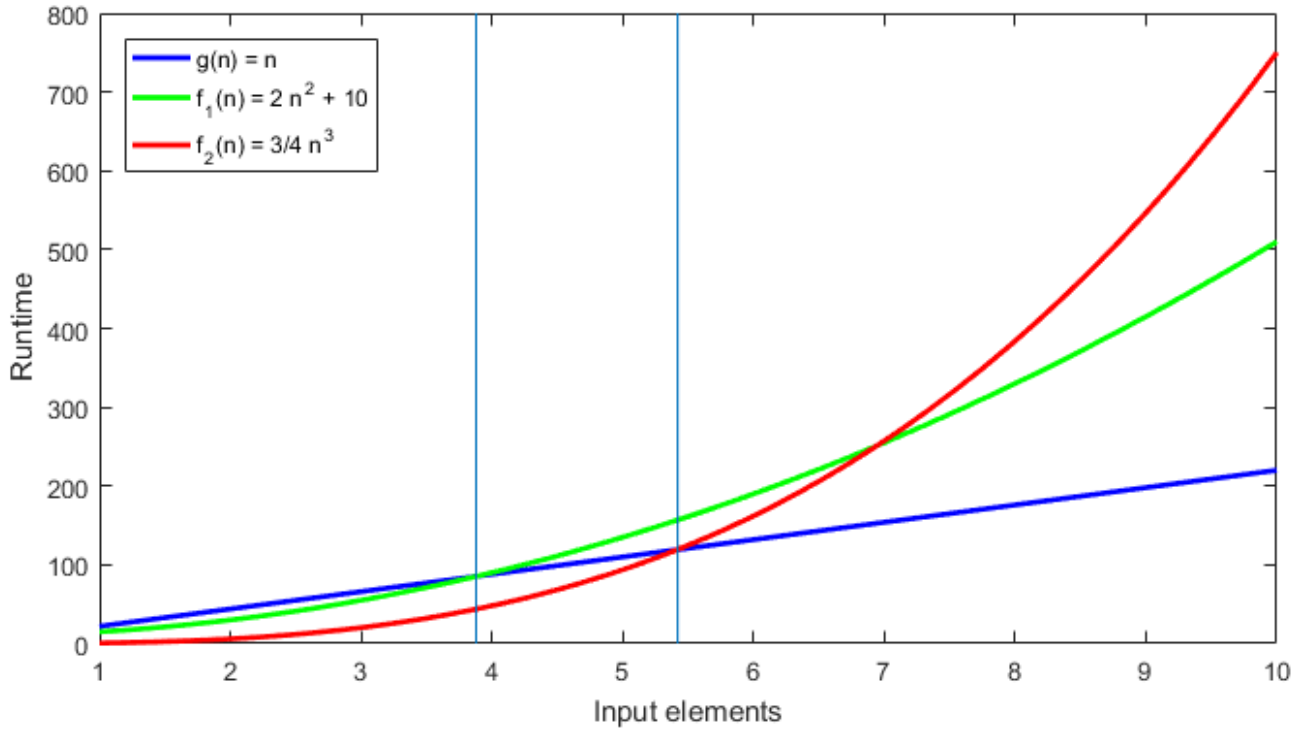
$$\underbrace{5\,n + 7}_{f(n)} \quad \geq \quad \underbrace{1 \cdot n}_{g(n)} \quad (\text{for } n \geq 1)$$

$\Rightarrow n_0 = 1,\ C = 1$ □

# Ω-Notation

## Examples

**Illustration of the Omega-Notation:**




$f_2 \in \Omega(g)$

$f_1 \in \Omega(g)$

$g(n)$

Figure: Runtime of two algorithms $f_1, f_2$

# Ω-Notation

## Examples

**Big Omega-Notation:**

- We are only interested in the term with the highest-order, the fastest growing summand, the others will be ignored
- $f(n)$ is limited from underneath by $c \cdot g(n)$

**Examples:**

$$2\,n^2 + 7\,n - 20 \in \Omega(n^2)$$

$$2\,n^2 + 7\,n \log n - 20 \in$$

$$7\,n \log n - 20 \in$$

$$5 \in$$

$$2\,n^2 + 7\,n \log n + n^3 \in$$

# Θ-Notation

## Definition

**Theta-Notation:**

- **Intuitive**: $f$ is Theta of $g$ ...
  - ... if $f$ is growing as much as $g$
  - $f \in \Theta(g)$, $f$ is growing at the same speed as $g$

