Algorithms and Datastructures

Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, January 2017

Structure

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Balanced Trees

Motivation

Binary search tree:

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  - if the keys are inserted in ascending / descending order $(20, 19, 18, \dots)$

Gnarley trees:

Balanced Trees

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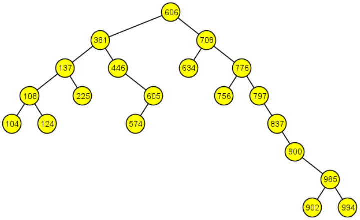

Figure: Binary search tree with random insert [Gna]

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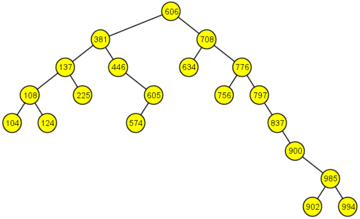

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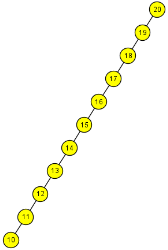

Figure: Binary search tree with descending insert [Gna]

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- We rebalance the tree from time to time

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Balanced Trees

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  - Used in C++ std::map, Java SortedMap

Structure

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- Height difference of left and right subtree is at maximum one
- With that the height of the search tree is always $O(\log n)$
- We can perform all basic operations in $O(\log n)$

Balanced Trees

AVL-Tree

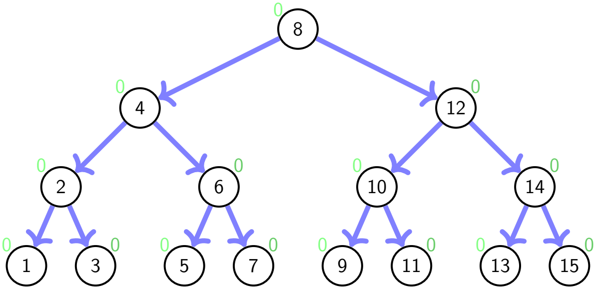

Figure: Example of an AVL-Tree

Balanced Trees

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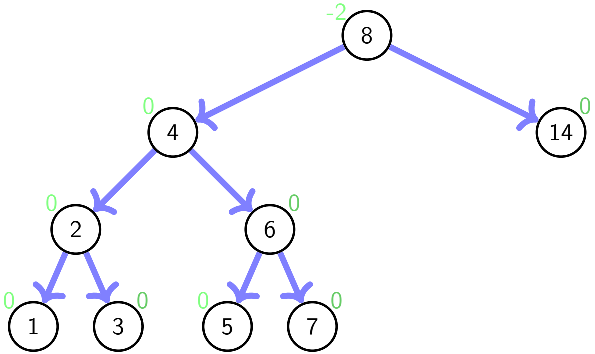


Figure: **Not** an AVL-Tree

Balanced Trees

AVL-Tree

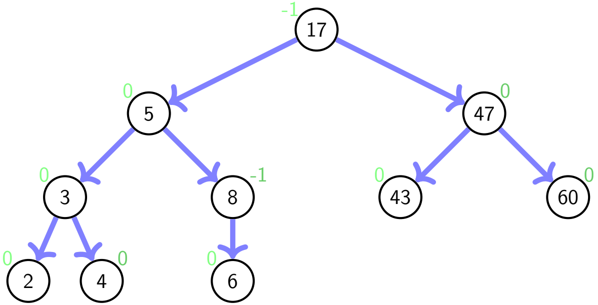


Figure: Another example of an AVL-Tree

Balanced Trees

AVL-Tree - Rebalancing

Rotation:

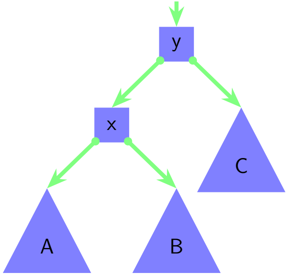
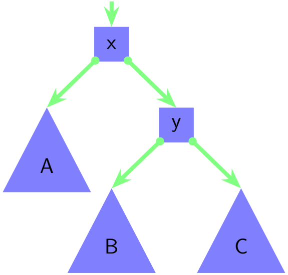


Figure: Before rotating

$\Rightarrow$



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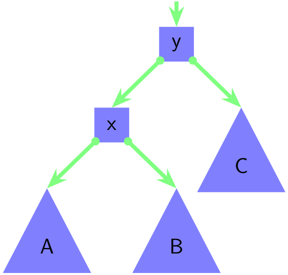

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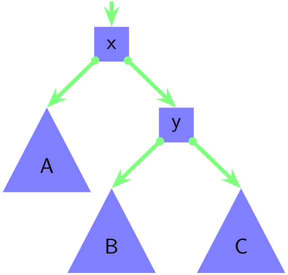

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- Central operation of rebalancing

