

# Algorithms and Datastructures

Cache Efficiency, Divide and Conquer

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# Structure

## Cache Efficiency

- Introduction

- Cache Organization

## Divide and Conquer

- Introduction

# Cache Efficiency

## Introduction

**Background:**

# Cache Efficiency

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- ▶ Assuming this is a good measure for the runtime of a algorithm/tool

# Cache Efficiency

## Introduction

### **Background:**

- ▶ Up to now we always counted the **number of operations**
- ▶ Assuming this is a good measure for the runtime of a algorithm/tool
- ▶ Today we will see examples where this is not suitable

# Cache Efficiency

## Introduction

### Example:

- ▶ We sum up all elements of a field  $a$  of size  $n$  in ...
  - ▶ natural order:

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

- ▶ random order:

$$\text{sum}(a) = a[21] + a[5] + \cdots + a[8]$$

# Cache Efficiency

## Linear Order - Python

### Python:

```
def init(size):  
    """Creates the dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

# Cache Efficiency

## Linear Order - Python

### Python:

```
def run(param):  
    """Processes the dataset."""  
  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```



# Cache Efficiency

## Linear Order

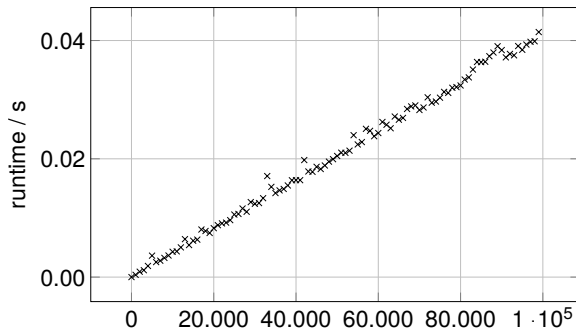


Figure: Summing elements in linear order

# Cache Efficiency

## Random Order - Python

```
def init(size):  
    """Creates a randomly ordered dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

# Cache Efficiency

## Random Order



Figure: Summing elements in random order

# Cache Efficiency

## Algorithm Comparision

**Conclusion:**

# Cache Efficiency

## Algorithm Comparision

### **Conclusion:**

- ▶ The number of operations are identical for both algorithms

# Cache Efficiency

## Algorithm Comparision

### Conclusion:

- ▶ The number of operations are identical for both algorithms
- ▶ Accessing elements in random order takes a lot longer (Factor 10) Why?
- ▶ The costs in terms of memory access are very different

# Structure

## Cache Efficiency

Introduction

Cache Organization

## Divide and Conquer

Introduction

# Cache Efficiency

## CPU Cache





# Cache Efficiency

## CPU Cache



**Principle / organization:**

# Cache Efficiency

## CPU Cache



## Principle / organization:

- ▶ Accessing one byte of the main memory takes  $\approx 100$  ns

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## CPU Cache



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- ▶ Accessing one byte of the main memory takes  $\approx 100$  ns
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## CPU Cache



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- ▶ Accessing one byte of (L1-)cache takes  $\approx 1$  ns
- ▶ Accessing one or more byte/s of main memory loads a whole block  $\approx 100$  B into the cache
- ▶ As long as this block is in the cache, it is not necessary to access the memory for bytes of this block

# Cache Efficiency

## CPU Cache



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**Cache organization:**

# Cache Efficiency

## CPU Cache



## Cache organization:

- The (L1-)cache can hold multiple memory blocks



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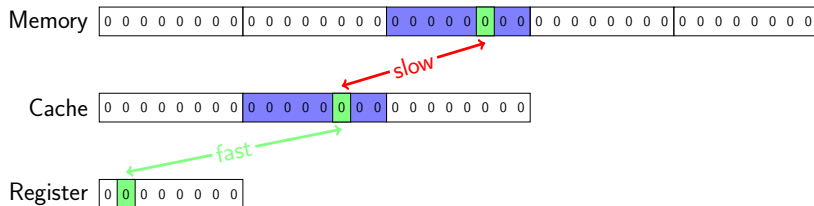


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  - ▶ Least frequently used (LFU)
  - ▶ First in first out (FIFO)
- ▶ Details of discarding are not the topic for today

# Cache Efficiency

## Block Operations



## Terminology:

# Cache Efficiency

## Block Operations



## Terminology:

- The system consists of slow and fast memory



# Cache Efficiency

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- ▶ The **slow memory** is divided in **blocks of size  $B$**

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- ▶ The **fast cache** has size  $M$  and can store  $M/B$  blocks

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## Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size  $B$**
- ▶ The **fast cache** has size  $M$  and can store  $M/B$  blocks
- ▶ If data is not in fast memory, the corresponding block is loaded into the **cache**

# Cache Efficiency

## Block Operations



## Terminology:

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- ▶ The program defines which blocks are held in the **cache**

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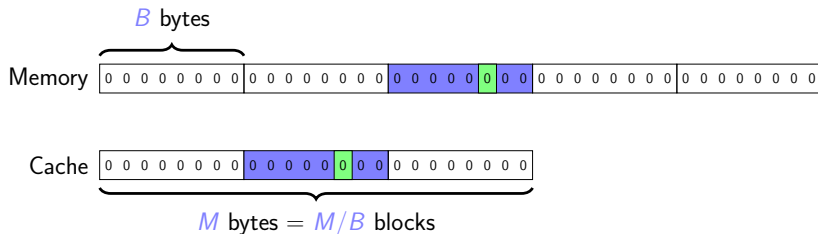