**Formal:** $f \in \Theta(g)$

$$\Theta(g) = \underbrace{\mathcal{O}(g) \cap \Omega(g)}_{\textit{Intersection}}$$

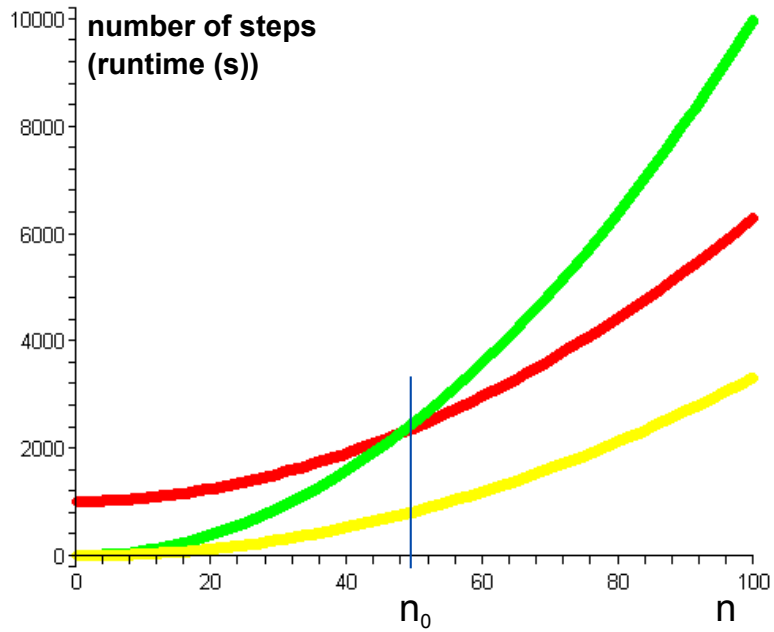
**Example:**

$$f(n) = 5n + 7,\ f(n) \in \mathcal{O}(n),\ f(n) \in \Omega(n)$$
$$\Rightarrow f(n) \in \Theta(n)$$

*Proof for $\mathcal{O}(g)$ and $\Omega(g)$ look at slides 11 and 17*

# Θ-Notation

## Graphs




$c_2 \cdot g(n)$ upper bound

$T(n)$ runtime of algorithm

$c_1 \cdot g(n)$ lower bound

- $f$ and $g$ have the same "growth"

# Runtime

## Landau-Symbol Summary

**Big O-Notation $\mathcal{O}(n)$:**

- $f$ is growing at most as fast as $g$
- $C \cdot g(n)$ is the upper bound

**Big Omega-Notation $\Omega(n)$:**

- $f$ is growing at least as fast as $g$
- $C \cdot g(n)$ is the lower bound

**Big Theta-Notation $\Theta(n)$:**

- $f$ is growing at the same speed as $g$
  - $C_1 \cdot g(n)$ is the lower bound
  - $C_2 \cdot g(n)$ is the upper bound

# Runtime

## Common Runtimes

Table: Common runtime types

| Runtime | Growth |
|---|---|
| $f \in \Theta(1)$ | constant time |
| $f \in \Theta(\log n) = \Theta(\log_k n)$ | logarithmic time |
| $f \in \Theta(n)$ | linear time |
| $f \in \Theta(n \log n)$ | n-log-n time (nearly linear) |
| $f \in \Theta(n^2)$ | squared time |
| $f \in \Theta(n^3)$ | cubic time |
| $f \in \Theta(n^k)$ | polynomial time |
| $f \in \Theta(k^n)$, $f \in \Theta(2^n)$ | exponential time |

- So far discussed:
  - Membership in $O(\ldots)$ proofed by hand:
    Explicit calculation of $n_0$ and $C$
  - **However:** Both hint at limits in calculus

**Definition of "Limit"**

- The limit $L$ exists for an infinite sequence $f_1, f_2, f_3, \ldots$ if for all $\epsilon > 0$ one $n_0 \in \mathbb{N}$ exists, such that for all $n \geq n_0$ the following holds true: $|f_n - L| \leq \epsilon$
- A function $f : \mathbb{N} \to \mathbb{R}$ can be written as a sequence $\Rightarrow \lim_{n \to \infty} f_n = L$