Balanced Trees

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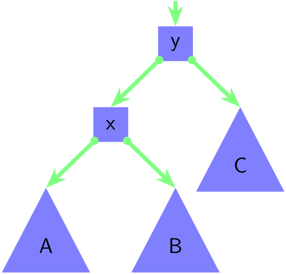

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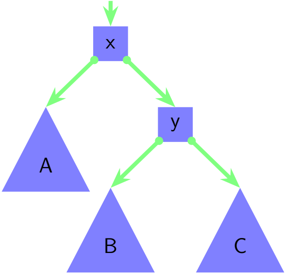

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- After rotation to the right:

Balanced Trees

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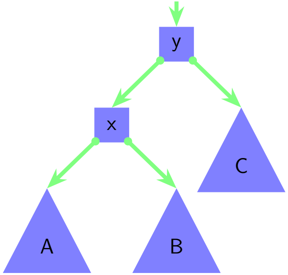

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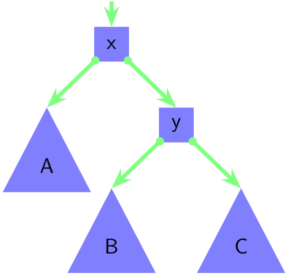

Figure: After rotating

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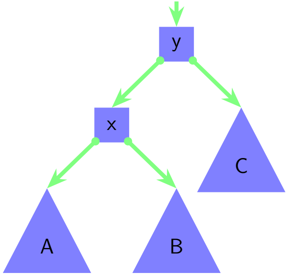

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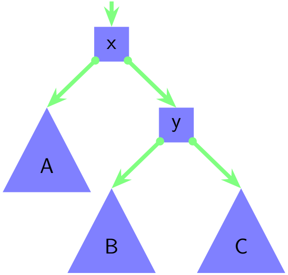

Figure: After rotating

- Central operation of rebalancing
- After rotation to the right:
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  - The parent child relations between nodes x and y have been swapped

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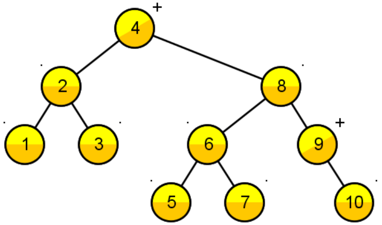

Figure: Inserting $1, \ldots, 10$ into an AVL-tree [Gna]

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- However not amortized update costs of $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are (a,b)-trees

Structure

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- So we have space for elements on an `insert` and balance operation

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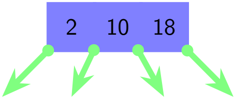

(a,b)-Trees
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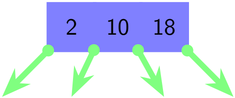

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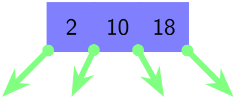

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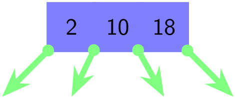

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(a,b)-Trees
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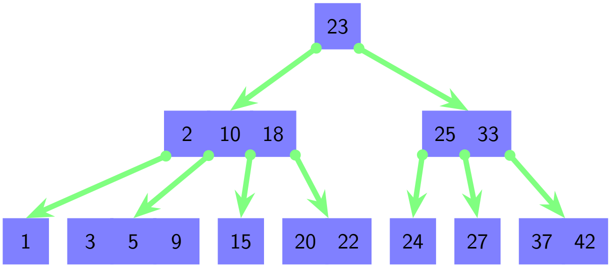

Figure: Example of an (2,4)-tree

(a,b)-Trees

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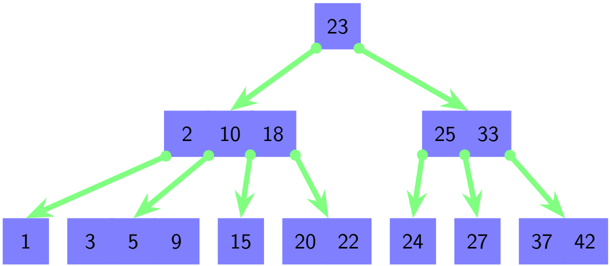

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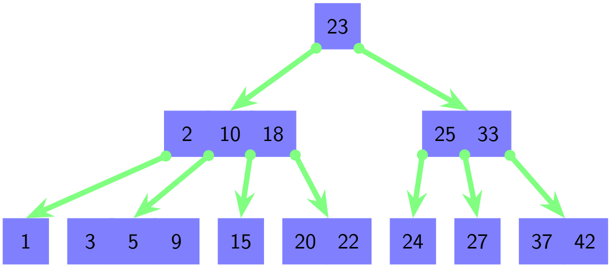

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- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

