Algorithms and Datastructures

Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, January 2017

Structure

Binary search tree:

Balanced Trees

Motivation

Binary search tree:

- With `BinarySearchTree` we could perform an `lookup` or `insert` in $O(d)$, with d being the depth of the tree

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- Best case: $d \in O(\log n)$, keys are inserted randomly
- Worst case: $d \in O(n)$, keys are inserted in ascending / descending order $(20, 19, 18, \ldots)$

Gnarley trees:

Balanced Trees

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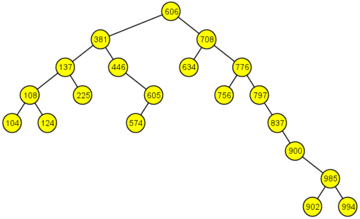

Figure: Binary search tree with random insert [Gna]

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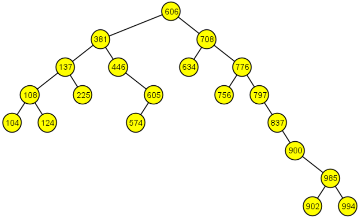

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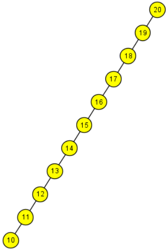

Figure: Binary search tree with descending insert [Gna]

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- We rebalance the tree from time to time

Balanced Trees

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  - Used in C++ `std::map` and Java `SortedMap`

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- We can perform all basic operations in $O(\log n)$

Balanced Trees

AVL-Tree

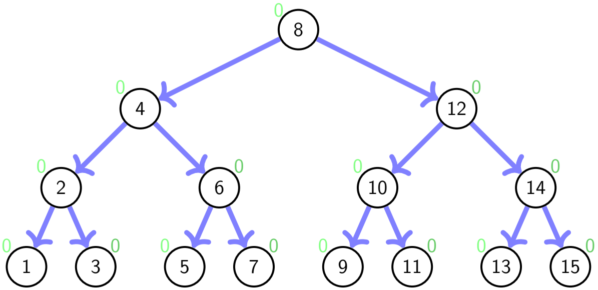

Figure: Example of an AVL-Tree

Balanced Trees

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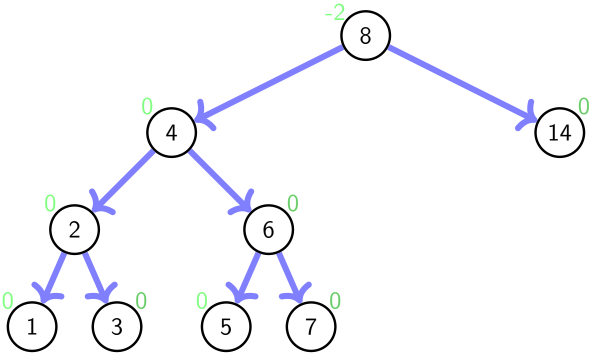


Figure: **Not** an AVL-Tree

Balanced Trees

AVL-Tree

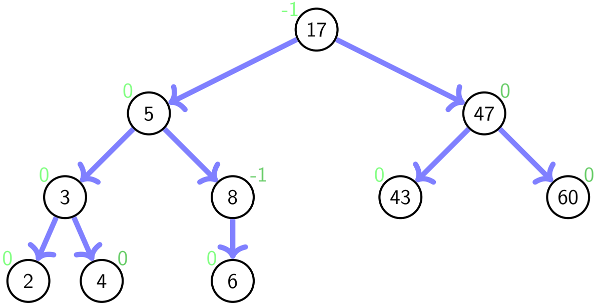


Figure: Another example of an AVL-Tree

Balanced Trees

AVL-Tree - Rebalancing

Rotation:

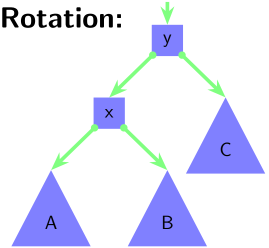
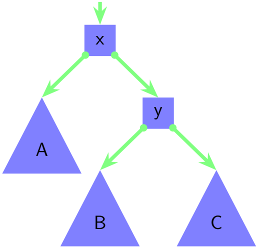


