Algorithms and Datastructures

Graphs, Depth-/Breadth-first Search, Graph-Connectivity

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, January 2017

Structure

Feedback

- Exercises
- Lecture

Graphs

- Introduction
- Implementation
- Application example

Structure

Feedback from the exercises

The upcoming exercise sheet 12 and 13 will be merged together (finding largest connected component + Dijkstra)

Some people were asking for more solution sheets for the exercises

We are working on it.

Feedback from the lecture

Code in the lecture will be a litte bit different from exercise sheet.

One person asked for additional explanations regarding proofs.

Structure

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Graphs - Overview:

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- Besides arrays, lists and trees the most common datastructure (Trees are a special type of graph)

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Graphs

Introduction

Terminology:

Graphs

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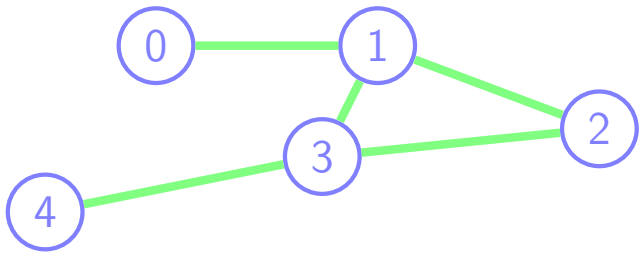

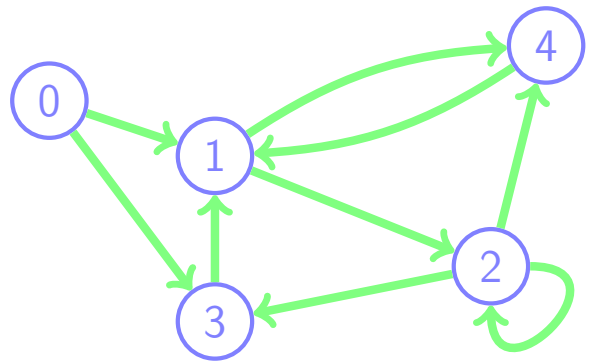

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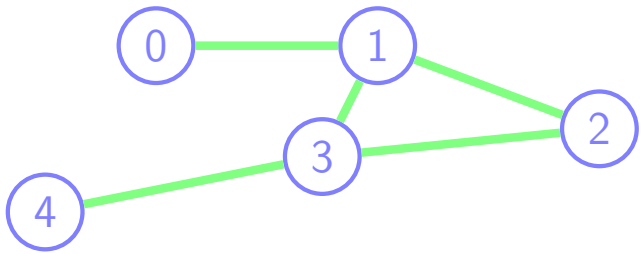

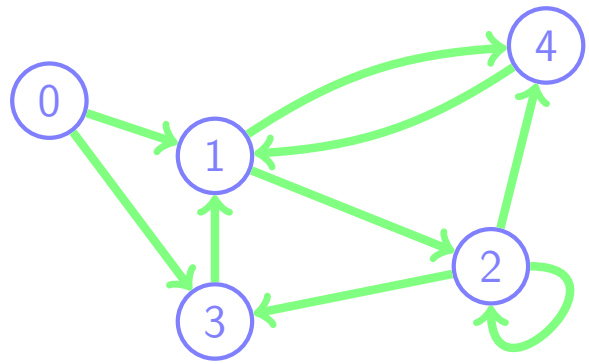

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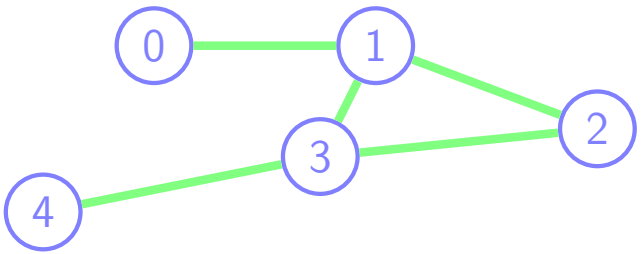

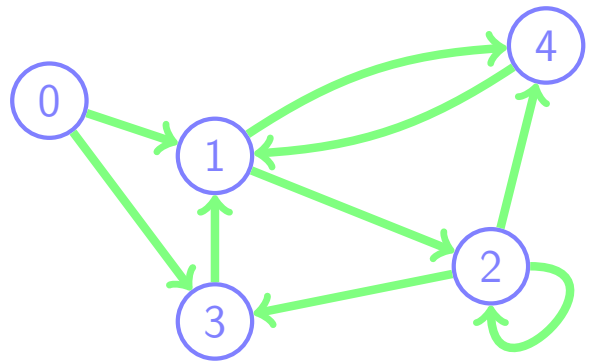

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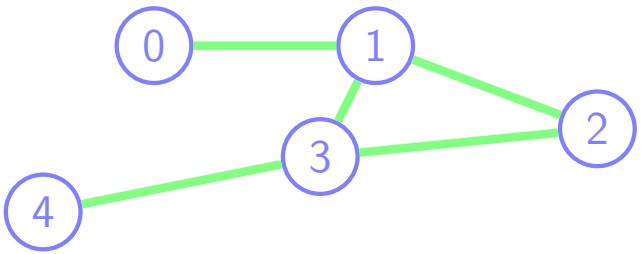

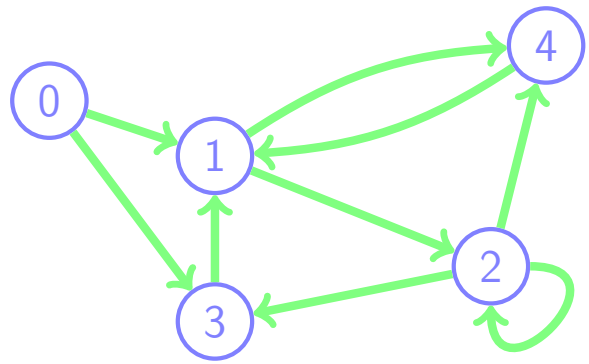

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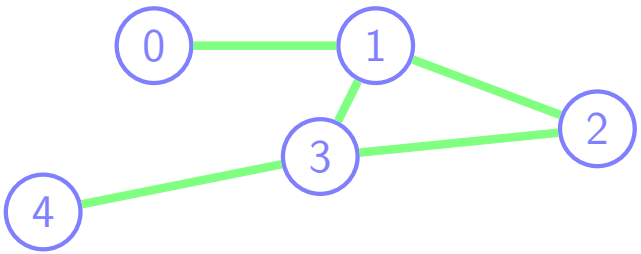

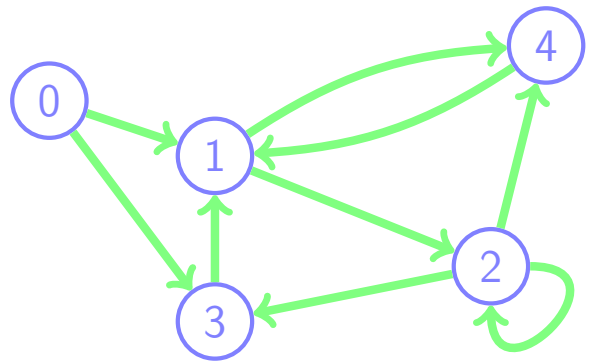

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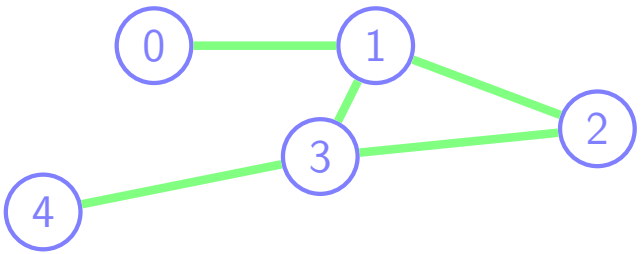

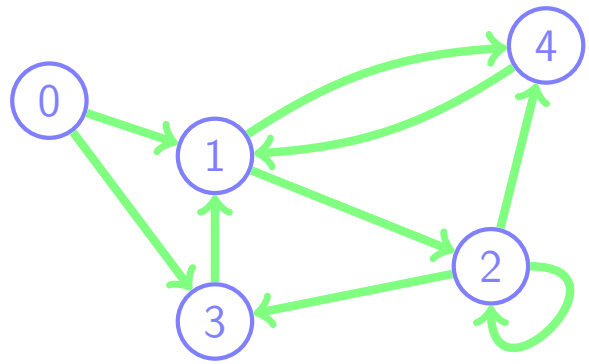

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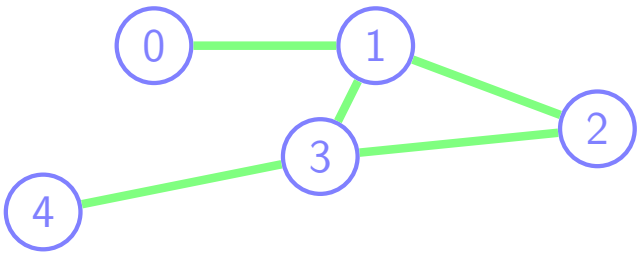

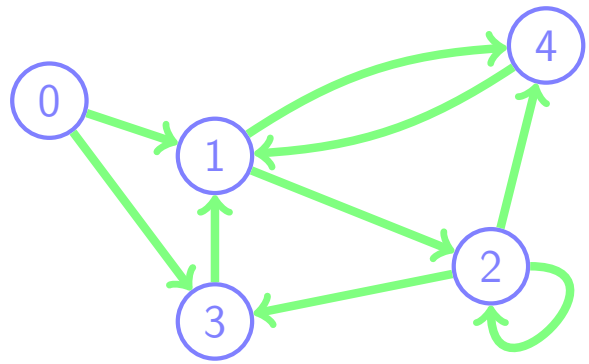

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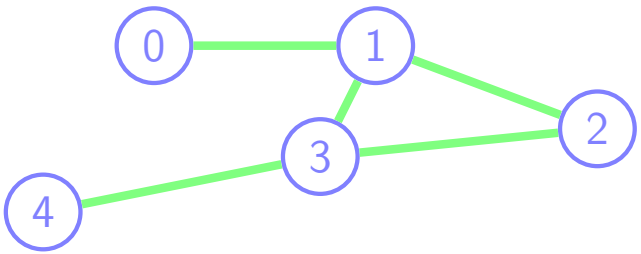

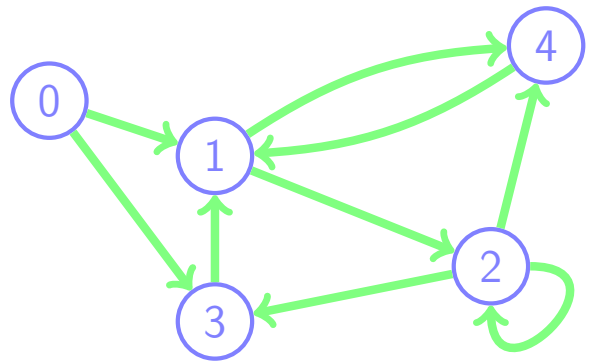

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- Self-loops are also possible: $e = (u, u)$ or $e = \{u, u\}$

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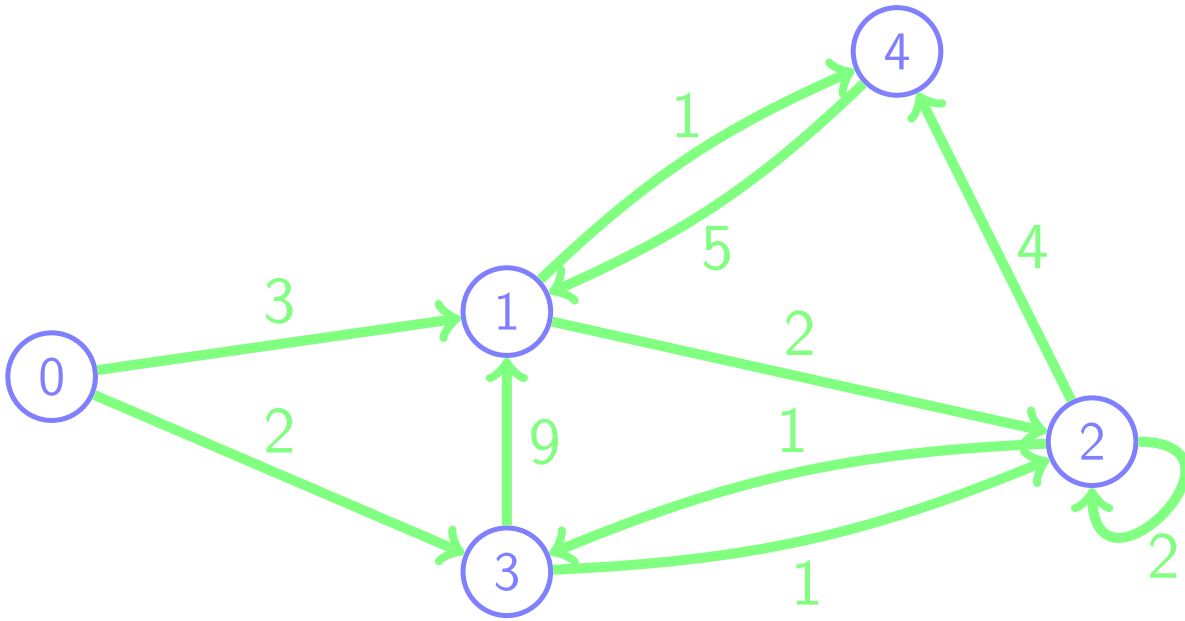

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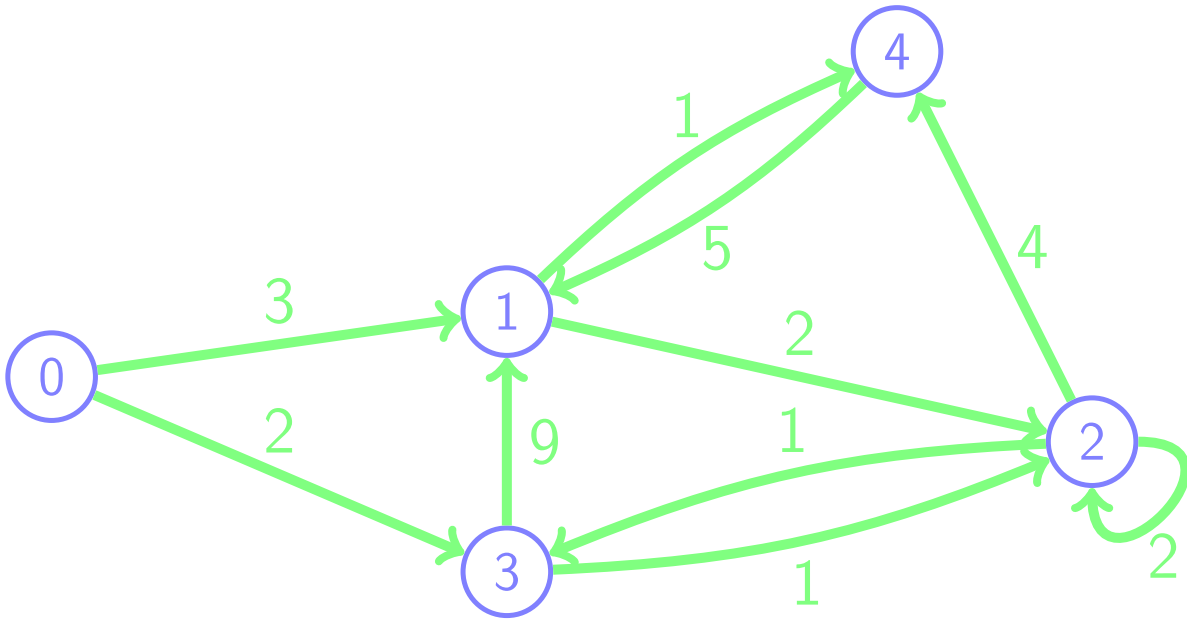

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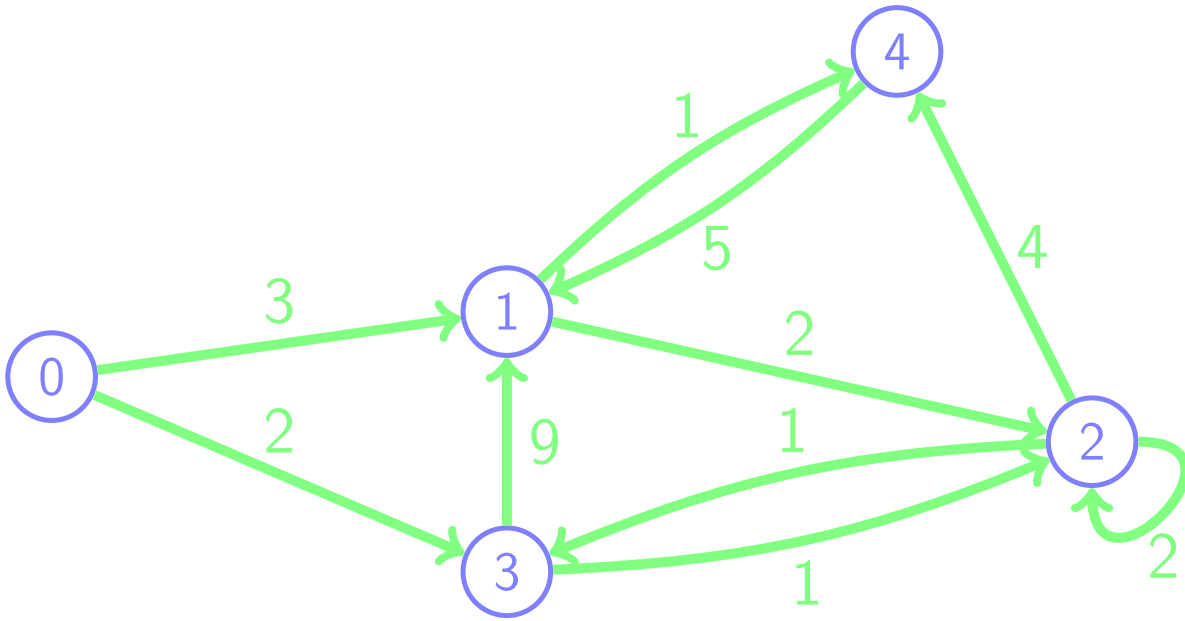

- Each edge is marked with a real number named weight
- The weight is also named length or cost of the edge depending on the application

Example: Road network

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- Intersections: vertices

Graphs

Introduction

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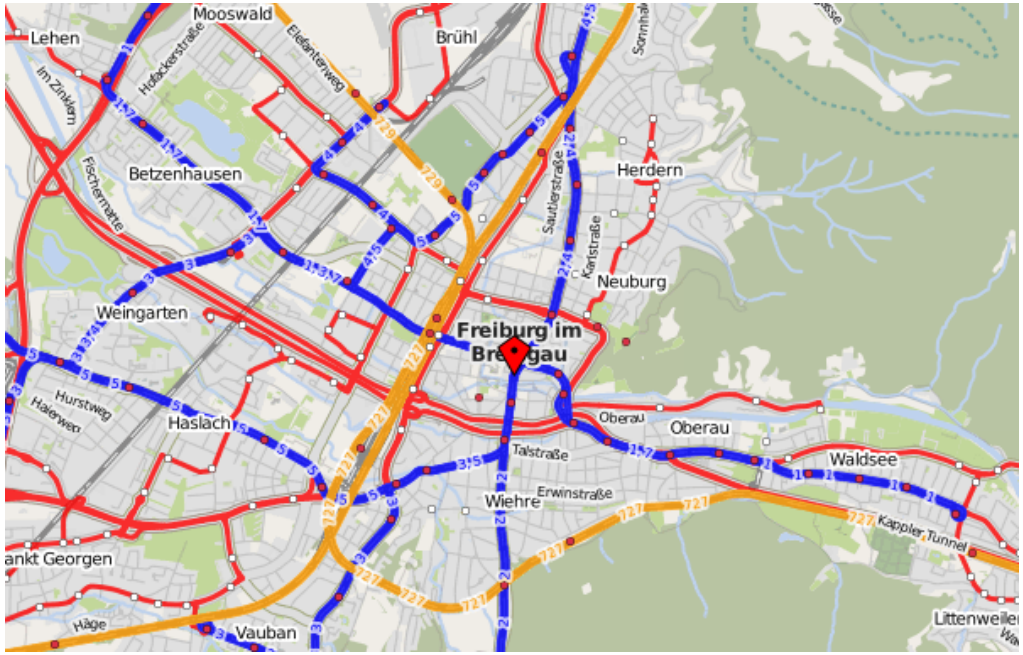

Figure: Map of Freiburg © OpenStreetMap

Structure

How to represent this graph computationally?