The limit is converging:

$$\forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} \;\; \forall n \geq n_0 : \;\; |f_n - L| \leq \epsilon$$

# $\mathcal{O}$-Notation

## Limit / Convergence

- Example for the proof of a limit
- Function $f(n) = 2 + \frac{1}{n}$ with limes $\lim_{n\to\infty} f(n) = 2$
- "Engineering" solution: use $n = \infty$

$$\frac{1}{\infty} = 0 \Rightarrow \lim_{n\to\infty} f(n) = \lim_{n\to\infty} 2 + \frac{1}{n} = 2$$

# $\mathcal{O}$-Notation

## Limit / Convergence

- Now a more formal proof for $\lim_{n\to\infty} 2 + \frac{1}{n} = 2$
- We need to show: for all given $\epsilon$ there is an $n_0$ such that for all $n \geq n_0$

$$\left|2 + \frac{1}{n} - 2\right| = \left|\frac{1}{n}\right| \leq \epsilon$$

- E.g.: for $\epsilon = 0.01$ we get $\frac{1}{n} \leq \epsilon$ for $n \geq 100$
- In general

$$n_0 = \left\lceil \frac{1}{\epsilon} \right\rceil$$

- Then we get:

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{n_0} = \frac{1}{\left\lceil \frac{1}{\epsilon} \right\rceil} \leq \frac{1}{\frac{1}{\epsilon}} = \epsilon \quad \square$$

# $\mathcal{O}$-Notation

## Limit / Convergence

Let $f, g\colon\ \mathbb{N} \to \mathbb{R}$ with an existing limit

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$$

Hence the following holds:

$$f \in \mathcal{O}(g) \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \tag{1}$$

$$f \in \Omega(g) \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \tag{2}$$

$$f \in \Theta(g) \quad \Leftrightarrow \quad 0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \tag{3}$$

# $\mathcal{O}$-Notation

## Limit / Convergence

$$f \in \mathcal{O}(g) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

Forward proof ($\Rightarrow$):

$$f \in \mathcal{O}(g) \overset{\text{def. of } \mathcal{O}(n)}{\Rightarrow} \exists n_0,\ C\ \forall n \geq n_0 :\ f(n) \leq C \cdot g(n)$$

$$\Rightarrow \exists n_0,\ C\ \forall n \geq n_0 : \frac{f(n)}{g(n)} \leq\ C$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} \leq\ C \quad \square$$

# $\mathcal{O}$-Notation

## Limit / Convergence

### Backward proof ($\Leftarrow$):

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = C \qquad \text{For some } C \in \mathbb{R} \text{ (Limit)}$$

$$\overset{\text{def. limes}}{\Rightarrow} \quad \exists n_0, \forall n \geq n_0 : \quad \frac{f(n)}{g(n)} \leq C + \varepsilon \quad (e.g.\ \varepsilon = 1)$$

$$\Rightarrow \quad \exists n_0, \forall n \geq n_0 : \quad f(n) \leq \underbrace{(C+1)}_{O-\textit{notation constant}} \cdot g(n)$$

$$\Rightarrow \quad f \in \mathcal{O}(g) \quad \square$$

# Limits with L'Hôpital

- **Intuitive:**

$$\lim_{n\to\infty} 2 + \frac{1}{n} = 2 + \frac{1}{\infty} = 2$$

- **With L'Hôpital:**
  - Let $f, g : \mathbb{N} \to \mathbb{R}$
  - If $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty/0$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

- **Holy inspiration**

you need a doctoral degree for that

# Limits with L'Hôpital

**The limit can not be determined in the way of an Engineer:**

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \frac{\lim_{n\to\infty} \ln(n)}{\lim_{n\to\infty} n} \quad \xrightarrow{\text{plugging in}} \quad \frac{\infty}{\infty}$$