(a,b)-Trees

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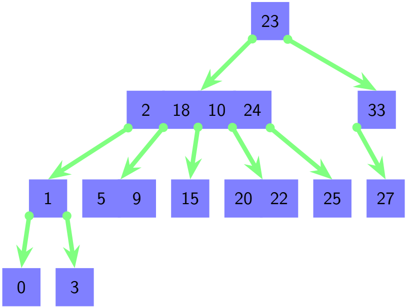

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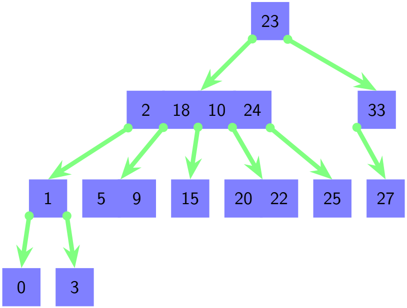

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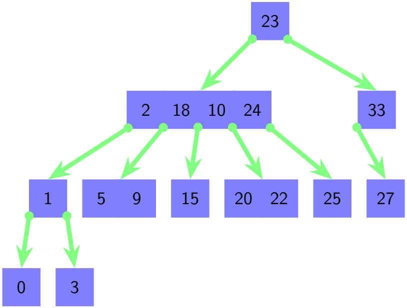

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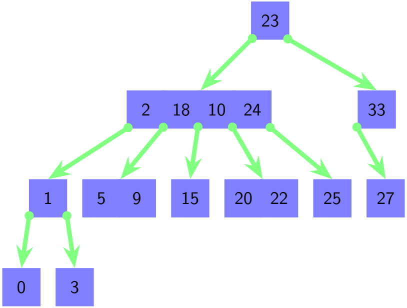

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- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels

(a,b)-Trees

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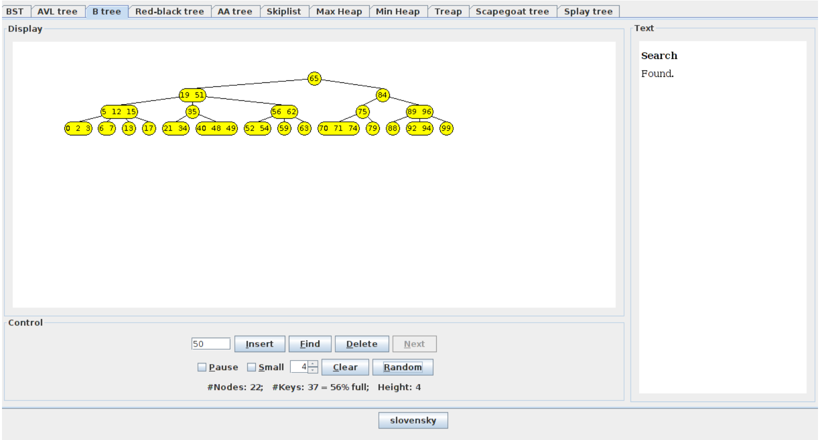

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- Then we **split** the node

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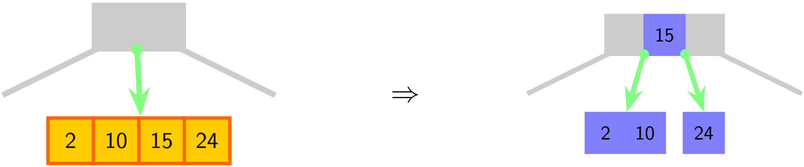

Figure: Splitting a node

(a,b)-Trees
Implementation - Insert

Inserting an element: (`insert`)

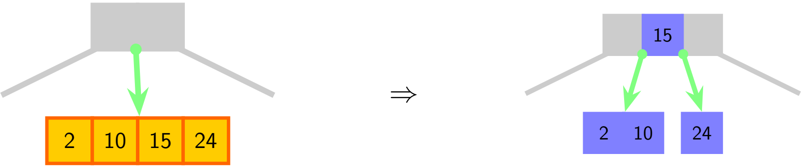

Figure: Splitting a node

- If the degree is higher than $b+1$ we split the node

(a,b)-Trees

Implementation - Insert

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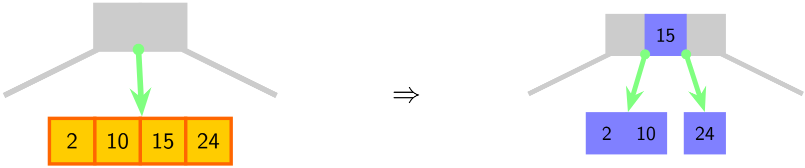

Figure: Splitting a node

- If the degree is higher than $b + 1$ we split the node
  - This results in a node with $\mathrm{ceil}\left(\frac{b-1}{2}\right)$ elements, a element for the parent node, and a node with $\mathrm{floor}\left(\frac{b-1}{2}\right)$ elements