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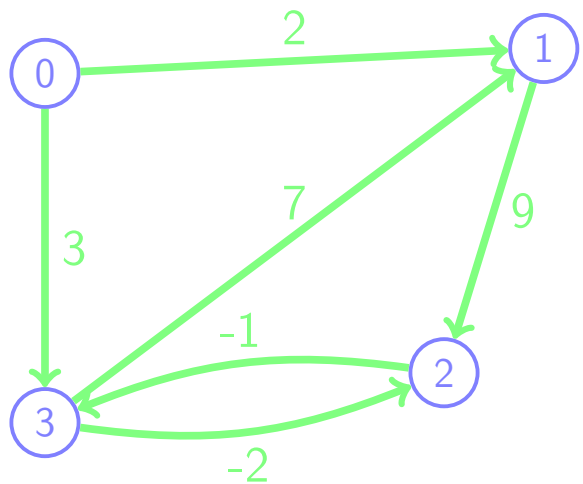

Figure: Weighted graph with $|V| = 4$, $|E| = 6$

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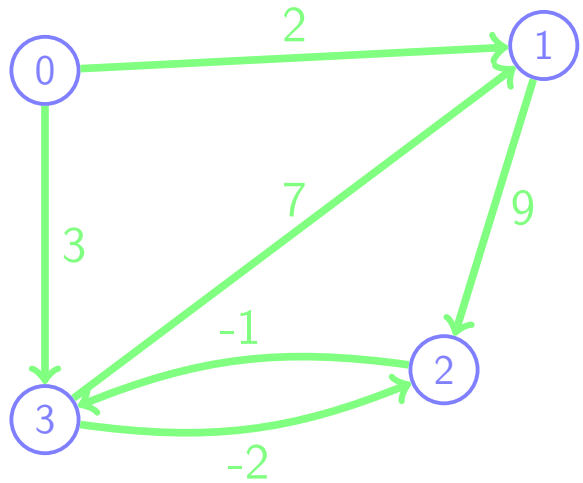

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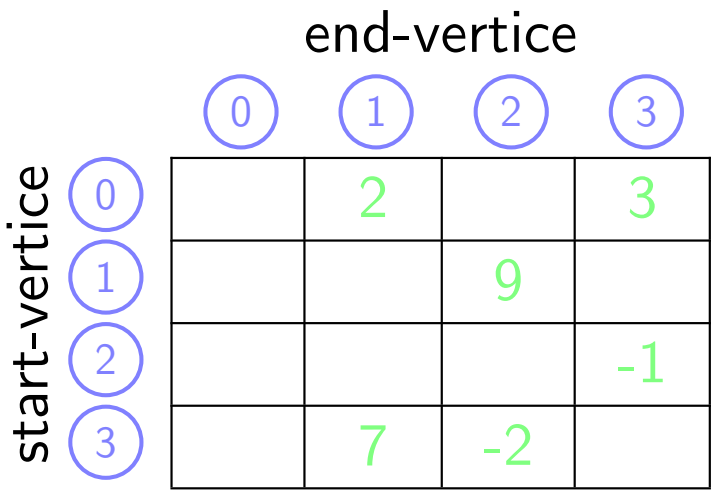

Figure: Adjacency matrix

Graphs

Implementation

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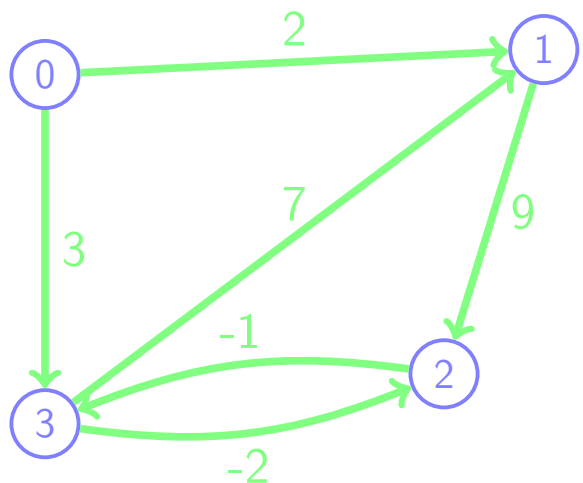

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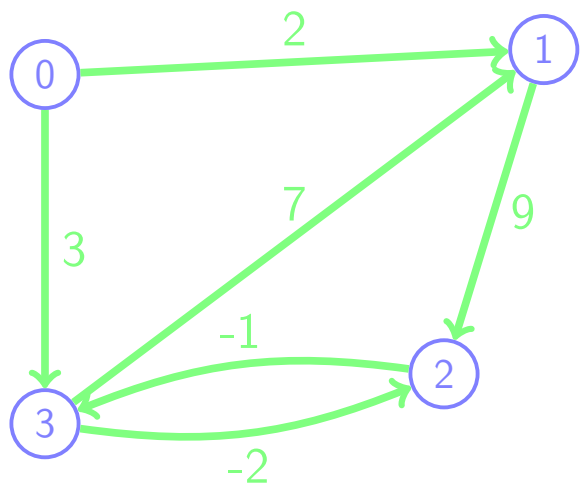

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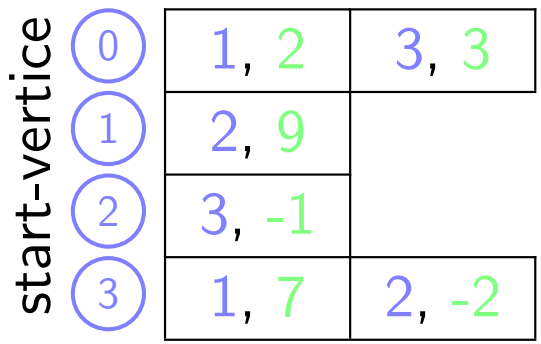

Figure: Adjacency list

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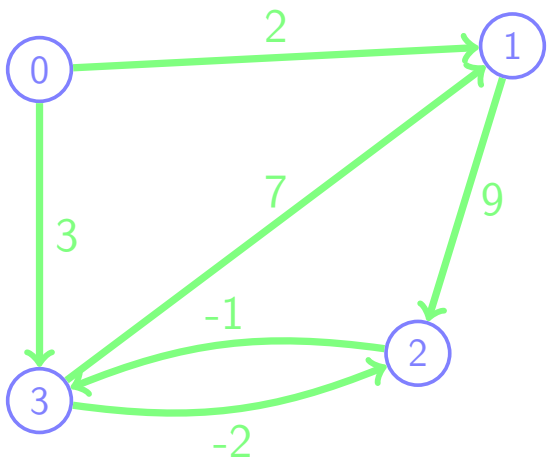

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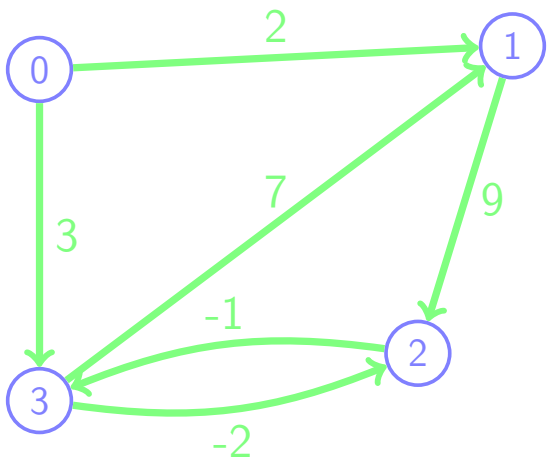

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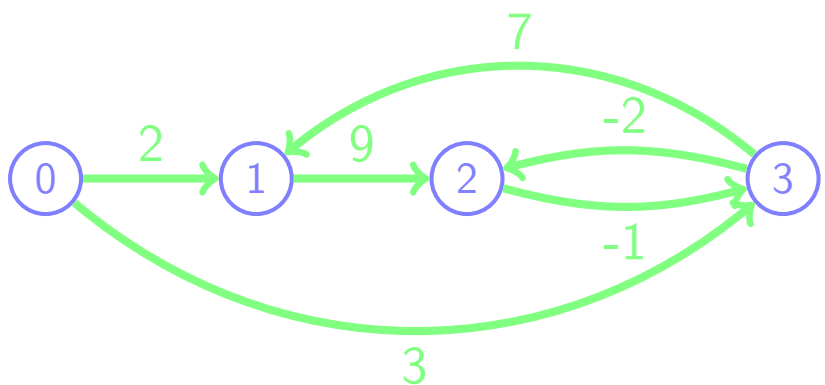

Figure: Same graph ordered by number - outer planar graph

Graphs

Implementation - Python

```
class Graph:
    def __init__(self):
        self.vertices = []
        self.edges = []

    def addVertice(self, vert):
        self.vertices.append(vert)

    def addEdge(self, fromVert, toVert):
        self.edges.append((fromVert, toVert))

    ...
```

