

# Algorithms and Datastructures

Open Addressing, Priority Queue

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# Structure

## Hashing

- Recapitulation

- Treatment of hash collisions

- Open Addressing

- Summary

## Priority Queue

- Introduction

# Hashing

## Recapitulation

### Hashing:

- ▶ No hash function is good for all key sets!
  - ▶ This cannot work, because a big universe is mapped onto a small set:  $|\mathcal{U}| > m$
- ▶ For random key sets also simple hash function work, e.g.

$$\Rightarrow h(x) = x \bmod m$$

- ▶ Then the random keys make sure that it is distributed evenly
- ▶ To find a good hash function for every key set universal hashing is needed
  - ▶ Then however, for a fixed set of keys not every hash function is suitable, but only some

# Hashing

## Recapitulation

### Rehashing:

- ▶ It is possible to get bad hash functions with universal hashing, but it is unlikely
- ▶ This is determinable by monitoring the maximum bucket size
- ▶ If a pre-defined level is exceeded, then a **rehash** is performed

### How to rehash?

- ▶ New hash table with a new random hash function
- ▶ Copy elements into the new table
  - ▶ Expensive but happens not often
  - ▶ Therefore the average cost is low
  - ▶ Look at **amortized analysis** in the next lecture

# Hashing

## Linked List

### Buckets as linked list:

- ▶ Each bucket is a linked list
- ▶ Colliding keys are inserted into the linked list of a bucket, either sorted or appended at the end



- ▶ Operations in  $O(1)$  are possible if a suitable table size and hash function is selected
- ▶ Worst case  $O(n)$ , e.g. table size of 1
- ▶ Dynamic number of elements is possible

# Hashing

## Open Addressing

- ▶ For colliding keys we choose a new free entry
- ▶ Static, fixed number of elements
- ▶ The **probe sequence** determines for each key, in which sequence all hash table entries are searched for a free bucket
  - ▶ If an entry is already occupied, then iteratively the **following entry** can be checked. If a free entry is found the element is inserted
  - ▶ If element is not found at the corresponding table entry, even if the entry is occupied, then probing has to be performed until the element or a free entry have been found

# Hashing

## Open Addressing

### Definitions:

$h(s)$  Hash function for key  $s$

$g(s, j)$  Probing function for key  $s$  with overflow positions  
 $j \in \{0, \dots, m-1\}$  e.g.  $g(s, j) = j$

- The **probe sequence** is calculated by

$$h(s, j) = (h(s) - g(s, j)) \bmod m \in \{0, \dots, m-1\}$$



# Hashing

## Open Addressing - Python

```
def insert(s, value):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
           is not None:  
        j += 1  
  
    t[(h(s) - g(s, j)) mod m] \  
      = (s, value)
```



# Hashing

## Open Addressing - Python

```
def lookup(s):  
    j = 0  
  
    while t[(h(s) - g(s, j)) mod m] \  
           is not None:  
  
        if t[(h(s) - g(s, j)) mod m][0] == s:  
            return t[(h(s) - g(s, j)) mod m]  
  
        j += 1  
  
    return None
```

# Hashing

## Open Addressing - Linear Probing



Figure: Linear probe sequence

- ▶ Check the element with lower index:  $g(s, j) := j$   
 $\Rightarrow$  Hash function:  $h(s, j) = (h(s) - j) \bmod m$
- ▶ This leads to the following probe sequence

$$h(s), h(s) - 1, h(s) - 2, \dots, \underbrace{0, m - 1}_{\text{clipping}}, m - 2, \dots, h(s) + 1$$

# Hashing

## Open Addressing - Linear Probing



Figure: Linear probe sequence

- ▶ Can result in primary clustering
- ▶ Dealing with a hash collision will result in a higher probability of hash collisions in close entries

# Hashing

## Open Addressing - Linear Probing

### Example:

- ▶ Keys: {12, 53, 5, 15, 2, 19}
- ▶ Hash function:  $h(s, j) = (s \bmod 7 - j) \bmod 7$
- ▶ t. insert (12, "A"),  $h(12, 0) = 5$