(a,b)-Trees

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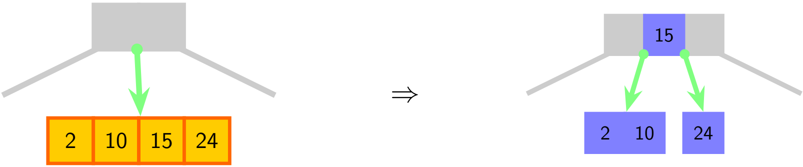

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  - This results in a node with $\mathrm{ceil}\left(\frac{b-1}{2}\right)$ elements, a element for the parent node, and a node with $\mathrm{floor}\left(\frac{b-1}{2}\right)$ elements
  - Thats why we have the limit $b \geq 2a-1$

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- If the node to split is the root we split it and create a new root node
  (The tree is now one level deeper)

Removing an element: (`remove`)

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- Search the element in $O(\log n)$ time

(a,b)-Trees

Implementation - Remove

Removing an element: (`remove`)

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(a,b)-Trees

Implementation - Remove

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- Attention: The leaf might be too small (degree of $a - 1$)
  $\Rightarrow$ We rebalance the tree

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(a,b)-Trees

Implementation - Remove

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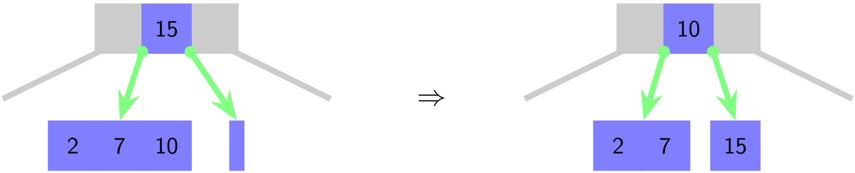

Figure: Borrowing an element

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  - **Case b:** We combine the node with its right or left neighbour

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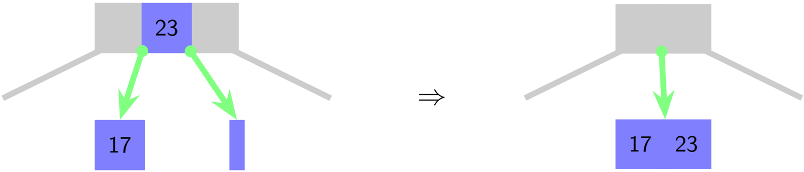

Figure: Combining two nodes

Removing an element: (`remove`)

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- Now the parent node can be of degree $a - 1$

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Removing an element: (`remove`)

- Now the parent node can be of degree $a - 1$
- We combine parent nodes the same way
- If the root has only one child left we take the child as new root (The tree shrinks one level)

Runtime complexity of `lookup`, `insert` and `remove`:

(a,b)-Trees

Runtime Complexity

Runtime complexity of `lookup`, `insert` and `remove`:

- All operations in $O(d)$ with d being the depth of the tree

(a,b)-Trees

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- All operations in $O(d)$ with d being the depth of the tree
- Each node (except the root) has more than a children
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(a,b)-Trees

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(a,b)-Trees
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  - Therefore instead of $b \geq 2\,a - 1$ we need $b \geq 2\,a$.

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  - Therefore instead of $b \geq 2\,a - 1$ we need $b \geq 2\,a$.
  - Here is a counter-example for (2,3)-trees, analysis of (2,4)-trees

(a,b)-Trees

Runtime Complexity - Counter-example for (2,3)-Tree

(2,3)-Tree:

(a,b)-Trees

Runtime Complexity - Counter-example for (2,3)-Tree

(2,3)-Tree:

- Before executing `delete(11)`

(a,b)-Trees

Runtime Complexity - Counter-example for (2,3)-Tree

(2,3)-Tree:

- Before executing `delete(11)`

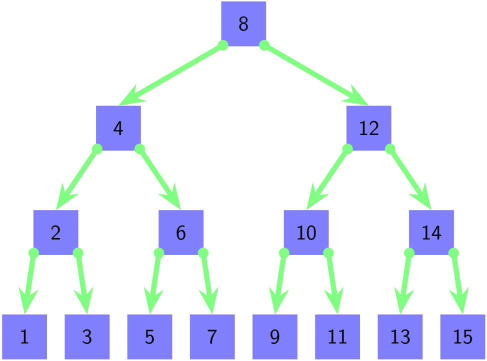

Figure: Normal (2,3)-Tree

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- Executing `delete(11)`

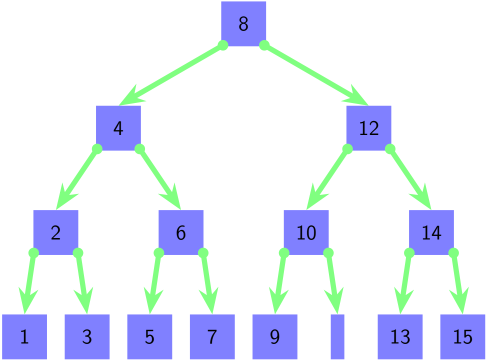

Figure: (2,3)-Tree - Delete step 1

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- Executing `delete(11)`

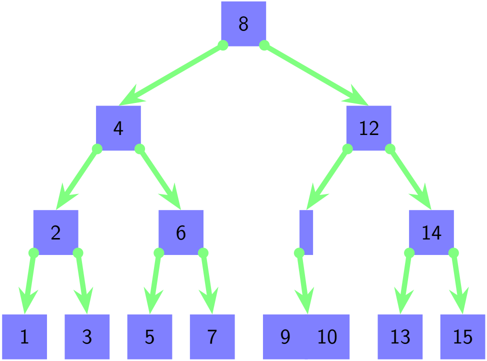

Figure: (2,3)-Tree - Delete step 2

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- Executing `delete(11)`

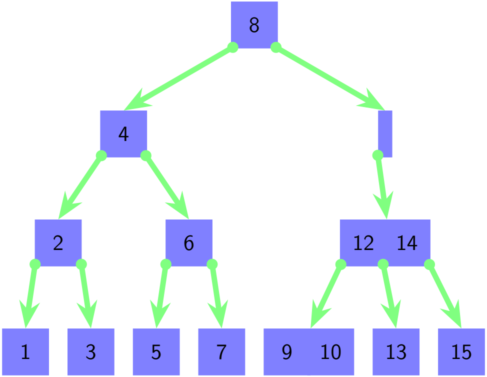

Figure: (2,3)-Tree - Delete step 3

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- Executed `delete(11)`

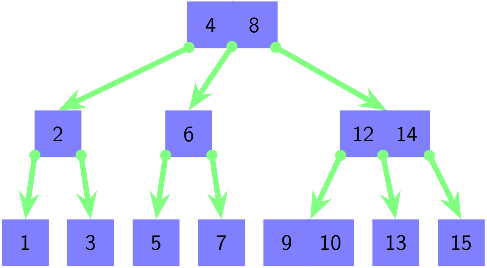

Figure: (2,3)-Tree - Delete step 4

(2,3)-Tree:

(2,3)-Tree:

- Executing `insert(11)`

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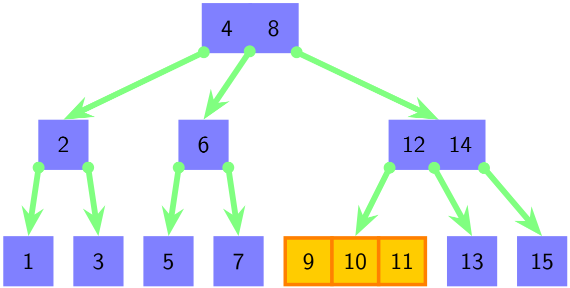

Figure: (2,3)-Tree - Insert step 1

(2,3)-Tree:

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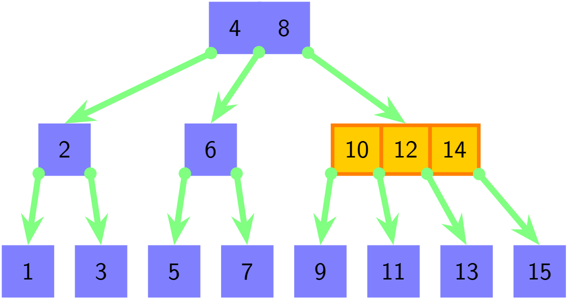

Figure: (2,3)-Tree - Insert step 2

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- Executing `insert(11)`

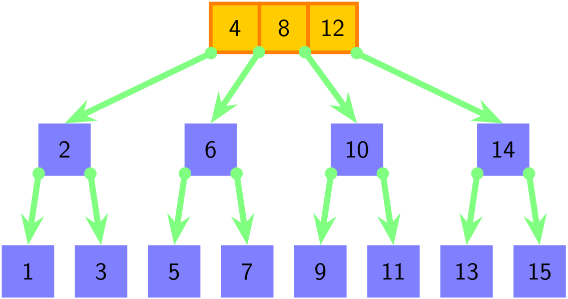

Figure: (2,3)-Tree - Insert step 3

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

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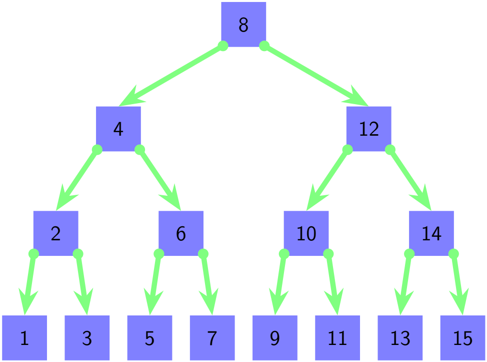