Figure: Before rotating

$\Rightarrow$



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Balanced Trees

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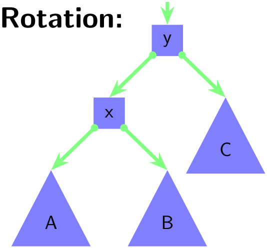

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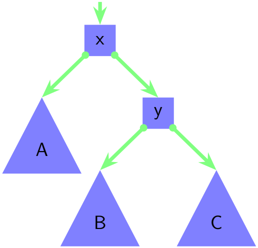

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- Central operation of rebalancing

Balanced Trees

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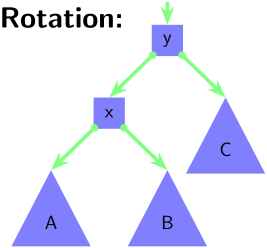

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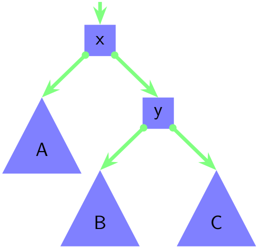

Figure: After rotating

- Central operation of rebalancing
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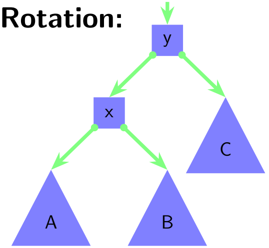

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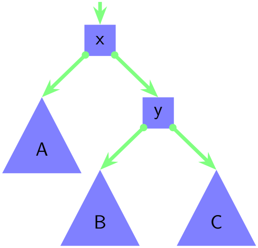

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Balanced Trees

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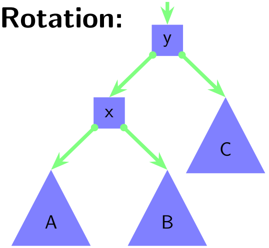

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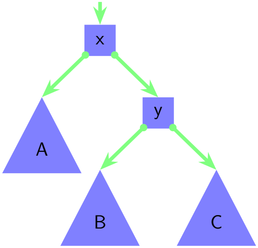

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- Central operation of rebalancing
- After rotation to the right:
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  - The parent child relations between nodes x and y have been swapped

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Balanced Trees

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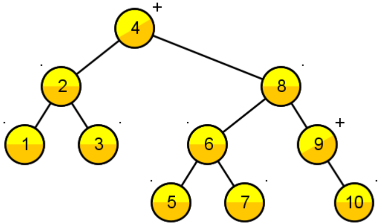

Figure: Inserting $1, \ldots, 10$ into an AVL-tree [Gna]

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- However not amortized update costs of $O(1)$
- Additional memory costs: We have to save a height difference for every node
- Better (and easier) to implement are (a,b)-trees

Structure

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- So we have space for elements on an `insert` and balance operation

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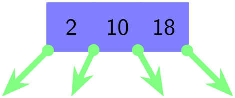

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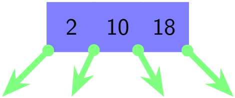

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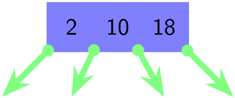

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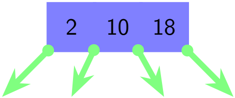

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(2,4)-Tree:

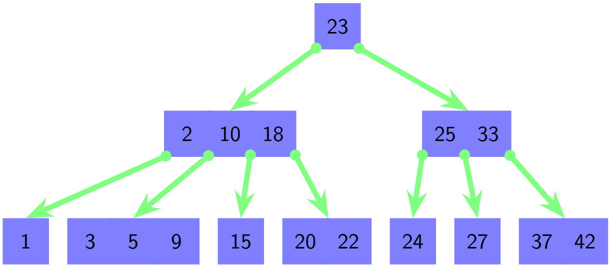

Figure: Example of an (2,4)-tree

(a,b)-Trees
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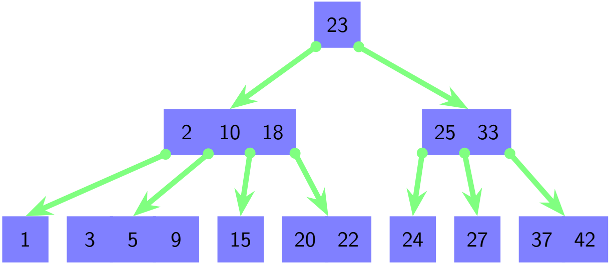