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## Block Operations



### Terminology:

- ▶ The program defines which blocks are held in the **cache**
- ▶ We use the number of **block operations** as runtime estimation
- ▶ We ignore runtime costs of cache accesses / management

## Cache Efficiency



Figure: Comparison good / bad locality

Accessing the cache  $B$  times:

- ▶ **Best case:** 1 block operation  $\rightarrow$  good locality
- ▶ **Worst case:**  $B$  block operations  $\rightarrow$  bad locality



# Cache Efficiency

## Block Operations

**Additional factors:**

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  - ▶ Partitioning of the slow memory into blocks
  - ▶ Regardless of the block size: 1 Bytes or 4 Bytes or 8 Bytes

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### **Note:**

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  - ▶ Partitioning of the slow memory into blocks
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### Note:

- ▶ If the input size is smaller than  $M$  we load the complete data chunk directly into the cache

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  - ▶ Partitioning of the slow memory into blocks
  - ▶ Regardless of the block size: 1 Bytes or 4 Bytes or 8 Bytes

### Note:

- ▶ If the input size is smaller than  $M$  we load the complete data chunk directly into the cache
- ▶ Cache handling is only interesting when the input size is greater than  $M$

# Cache Efficiency

## Block Operations

**Typical values:** (Intel® i7-4770 Haswell, WD® Blue 2 TB)



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- ▶ CPU L1 Cache:  $B = 64 \text{ B}$ ,  $M = 4 \times (32 \text{ kB} + 32 \text{ kB})$

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- ▶ CPU L3 Cache:  $B = 64 \text{ B}$ ,  $M = 8 \text{ MB}$
- ▶ Disk Cache:  $B = 64 \text{ kB}$ ,  $M = 64 \text{ MB}$ 
  - ▶ Many operating systems use free system memory as disk cache

# Cache Efficiency

## Block Operations

### **Terminology:**

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- ▶ Block loads on CPU-cache are called **cache misses**

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(input / output operations)

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## Block Operations

### Terminology:

- ▶ Block loads on CPU-cache are called **cache misses**
- ▶ Block operations on disk-cache are called **IOs**  
(input / output operations)
- ▶ These also fall under the term **cache efficiency** or **IO efficiency**



# Cache Efficiency

Block Operations - Linear Order

**Example 1 - Linear order:**

# Cache Efficiency

## Block Operations - Linear Order

### **Example 1 - Linear order:**

- ▶ We sum up all elements in **natural order**

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

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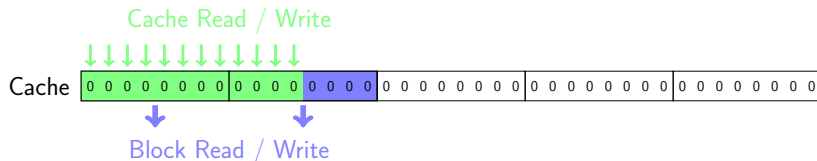


Figure: Good locality of sum operation

# Cache Efficiency

Block Operations - Random Order

**Example 2 - Random order:**

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Figure: Bad locality of sum operation

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## Block Operations

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## Block Operations

**Generally the factor is substantially  $< B$**

- ▶ Even with a **random order** we access 4 neighboring bytes at once per `int` (`int32_t`)
- ▶ The next element might already be loaded in the cache
- ▶ If **not**  $n \gg M$  this might occur with a high probability

# Cache Efficiency

Block Operations - QuickSort

**QuickSort:**

# Cache Efficiency

## Block Operations - QuickSort

### **QuickSort:**

- ▶ **Strategy:** Divide and conquer

# Cache Efficiency

## Block Operations - QuickSort

### QuickSort:

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- ▶ Divide the data into two parts where the “left” part contains all values  $\leq$  those in the right part



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Figure: QuickSort with pivot-element

# Idea of Quicksort

- ▶ **At start:** Pivot in first position, first re-arrange list such that left part contains small, right part larger elements
- ▶ Do required changes *in place*



- ▶ **End point:**  $k$  is left to left-most element greater than pivot  
*swap position 0 (pivot) with  $k$  (smaller than pivot)*

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# Cache Efficiency

## Block Operations - QuickSort - Python

### Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[0]  
  
    ...
```