Graphs

Implementation - Python

```
...

def toString(self):
    return '{'
        + ', '.join( \
            [str(len(self.vertices)), \
                str(len(self.edges))] \
            + ["(%s, %s)" % tup \
                for tup in self.edges]) \
        + '}'
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Degree of a vertex: Directed graph: $G = (V, E)$

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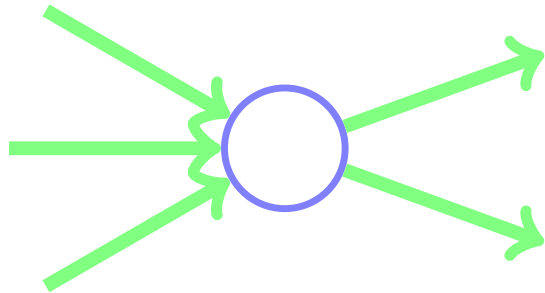

Figure: Vertex with in- / outdegree of 3 / 2

Graphs

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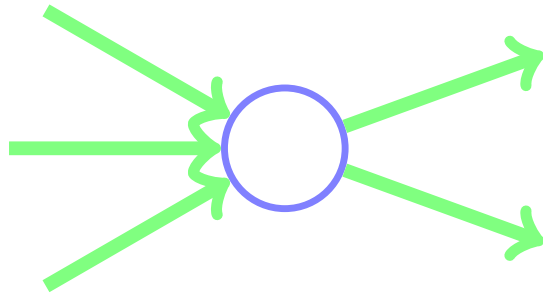

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- Indegree of a vertex u is the number of edge heads adjacent to the vertex

$$\deg^+(u) = |\{(v, u) : (v, u) \in E\}|$$

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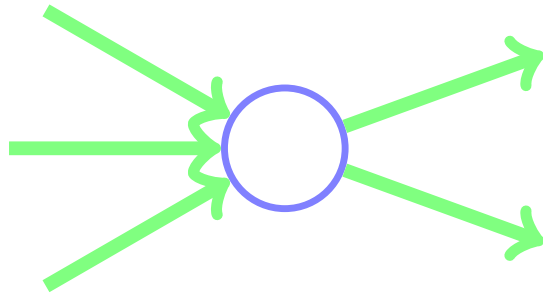

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- Outdegree of a vertex u is the number of edge tails adjacent to the vertex

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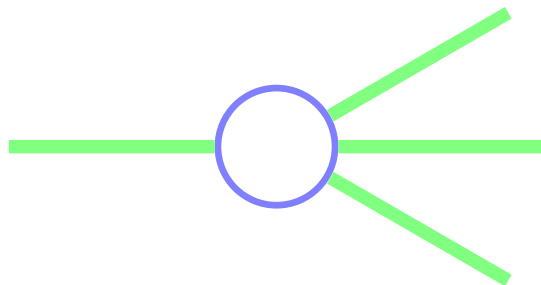
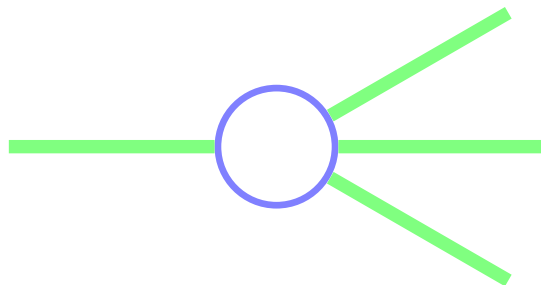


Figure: Vertex with degree of 4

Degree of a vertex: Undirected graph: $G = (V, E)$



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Paths in a graph: $G = (V, E)$

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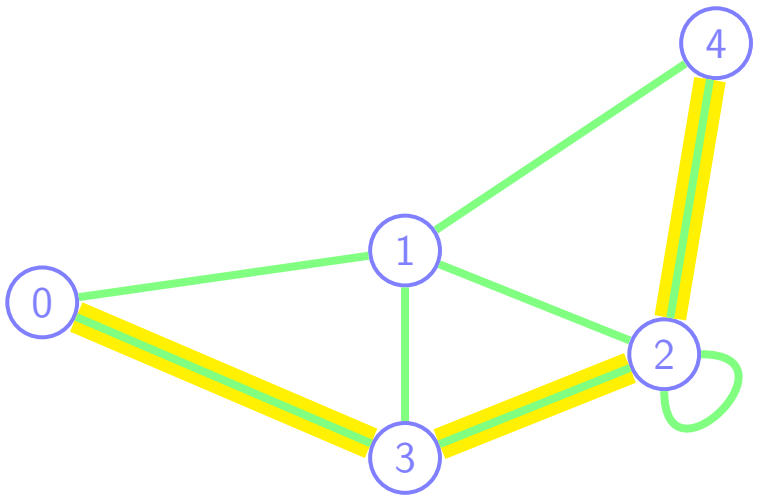

Figure: Undirected path of length 3
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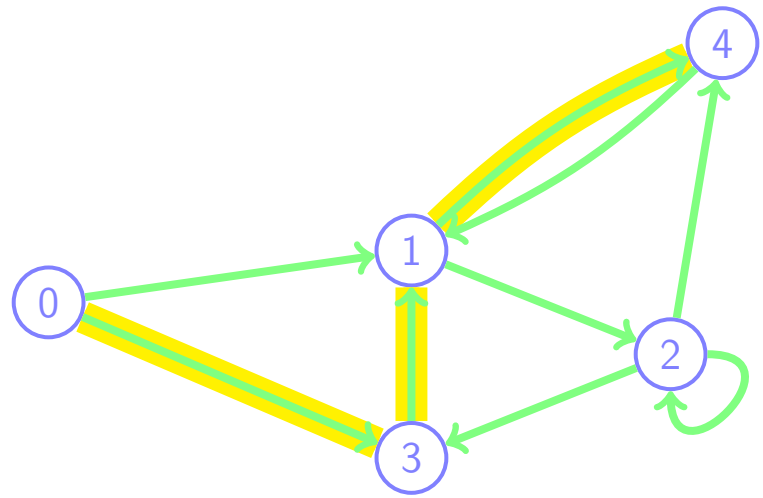

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Graphs

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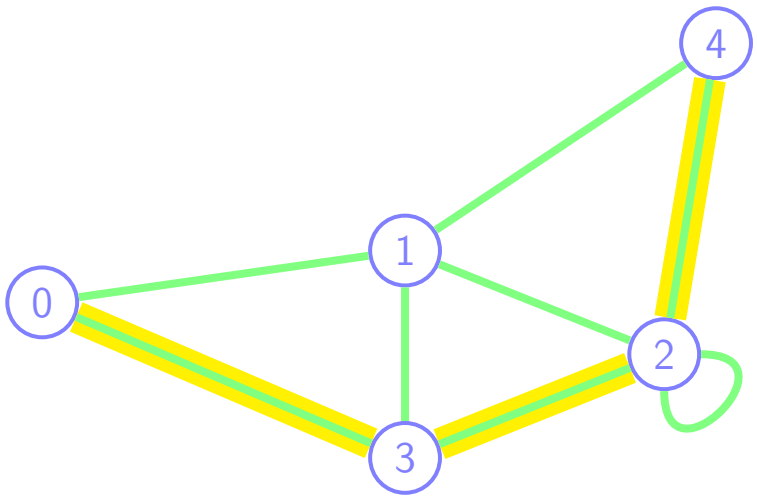

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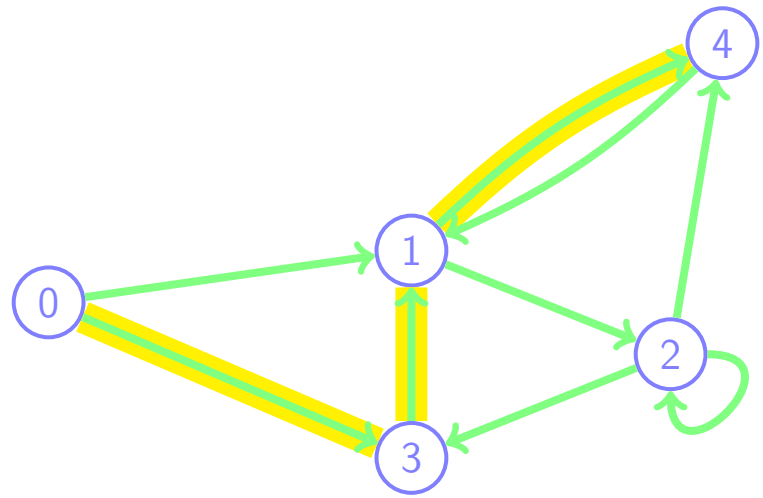

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Graphs

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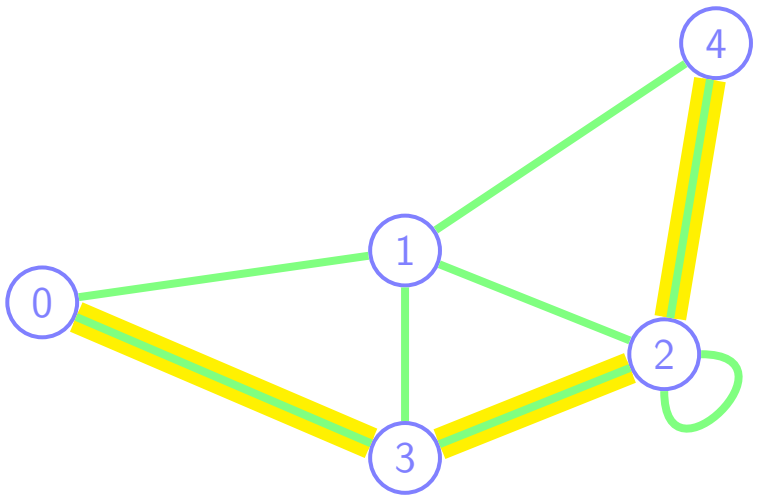

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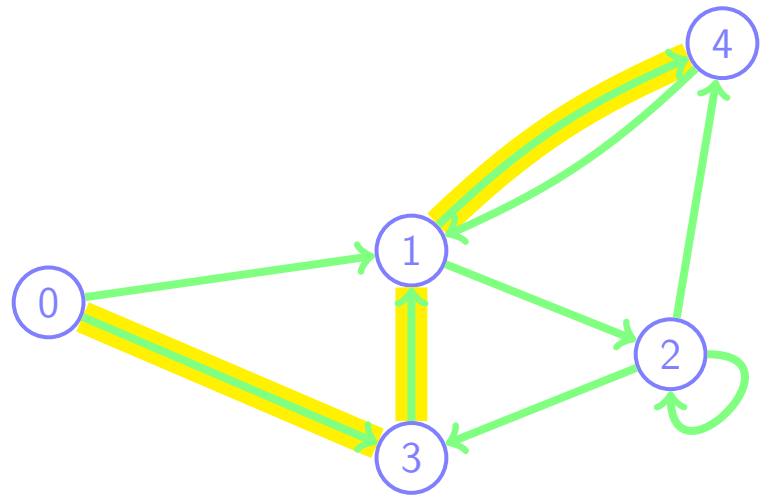

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Graphs

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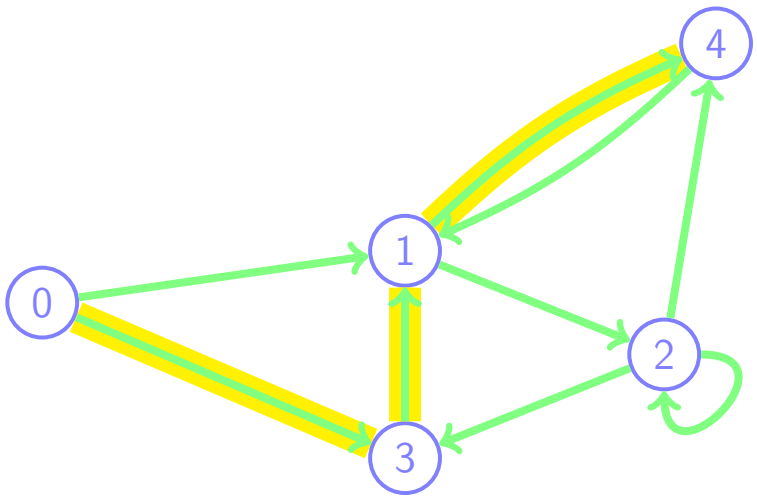

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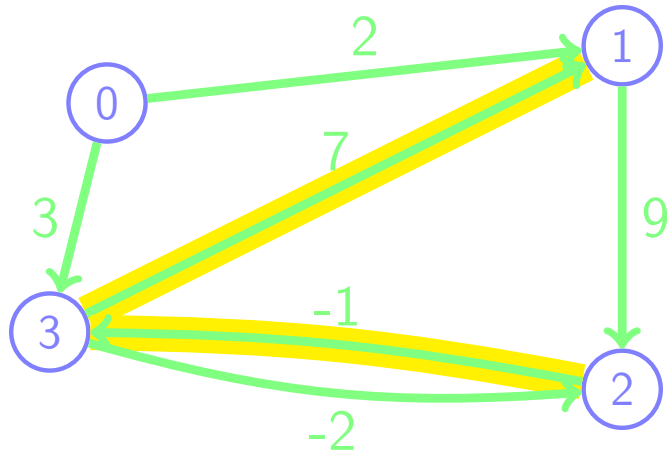

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Graphs