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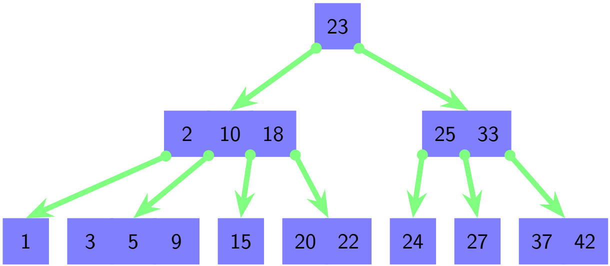

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- (2,4)-tree with depth of 3
- Each node has between 2 and 4 children (1 to 3 elements)

(a,b)-Trees

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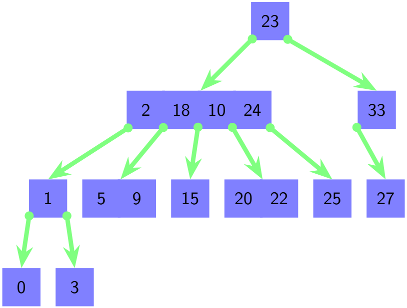

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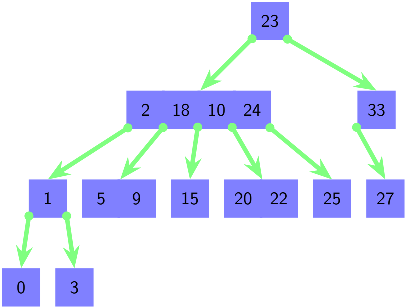

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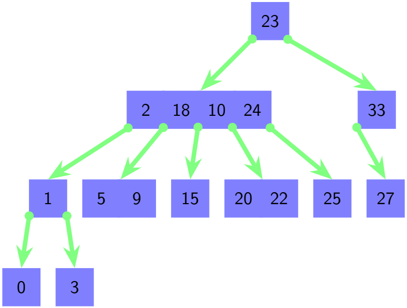

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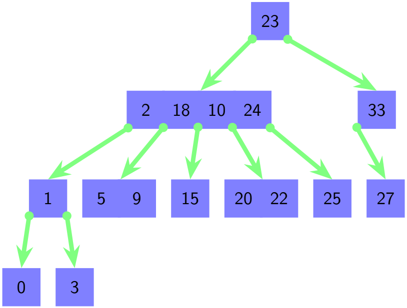

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- Invalid sorting
- Degree of node too large / too small
- Leaves on different levels

Searching an element: (`lookup`)

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(a,b)-Trees

Implementation - Lookup

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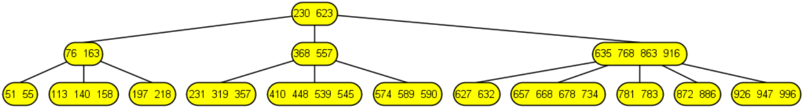

Figure: (3,5)-Tree [Gna]

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- Then we **split** the node

(a,b)-Trees

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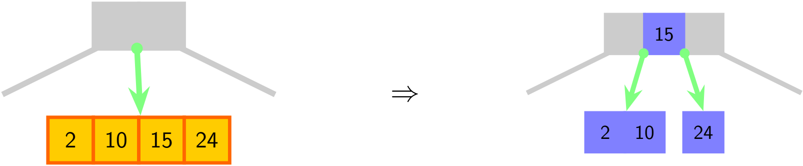

Figure: Splitting a node

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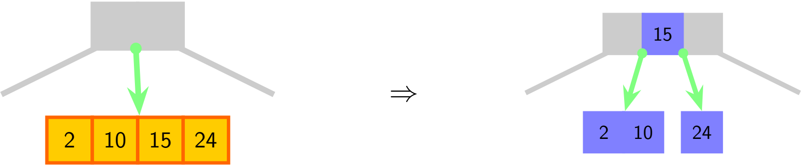