# Cache Efficiency

## Block Operations - QuickSort - Python

```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```

# Cache Efficiency

## Block Operations - QuickSort

**Number of operations for Quicksort:**

# Cache Efficiency

## Block Operations - QuickSort

### Number of operations for Quicksort:

- ▶ Let  $T(n)$  be the runtime for the input size  $n$

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## Block Operations - QuickSort

### Number of operations for Quicksort:

- ▶ Let  $T(n)$  be the runtime for the input size  $n$

### Assumptions:

- ▶ Fields are always separated perfectly in the middle
- ▶  $n$  is a power of two and recursion depth is  $k = \log_2 n$

# Cache Efficiency

## Block Operations - QuickSort

$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left( A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$

# Cache Efficiency

## Block Operations - QuickSort



Figure: Locality of quicksort

# Cache Efficiency

## Block Operations - QuickSort

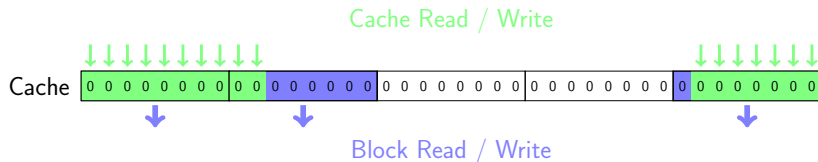


Figure: Locality of quicksort

- Let  $IO(n)$  be the number of **block operations** for input size  $n$

# Cache Efficiency

## Block Operations - QuickSort



Figure: Locality of quicksort

- ▶ Let  $IO(n)$  be the number of **block operations** for input size  $n$
- ▶ Assumptions as before but recursion depth is  $k = \log_2 \frac{n}{B}$   
Why?

# Cache Efficiency

## Block Operations - QuickSort

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in \mathcal{O}\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

# Structure

## Cache Efficiency

Introduction

Cache Organization

## Divide and Conquer

Introduction



# Divide and Conquer

## Introduction

**Concept:**

# Divide and Conquer

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### **Concept:**

- ▶ **Divide** the problem into smaller subproblems

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- ▶ **Conquer** the subproblems through recursive solving.  
If subproblems are small enough solve them directly

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### Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.  
If subproblems are small enough solve them directly
- ▶ **Connect** all solutions of the subproblems to a solution of the full problem
- ▶ **Recursive** application of the algorithm to ever smaller subproblems
- ▶ **Direct** solving of sufficiently small subproblems

# Divide and Conquer

## Introduction - Python

# Divide and Conquer

## Introduction - Python

- ▶ Function `solve` for solving a `problem` of size `n`



# Divide and Conquer

## Introduction - Python

- Function `solve` for solving a `problem` of size `n`