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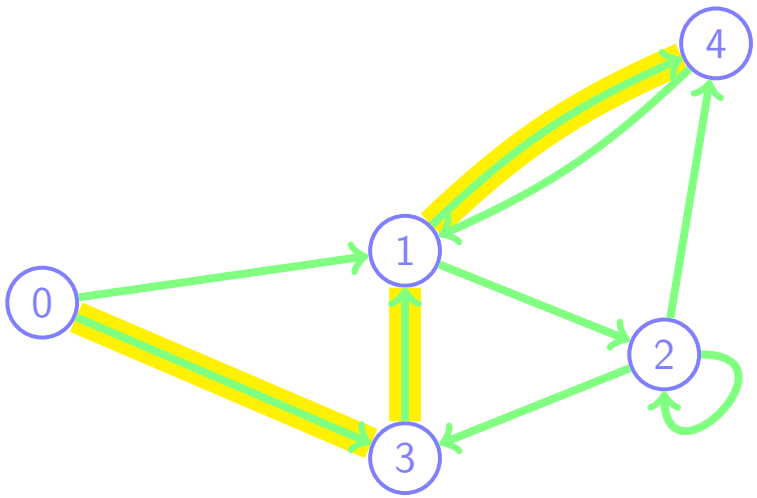

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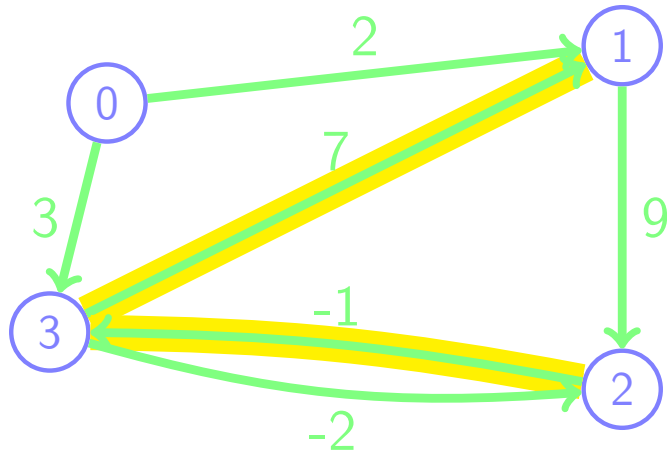

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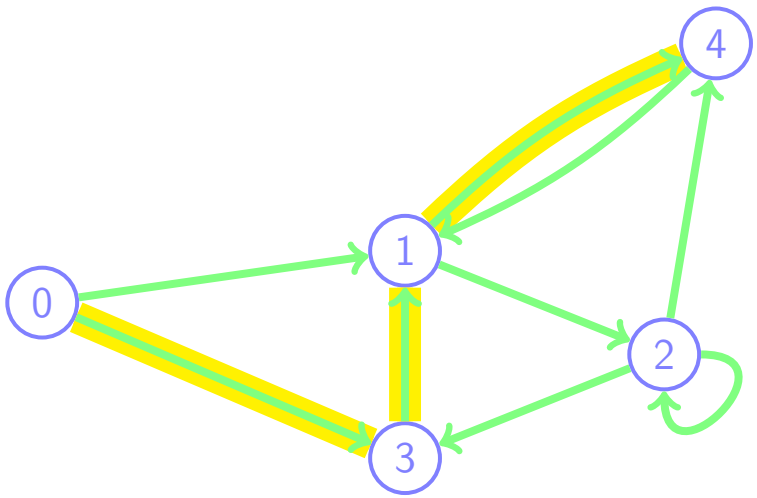

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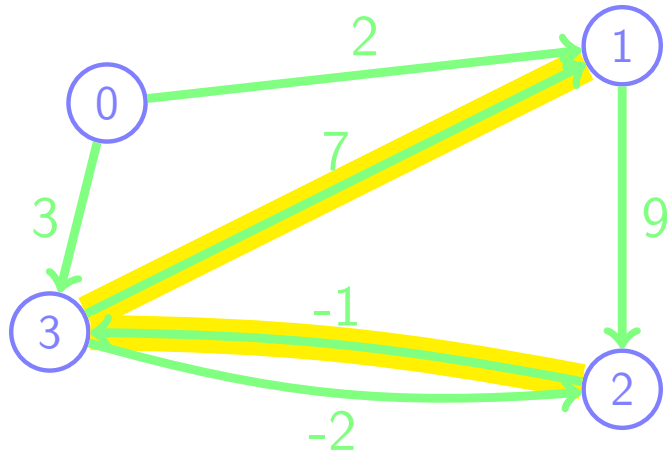

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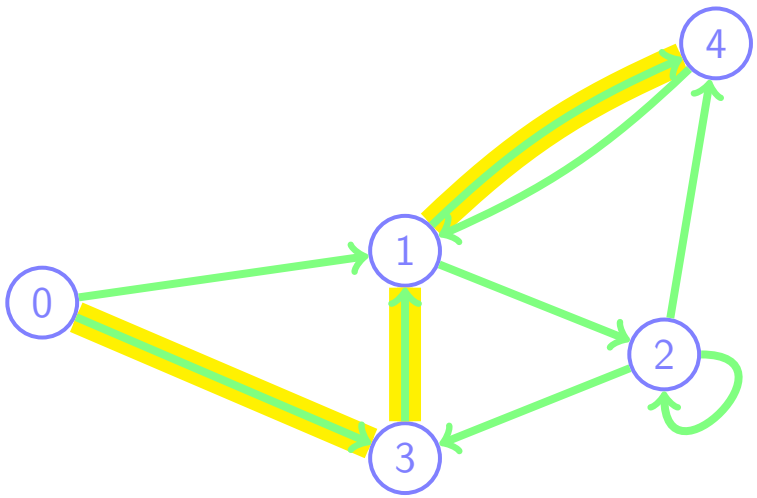

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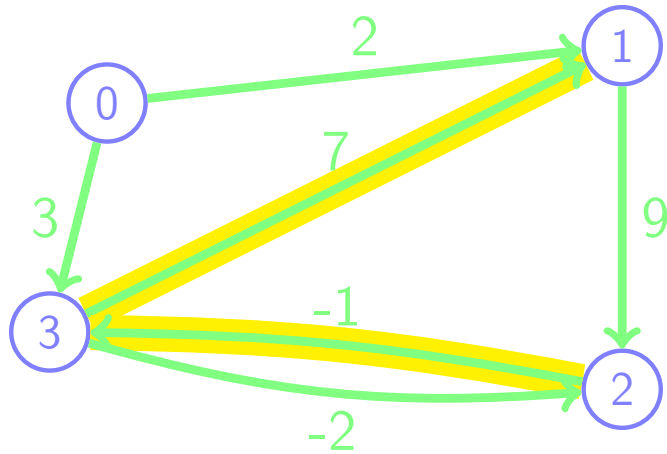

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Graphs

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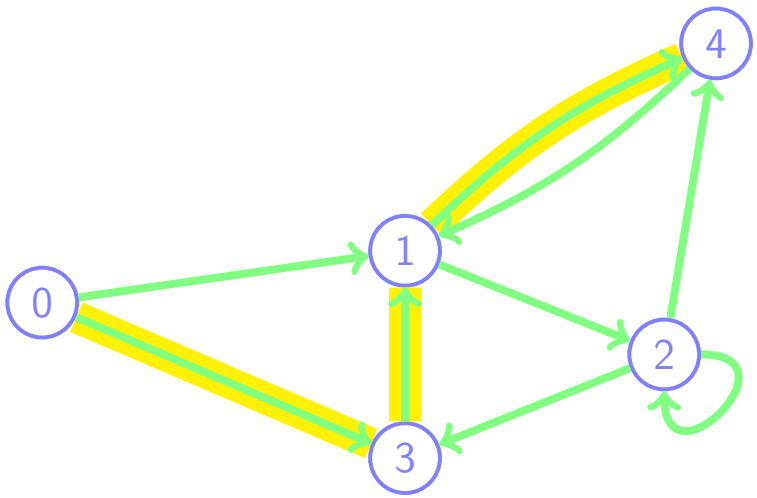

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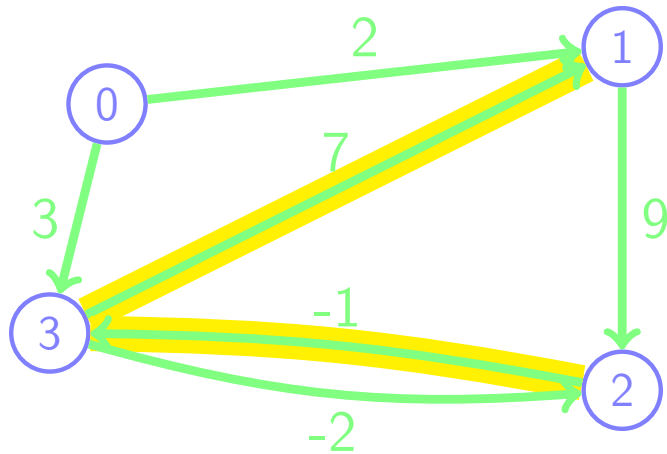

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Graphs

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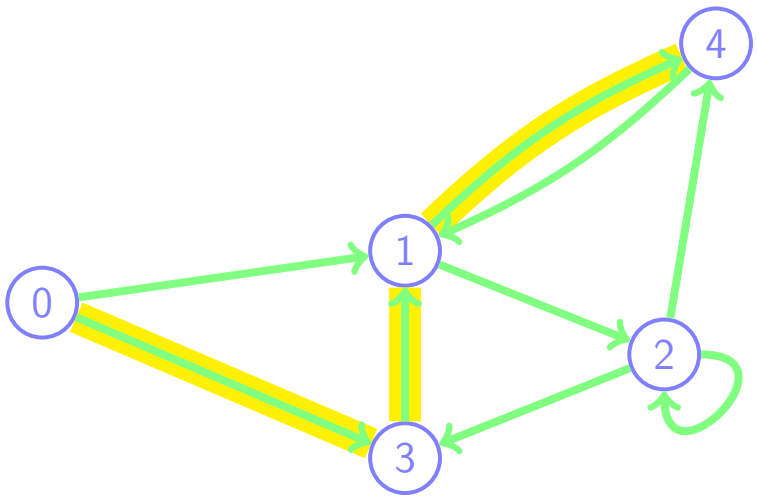

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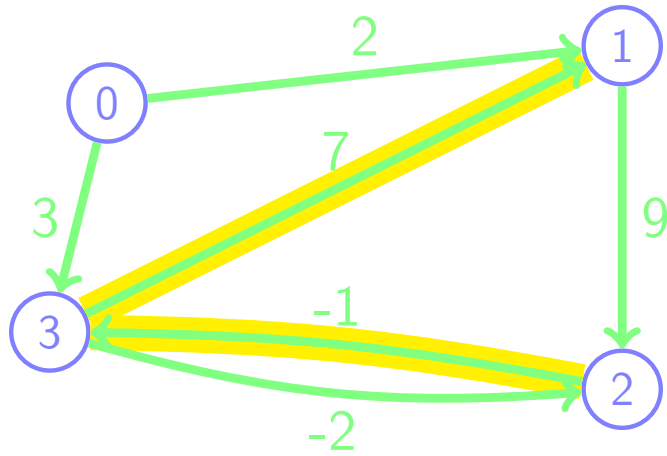

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- The length of a path is: (also costs of a path)
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  - With weights: sum of weigths of edges taken

Graphs

Paths

Shortest path in a graph: $G = (V, E)$

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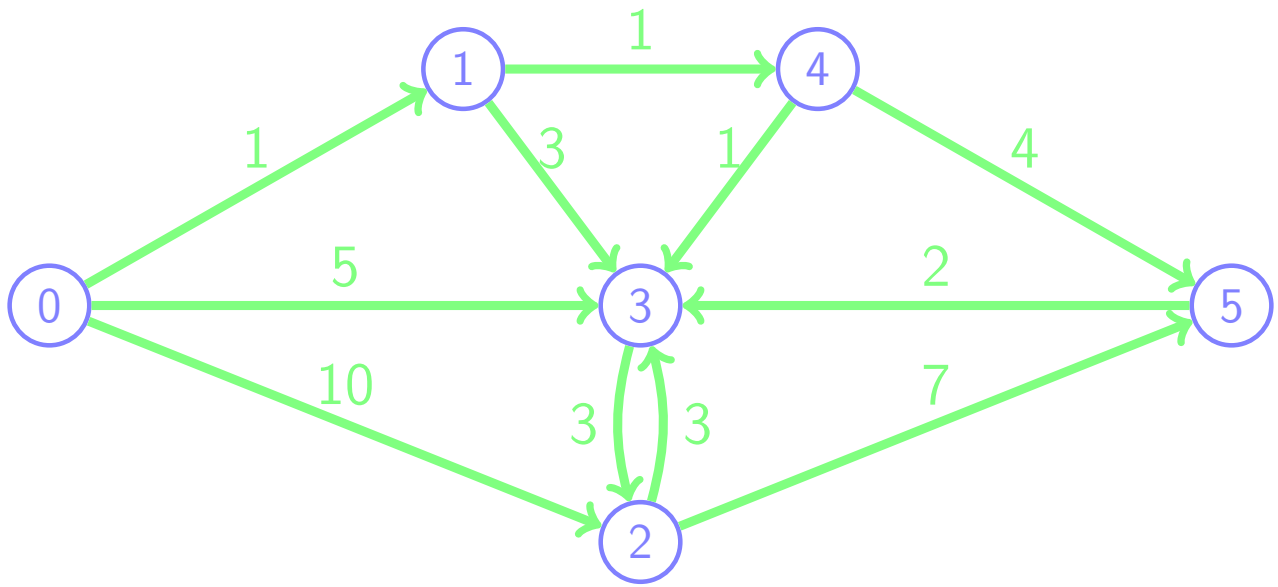
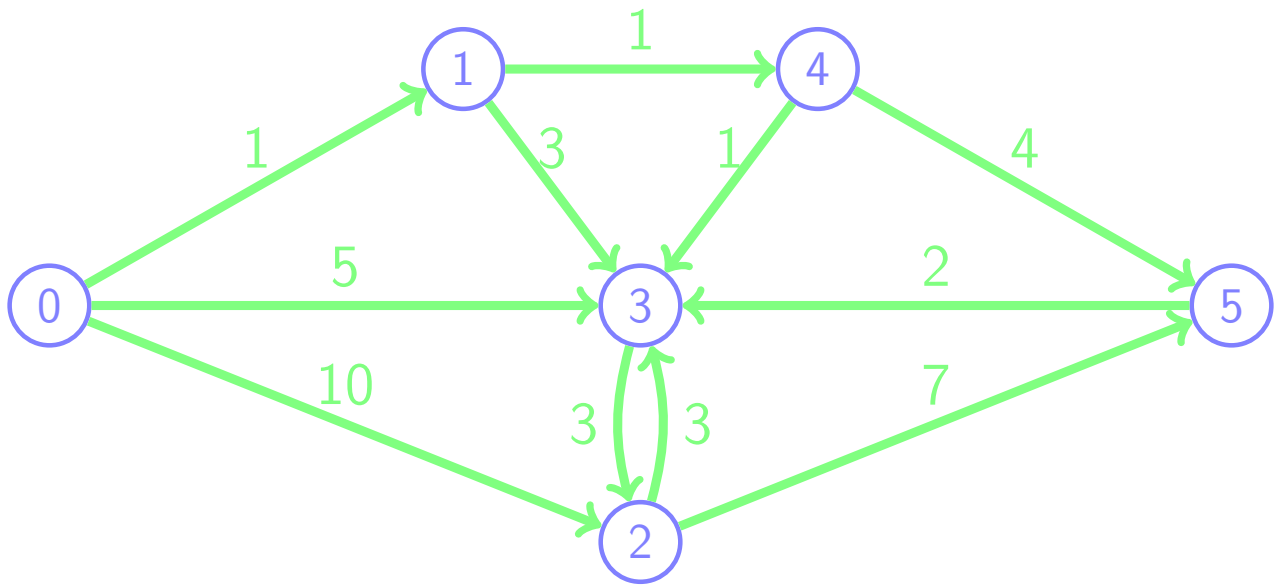


Figure: Shortest path from 0 to 2 with cost / distance $d(0, 2) = ?$

Shortest path in a graph: $G = (V, E)$



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- The shortest path between two vertices u, v is the path $P = (u, \ldots, v)$ with the shortest length $d(u, v)$ or lowest costs

Graphs

Paths

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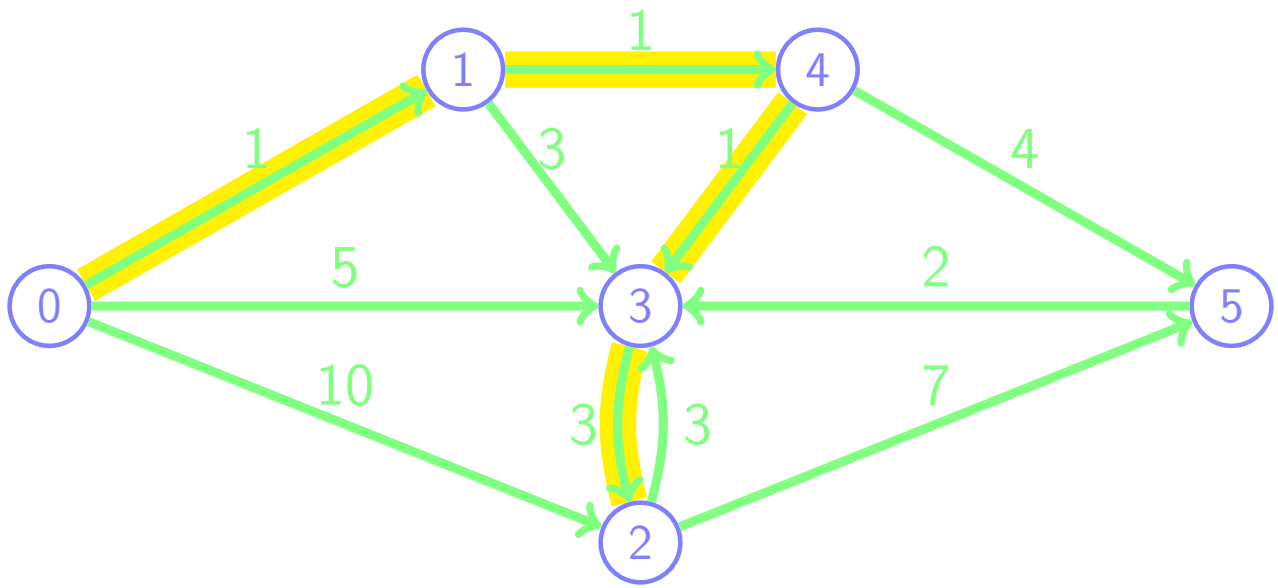

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$P = (0, 1, 4, 3, 2)$

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Diameter of a graph: $G = (V, E)$

Graphs

Paths

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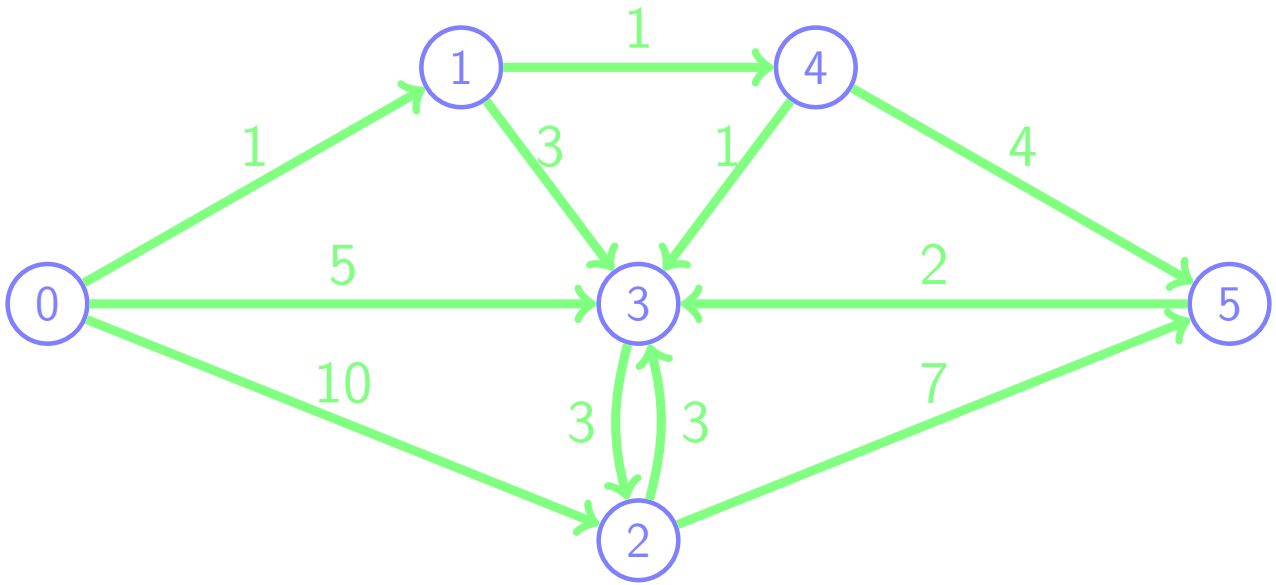

Figure: Diameter of graph is $d =$?

Graphs

Paths

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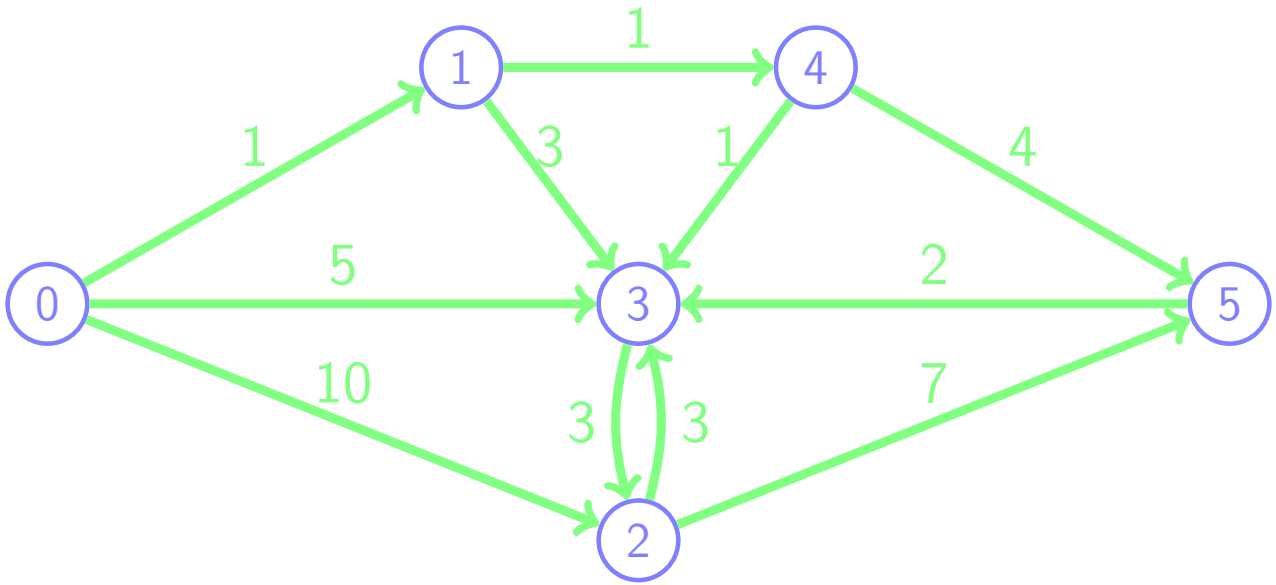

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Graphs