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(a,b)-Trees

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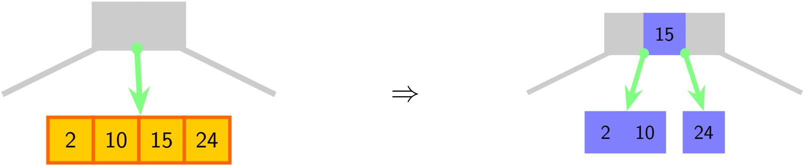

Figure: Splitting a node

- If the degree is higher than $b + 1$ we split the node
- This results in a node with $\text{ceil}\left(\frac{b-1}{2}\right)$ elements, a node with $\text{floor}\left(\frac{b-1}{2}\right)$ elements and one element for the parent node

(a,b)-Trees

Implementation - Insert

Inserting an element: (`insert`)

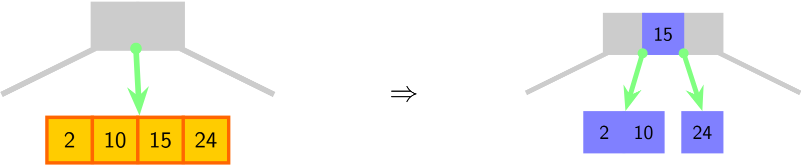

Figure: Splitting a node

- If the degree is higher than $b + 1$ we split the node
- This results in a node with $\text{ceil}\left(\frac{b-1}{2}\right)$ elements, a node with $\text{floor}\left(\frac{b-1}{2}\right)$ elements and one element for the parent node
- Thats why we have the limit $b \geq 2\,a - 1$

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- If we split the root node we create a new parent root node (The tree is now one level deeper)

Removing an element: (`remove`)

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- Search the element in $O(\log n)$ time

Removing an element: (`remove`)

- Search the element in $O(\log n)$ time
- **Case 1:** The element is contained by a leaf
  - Remove element

(a,b)-Trees

Implementation - Remove

Removing an element: (`remove`)

- Search the element in $O(\log n)$ time
- **Case 1:** The element is contained by a leaf
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- **Case 2:** The element is contained by an inner node

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- **Case 2:** The element is contained by an inner node
  - Search the successor in the right subtree
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  - Replace the element with its successor and delete the successor from the leaf

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- **Case 1:** The element is contained by a leaf
  - Remove element
- **Case 2:** The element is contained by an inner node
  - Search the successor in the right subtree
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- **Attention:** The leaf might be too small (degree of $a - 1$)
  $\Rightarrow$ We rebalance the tree

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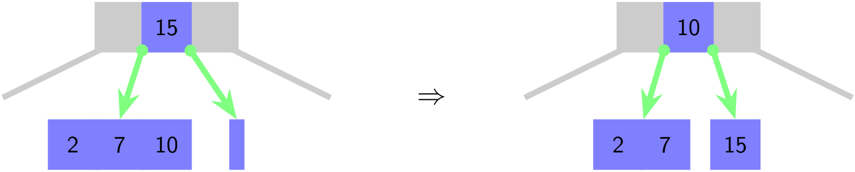

Figure: Borrow an element

Removing an element: (`remove`)

(a,b)-Trees

Implementation - Remove

Removing an element: (`remove`)

- **Attention:** The leaf might be too small (degree of $a-1$)
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  - **Case b:** We merge the node with its right or left neighbour

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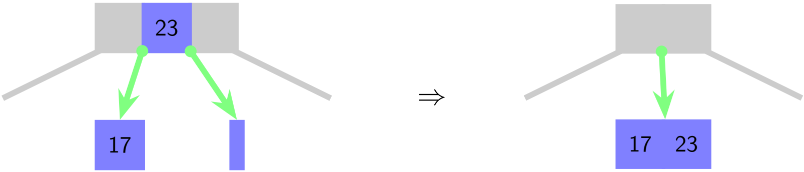

Figure: Merge two nodes

Removing an element: (`remove`)

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- Now the parent node can be of degree $a - 1$

Removing an element: (`remove`)

- Now the parent node can be of degree $a - 1$
- We merge parent nodes the same way

(a,b)-Trees

Implementation - Remove

Removing an element: (`remove`)