Figure: (2,3)-Tree - Insert step 4

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

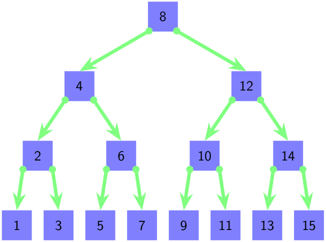


Figure: (2,3)-Tree

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- We are exactly where we started

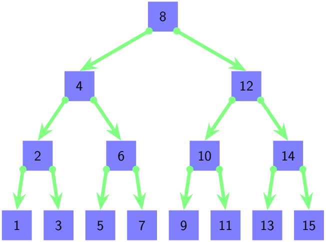


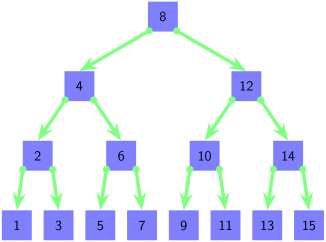
Figure: (2,3)-Tree

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- We are exactly where we started
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$



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(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

(2,3)-Tree:

- We are exactly where we started
- If $b = 2\,a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$
- We need $b \geq 2\,a$ instead of $b \geq 2\,a - 1$

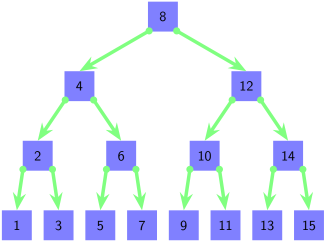

Figure: (2,3)-Tree

(2,4)-Tree:

(2,4)-Tree:

- If all nodes have 2 children we have to combine the nodes up to the root on a `remove` operation

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

(2,4)-Tree:

- If all nodes have 2 children we have to combine the nodes up to the root on a `remove` operation
- If all nodes have 4 children we have to split the nodes up to the root on a `insert` operation

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- If all nodes have 2 children we have to combine the nodes up to the root on a `remove` operation
- If all nodes have 4 children we have to split the nodes up to the root on a `insert` operation
- If all nodes have 3 children it takes some time to reach one of the previous two states

(2,4)-Tree:

- If all nodes have 2 children we have to combine the nodes up to the root on a `remove` operation
- If all nodes have 4 children we have to split the nodes up to the root on a `insert` operation
- If all nodes have 3 children it takes some time to reach one of the previous two states

$\Rightarrow$ **Nodes of degree 3 are harmless**
Neither an `insert` nor a `remove` operation trigger rebalancing operations

(2,4)-Tree:

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- **Idea:**

(2,4)-Tree:

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  - After an expensive operation the tree is in a stable state

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- **Idea:**
  - After an expensive operation the tree is in a stable state
  - It takes some time until the next expensive operation occurs
- Like with dynamic arrays:
  - Reallocation is expensive but it takes some time until the next expensive operation occurs
  - If we overallocate clever we have an amortized runtime of $O(1)$

Terminology:

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- We analyze a sequence of n operations

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- Let Φ_i be the potential of the tree after the i-*th* operation

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- We analyze a sequence of n operations
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- $=$ is the number of nodes with degree 3

Example:

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- Nodes of degree 3 are highlighted

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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- Nodes of degree 3 are highlighted

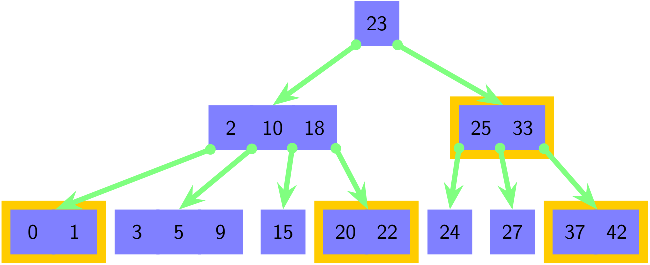

Figure: Tree with potential $\Phi = 4$

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  - For each further step, that incurs cost, the operation creates a further harmless node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Terminology:

- Let c_i be the costs = runtime of the i-th operation
- We will show:
  - Each operation can maximally destroy one harmless node
  - For each further step, that incurs cost, the operation creates a further harmless node
- The costs for operation i are coupled to the difference of the potential levels

$$c_i \leq A \cdot \underbrace{(\Phi_i - \Phi_{i-1})} + B, \qquad A > 0 \text{ and } B > A$$

Number of harmless (degree 3) nodes at operation i. Can be -1, but not smaller than -1

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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Number of harmless (degree 3) nodes at operation i. Can be -1, but not smaller than -1

- With that each operation has an amortized cost of $O(1)$

Case 1: *i-th* operation is an `insert` operation on a full node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: *i-th* operation is an `insert` operation on a full node