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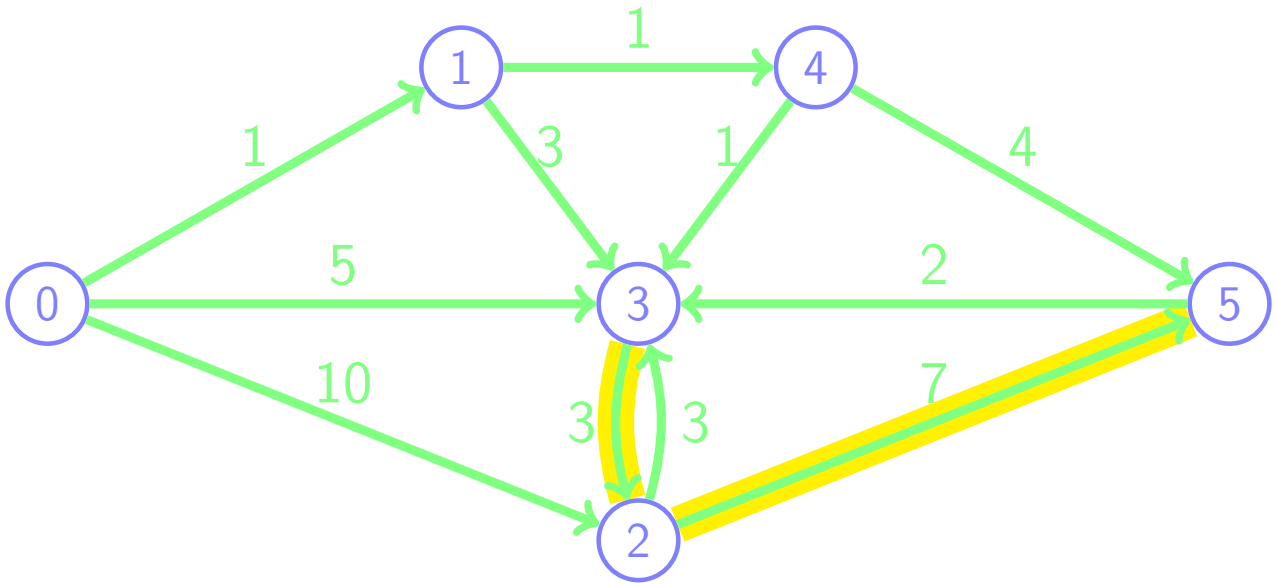

Figure: Diameter of graph is $d = 10$, $P = (3, 2, 5)$

- The diameter of a graph is the length / the costs of the longest shortest path

Graphs

Connected Components

Connected components: $G = (V, E)$

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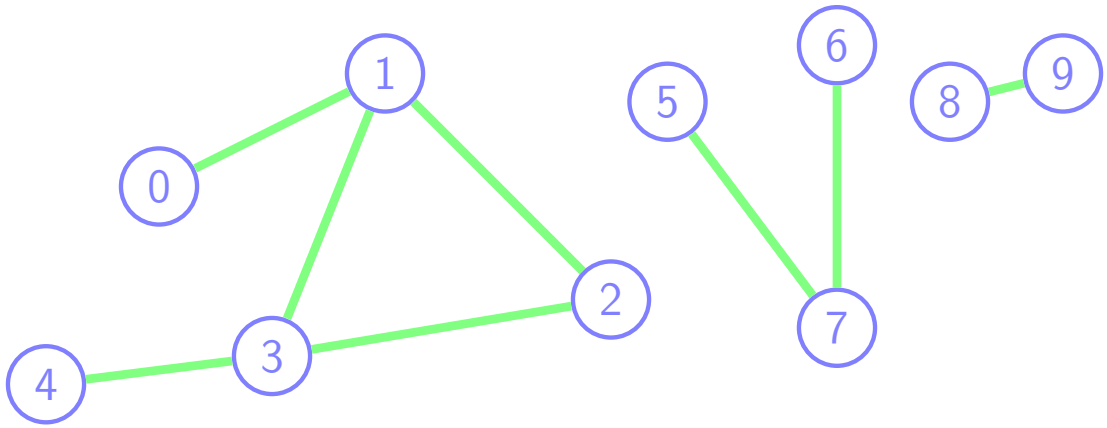

Figure: Three connected components

- Undirected graph:

Graphs

Connected Components

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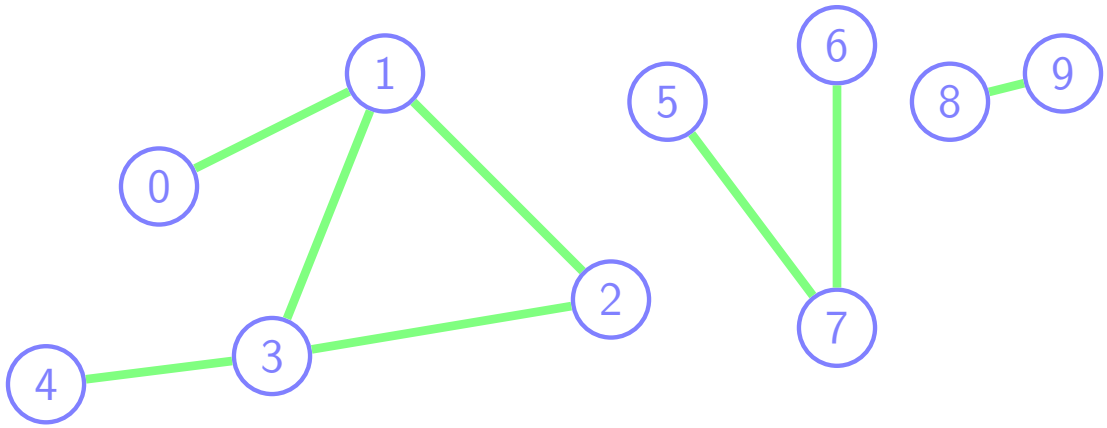

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$$V = V_1 \cup \cdots \cup V_k$$

Graphs

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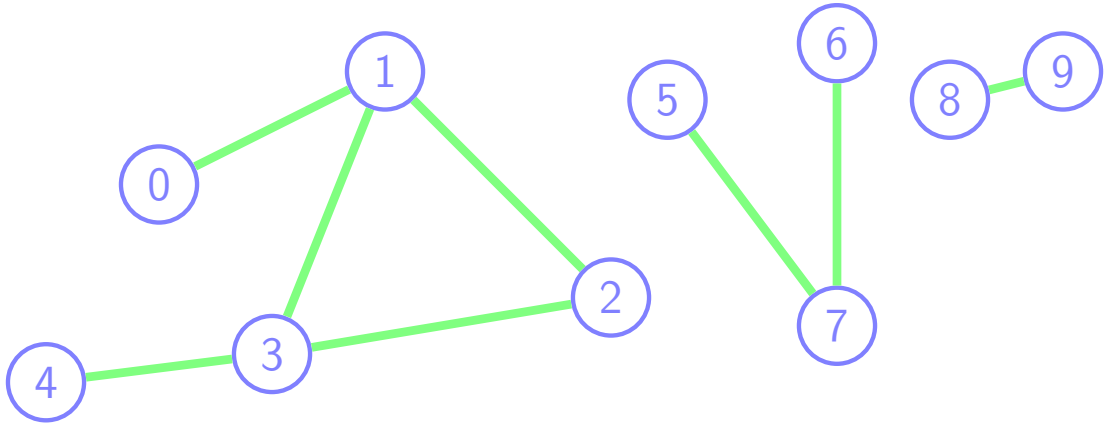

Figure: Three connected components

- Undirected graph:
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$$V = V_1 \cup \cdots \cup V_k$$

  - Two vertices u, v are in the same connected component if a path between u and v exists

Graphs

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  - Not part of this lecture

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- Let $G = (V, E)$ be a graph and $s \in V$ a start vertex

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- We visit each reachable vertex connected to s
- Breadth-first search: in sequence of the smallest distance to s
- Depth-first search: in sequence of the largest distance to s
- Not a problem on its own but is often used as subroutine of other algorithms

Graphs

Connected Components - Breadth-First Search

Idea:

Graphs

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5. Iteratively mark reachable vertices for all levels
6. All connected nodes are now marked and in the same connected component as the start vertex s

Graphs

Connected Components - Breadth-First Search

Graphs

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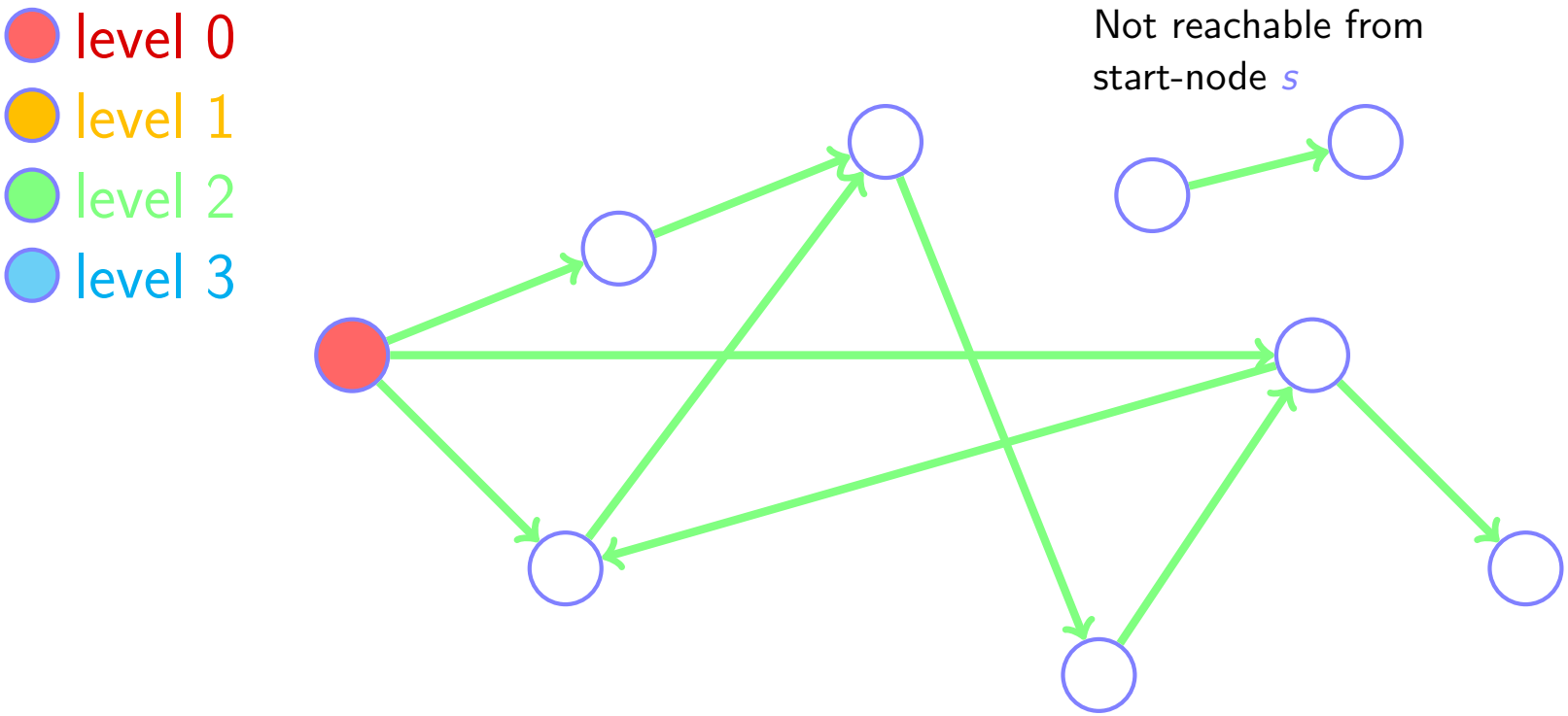

Figure: spanning tree of a breadth-first search

Graphs

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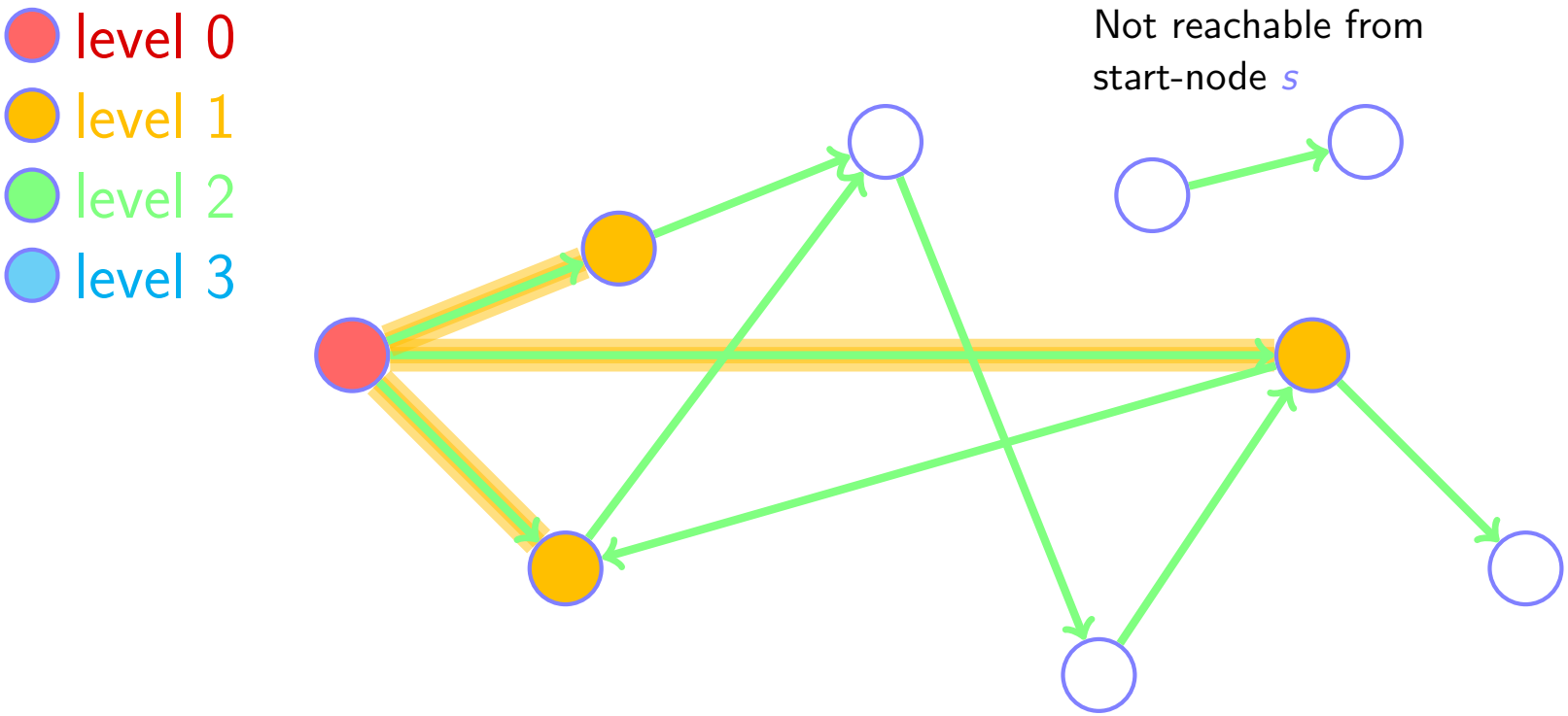

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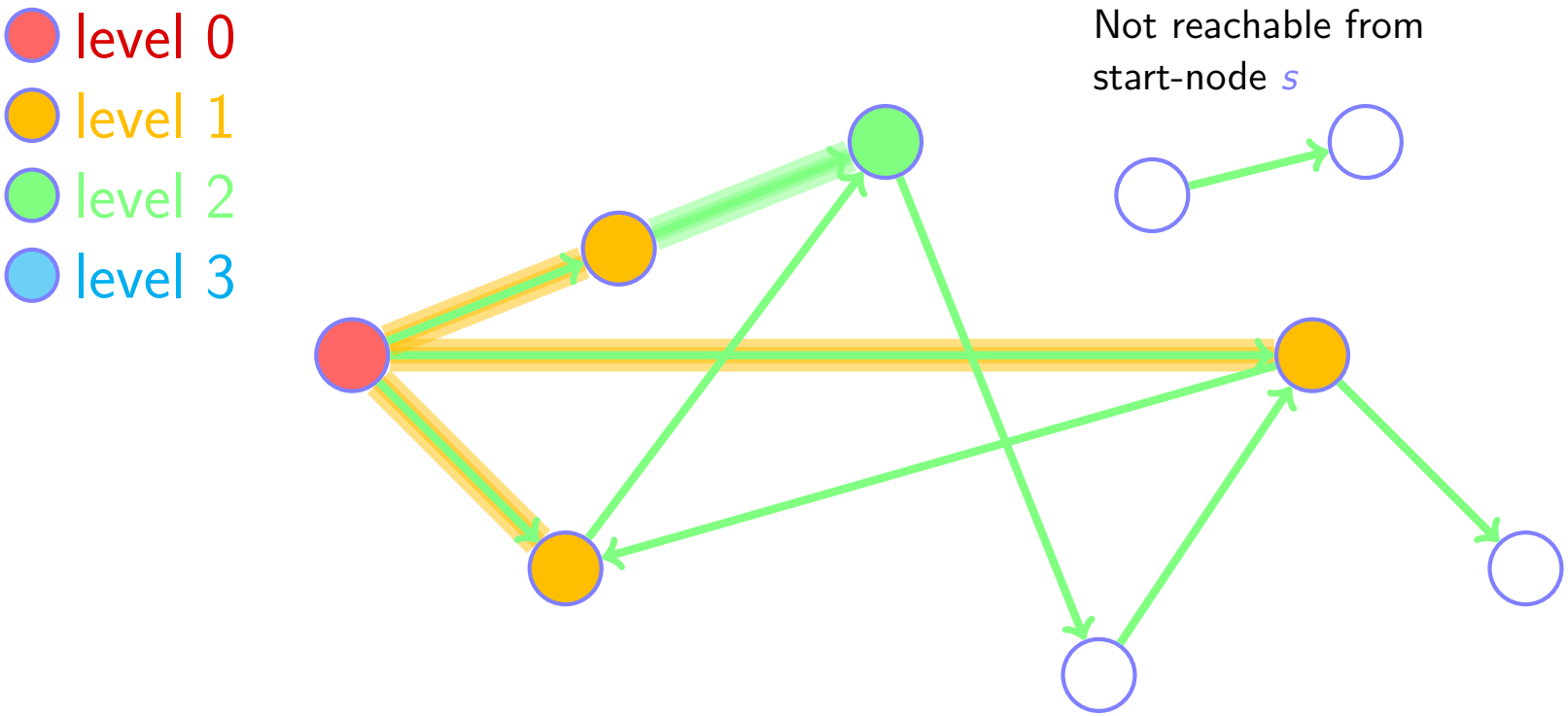

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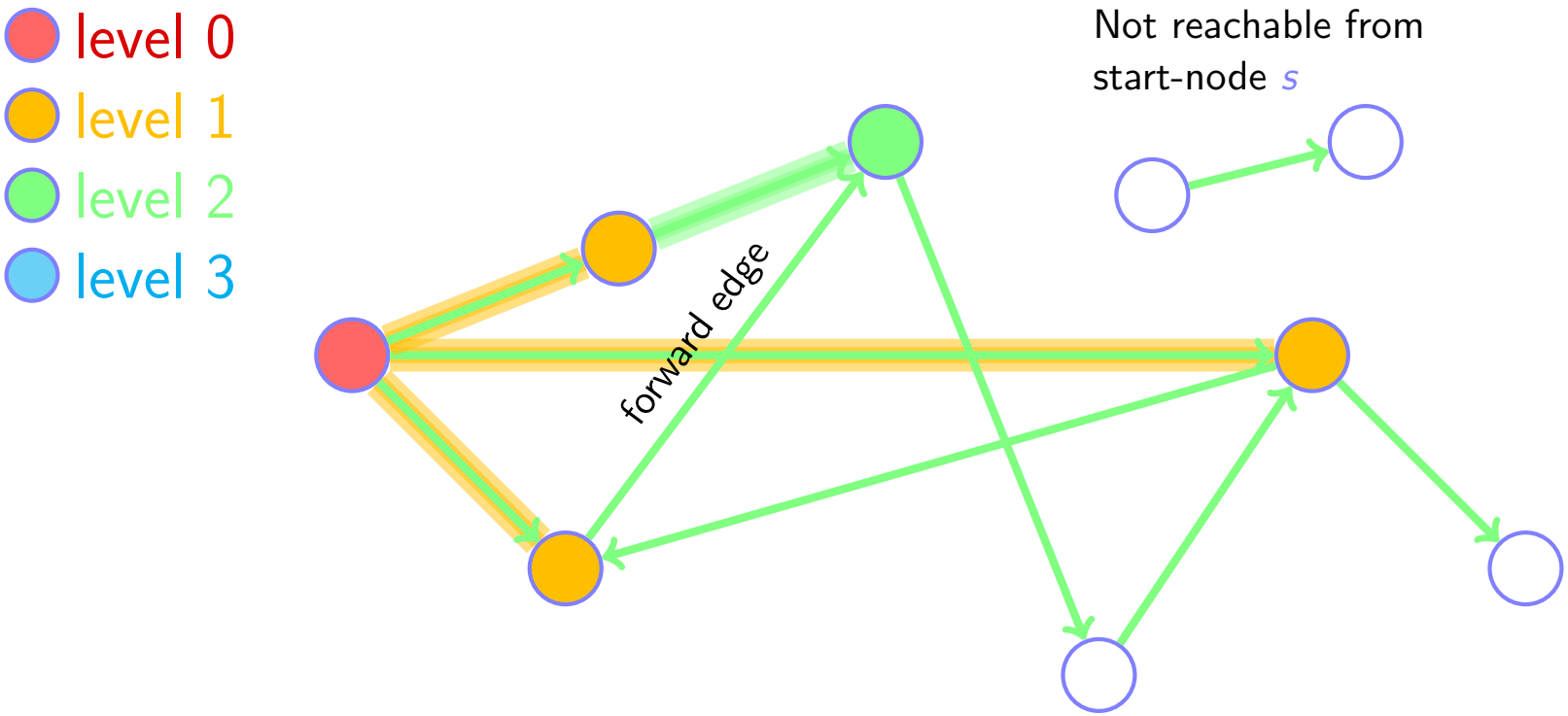

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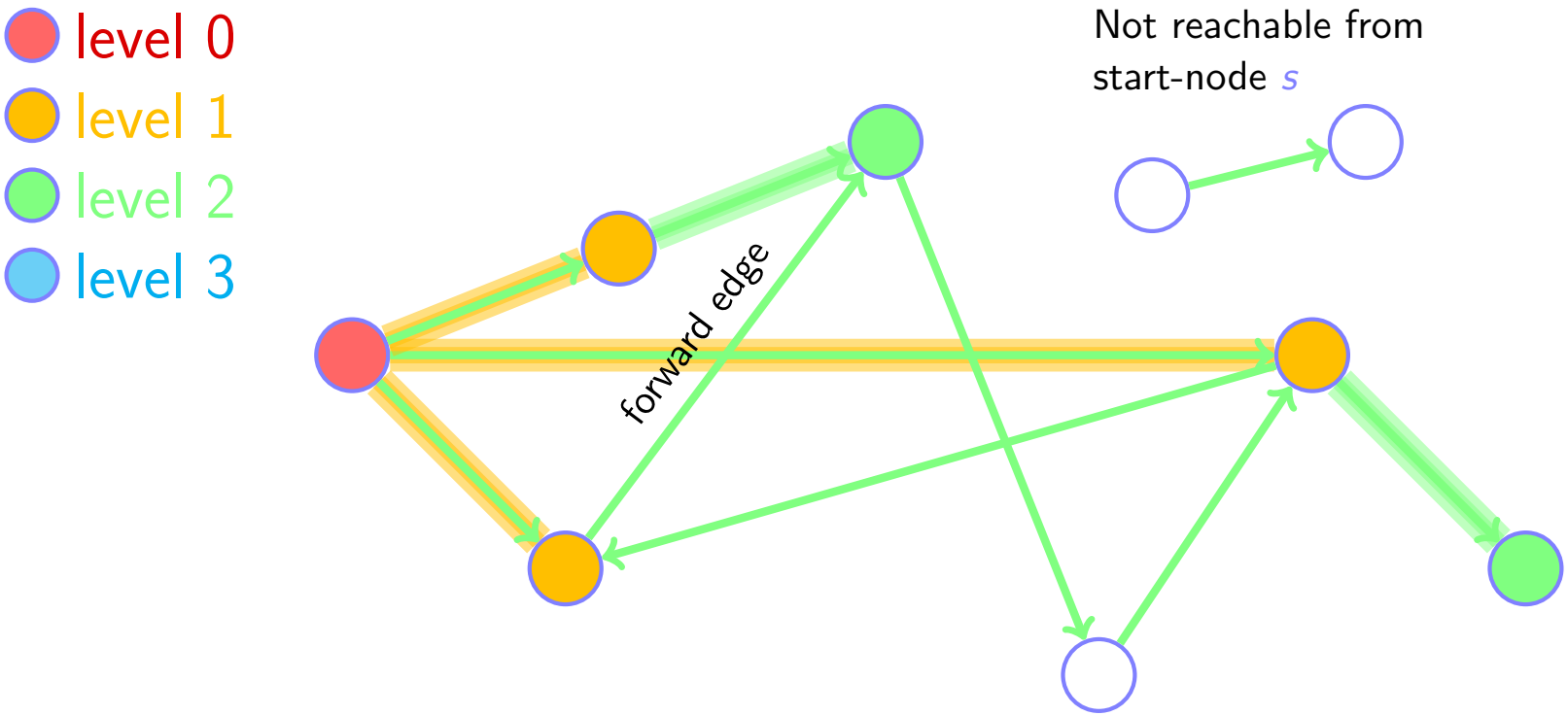