**Determine the limit with using L'Hôpital:**

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

# Limits with L'Hôpital

## Using L'Hôpital:

Numerator: $\mathbf{f(n)}: n \mapsto \ln(n)$

Denominator: $\mathbf{g(n)}: n \mapsto n$

$$\Rightarrow f'(n) = \frac{1}{n} \quad \text{(derivation from Numerator)}$$

$$\Rightarrow g'(n) = 1 \quad \text{(derivation from Denominator)}$$

$$\lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0$$

# Limits with L'Hôpital

**What can we take for granted without proofing?**

- Only things that are trivial
- It is always better to proof it

**Examples:**

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{is trivial}$$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \qquad \text{is trivial}$$

$$\lim_{n\to\infty} \frac{\log(n)}{n} = 0 \qquad \text{use L'Hopital}$$

**Practical use:**

- It is much easier to determine the runtime of an algorithm by using the $\mathcal{O}$-Notation
  1. Computing rules
  2. Practical use

# $\mathcal{O}$-Notation

## Characteristics

- Transitivity:

$$\begin{aligned} f \in \Theta(g) \ \wedge \ g \in \Theta(h) \quad &\rightarrow \quad f \in \Theta(h) \\ f \in \mathcal{O}(g) \ \wedge \ g \in \mathcal{O}(h) \quad &\rightarrow \quad f \in \mathcal{O}(h) \\ f \in \Omega(g) \ \wedge \ g \in \Omega(h) \quad &\rightarrow \quad f \in \Omega(h) \end{aligned}$$

- Symmetry:

$$\begin{aligned} f \in \Theta(g) \quad &\leftrightarrow \quad g \in \Theta(f) \\ f \in \mathcal{O}(g) \quad &\leftrightarrow \quad g \in \Omega(f) \end{aligned}$$

- Reflexivity:

$$f \in \Theta(f) \quad f \in \Omega(f) \quad f \in \mathcal{O}(f)$$

# $\mathcal{O}$-Notation

## Calculation Rules

- Trivial:

$$\begin{aligned} f &\in \mathcal{O}(f) \\ k \cdot \mathcal{O}(f) &= \mathcal{O}(f) \\ \mathcal{O}(f + k) &= \mathcal{O}(f) \end{aligned}$$

- Addition:

$$\mathcal{O}(f) + \mathcal{O}(g) = \mathcal{O}(\max\{f,\ g\})$$

- Multiplication:

$$\mathcal{O}(f) \cdot \mathcal{O}(g) = \mathcal{O}(f \cdot g)$$

# $\mathcal{O}$-Notation

## Runtime Complexity

- The input size for all examples is $n$
- Basic operations

| | |
|---|---|
| i1 $= 0$ | $\mathcal{O}(1)$ |

- Sequences of basic operations

| | | |
|---|---|---|
| i1 $= 0$ | $\mathcal{O}(1)$ | $327 \cdot \mathcal{O}(1) = \mathcal{O}(1)$ |
| i2 $= 0$ | $\mathcal{O}(1)$ | |
| $\ldots$ | $\ldots$ | |
| i327 $= 0$ | $\mathcal{O}(1)$ | |

# $\mathcal{O}$-Notation

## Runtime Complexity

- Loops

```
for i in range(0, n):
    a[i] = 0
```

$$\left.\begin{array}{l}\mathcal{O}(n)\\ \hline \mathcal{O}(1)\end{array}\right\} \quad \mathcal{O}(1) \cdot \mathcal{O}(n) = \mathcal{O}(n)$$

```
for i in range(0, n):
    a1[i] = 0
    ...
    a137[i] = 0
```

$$\left.\begin{array}{l}\mathcal{O}(n)\\ \hline \left.\begin{array}{l}\mathcal{O}(1)\\ \cdots\\ \mathcal{O}(1)\end{array}\right\} \; 137 \cdot \mathcal{O}(1) = \mathcal{O}(1)\end{array}\right\} \quad \mathcal{O}(1) \cdot \mathcal{O}(n) = \mathcal{O}(n)$$

# $\mathcal{O}$-Notation

## Runtime Complexity

- Loops

```
for i in range(0, n):
    for j in range(0, n-1):
        a1[i][j] = 0
        ...
        a137[i][j] = 0
```

$$\left.\begin{array}{l} \left.\begin{array}{l} \mathcal{O}(n) \\ \hline \mathcal{O}(n-1) \end{array}\right\} \mathcal{O}(n) \cdot \mathcal{O}(n) = \mathcal{O}(n^2) \\ \left.\begin{array}{l} \mathcal{O}(1) \\ \dots \\ \mathcal{O}(1) \end{array}\right\} 137 \cdot \mathcal{O}(1) = \mathcal{O}(1) \end{array}\right\} \mathcal{O}(1) \cdot \mathcal{O}(n^2) = \mathcal{O}(n^2)$$