- Now the parent node can be of degree $a - 1$
- We merge parent nodes the same way
- If the root has only a single child
  - Remove the root
  - Define sole child as new root
  - The tree shrinks by one level

Runtime complexity of `lookup`, `insert` and `remove`:

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- All operations in $O(d)$ with d being the depth of the tree

(a,b)-Trees
Runtime Complexity

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- All operations in $O(d)$ with d being the depth of the tree
- Each node (except the root) has more than a children
  $\Rightarrow n \geq a^{d-1}$ and $d \leq 1 + \log_a n = O(log_a n)$

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(a,b)-Trees

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- `lookup` always takes $\Theta(d)$
- `insert` and `remove` often require only $O(1)$ time
- **Worst case:** split or merge all nodes on path up to the root
- Therefore instead of $b \geq 2a - 1$ we need $b \geq 2a$

Counter example (2,3)-Tree:

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- Before executing `delete(11)`

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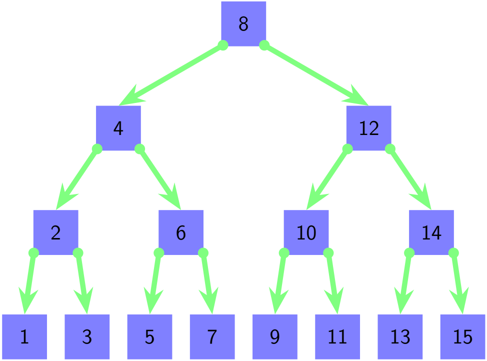

Figure: Normal (2,3)-Tree

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `delete(11)`

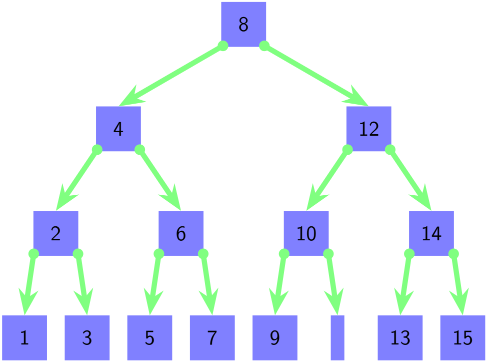

Figure: (2,3)-Tree - Delete step 1

Counter example (2,3)-Tree:

- Executing `delete(11)`

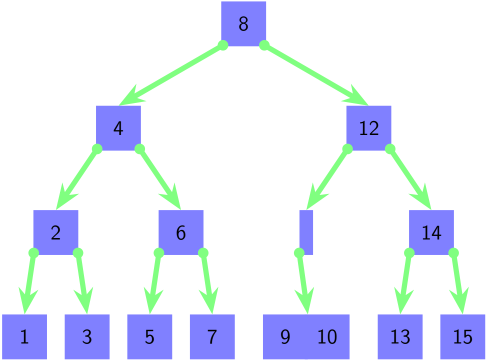

Figure: (2,3)-Tree - Delete step 2

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- Executing `delete(11)`

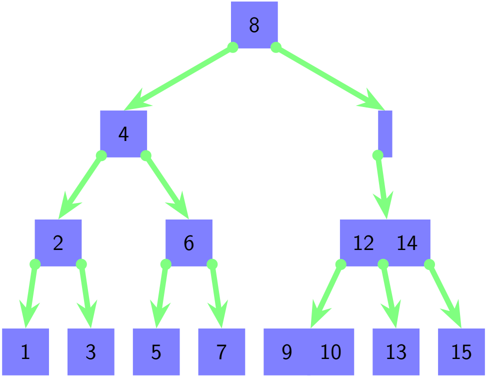

Figure: (2,3)-Tree - Delete step 3

Counter example (2,3)-Tree:

- Executed `delete(11)`

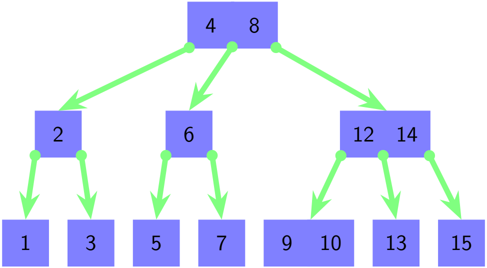

Figure: (2,3)-Tree - Delete step 4

Counter example (2,3)-Tree:

- Executing `insert(11)`

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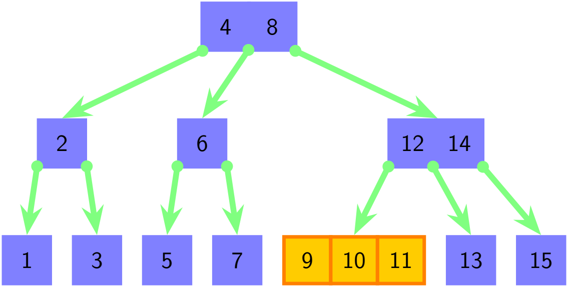

Figure: (2,3)-Tree - Insert step 1

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

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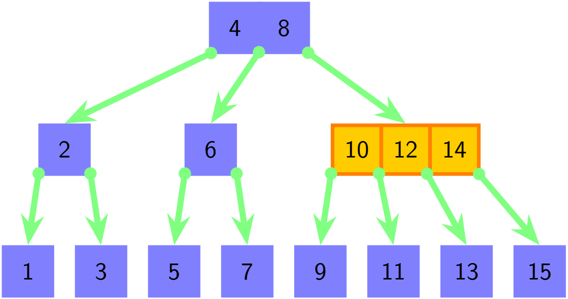

Figure: (2,3)-Tree - Insert step 2

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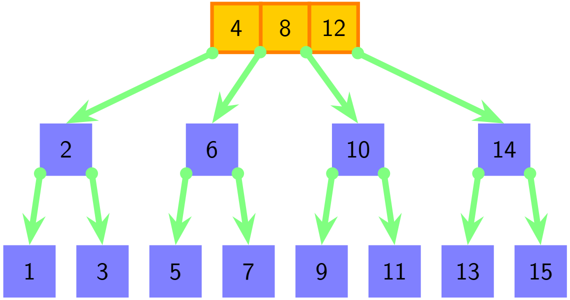

Figure: (2,3)-Tree - Insert step 3

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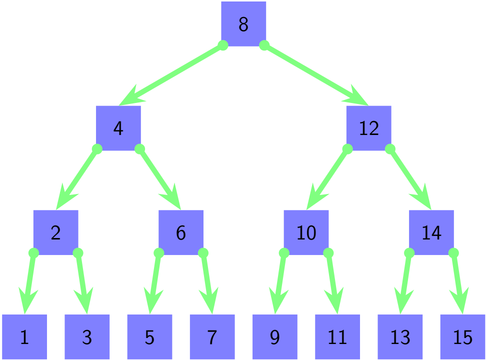

Figure: (2,3)-Tree - Insert step 4

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- We are exactly where we started

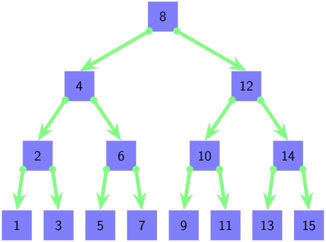

Figure: (2,3)-Tree

(a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

Counter example (2,3)-Tree:

- We are exactly where we started
- If $b = 2a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$

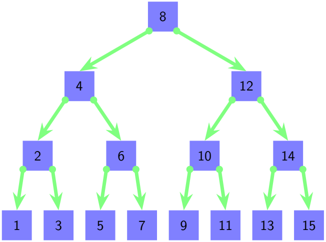

Figure: (2,3)-Tree

Counter example (2,3)-Tree:

- We are exactly where we started
- If $b = 2\,a - 1$ then we can create a sequence of `insert` and `remove` operations where each operation costs $O(\log n)$
- We need $b \geq 2\,a$ instead of $b \geq 2\,a - 1$

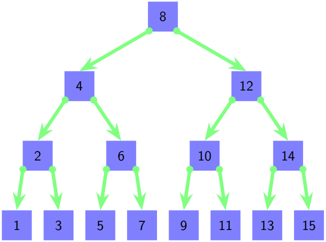

Figure: (2,3)-Tree

(2,4)-Tree:

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(2,4)-Tree:

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⇒ **Nodes of degree 3 are stable**
Neither an `insert` nor a `remove` operation trigger rebalancing operations