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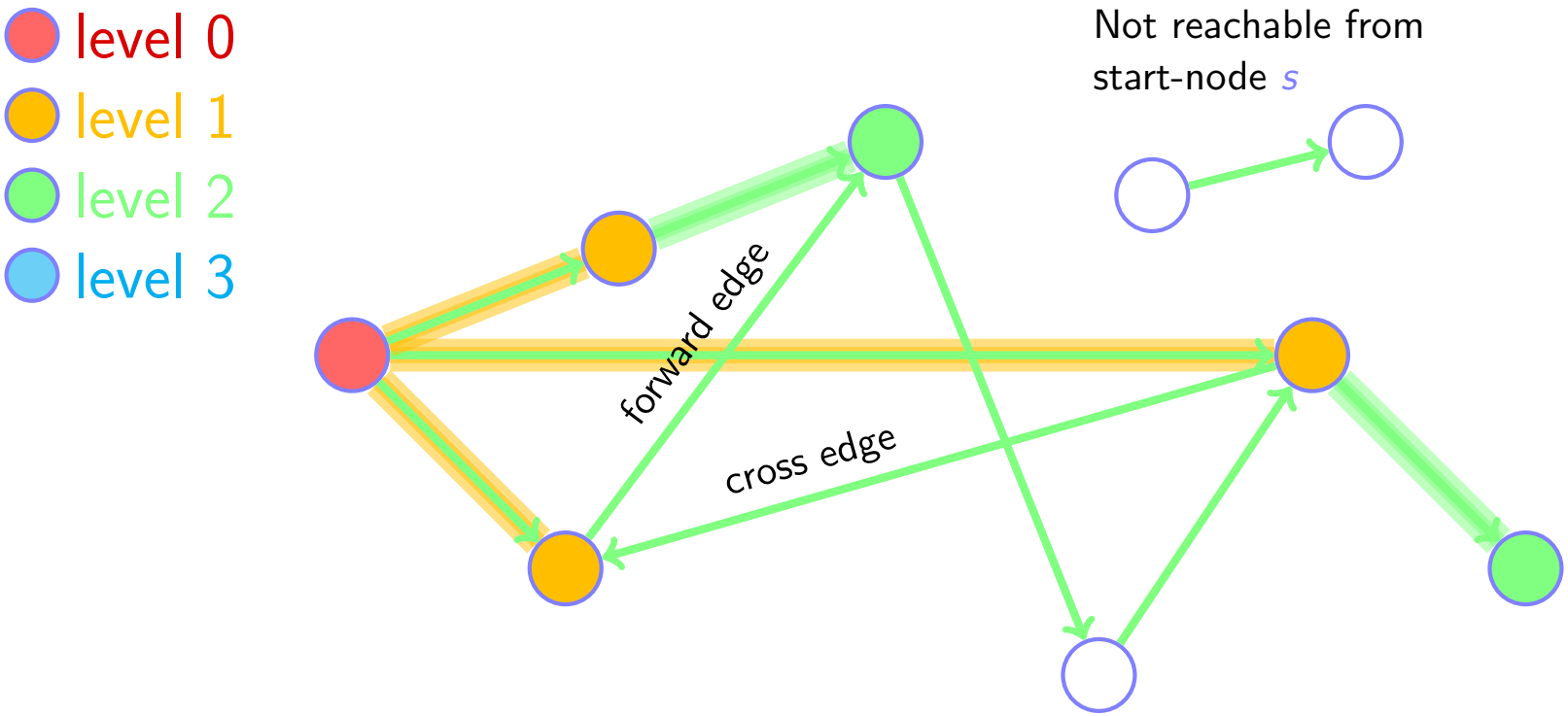

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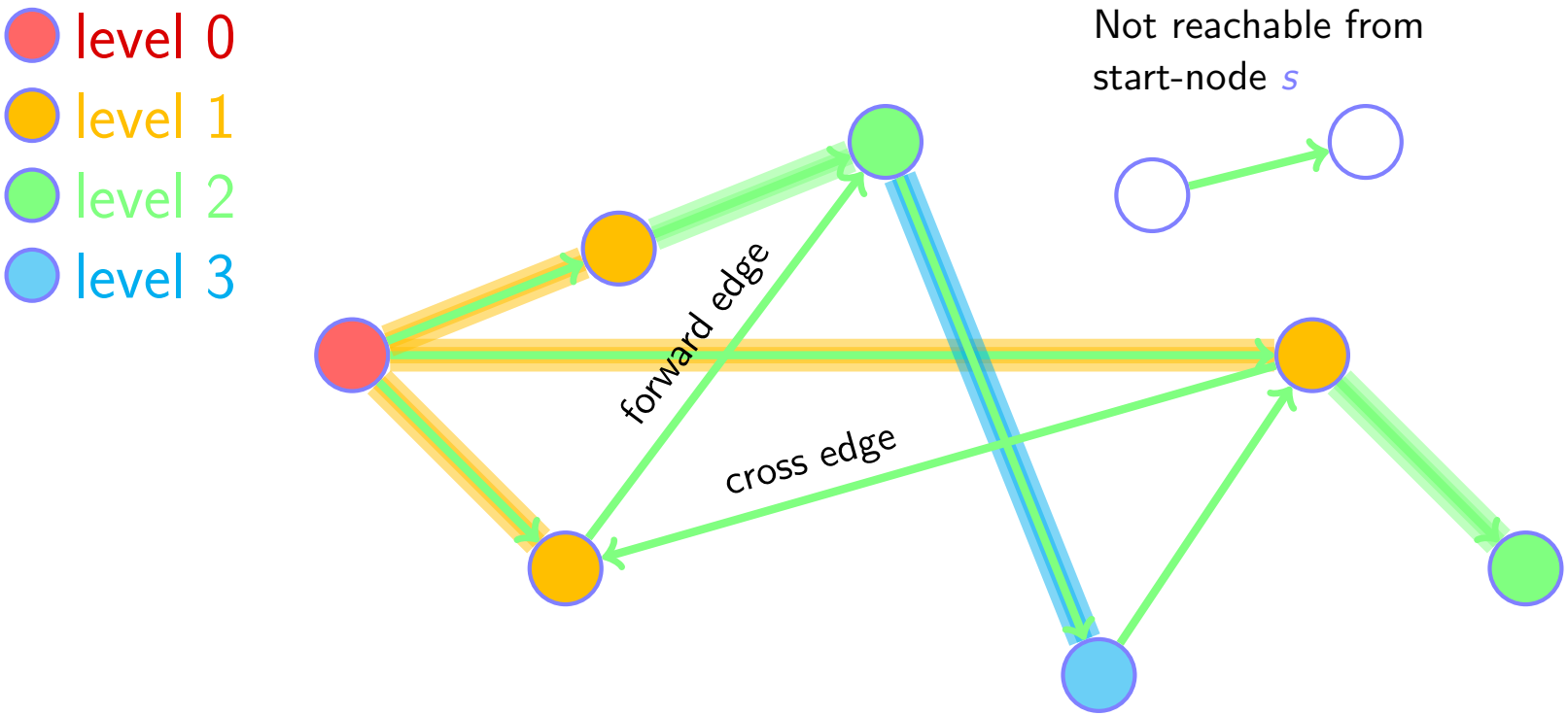

Figure: spanning tree of a breadth-first search

Graphs

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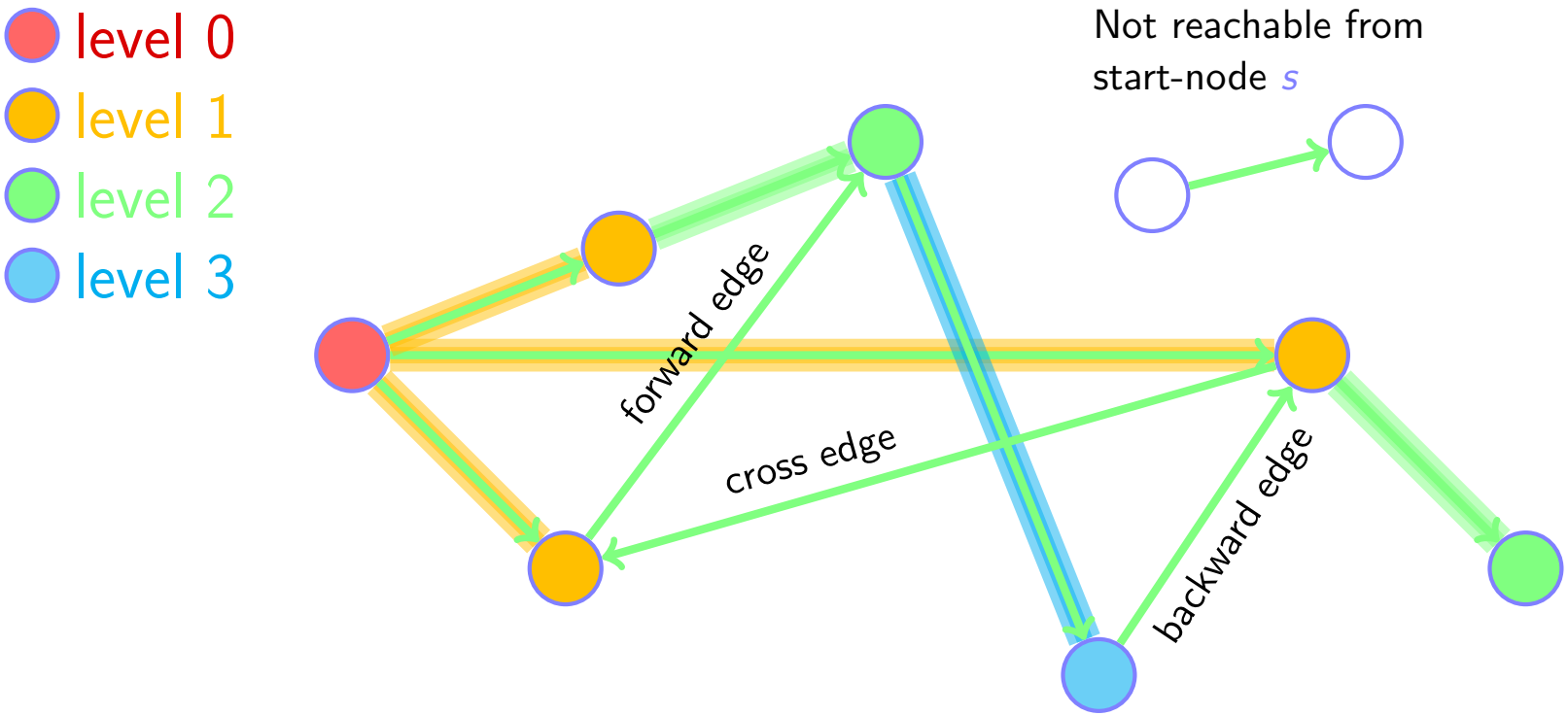

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Graphs

Connected Components - Depth-First Search

Idea:

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1. We start with all vertices unmarked and mark visited vertices

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1. We start with all vertices unmarked and mark visited vertices
2. Mark the start vertex s
3. Pick an unmarked connected vertex and start a recursive depth-first search with the vertex as start vertex
   (continue on step 2)
4. If no unmarked connected vertex exists go one vertex back
   (reduce the recursion level by one)

Depth-first search:

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- Search starts with long paths (searching with depth)

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Graphs

Connected Components - Depth-First Search

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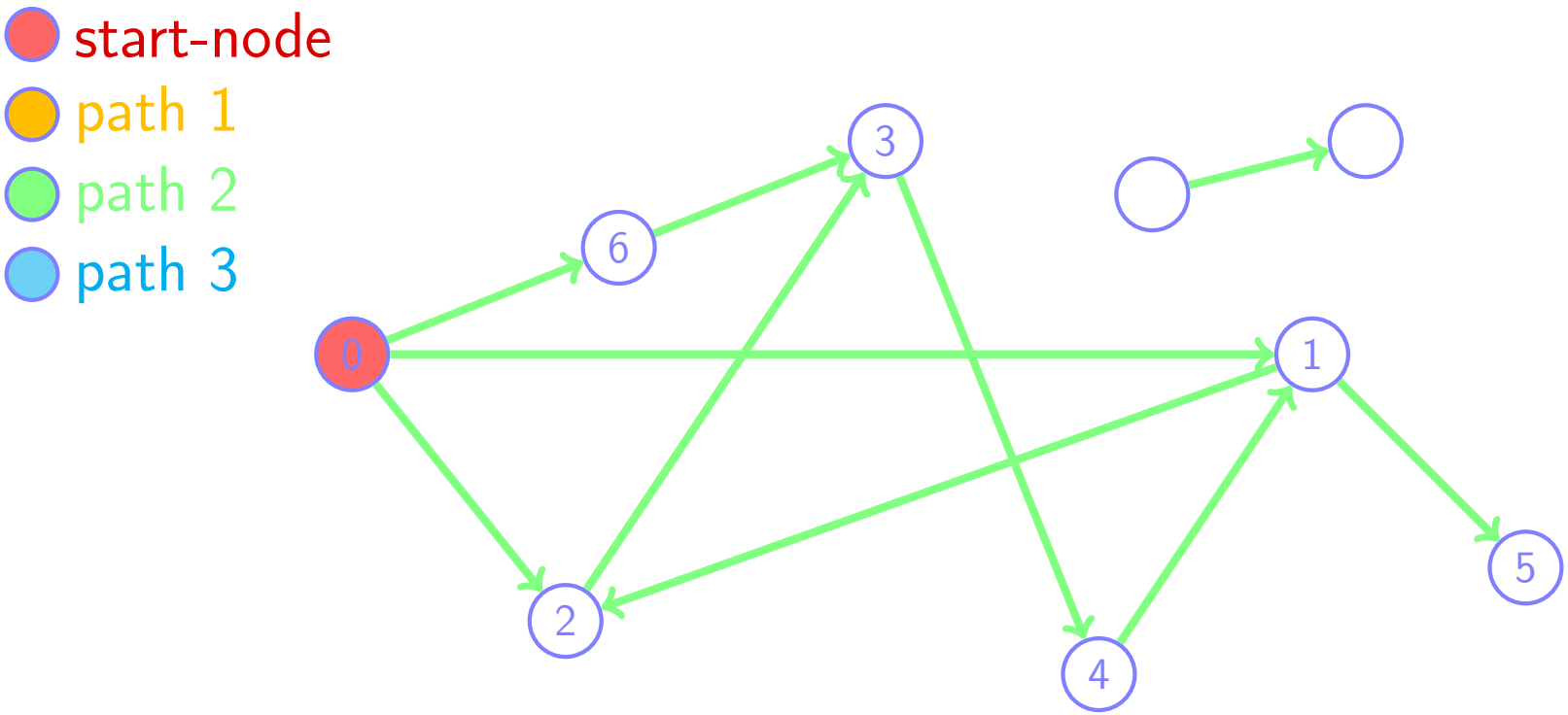

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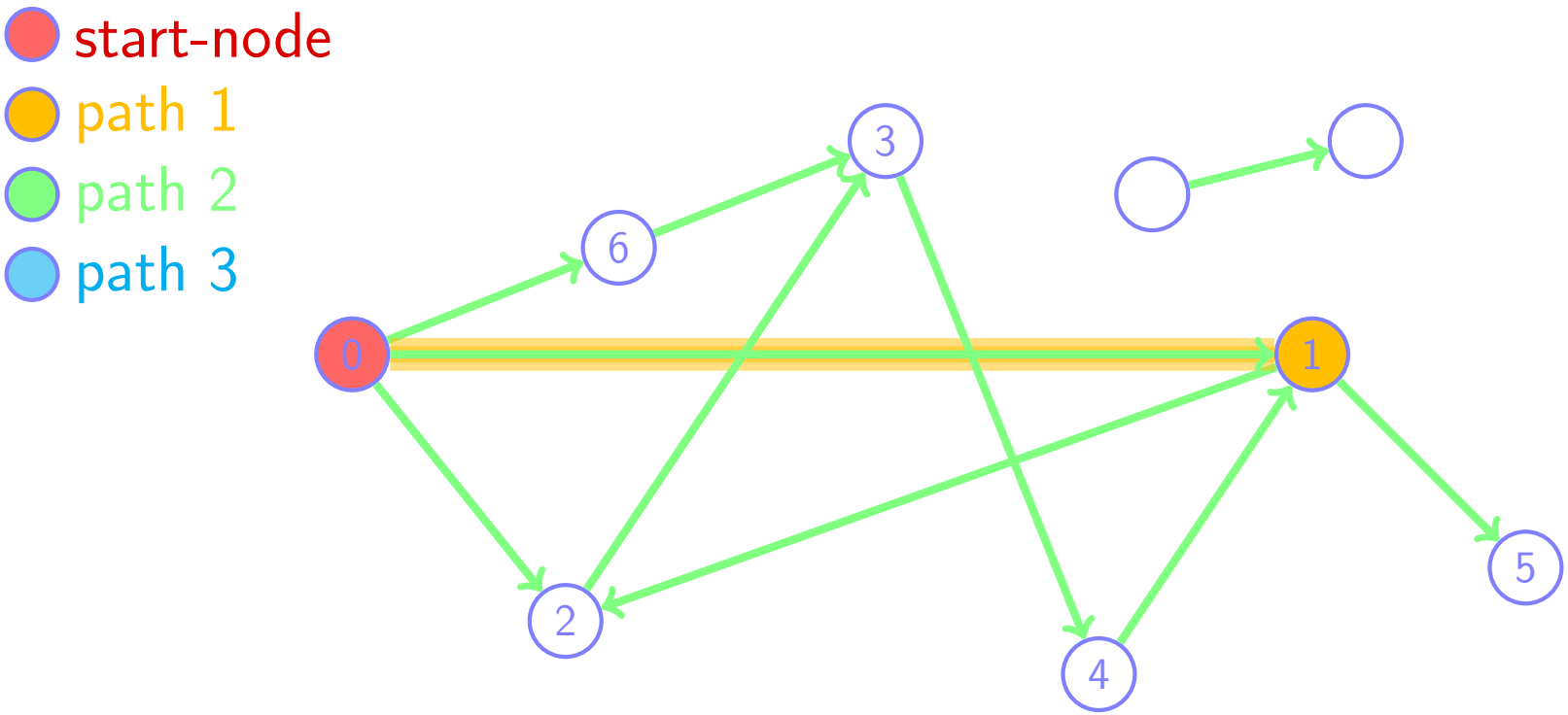

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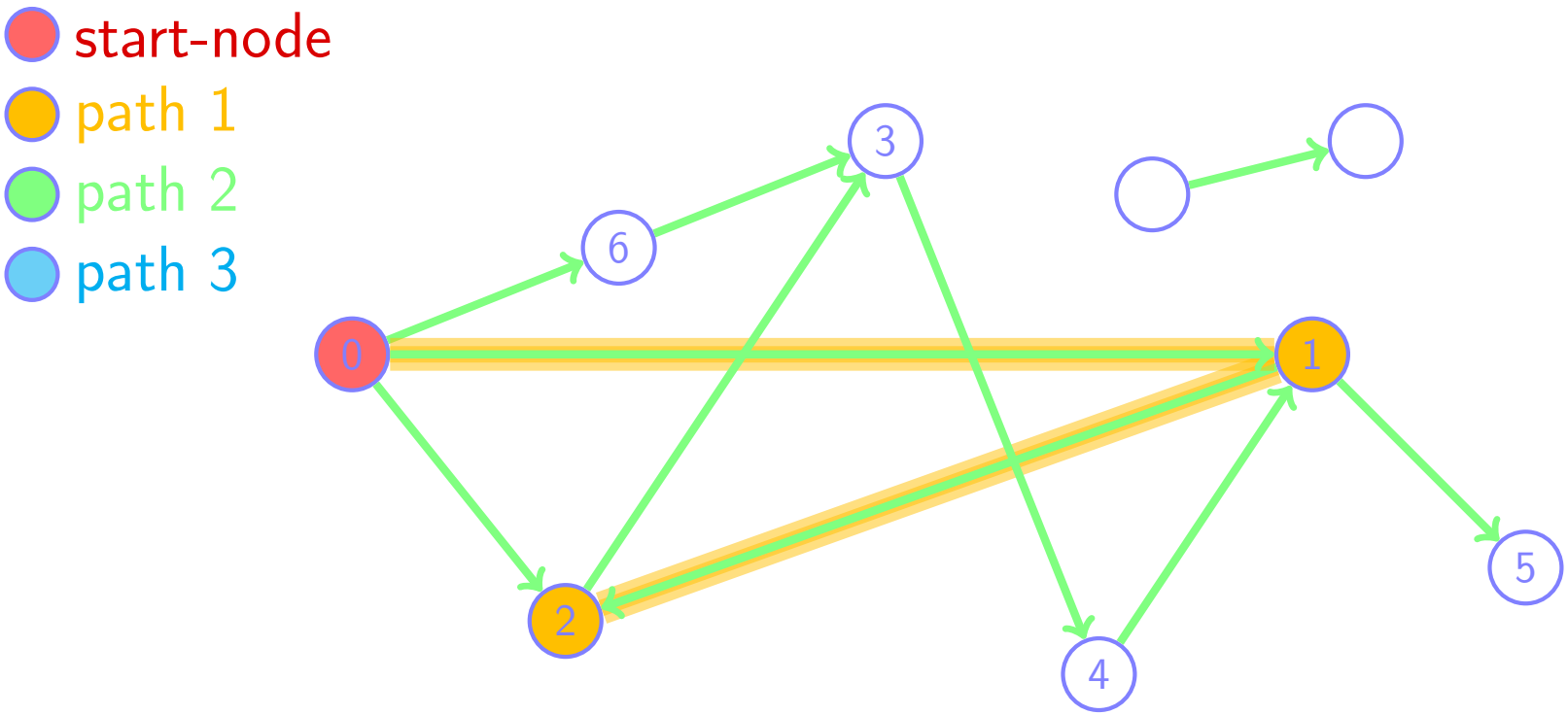

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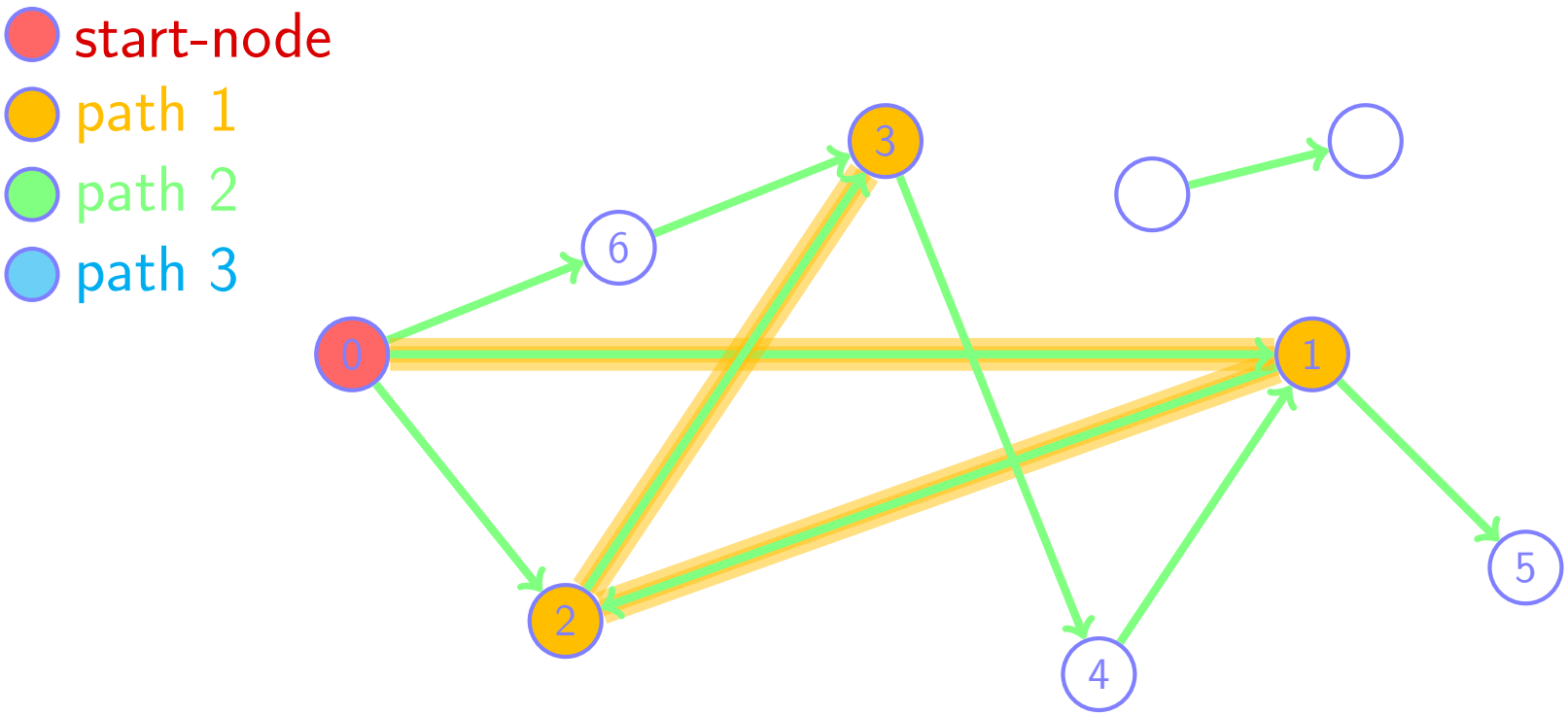

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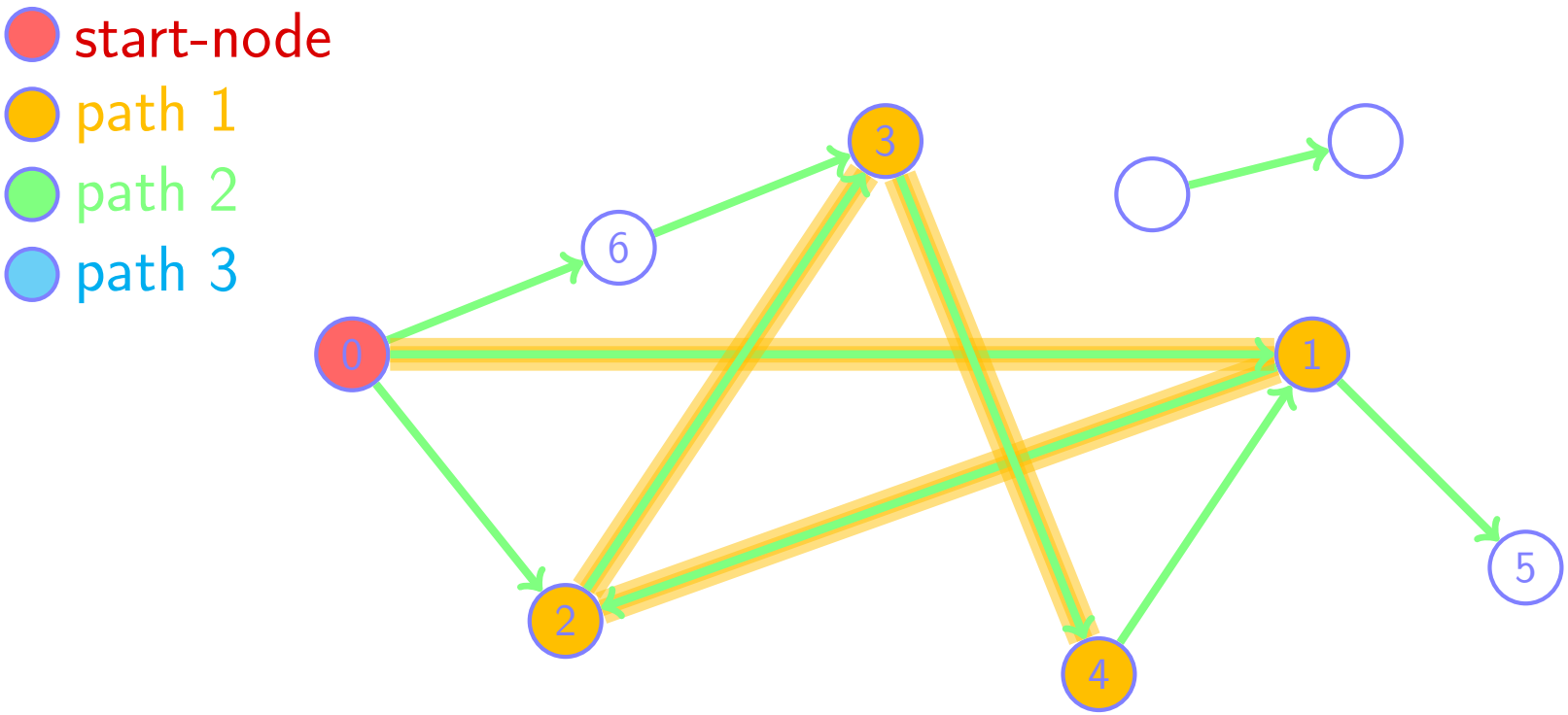

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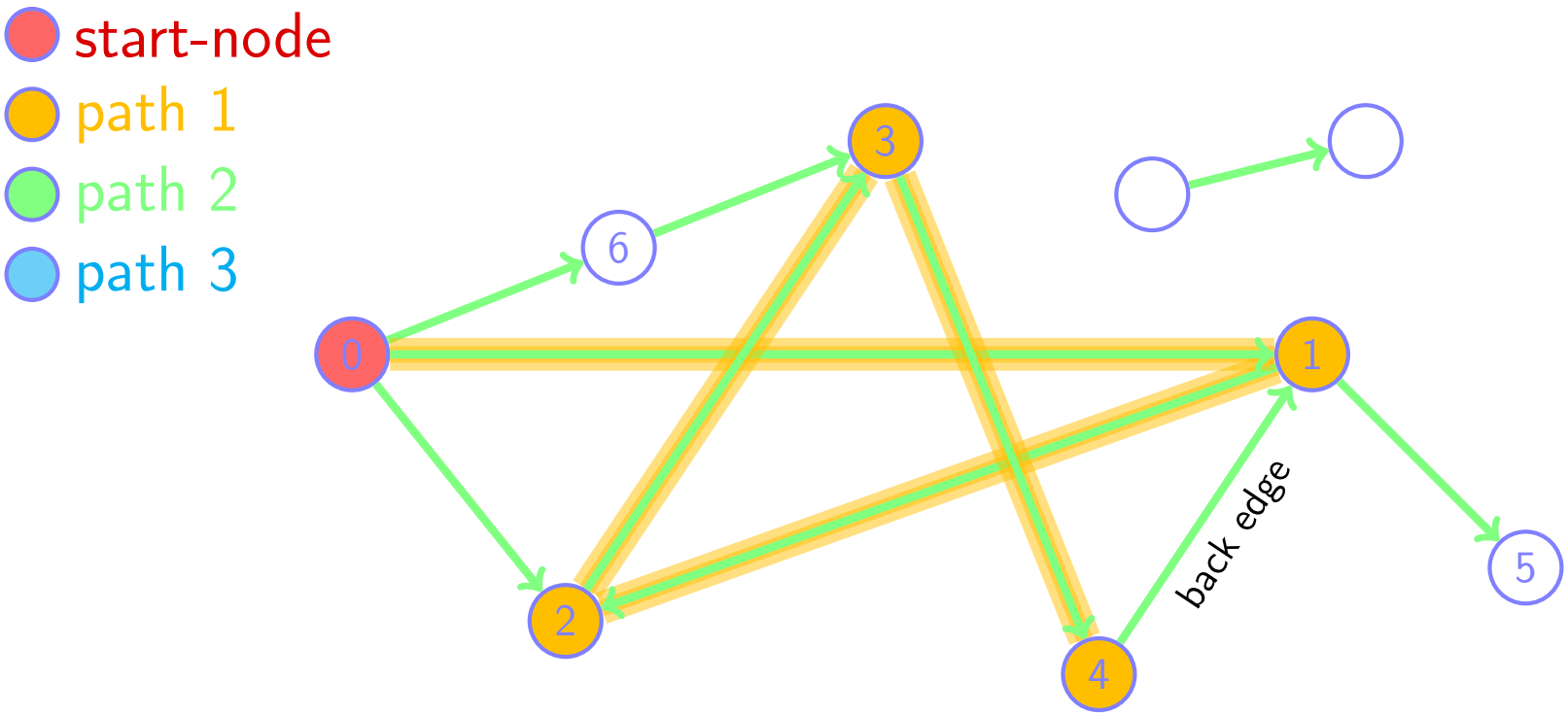

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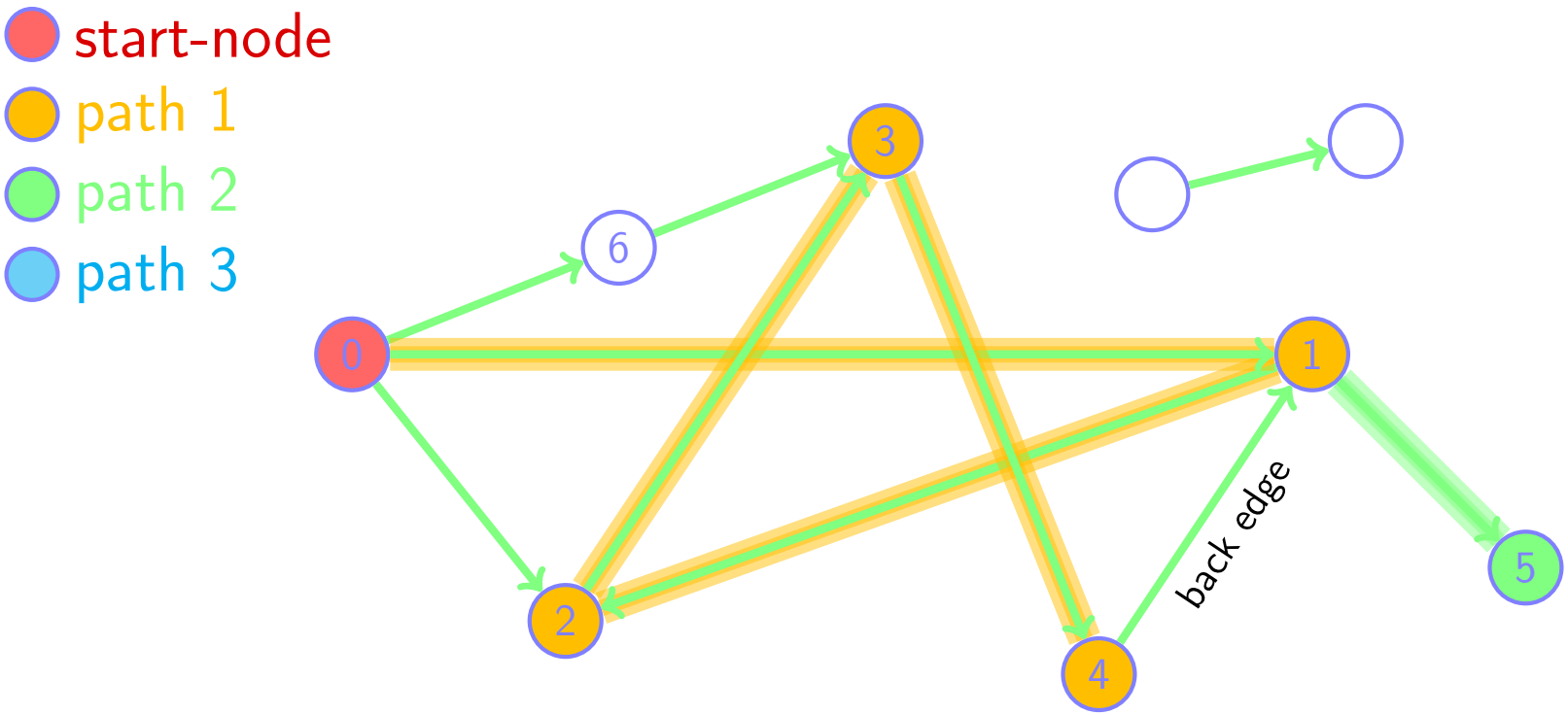

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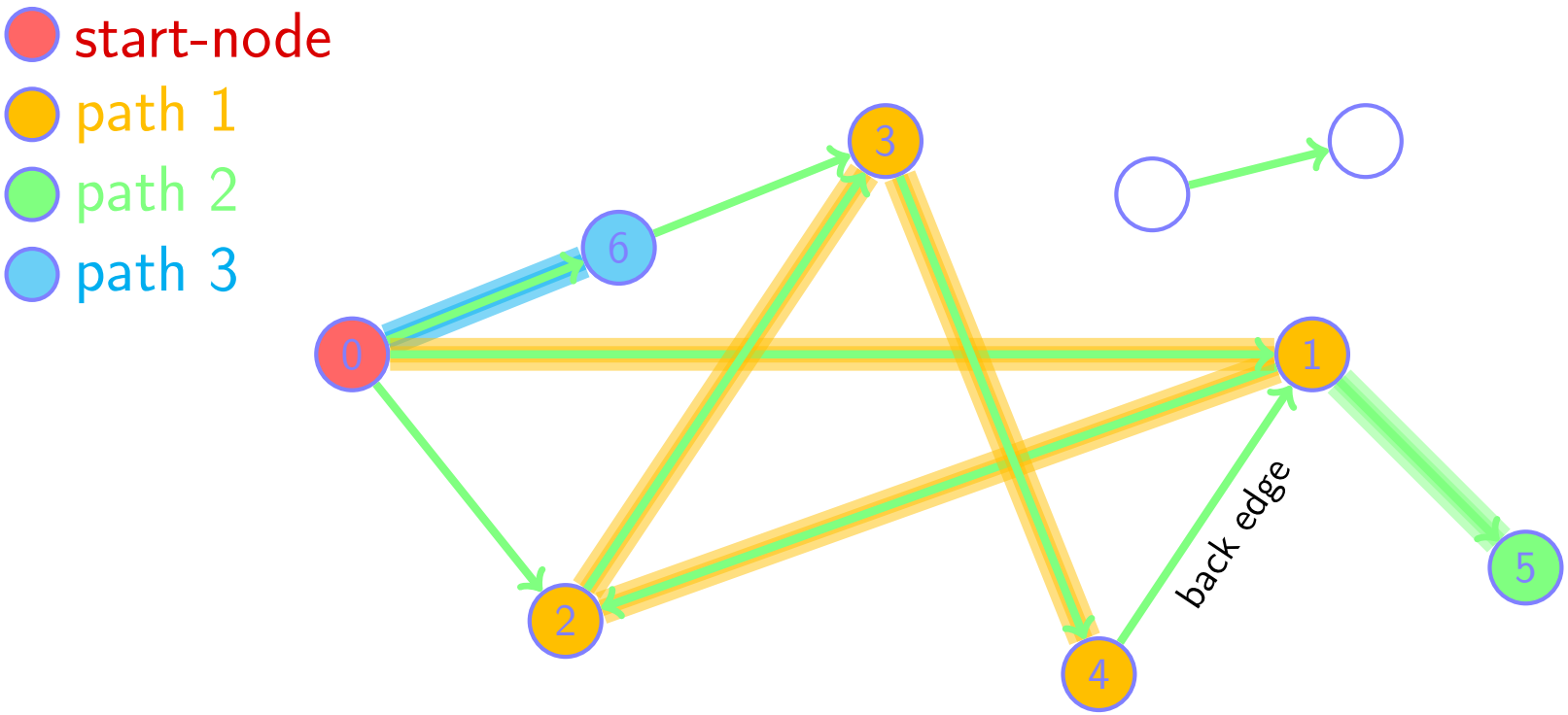

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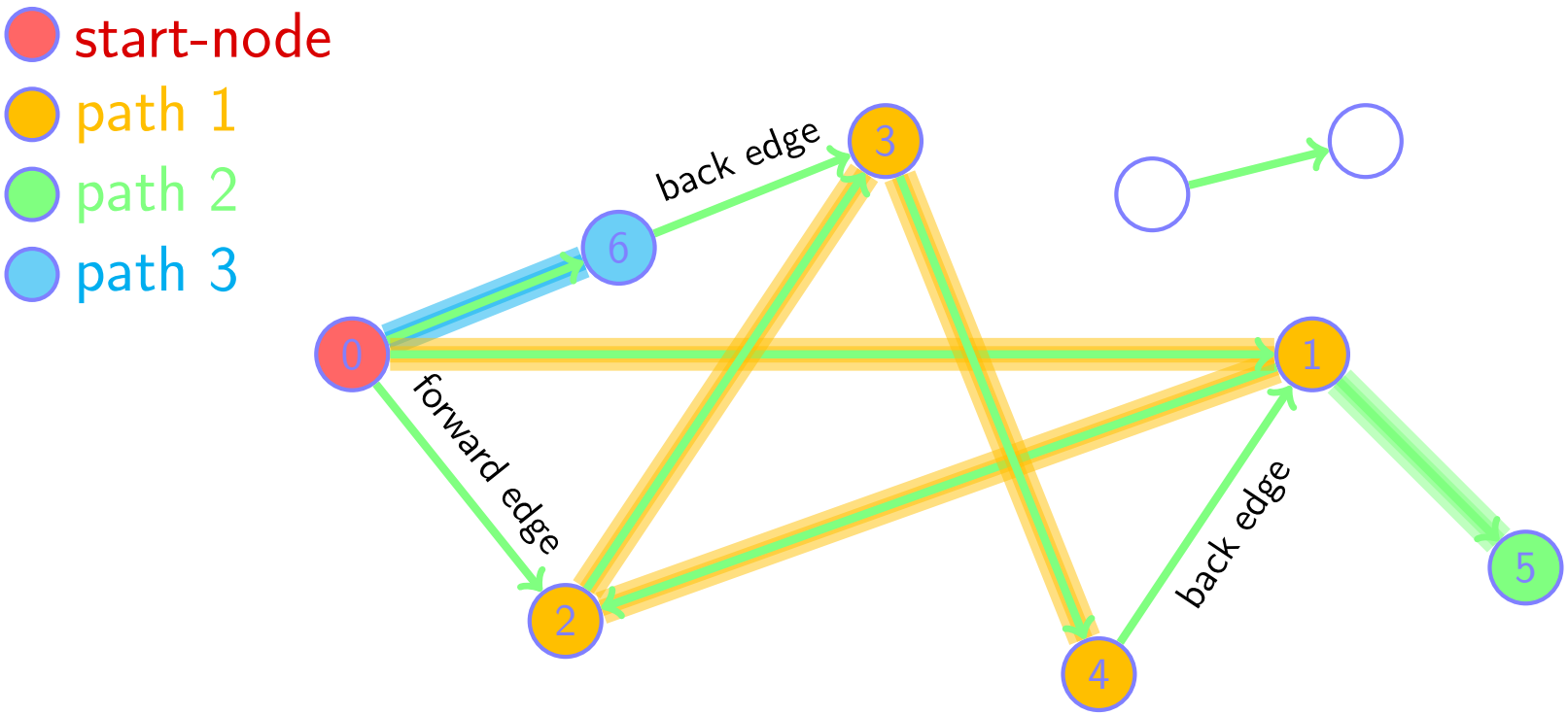

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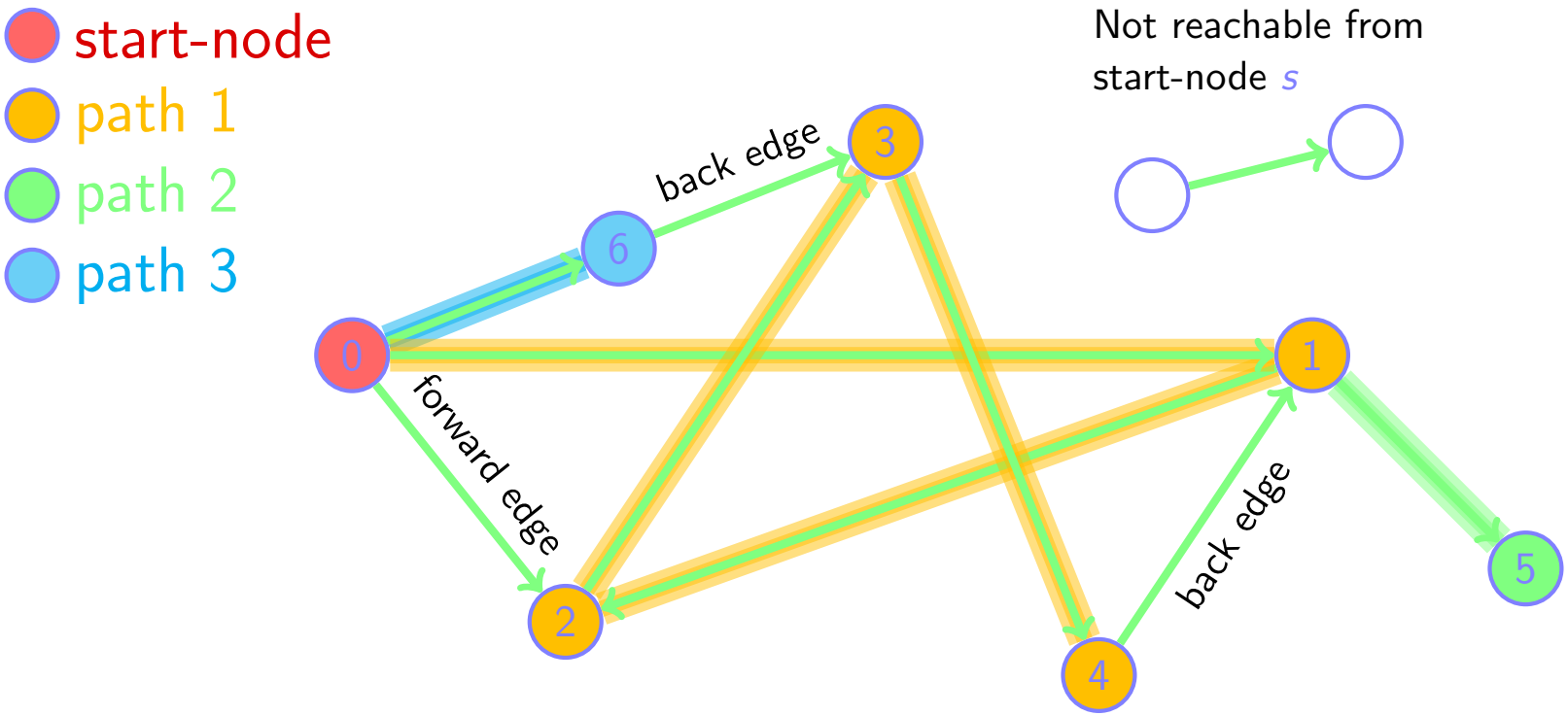

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Graphs

Why is this called Breadth - and Depth First Search?

Runtime complexity:

Runtime complexity:

- Constant costs for each visited vertex and edge