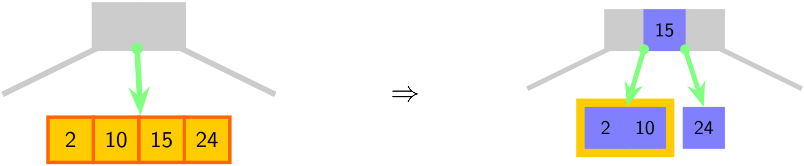

Figure: Splitting a node on `insert`

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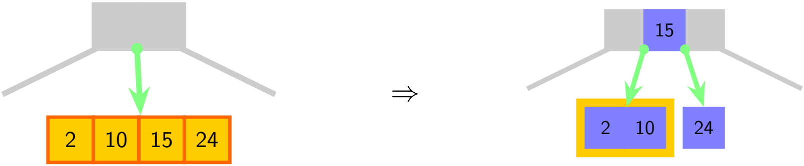

Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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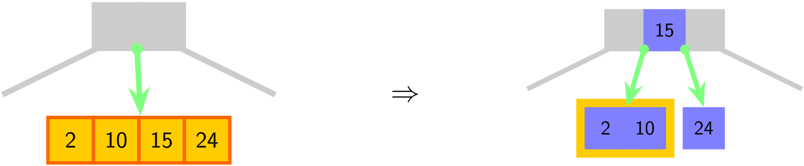

Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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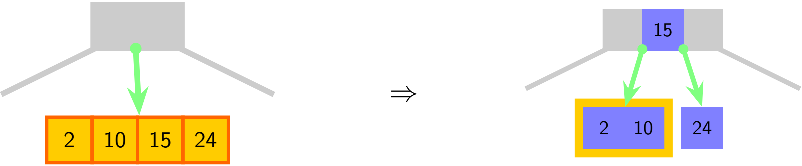

Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: *i-th* operation is an `insert` operation on a full node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: i-*th* operation is an `insert` operation on a full node

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$$\Phi_i \geq \Phi_{i-1} + m - 1$$
$$\Rightarrow m \leq \Phi_i - \Phi_{i-1} + 1$$

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: *i-th* operation is an `insert` operation on a full node

- Let m be the number of nodes split
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$$\Phi_i \geq \Phi_{i-1} + m - 1$$
$$\Rightarrow m \leq \Phi_i - \Phi_{i-1} + 1$$

Costs: $c_i \leq A \cdot m + B$

$$\Rightarrow c_i \leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B$$
$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}$$

Case 2: i-*th* operation is an `remove` operation

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- **Case 2.1:** Inner node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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(a,b)-Trees

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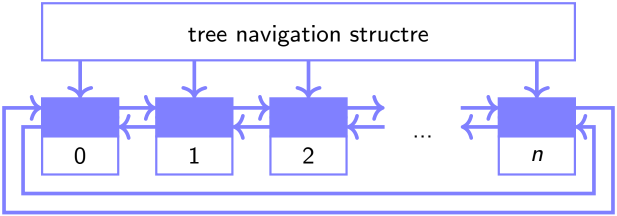

Figure: Tree with doubly linked list

Case 2: i-*th* operation is an `remove` operation

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrowing a node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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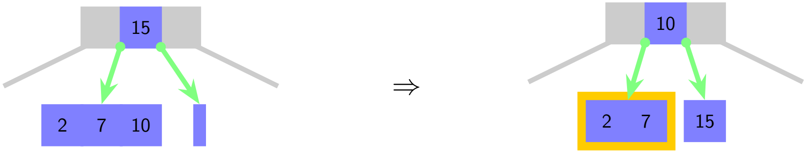

Figure: Borrowing an element case 2.1.1

Case 2: i-*th* operation is an `remove` operation

Case 2: i-*th* operation is an `remove` operation

- **Case 2.1:** Borrowing a node

Case 2: *i-th* operation is an `remove` operation

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(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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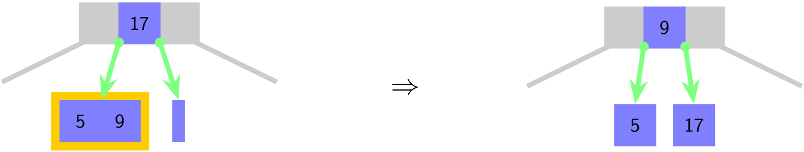

Figure: Borrowing an element case 2.1.2

Case 2: i-*th* operation is an `remove` operation

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging a node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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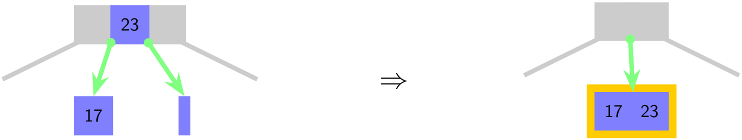