0	1	2	3	4	5	6
					12, A	

- ▶ t. insert (53, "B"),  $h(53, 0) = 4$

				53, B	12, A	
--	--	--	--	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

# Hashing

## Open Addressing - Linear Probing

### Example:

- ▶ Hash function:  $h(s, j) = (s \bmod 7 - j) \bmod 7$
- ▶ t. insert (5, "C"),  $h(5, 0) = 5$ ,  $h(5, 1) = 4$ ,  $h(5, 2) = 3$

0	1	2	3	4	5	6
			5, C	53, B	12, A	

- ▶ t. insert (15, "D"),  $h(15, 0) = 1$

	15, D		5, C	53, B	12, A	
--	-------	--	------	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

# Hashing

## Open Addressing - Linear Probing

### Example:

► Hash function:  $h(s, j) = (s \bmod 7 - j) \bmod 7$

► t. insert (2, "E"),  $h(2, 0) = 2$

0	1	2	3	4	5	6
	15, D	2, E	5, C	53, B	12, A	

► t. insert (19, "F"),  $h(19, 0) = 5$ ,  $h(19, 1) = 4$ ,  
 $h(19, 2) = 3$ ,  $h(19, 3) = 2$ ,  $h(19, 4) = 1$ ,  $h(19, 5) = 0$

19, F	15, D	2, E	5, C	53, B	12, A	
-------	-------	------	------	-------	-------	--

Figure: Probe/Insertion sequence on a hash map

# Hashing

## Open Addressing - Squared Probing

### Squared probing:

- Motivation: Avoid local clustering

$$g(s, j) := (-1)^j \left\lceil \frac{j}{2} \right\rceil^2$$



Figure: Squared probe sequence

- This leads to the following probe sequence

$$h(s), h(s) + 1, h(s) - 1, h(s) + 4, h(s) - 4, h(s) + 9, h(s) - 9, \dots$$

# Hashing

## Open Addressing - Squared Probing

### Squared probing:

$$g(s, j) := (-1)^j \left\lceil \frac{j}{2} \right\rceil^2$$

- ▶ If  $m$  is a prime number for which  $m = 4 \cdot k + 3$  then the probe sequence is a permutation of the indices of the hash tables
- ▶ Alternatively:  $h(s, j) := (h(s) - c_1 \cdot j + c_2 \cdot j^2) \bmod m$
- ▶ Problem of secondary clustering  
No local clustering anymore, but keys with same hash value have similar probe sequence



# Hashing

## Open Addressing - Uniform Probing

### Uniform Probing:

- ▶ Motivation: So far uses function  $g(s, j)$  only the step counter  $j$  for linear and squared probing  
⇒ The probe sequence is independent of the key  $s$
- ▶ Uniform probing computes the sequence  $g(s, j)$  of permutations of all possible indices in dependency on key  $s$
- ▶ **Advantage:** Prevents clustering because different keys with the same hash value do not produce the same probe sequence
- ▶ **Disadvantage:** Hard to implement

# Hashing

## Open Addressing - Double Hashing

### Double Hashing:



Figure: Double hashing probe sequence

- ▶ Motivation: Consider key  $s$  in probe sequence
- ▶ Use two independent hash functions  $h_1(s)$ ,  $h_2(s)$
- ▶ Hash function:  $h(s, j) = (h_1(s) + j \cdot h_2(s)) \bmod m$

# Hashing

## Open Addressing - Double Hashing

### Double Hashing:

- ▶ Hash function:  $h(s, j) = (h_1(s) + j \cdot h_2(s)) \bmod m$
- ▶ probe sequence:

$$h_1(s), h_1(s) + h_2(s), h_1(s) + 2 \cdot h_2(s), h_1(s) + 3 \cdot h_2(s), \dots$$

- ▶ Works well in practical use
- ▶ This method is an approximation of uniform probing