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- We get a runtime complexity of $\Theta(|V'| + |E'|)$
- Let V' and E' be the reachable vertices and edges
- All vertices of V' are in the same connected component as our start vertex s
- This can only be improved by a constant factor

Structure

Application example

Image processing

Application example

Image processing

- Connected component labeling

Application example

Image processing

- Connected component labeling
- Counting of objects in an image

Application example

Image processing

- Connected component labeling
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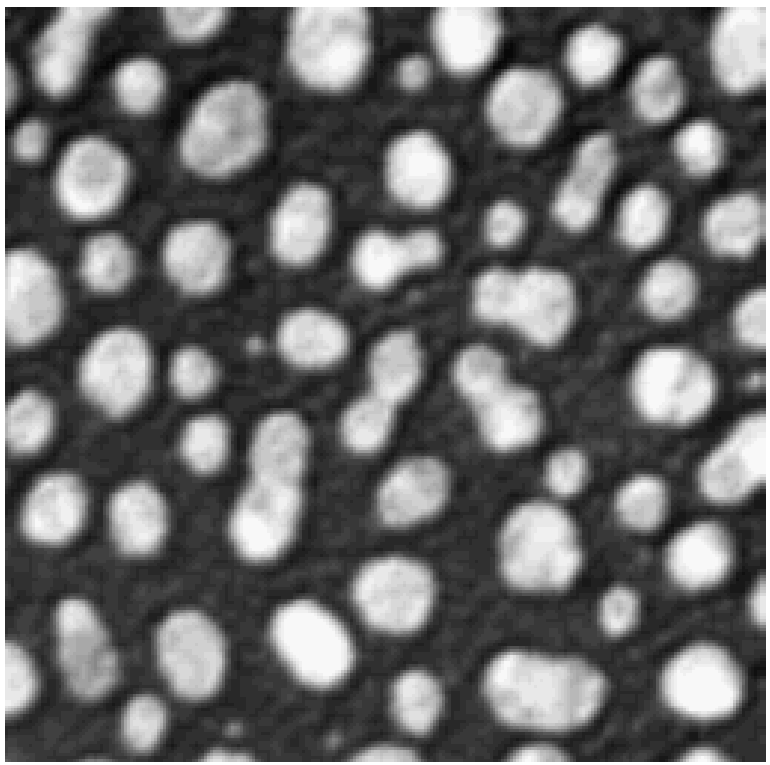
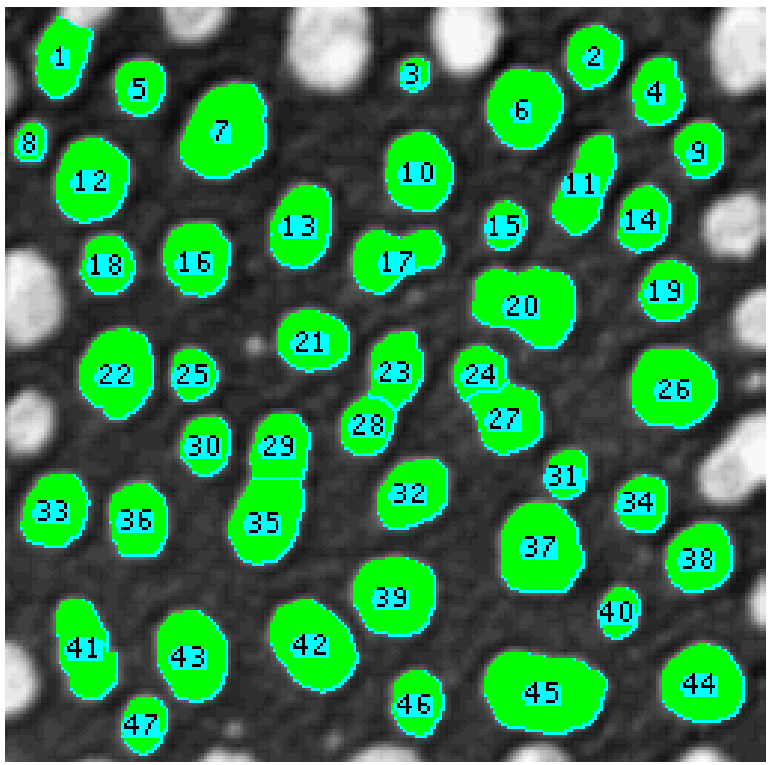


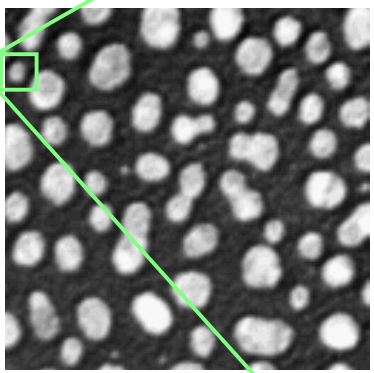
$\Rightarrow$



Application example

Image processing

What's object, what's background?



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
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36	80	64	48	32	16	16	16	24	32	40	40	40	40	40	40	40	32	32	24	24	24
37	56	48	32	24	8	16	16	32	40	48	48	48	40	40	40	40	32	32	24	24	24
38	40	32	24	24	16	32	48	64	72	80	80	72	56	56	48	48	40	40	32	32	32
39	16	16	16	24	24	48	72	88	104	112	112	96	72	64	56	48	40	40	40	40	40
40	16	16	24	40	56	88	120	128	136	144	144	120	96	88	72	56	48	48	40	40	40
41	8	16	24	56	80	120	160	168	168	168	168	144	120	104	80	64	48	48	40	40	32
42	16	32	40	80	112	144	176	176	176	176	168	152	128	112	88	64	48	40	32	32	24
43	24	40	56	96	136	160	184	184	176	176	168	152	136	112	88	64	40	32	24	24	16
44	40	56	80	112	152	168	184	184	176	176	168	152	136	112	80	64	40	32	16	16	16
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46	48	72	96	136	168	176	192	192	184	184	176	160	136	104	72	56	32	24	16	24	32
47	48	72	96	136	168	184	192	192	192	192	184	160	136	104	72	48	24	24	16	32	48
48	48	72	96	128	168	184	200	200	200	192	184	160	128	96	64	48	24	32	32	56	72
49	48	72	88	128	160	184	200	200	200	192	184	152	120	88	56	40	24	32	40	72	96
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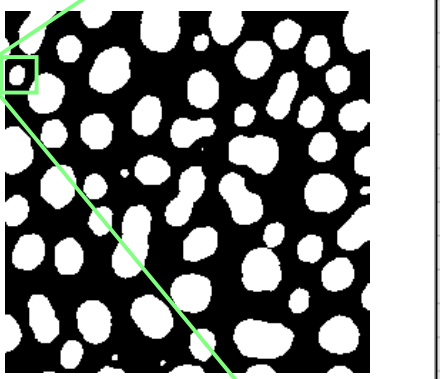
51	48	64	72	96	112	128	144	152	152	144	136	112	88	64	40	40	32	48	64	112	152
52	48	56	64	80	88	104	112	112	120	112	104	88	72	56	32	32	32	64	88	128	168
53	40	48	48	56	64	72	72	80	80	80	72	64	48	40	24	32	32	72	104	144	184
54	48	48	48	48	48	56	56	56	64	56	56	48	40	32	24	40	48	88	128	160	200

Application example

Image processing

Convert to black white using threshold:

```
value = 255 if value > 100 else 0
```



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0
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44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0
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52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
57	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Application example

Image processing

Interpret image as graph:

Application example

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Interpret image as graph:

- Each white pixel is a node

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Image processing

Interpret image as graph:

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- Edges are not saved externally, algorithm works directly on array
- Breadth- / depth-first search find all connected components (particles)

Application example

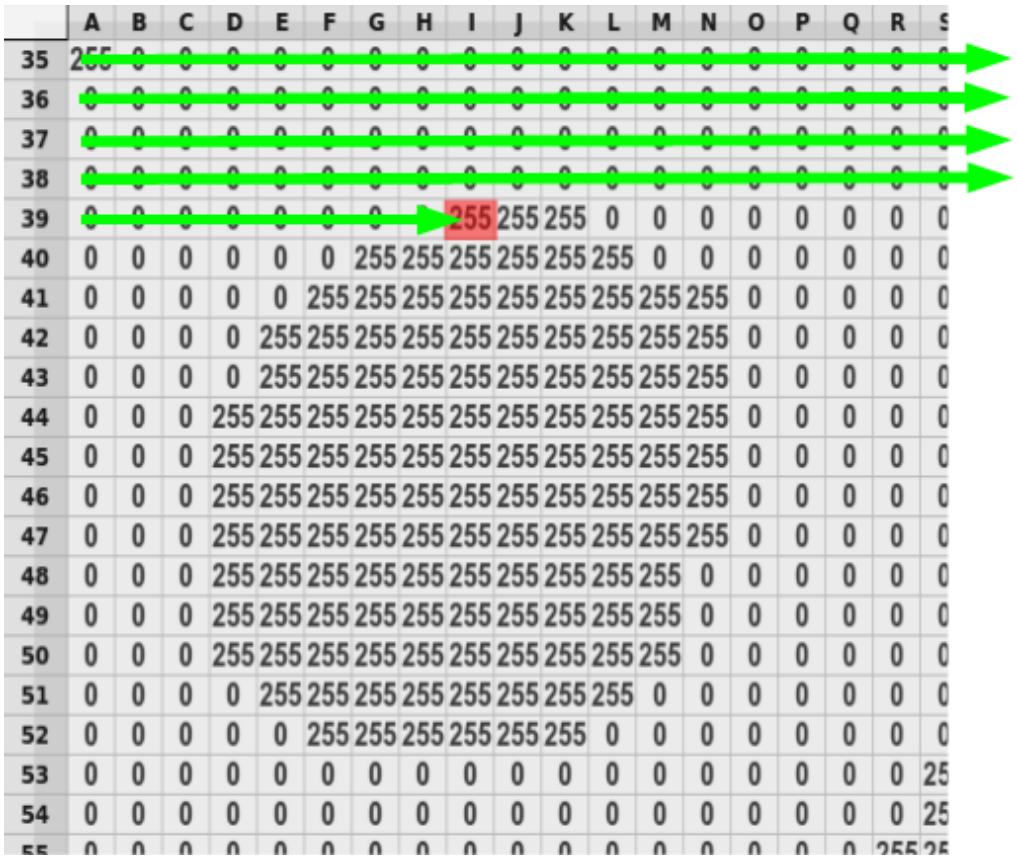
Image processing

Find connected components:

Application example

Image processing

Find connected components:



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	0	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

Application example

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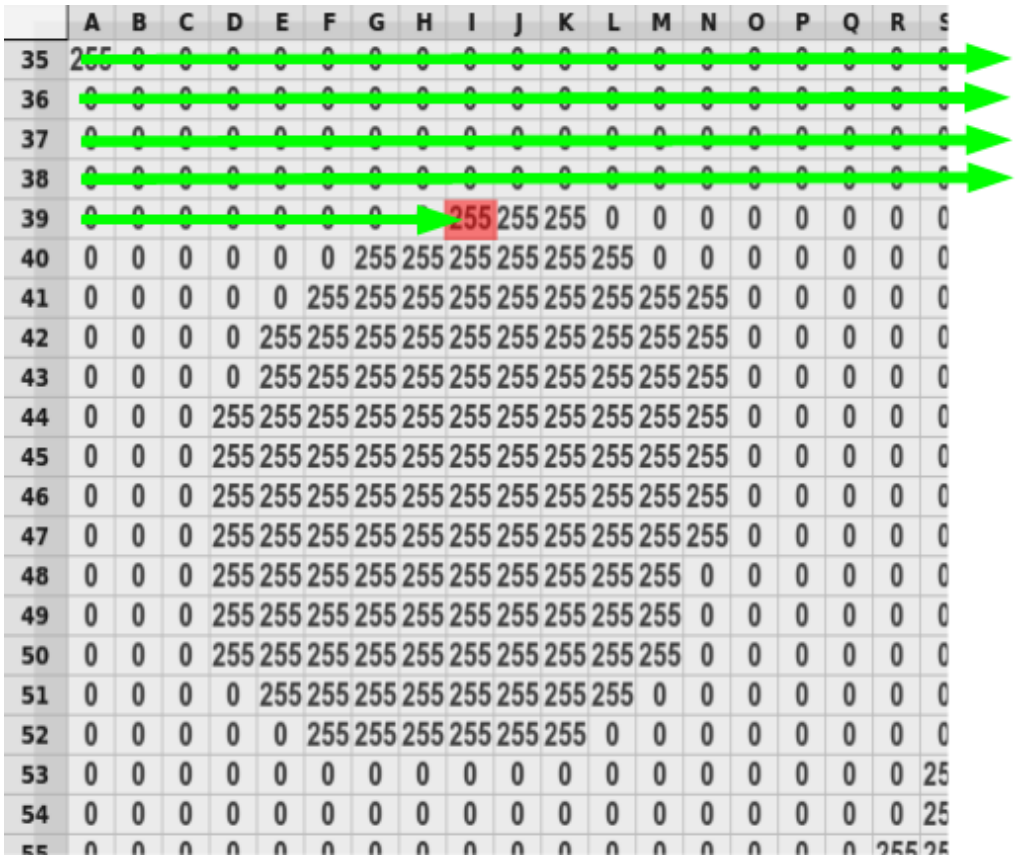

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
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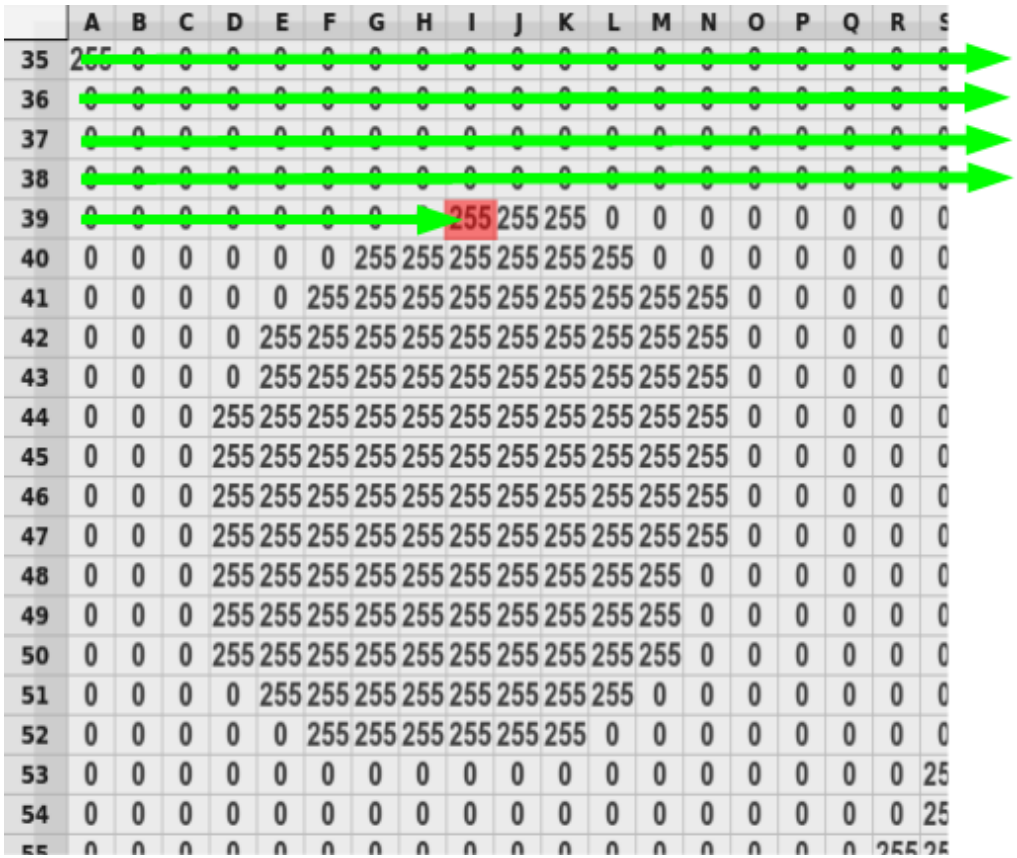
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	0	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity

Application example

Image processing

Find connected components:



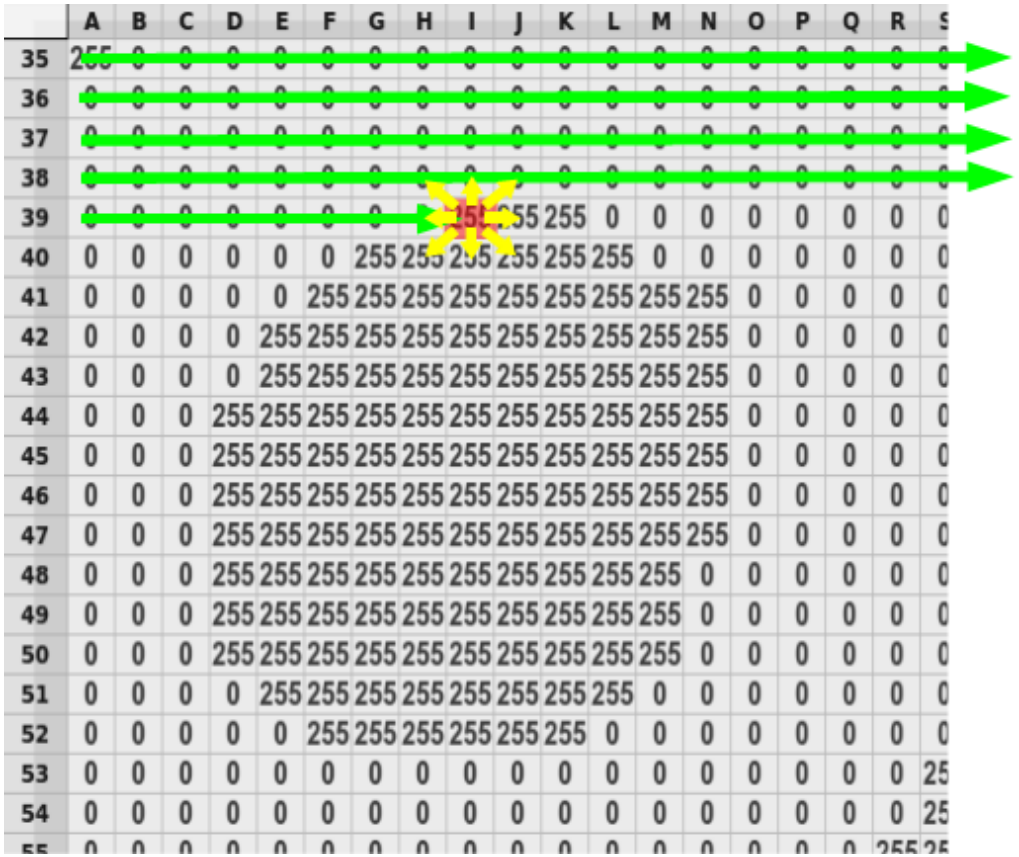
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	0	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1

Application example

Image processing

Find connected components:



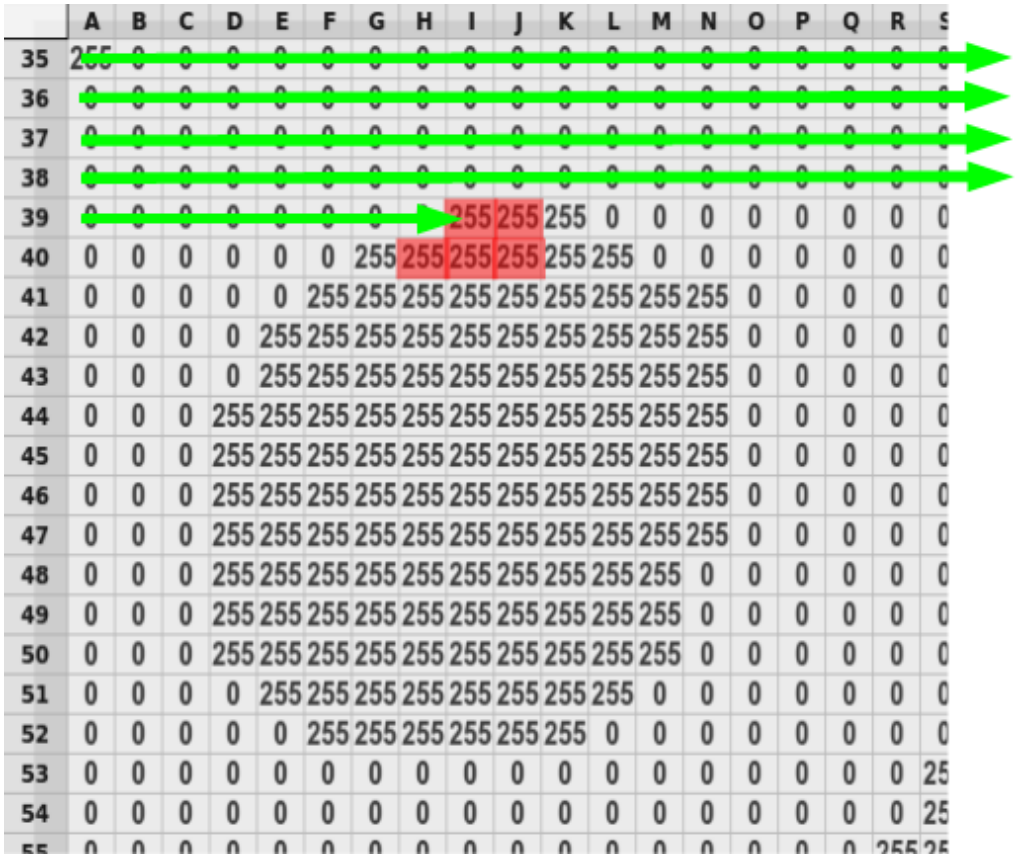
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	255	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	0	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels

Application example

Image processing

Find connected components:



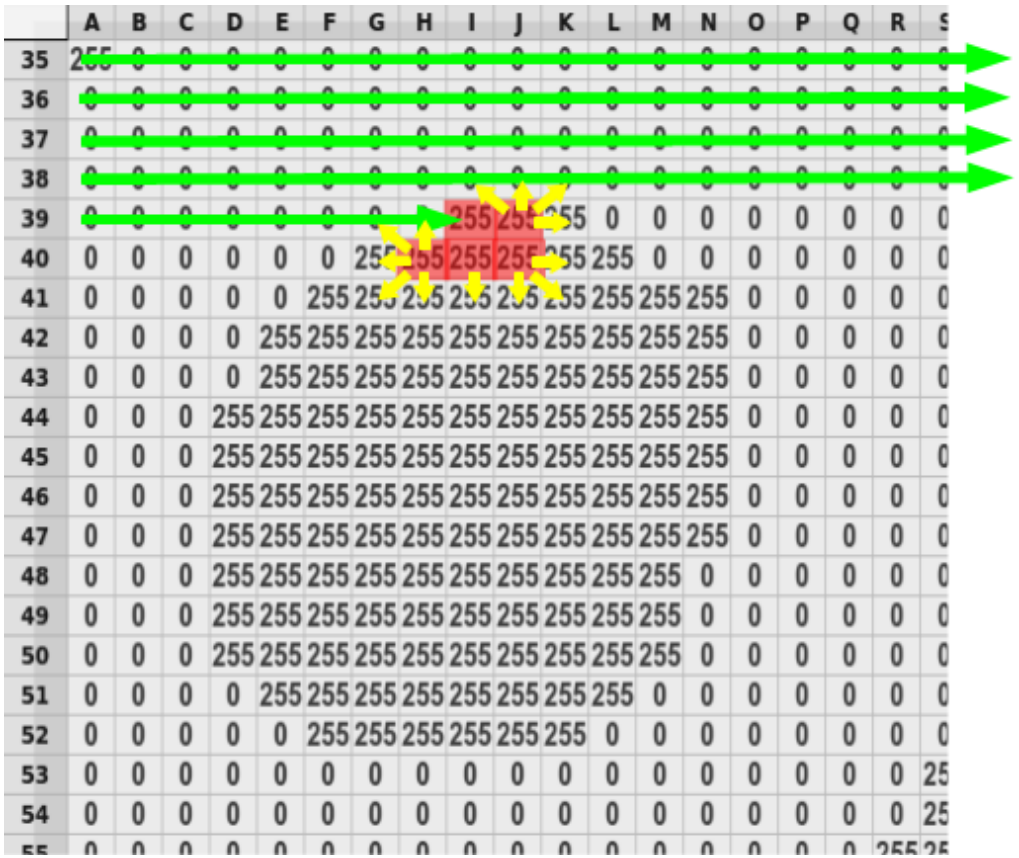
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	0	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:

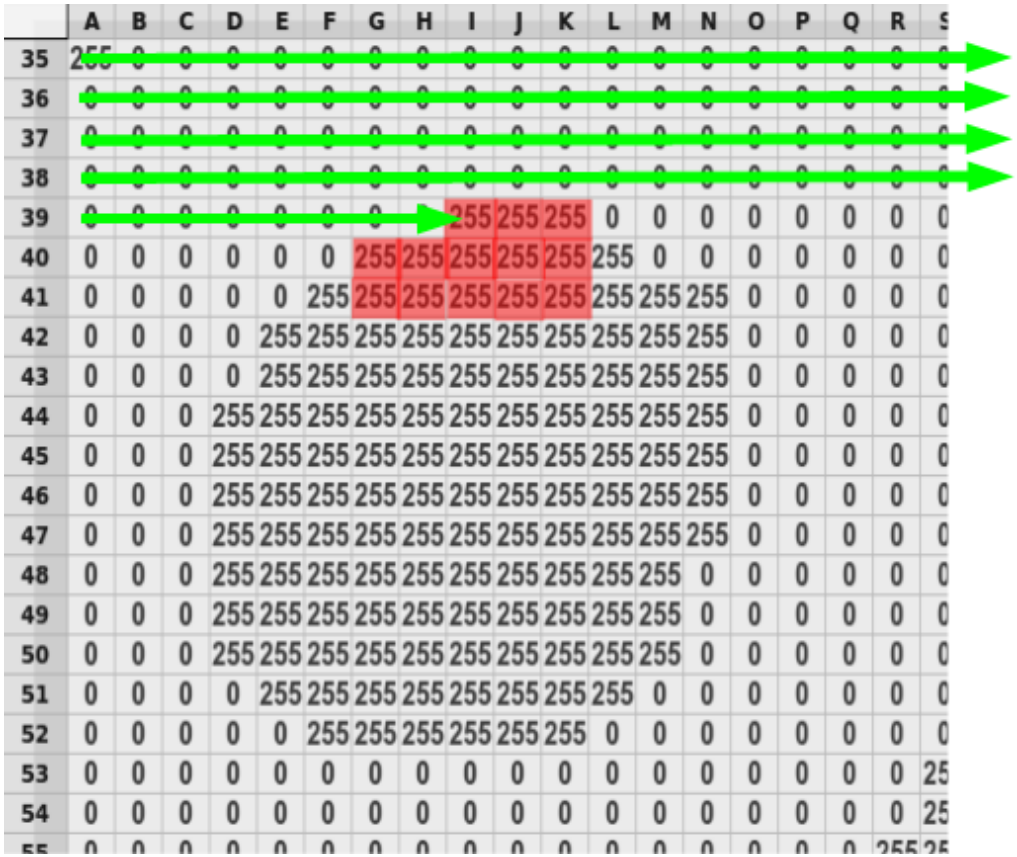


- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:



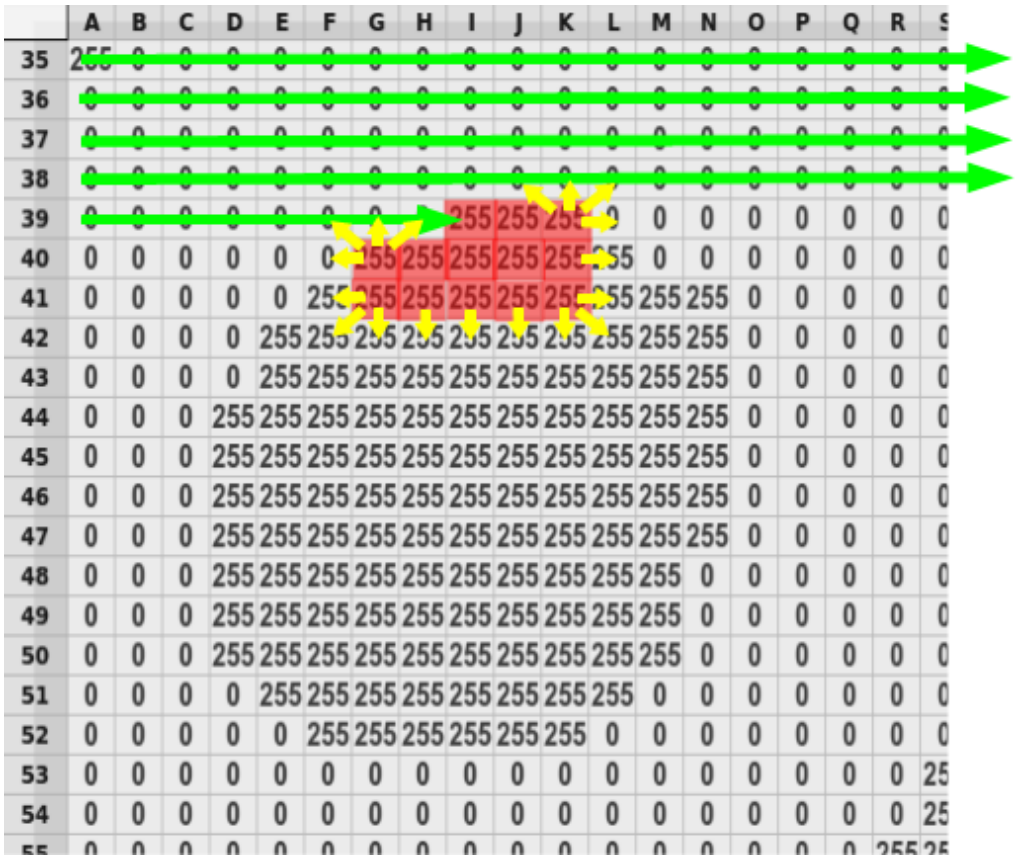
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	0	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:

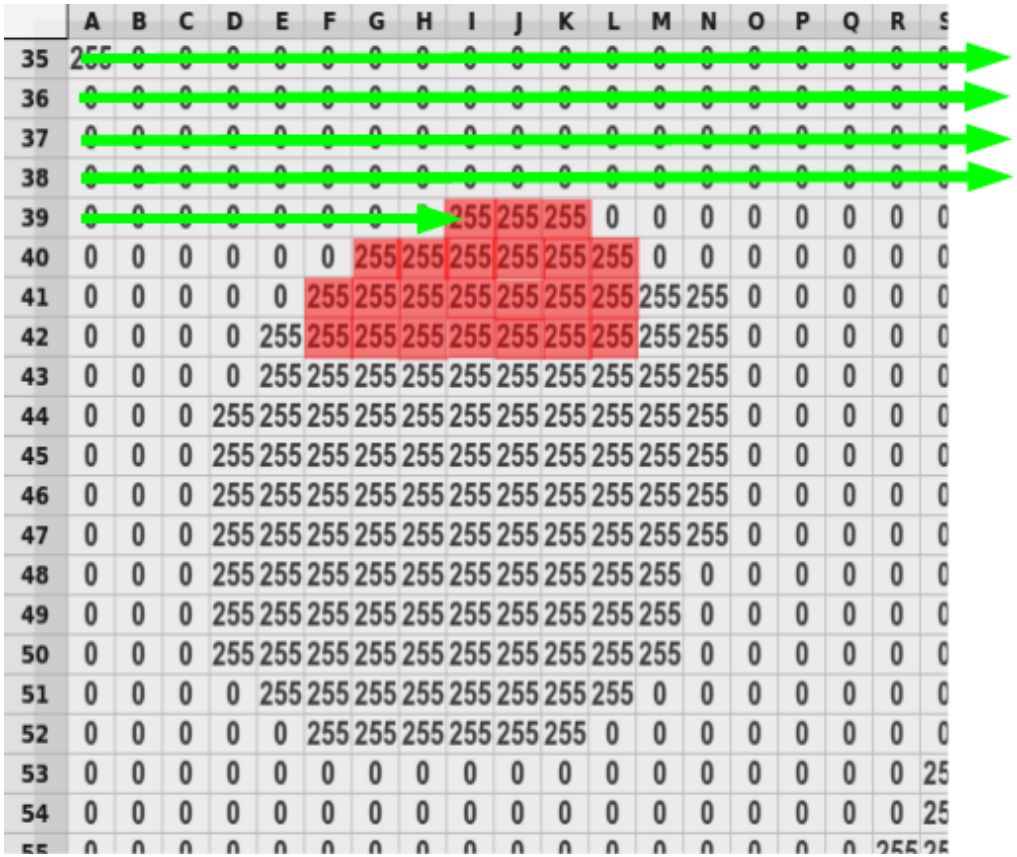


- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:



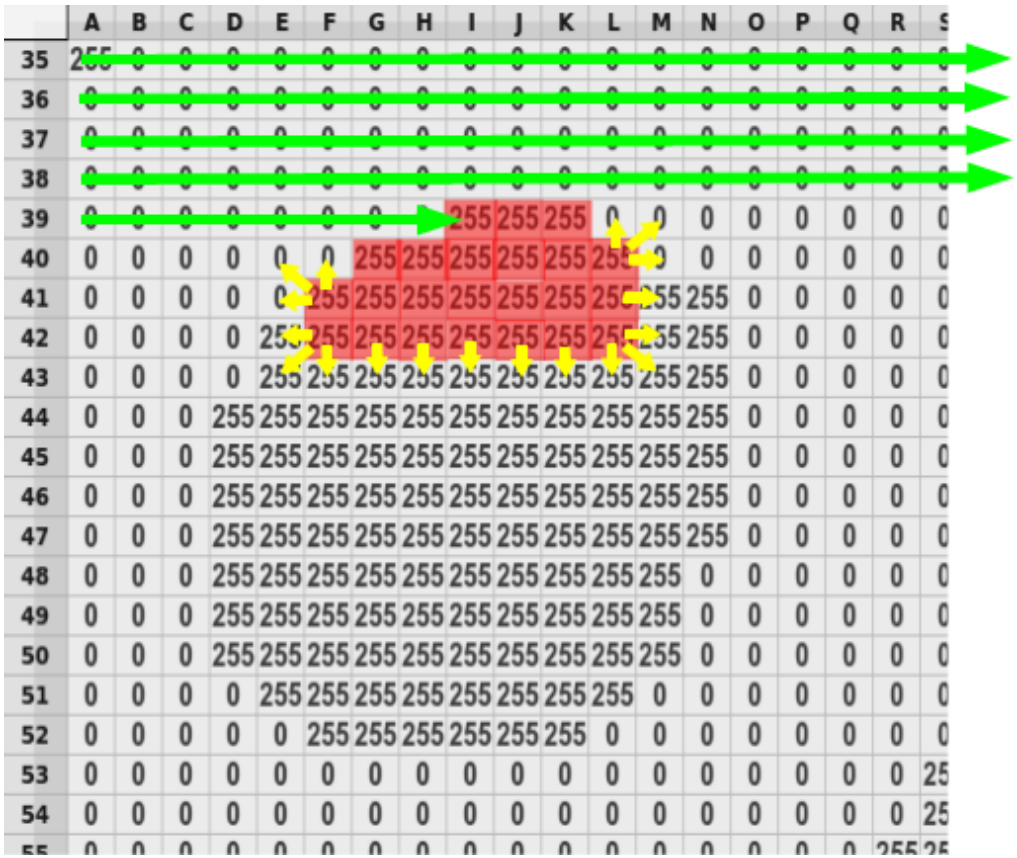
	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
35	255	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	255	255	255	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0
41	0	0	0	0	0	255	255	255	255	255	255	255	255	255	0	0	0	0	0
42	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
43	0	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
44	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
45	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
46	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
47	0	0	0	255	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0
48	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
49	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
50	0	0	0	255	255	255	255	255	255	255	255	255	255	0	0	0	0	0	0
51	0	0	0	0	255	255	255	255	255	255	255	255	0	0	0	0	0	0	0
52	0	0	0	0	0	255	255	255	255	255	255	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	25
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	255	25

- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:

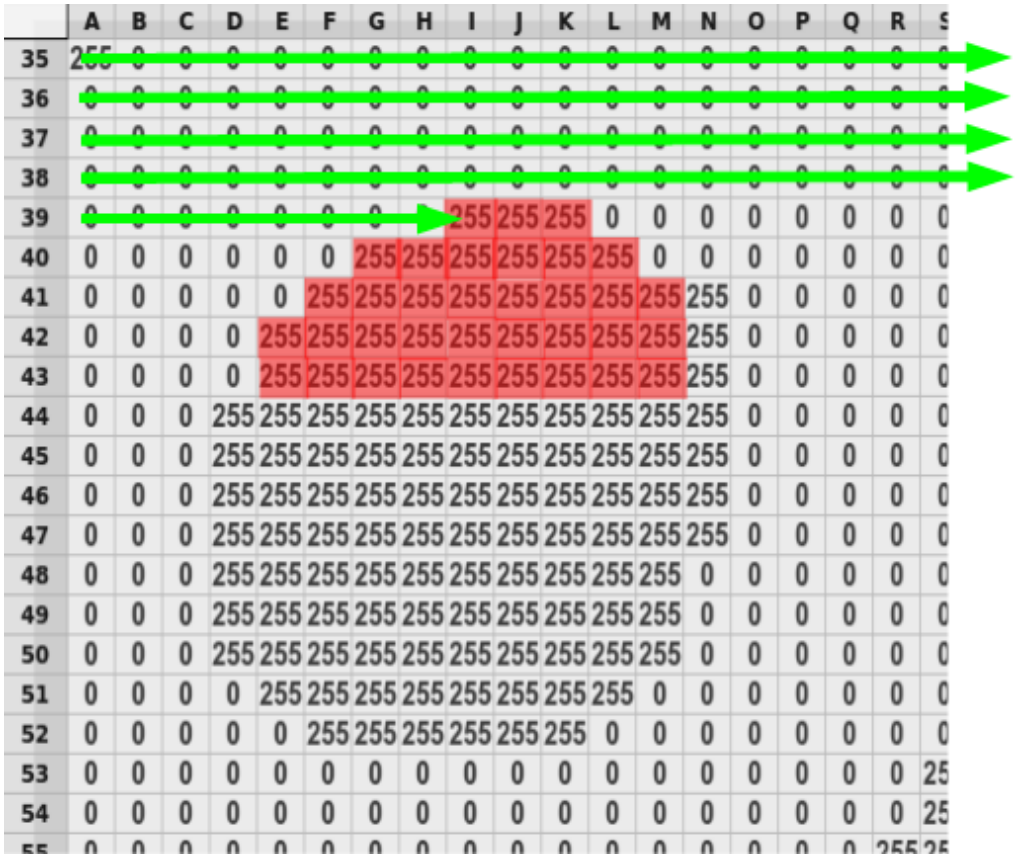


- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:

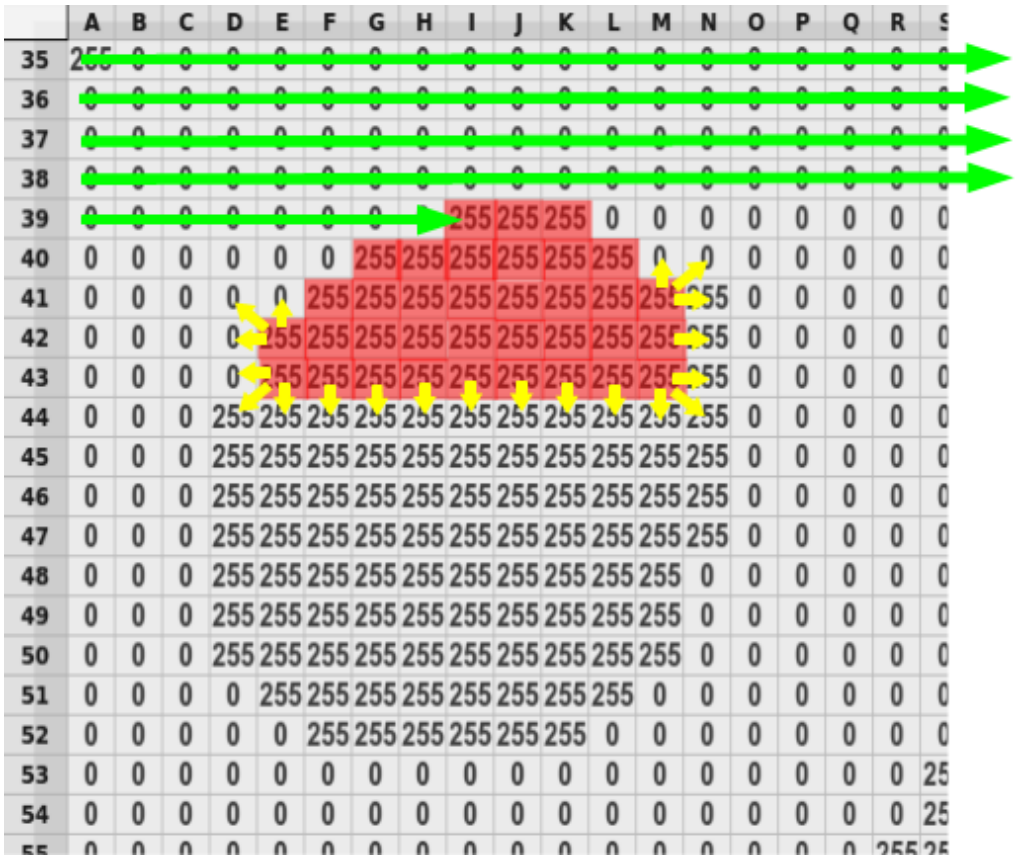


- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:

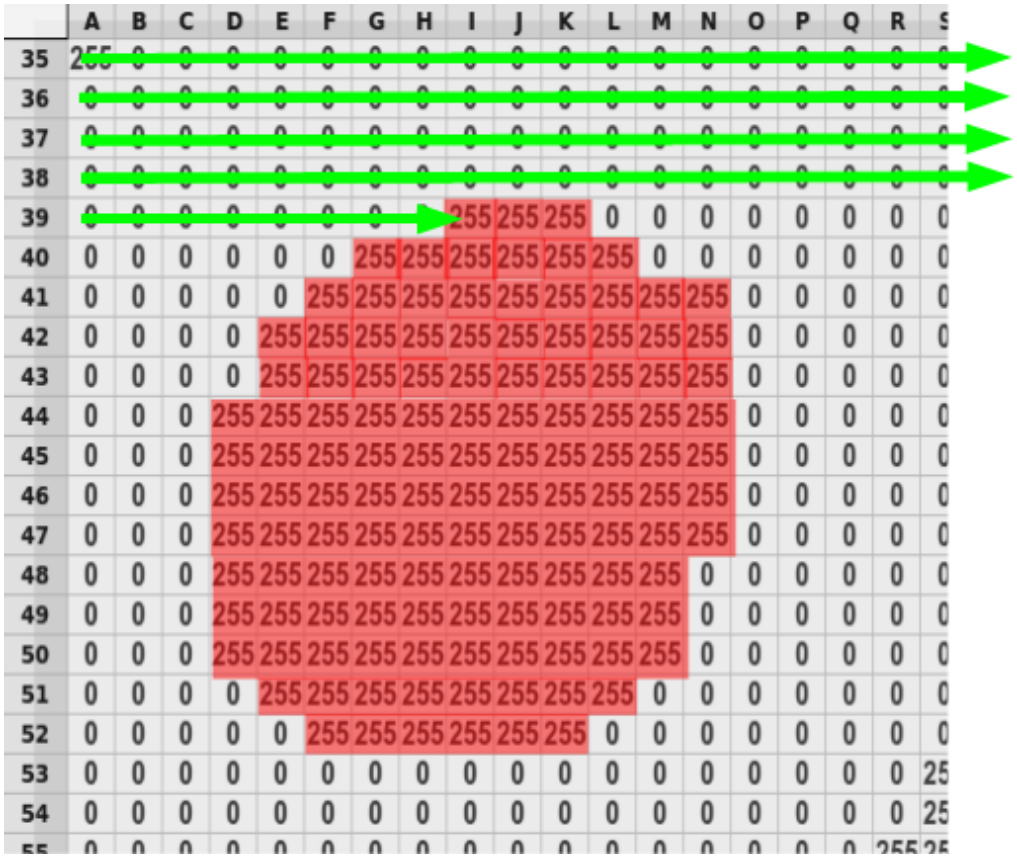


- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:

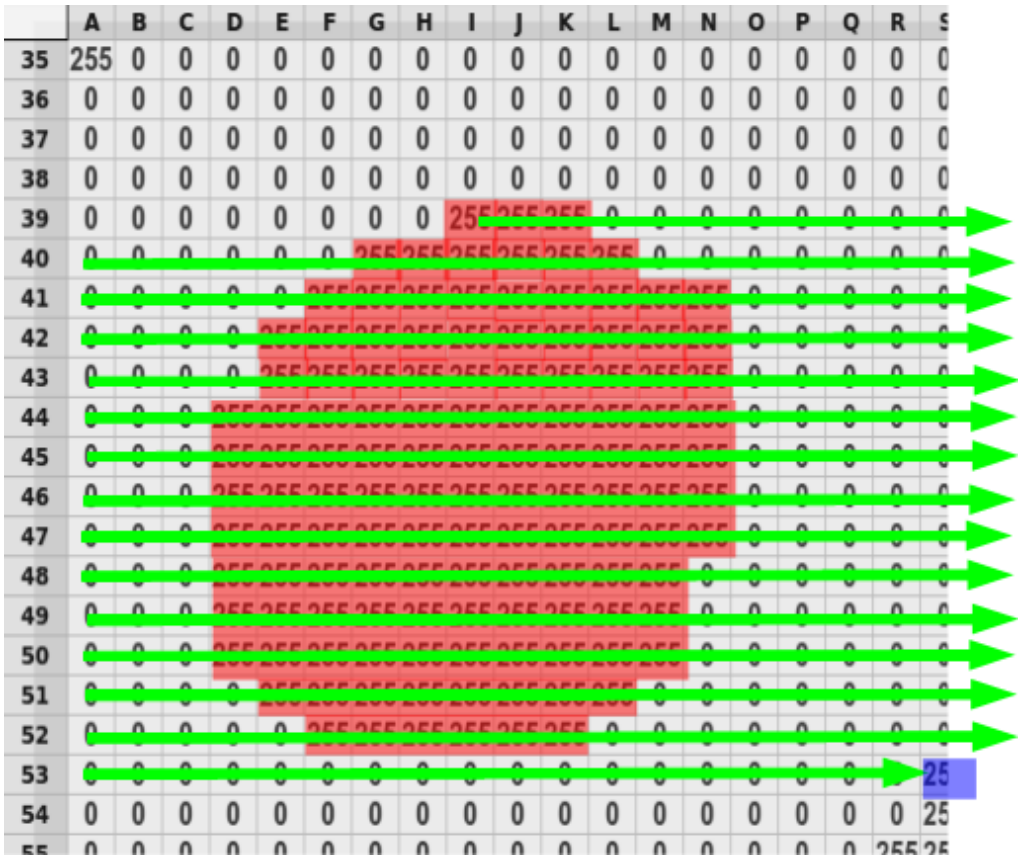


- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 1
- Check neighbors of all new labeled pixels
- Label non-zero pixels as component 1

Application example

Image processing

Find connected components:



- Search pixel-by-pixel for non-zero intensity
- Label found pixel as component 2
- ...

Application example

Image processing

Result of connected component labeling:

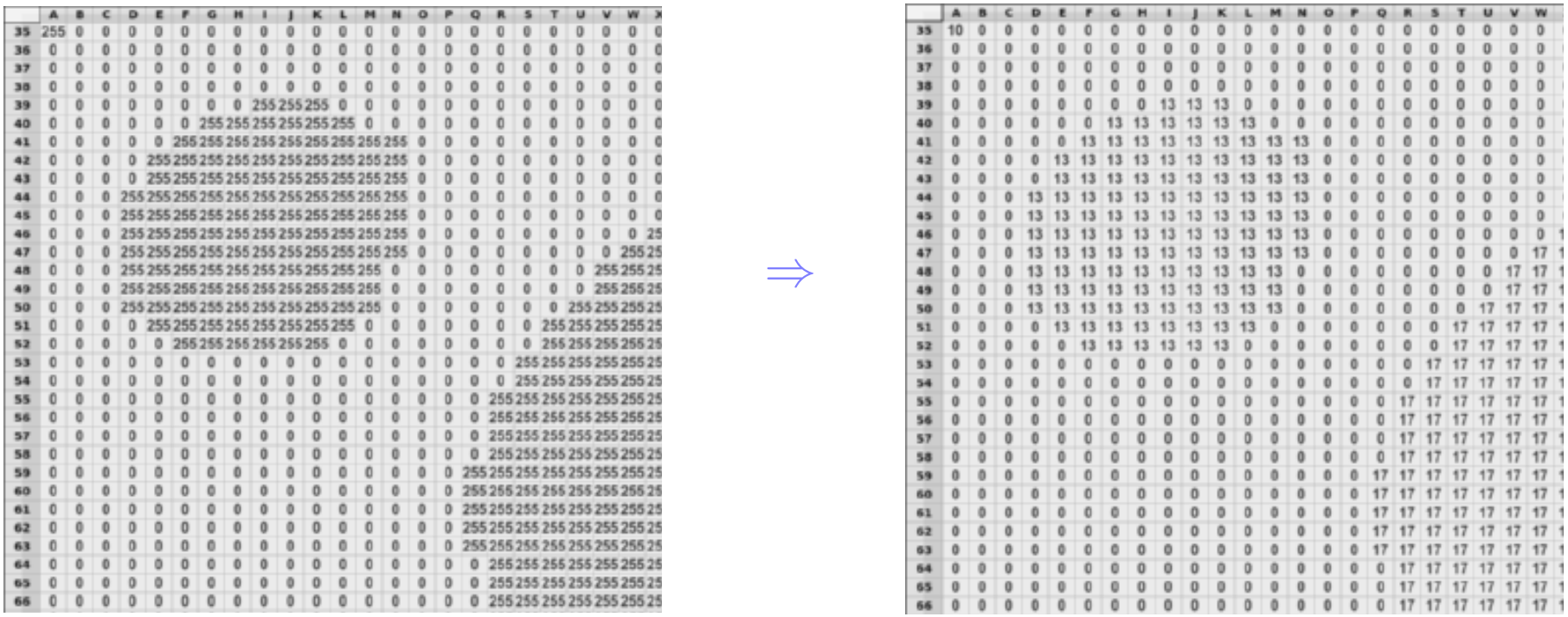


Figure: Result: particle indices instead of intensities

Further Literature

- **General**

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.
Introduction to Algorithms.
MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.
Algorithms and data structures, 2008.
https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf.

Further Literature

- **Graph-Search**

  [Wika] Breadth-first search
  `https://en.wikipedia.org/wiki/Breadth-first_search`

  [Wikb] Depth-first search
  `https://en.wikipedia.org/wiki/Depth-first_search`

- **Graph-Connectivity**

  [Wik] Connectivity (graph theory)
  `https://en.wikipedia.org/wiki/Connectivity_(graph_theory)`