```
def solve(problem):  
    if n < threshold:  
        return solution # solve directly  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
        return S1 + S2 + ... + Sk
```

# Divide and Conquer

## Features

**Divide and Conquer:**

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- ▶ Can help with conceptual hard problems

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- ▶ **Solution** of the trivial problems has to be known

# Divide and Conquer

## Features

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- ▶ **Dividing** in subproblems has to be possible

# Divide and Conquer

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- ▶ **Combination** of solutions has to be possible

# Divide and Conquer

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# Divide and Conquer

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# Divide and Conquer

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  - ▶ And the number of subproblems is limited
  - ▶ The runtime is  $\in O(n \cdot \log n)$
- ▶ Suitable for parallel processing
  - ▶ Subproblems are **independent** of each other
  - ▶ Only needed input for each subproblem has to be known

# Divide and Conquer

## Implementation

**Definition of the trivial case:**

# Divide and Conquer

## Implementation

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- ▶ Smaller subproblems are elegant and simple



# Divide and Conquer

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# Divide and Conquer

## Implementation

### **Definition of the trivial case:**

- ▶ Smaller subproblems are elegant and simple
- ▶ Otherwise the efficiency will be improved if relative big subproblems can be solved directly
- ▶ Recursion depth should not get too big (stack / memory overhead)

# Divide and Conquer

## Implementation

**Division in subproblems:**

# Divide and Conquer

## Implementation

### **Division in subproblems:**

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

# Divide and Conquer

## Implementation

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### **Combination of solutions:**

# Divide and Conquer

## Implementation

### **Division in subproblems:**

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

### **Combination of solutions:**

- ▶ Typically conceptual demanding

# Divide and Conquer

Example - Maximum Subtotal

**Example - Maximum Subtotal**

# Divide and Conquer

## Example - Maximum Subtotal

**Example - Maximum Subtotal Input:**



# Divide and Conquer

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# Divide and Conquer

## Example - Maximum Subtotal

### **Example - Maximum Subtotal Input:**

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### **Output:**

- ▶ Maximum sum of related subsequence and its index boundary

# Divide and Conquer

## Example - Maximum Subtotal

### Example - Maximum Subtotal Input:

- ▶ Sequence  $X$  of  $n$  integers

### Output:

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Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

# Divide and Conquer

## Example - Maximum Subtotal

### Example - Maximum Subtotal Input:

- ▶ Sequence  $X$  of  $n$  integers

### Output:

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Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

**Output:** Sum: 187, Start: 2, End: 6

# Divide and Conquer

## Example - Maximum Subtotal

### Application:

- ▶ Maximum profit of buying and selling shares



Figure: Stock value over time

# Divide and Conquer

Example - Maximum Subtotal - Python

**Naive solution (brute force)**



# Divide and Conquer

## Example - Maximum Subtotal - Python

### Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

Example - Maximum Subtotal - Python

**Runtime - Upper bound**

# Divide and Conquer

## Example - Maximum Subtotal - Python

### Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

Example - Maximum Subtotal

**Upper bound:**

# Divide and Conquer

## Example - Maximum Subtotal

### **Upper bound:**

- ▶ Three interleaved loops

# Divide and Conquer

## Example - Maximum Subtotal

### Upper bound:

- ▶ Three interleaved loops
- ▶ Each loop with runtime  $O(n)$

# Divide and Conquer

## Example - Maximum Subtotal

### Upper bound:

- ▶ Three interleaved loops
- ▶ Each loop with runtime  $O(n)$
- ▶ Algorithm runtime of  $O(n^3)$

# Divide and Conquer

## Example - Maximum Subtotal - Runtime

**Lower bound:**

Table: Operations

$i$	Additions	$j$
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$



# Divide and Conquer

## Example - Maximum Subtotal - Runtime

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Table: Operations

$i$	Additions	$j$
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- We iterate at least  $\frac{n}{3}$  values for  $i$

# Divide and Conquer

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- ▶ We iterate at least  $\frac{n}{3}$  values for  $i$
- ▶ For each  $i$  we iterate at least  $\frac{n}{3}$  values for  $j$

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Table: Operations

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- ▶ We iterate at least  $\frac{n}{3}$  values for  $i$
- ▶ For each  $i$  we iterate at least  $\frac{n}{3}$  values for  $j$
- ▶ For each  $j$  we have at least  $\frac{n}{3}$  additions

# Divide and Conquer

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- ▶ We iterate at least  $\frac{n}{3}$  values for  $i$
- ▶ For each  $i$  we iterate at least  $\frac{n}{3}$  values for  $j$
- ▶ For each  $j$  we have at least  $\frac{n}{3}$  additions
- ▶ We need at least  $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$  steps

# Divide and Conquer

Example - Maximum Subtotal - Runtime

**Runtime:**

# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Runtime:

- ▶ With  $T(n) \in O(n^3)$  and  $T(n) \in \Omega(n^3)$  we know:

$$T(n) \in \Theta(n^3)$$

# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Runtime:

- ▶ With  $T(n) \in O(n^3)$  and  $T(n) \in \Omega(n^3)$  we know:

$$T(n) \in \Theta(n^3)$$

- ▶ It is hard to solve the problem in a worse way ...

# Divide and Conquer

Example - Maximum Subtotal - Runtime

**Current approach:**



# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Current approach:

- ▶ Calculating the sum for range from  $i$  to  $j$  with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Current approach:

- ▶ Calculating the sum for range from  $i$  to  $j$  with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

### Better approach:

# Divide and Conquer

## Example - Maximum Subtotal - Runtime

### Current approach:

- ▶ Calculating the sum for range from  $i$  to  $j$  with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

### Better approach:

- ▶ Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i + 1] + \cdots + X[j] + X[j + 1]$$

$$S_{i,j+1} = S_{i,j} + X[j + 1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

# Divide and Conquer

Example - Maximum Subtotal - Python

**Better solution:**

# Divide and Conquer

## Example - Maximum Subtotal - Python

### Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

# Divide and Conquer

## Example - Maximum Subtotal - Python

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    return result
```