# Hashing

## Open Addressing - Double Hashing - Example

### Example:

$$h_1(s) = s \bmod 7$$

$$h_2(s) = (s \bmod 5) + 1$$

$$h(s, j) = h_1(s) + j \cdot h_2(s) \bmod 7$$

Table: Comparing both hash functions

s	10	19	31	22	14	16
$h_1(s)$	3	5	3	1	0	2
$h_2(s)$	1	5	2	3	5	2

- The efficiency of double hashing is dependent on  $h_1(s) \neq h_2(s)$

# Hashing

## Open Addressing - Double Hashing - Optimization



Figure: Double hashing

### Double hashing by Brent:

- Motivation:

Because different keys have different probe sequences, the sequence of the insertions has impact on efficiency of a successful search

# Hashing

## Open Addressing - Double Hashing - Optimization



Figure: Double hashing

### Example:

- ▶ The key  $s_1$  is inserted at position  $p_1 = h(s_1, 0)$
- ▶ The hash function for  $s_2$  also results in  $p_2 = h(s_2, 0) = p_1$
- ▶ The locations  $h(s_2, j)$ ,  $j \in \{1, \dots, n\}$  are also occupied
- ▶ If we insert  $s_2$  at position  $h(s_2, n + 1)$  the search will be inefficient

# Hashing

## Open Addressing - Double Hashing - Optimization



Figure: Double hashing by Brent

- ▶ Reversed sequence of keys would have been better
- ▶ **Brents Idea:**
  - ▶ Test if location  $h(s_1, 1)$  is free
  - ▶ If yes, move  $s_1$  from  $h(s_1, 0)$  to  $h(s_1, 1)$  and insert  $s_2$  at  $h(s_2, 0)$

# Hashing

## Open Addressing - Ordered Hashing

### Idea:

- ▶ Motivation: Colliding elements are inserted in the hashtable sorted.
- ▶ Therefore, in case of an unsuccessful search of elements in combination with linear probing or double hashing, aborting is earlier possible because single probing steps have a fixed length

### Implementation:

- ▶ Compare both keys if a collision occurs
- ▶ Insert the smaller key at  $p_1$
- ▶ Search a position based on the diversion order for the bigger key



# Hashing

## Open Addressing - Ordered Hashing

### Example:

- ▶ The key 12 is saved at position  $p_1 = h(12, 0)$
- ▶ We insert the key 5 into the hash map
- ▶ We assume  $h(5, 0)$  results in location  $p_1$
- ▶ Because  $5 < 12$  we insert the key 5 at position  $p_1$
- ▶ For the key 12 we iterate through the sequence

$$h(12, 1), h(12, 2), h(12, 3), \dots$$

# Hashing

## Open Addressing - Robin-Hood Hashing

### Motivation:

- ▶ Having similar length of probe sequences for all elements.  
Total costs stay the same, but they are distributed evenly.  
Results in approximately similar search times for all elements

### Implementation:

- ▶ If two keys  $s_1, s_2$  collide ( $p_1 = h(s_1, j_1) = h(s_2, j_2)$ ) we compare the length of the sequence ( $j_1$  or  $j_2$ )
- ▶ The key with the bigger search sequence is inserted at  $p_1$ . The other key is assigned a new location based on the sequence

# Hashing

## Open Addressing - Robin-Hood Hashing

### Example:

- ▶ The key 12 is saved at position  $p_1 = h(12, 7)$
- ▶ We insert the key 5 into the hash map
- ▶ We assume  $h(5, 0)$  results in location  $p_1$
- ▶ Because  $j_1 < j_2$  ( $0 < 7$ ) the key 12 stays at position  $p_1$
- ▶ For the key 5 we iterate through the sequence

$$h(5, 1), h(5, 2), h(5, 3), \dots$$

# Hashing

## Open Addressing - Implement Insert / Remove

### Problem:

- ▶ The key  $s_1$  is inserted at position  $p_1$
- ▶ The key  $s_2$  returns the same hash value, but is inserted at position  $p_2$  because of the probing order
- ▶ If  $s_1$  is removed, it is impossible to find  $s_2$