Figure: Merging two nodes

  - Potential rises by one

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

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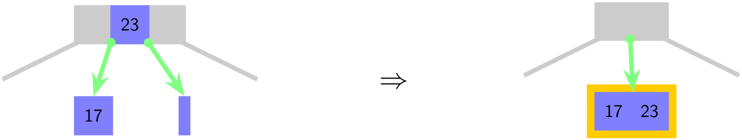

Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation

Case 2: *i-th* operation is an `remove` operation

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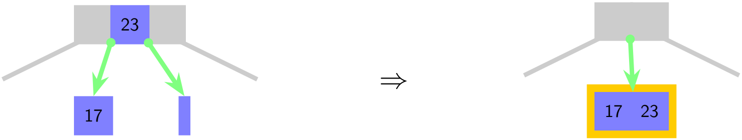

Figure: Merging two nodes

- Potential rises by one
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- This operation propagates upwards until a node of degree > 2 or a degree 2 node, which can borrow from a neighbour

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: i*-th* operation is an `remove` operation

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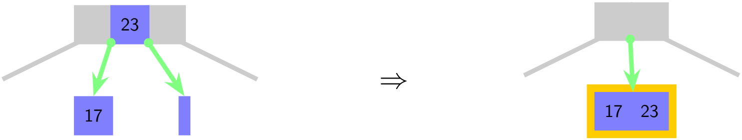

Figure: Merging two nodes

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- The potential rises by m

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: i-*th* operation is an `remove` operation

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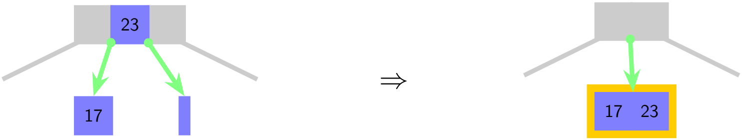

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(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging a node

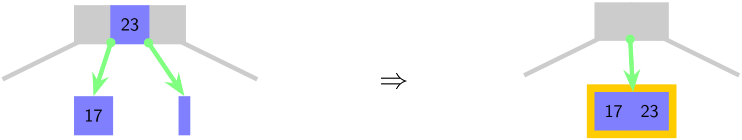

Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree > 2 or a degree 2 node, which can borrow from a neighbour
- The potential rises by m
- If the "stop-node" is of degree 2 then the potential eventually goes down by one
- Same costs as `insert`

Lemma:

Lemma:

- We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

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$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^{n} c_i = O(n)$$

(a,b)-Trees

Runtime Complexity - (2,4)-Tree - Lemma - Proof

Proof:

$$\sum_{i=0}^{n} c_i \leq \underbrace{A \cdot (\Phi_1 - \Phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\Phi_2 - \Phi_1) + B}_{\leq c_1} + \cdots + \underbrace{A \cdot (\Phi_n - \Phi_{n-1})}_{\leq c_n}$$

$$= A \cdot (\Phi_n - \Phi_0) + B \cdot n \qquad | \text{ telescope sum}$$

$$= A \cdot \Phi_n + B \cdot n \qquad | \text{ we start with an empty tree}$$

$$< A \cdot n + B \cdot n = O(n) \qquad | \text{ number of degree 3 nodes}$$

¡ number of nodes

Structure

Red-Black Tree:

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- Binary tree with red and black nodes

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Red-Black Tree:

- Binary tree with red and black nodes
- Number of black nodes on path to leaves is equal
- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- Each (2,4)-tree-node is a small red-black-tree with a black root node

Red-Black-Trees

Introduction

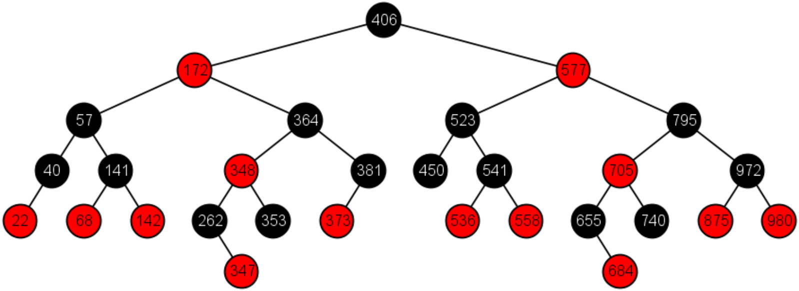

Figure: Example of an red-black-tree [Gna]

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- **Gnarley Trees**

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- **AVL-Tree**

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  https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934_tree

  [Wikb] (a,b)-tree
  https://en.wikipedia.org/wiki/(a,b)-tree

- **Red-Black-Tree**

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https://en.wikipedia.org/wiki/Red%E2%80%93black_tree