(2,4)-Tree:

- **Idea:**

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  - After an expensive operation the tree is in a stable state

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- **Idea:**
  - After an expensive operation the tree is in a stable state
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- Like with dynamic arrays:
  - Reallocation is expensive but it takes some time until the next expensive operation occurs
  - If we overallocate clever we have an amortized runtime of $O(1)$

Terminology:

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- We analyze a sequence of n operations

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- Let Φ_i be the potential of the tree after the i-*th* operation

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- Empty tree has 0 nodes: $\Phi = 0$

Example:

(a,b)-Trees
Runtime Complexity - (2,4)-Tree

Example:

- Nodes of degree 3 are highlighted

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Example:

- Nodes of degree 3 are highlighted

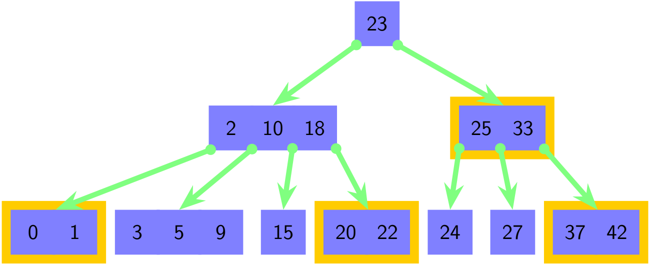

Figure: Tree with potential $\Phi = 4$

Terminology:

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- Let c_i be the costs = runtime of the i-th operation

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$$c_i \leq A \cdot \underbrace{(\Phi_i - \Phi_{i-1})} + B, \qquad A > 0 \text{ and } B > A$$

Number of gained stable nodes (degree 3) ≥ -1

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Number of gained stable nodes (degree 3) ≥ -1

- Each operation has an amortitzed cost of $O(1)$ summing up to $O(n)$ in total

Case 1: i-*th* operation is an `insert` operation on a full node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: *i-th* operation is an `insert` operation on a full node

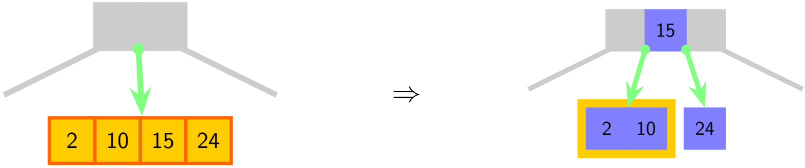

Figure: Splitting a node on `insert`

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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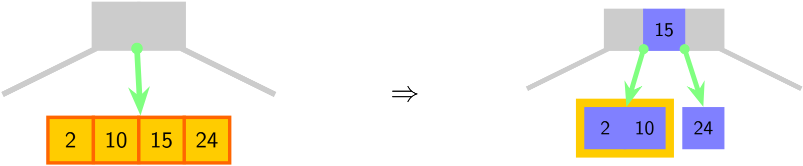

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- Each splitted node creates a node of degree 3

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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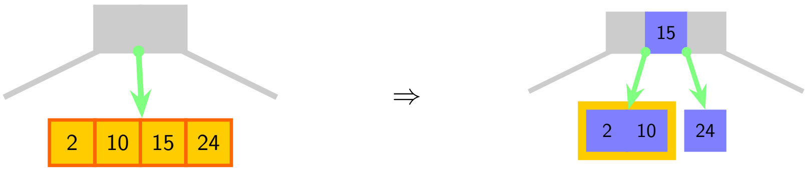

Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

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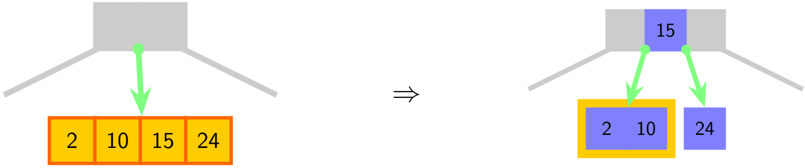

Figure: Splitting a node on `insert`

- Each splitted node creates a node of degree 3
- The parent node receives an element from the splitted node
- If the parent node is also full we have to split it too

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: *i-th* operation is an `insert` operation on a full node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: i-*th* operation is an `insert` operation on a full node