### Solution:

- ▶ **Remove:** Elements are marked as removed, but not deleted
- ▶ **Inserting:** Elements marked as removed will be overwritten

# Hashing

## Open Addressing - Summary Collision Handling

**Bucket as linked list:** (dynamic, number of elements variable)

- ▶ Save colliding elements as linked list

**Open hashing:** (static, number of elements fixed)

- ▶ Determine a probe sequence, permutation of all hash values
- ▶ Linear, quadratic probing:
  - ▶ Easy to implement
  - ▶ Raise the probability of collisions because probing order does not depend on the key

# Hashing

## Open Addressing - Summary Collision Handling

### **Open hashing:** (static, number of elements fixed)

- ▶ Uniform probing, double hashing:
  - ▶ Different probing orders for different keys
  - ▶ Avoids clustering of elements

### **Improving efficiency:** (Brent, Ordered Hashing)

- ▶ Improve search efficiency by sorting colliding insertions
  - ▶ Abortion of unsuccessful search
  - ▶ Search sequence length balancing

# Hashing

## Open Addressing - Summary Hashing

### Hashing:

- ▶ Efficient for dictionary operations:
  - Insert:  $O(1) \dots O(n)$
  - Search:  $O(1) \dots O(n)$
  - Remove:  $O(1) \dots O(n)$
- ▶ Direct access of all elements in a hash table
- ▶ Using a hash function to find the position (hash value) in the hash table
- ▶ Hash function, size of the hash table and strategy to avoid hash collisions influence the efficiency of the datastructure

# Priority Queue

## Introduction

### Definition:

- ▶ A priority queue saves a set of elements
- ▶ Each element contains a key and a value like a map
- ▶ There is a total order (like  $\leq$ ) defined on the keys



# Priority Queue

## Introduction

### Definition:

- ▶ The priority queue supports the following operations:

`insert(key, value):` Inserts a new element into the queue

`getMin():` Returns the element with the smallest key

`deleteMin():` Removes the element with the smallest key

- ▶ Sometimes additional operations are defined:

`changeKey(item, key):` Changes the key of the element

`remove(item):` Removes the element from the queue

# Priority Queue

## Introduction

### Special features:

- ▶ Multiple elements with the same key
  - ▶ No problem and for many applications necessary
  - ▶ If there is more than one element with the smallest key
    - `getMin()`: Returns just one of the possible elements
    - `deleteMin()`: Deletes the element returned by `getMin`
- ▶ Argument of `changeKey` and `remove` operations
  - ▶ There is no **quick-access** to a element in the queue
  - ▶ That's why `insert` and `getMin` return a reference (handle, accessor object)
  - ▶ `changeKey` and `remove` take this reference as argument
  - ▶ Therefore each element has to store its current position in the heap.

# Priority Queue

Python

```
from queue import PriorityQueue

q = PriorityQueue()

e1 = (5, "A") # element with priority 5
q.put(e1); # insert element e1

# remove and return the lowest item
e2 = q.get()
```

# Priority Queue

## Application Example

### Example 1:

- ▶ Calculation of the sorted union of  $k$  sorted lists  
(multi-way merge or  $k$ -way merge)

$L_1$  : 


3	5	8	12	...
---	---	---	----	-----

$L_3$  : 

1	10	11	24	...
---	----	----	----	-----

$L_2$  : 

4	5	6	7	...
---	---	---	---	-----

  
 $\Rightarrow R$  : 

1	3	4	5	5	6	7	8	10	...
---	---	---	---	---	---	---	---	----	-----

Figure: 3-way merge

# Priority Queue

## Application Example

### Example 1:

- ▶ Calculation of the sorted union of  $k$  sorted lists (multi-way merge or  $k$ -way merge)
- ▶ Runtime:  $N$  = length of resulting list
  - ▶ Trivial:  $\Theta(N \cdot k)$ , minimum calculation  $\Theta(k)$
  - ▶ Priority queue:  $\Theta(N \cdot \log k)$ , minimum calculation  $\Theta(\log k)$