# $\mathcal{O}$-Notation

## Runtime Complexity

- Conditions

```
if x < 100:
    y = x
else:
    for i in range(0, n):
        if a[i] > y:
            y = a[i]
```

$$\left.\begin{array}{l} \dfrac{\mathcal{O}(1)}{\mathcal{O}(1)} \\ \mathcal{O}(1) \end{array}\right\} \mathcal{O}(1)$$

$$\left.\begin{array}{l} \dfrac{\mathcal{O}(n)}{\dfrac{\mathcal{O}(1)}{\mathcal{O}(1)}} \\ \mathcal{O}(n) \cdot \mathcal{O}(1) = \mathcal{O}(n) \end{array}\right\}$$

$$\left.\begin{array}{l} \mathcal{O}(1) \\ \mathcal{O}(n) \end{array}\right\} \mathcal{O}(\max\{1, n\}) = \mathcal{O}(n)$$

# $\mathcal{O}$-Notation

## Arithmetic mean

- Input: List $x$ with $n$ numbers
- Output: $a[i]$ is the arithmetic mean of $x[0]$ to $x[i]$

```
def arithMean(x):
    a = [0] * len(x)
    for i in range(0, len(x)):
        s = 0
        for j in range(0, i+1):
            s = s + x[j]

        a[i] = s / (i+1)

    return a
```

# $\mathcal{O}$-Notation Runtime complexity

```
for i in range(0, len(x)):
    s = 0
    for j in range(0, i+1):
        s = s + x[j]
    a[i] = s / (i+1)
```

$$\left.\begin{array}{l} \left.\begin{array}{l} \mathcal{O}(n) \\ \hline \mathcal{O}(1) \end{array}\right\} \mathcal{O}(n) \\ \hline \left.\begin{array}{l} \mathcal{O}(i+1) \\ \hline \mathcal{O}(1) \end{array}\right\} \mathcal{O}(i) \\ \hline \mathcal{O}(1) \end{array}\right\} \mathcal{O}(n) \cdot \mathcal{O}(i) = \mathcal{O}(n^2)$$

- How often will the instructions in the loop be executed, when the problem has size $n$?

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2} \in \mathcal{O}(n^2)$$

# $\mathcal{O}$-Notation

## Discussion

**Way of speaking:**

- With the $\mathcal{O}$-Notation we look at the behavior of a function when $n \to \infty$
- We only analyze the runtime when $n \geq n_0$
- We talk about asymptotic analysis, when we discuss cost, runtime, etc. as $\mathcal{O}(\ldots)$, $\Omega(\ldots)$ or $\Theta(\ldots)$

# $\mathcal{O}$-Notation

## Discussion

**Attention:**

- If you are using **asymptotic analysis**, you can not make any predictions about the runtime of smaller input sizes ($n < n_0$)
- For small input sizes (mostly $n < 10$), the runtime is predictably small
- $n_0$ does not necessarily have to be small

# $\mathcal{O}$-Notation

## Discussion

**Examples:**

- Let A and B be algorithms
  - A has the runtime $f(n) = 80\,n$
  - B has the runtime $f(n) = 2\,n \log_2 n$
- So $f = \mathcal{O}(g)$ but **not** $\Theta(g)$
  - $\Rightarrow$ A is asymptotic faster than B
  - $\Rightarrow$ There is a $n_0$ for that $n \geq n_0$: $f(n) \leq g(n)$

## When is A faster then B?

We search the minimal $n_0$:

$$\begin{aligned} f(n_0) &= g(n_0) \\ 80\, n_0 &= 2\, n_0\, \log_2 n_0 \\ 40 &= \log_2 n_0 \\ n_0 &= 2^{40} \\ &= (2^{10})^4 = (1024)^4 \\ &\approx (10^3)^4 = 10^{12} \\ &\approx 1\,\text{trillion} \end{aligned}$$