- Runtime  $\in O(n^2)$

# Divide and Conquer

Example - Maximum Subtotal

## Divide and Conquer:



## Divide and Conquer Idea to solve:

- ▶ Split the sequence in the middle

# Divide and Conquer

## Example - Maximum Subtotal

### Divide and Conquer:



### Divide and Conquer Idea to solve:

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# Divide and Conquer

## Example - Maximum Subtotal

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## Example - Maximum Subtotal

### Divide and Conquer:



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- ▶ Maximum might be located in **left half (A)** or **right half (B)**

# Divide and Conquer

## Example - Maximum Subtotal

### Divide and Conquer:



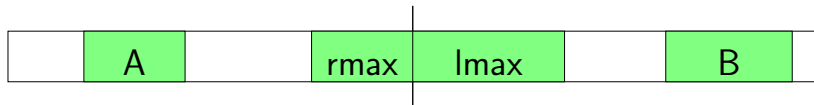
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- ▶ Problem: Maximum can overlap the split

# Divide and Conquer

## Example - Maximum Subtotal

### Divide and Conquer:



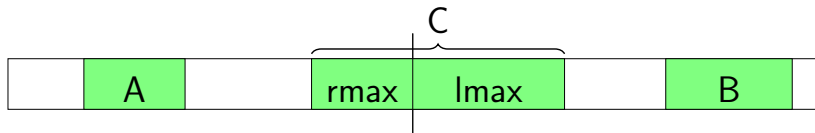
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- ▶ To solve this case we have to calculate *rmax* and *lmax*

# Divide and Conquer

## Example - Maximum Subtotal

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- ▶ The overall solution is the maximum of *A*, *B* and *C*

# Divide and Conquer

Example - Maximum Subtotal

**Principle - Divide and Conquer:**

# Divide and Conquer

## Example - Maximum Subtotal

### Principle - Divide and Conquer:

- ▶ Small problems are solved directly:  $n = 1 \Rightarrow \max = X[0]$

# Divide and Conquer

## Example - Maximum Subtotal

### Principle - Divide and Conquer:

- ▶ Small problems are solved directly:  $n = 1 \Rightarrow \max = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions  $A$  and  $B$  are returned



# Divide and Conquer

## Example - Maximum Subtotal

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- ▶ To determine subsolution  $C$ ,  $\text{rmax}$  and  $\text{lmax}$  for the subproblems are computed

# Divide and Conquer

## Example - Maximum Subtotal

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- ▶ To determine subsolution  $C$ ,  $\text{rmax}$  and  $\text{lmax}$  for the subproblems are computed
- ▶ The overall solution is the maximum of  $A$ ,  $B$  and  $C$

# Divide and Conquer

## Example - Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

# Further Literature

## ► General

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*Introduction to Algorithms.*

MIT Press, Cambridge, Mass, 2001.

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Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

# Further Literature

- ▶ **Caching**

[Wik] [Cache](#)

`https://en.wikipedia.org/wiki/Cache`