### Example 2:

- ▶ For example Dijkstra's algorithm for computing the shortest path ( $\leftarrow$  following lecture)
- ▶ Among other applications it can be used for sorting

# Priority Queue

## Implementation

### Idea:

- ▶ Save elements as tuples in a binary heap
- ▶ Summary from lecture 1 (*HeapSort*):
  - ▶ Nearly complete binary tree
  - ▶ **Heap condition:**  
The key of each node  $\leq$  the keys of the children

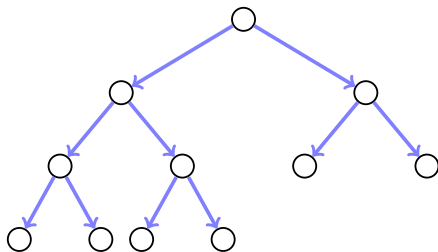


Figure: Heap with 11 nodes

# Priority Queue

## Implementation



Figure: Min heap stored in array

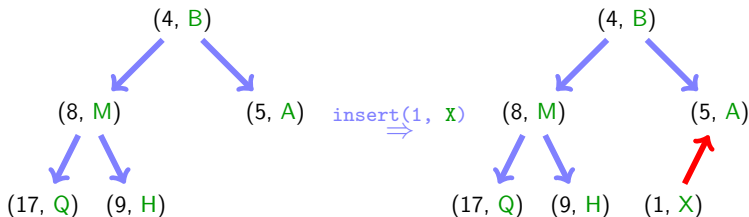
### Storing a binary heap:

- ▶ Number nodes from top to bottom and left to right starting with 0 and store entries in array
- ▶ Children of node  $i$  are the nodes  $2i + 1$  and  $2i + 2$
- ▶ Parent node of node  $i$  is  $\text{floor}((i - 1)/2)$

# Priority Queue

## Implementation - Insertion

Inserting an element: `insert(key, item)`



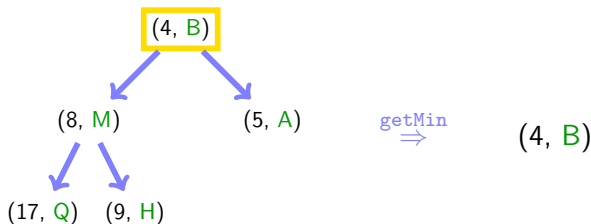
- ▶ Append the element at the end of the array
- ▶ The **heap condition** may be violated, but only at the last index
- ▶ Repair **heap condition**  $\Rightarrow$  We will see later how to do this



# Priority Queue

## Implementation

Returning the minimum: `getMin()`



- ▶ Else return the first element
- ▶ If the heap is empty return `None`

# Priority Queue

## Implementation

Removing the minimum: `deleteMin()`



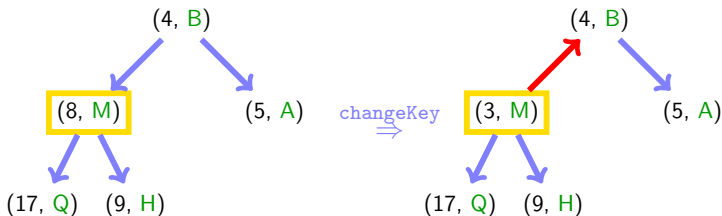
- ▶ Deleting the element with the lowest key
- ▶ Swap the last element with the first element and shrink the heap by one
- ▶ The **heap condition** may be violated, but only at the first index
- ▶ Repair **heap condition**

# Priority Queue

## Implementation

**Changing the key (priority):** `changeKey(item, key)`

- ▶ The element (queue item) is given as argument
- ▶ Replace the key of the element
- ▶ The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- ▶ Repair **heap condition**



# Priority Queue

## Implementation

**Changing the key (priority):** `changeKey(item, key)`



- ▶ The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- ▶ Repair **heap condition**