A ist faster than B if $n_0$ has more than 1 trillion elements

# Runtime Examples

## Continued

- Logarithm of different bases differ only by a constant

$$\log_a n = \frac{\log_b n}{\log_b a} = \frac{1}{\log_b a} \cdot \log_b n$$

- Hence: $\log_a n \in \Theta(\log_b n)$
- For exponent this does not hold

$$3^n \notin \Theta(2^n)$$

- Proof: Use equation (1) from Slide 31

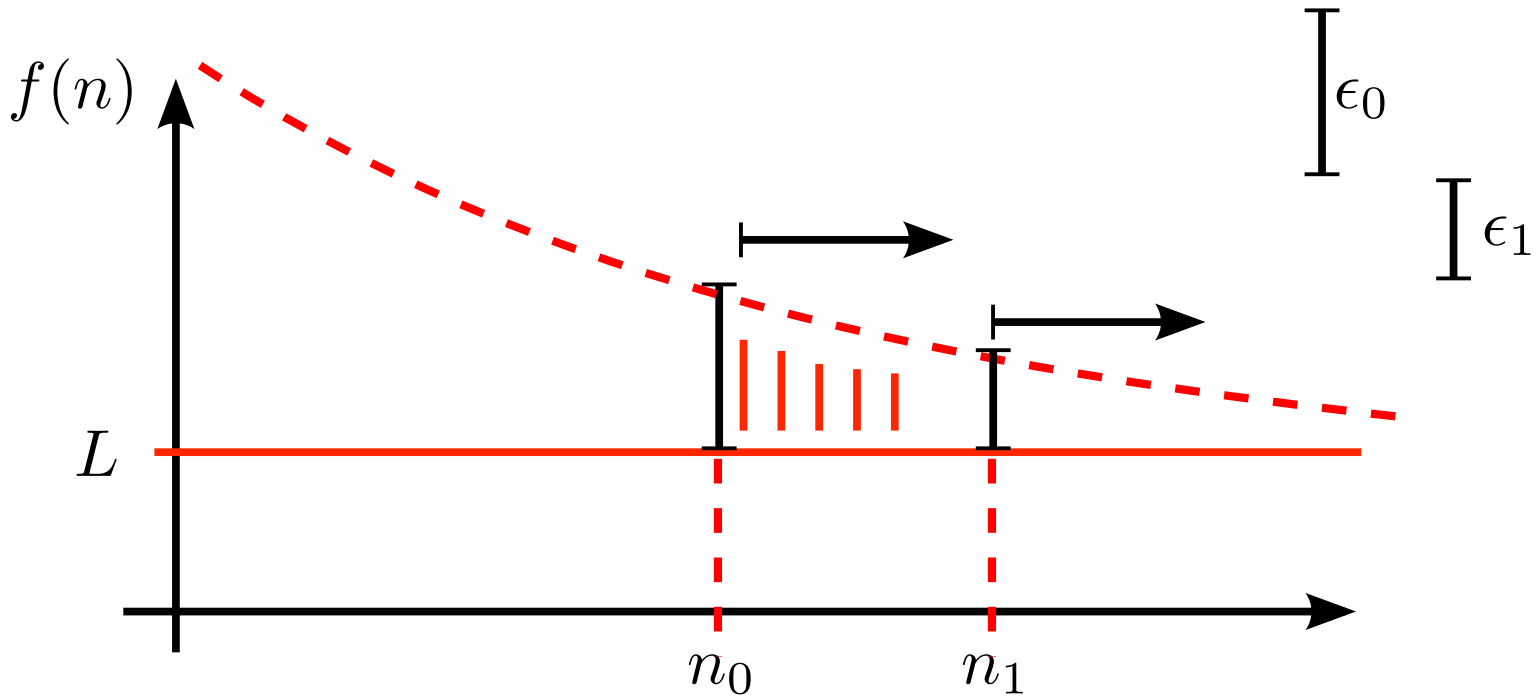
$$3^n \in \mathcal{O}(2^n) \Leftrightarrow \lim_{n\to\infty} \frac{3^n}{2^n} < \infty$$

- However:

$$\lim_{n\to\infty} \frac{3^n}{2^n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$$

# Additional Figure

- Figure for slide 28



# Further Literature

- **General**

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

# Further Literature

- **Big O notation**

  [Wik] Big O notation
  `https://en.wikipedia.org/wiki/Big_O_notation`