# Priority Queue

## Implementation

**Removing an element:** `remove(item)`



- ▶ The element (queue item) is given as argument
- ▶ Replace the element with the last element and shrink the heap by one
- ▶ The **heap condition** may be violated, but only at the element index and only in one direction (up / down)
- ▶ Repair **heap condition**

# Priority Queue

## Implementation - Repairing the Heap

### Repairing after modifying operations:

- ▶ The heap condition can be violated after using `insert`, `deleteMin`, `changeKey`, `remove`, but only at one known position with index  $i$
- ▶ Heap conditions can be violated in two directions:
  - ▶ Downwards: The key at index  $i$  is not  $\leq$  than the value of its children
  - ▶ Upwards: The key at index  $i$  is not  $\geq$  than the value of its parent
- ▶ We need two repair methods: `repairHeapUp`, `repairHeapDown`

# Priority Queue

## Implementation - Repairing the Heap

repairHeapDown:

- ▶ Sift the element until the **heap condition** is valid
  - ▶ Change node with child, which has the lower key of both children
  - ▶ If the **heap condition** is violated repeat for the child node



Figure: Repairing the heap downwards

# Priority Queue

## Implementation - Repairing the Heap

`repairHeapDown:`

- ▶ Sift the element until the **heap condition** is valid
  - ▶ Change node with child, which has the lower key of both children
  - ▶ If the **heap condition** is violated repeat for the child node



Figure: Repairing the heap downwards



# Priority Queue

## Implementation - Repairing the Heap

repairHeapUp:

- Change node with parent
- If the **heap condition** is violated repeat for parent node



Figure: Repairing the heap upwards

# Priority Queue

## Implementation - Repairing the Heap

`repairHeapUp:`

- Change node with parent
- If the **heap condition** is violated repeat for parent node



Figure: Repairing the heap upwards

# Priority Queue

## Implementation - Priority Queue Item

### Index of a priority queue item:

- ▶ **Attention:** For `changeKey` and `remove` the item has to “know” where it is located in the heap
- ▶ Remember for `repairHeapUp` and `repairHeapDown`:  
Update the index if moving an heap element

# Priority Queue

## Implementation - Priority Queue Item - Python

```
class PriorityQueueItem:

    """Provides a handle for a queue item.

    This handle can be used to remove or
    update the queue item.
    """

    def __init__(self, key, value, index):
        self.key = key
        self.value = value
        self.index = index
```

# Priority Queue

## Complexity

### Summary lecture 1:

- ▶ A full binary tree with  $n$  elements, has a depth of  $O(\log n)$
- ▶ The maximum distance from the root to a leaf can be  $O(\log n)$  elements
- ▶ Repairing the heap upwards and downwards:  
We have only one path to traverse:  $O(\log n)$

### Runtime for methods

- ▶ `insert`, `deleteMin`, `changeKey`, `remove`:  
We have to repair the heap:  $O(\log n)$
- ▶ `getMin`: Return the element at index 0:  $O(1)$

# Priority Queue

## Complexity

### Improvements (Fibonacci heaps):

- ▶ `getMin`, `insert` and `decreaseKey` in amortized time of  $O(1)$
- ▶ `deleteMin` in amortized time  $O(\log n)$

### Practical experience:

- ▶ The binary heap is simpler: Costs for managing the structure are low
- ▶ If the number of elements is relatively small so the difference is negligible
- ▶ Example:
  - ▶ For  $n = 2^{10} \approx 1,000$  is the the `depth`  $\log_2 n$  only 10
  - ▶ For  $n = 2^{20} \approx 1,000,000$  is the `depth`  $\log_2 n$  only 20

# Further Literature

## ► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

# Further Literature

## ► Priority Queue - Implementations / API

[Cpp] [C++ - priority\\_queue](#)

`http:`

`//www.sgi.com/tech/stl/priority_queue.html`

[Jav] [Java - PriorityQueue](#)

`https://docs.oracle.com/javase/7/docs/api/  
java/util/PriorityQueue.html`

[Pyt] [Python - PriorityQueue](#)

`https://docs.python.org/3/library/queue.  
html#queue.PriorityQueue`