- Let m be the number of nodes split

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: i-*th* operation is an `insert` operation on a full node

- Let m be the number of nodes split
- The potential rises by m

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 1: *i-th* operation is an `insert` operation on a full node

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$$\Phi_i \geq \Phi_{i-1} + m - 1$$
$$\Rightarrow m \leq \Phi_i - \Phi_{i-1} + 1$$

Costs: $c_i \leq A \cdot m + B$

$$\Rightarrow c_i \leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B$$
$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}$$

Case 2: *i-th* operation is an `remove` operation

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Inner node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

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Case 2: *i-th* operation is an `remove` operation

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Case 2: *i-th* operation is an `remove` operation

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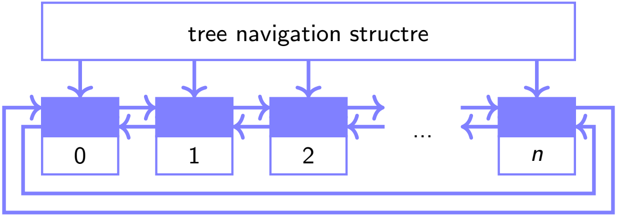

Figure: Tree with doubly linked list

Case 2: i-*th* operation is an `remove` operation

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
  - Case 2.1.1: Potential rises by one

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
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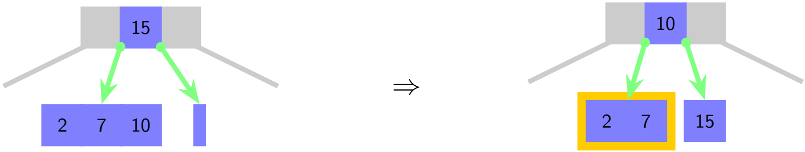

Figure: Case 2.1.1: Borrow an element

Case 2: i-*th* operation is an `remove` operation

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
  - Case 2.1.2: Potential is lowered by one

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.1:** Borrow a node
  - Creates no additional operations
  - Case 2.1.2: Potential is lowered by one

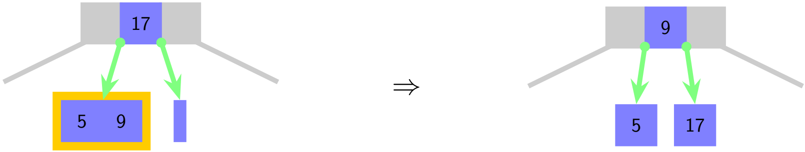

Figure: Case 2.1.2: Borrow an element

Case 2: *i-th* operation is an `remove` operation

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

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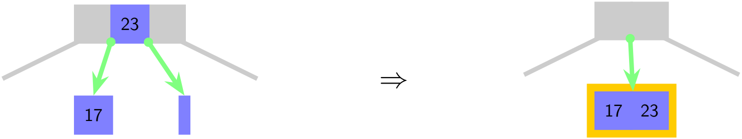

Figure: Merging two nodes

- Potential rises by one

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

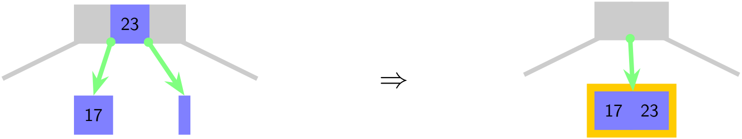

Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

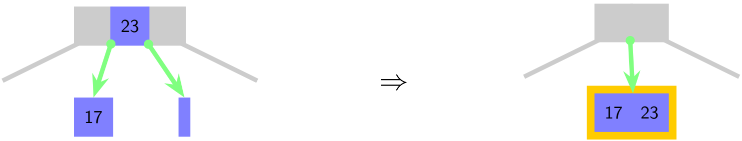

Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree > 2 or a node of degree 2, which can borrow from a neighbour

(a,b)-Trees

Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

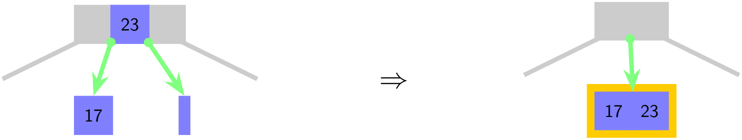

Figure: Merging two nodes

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

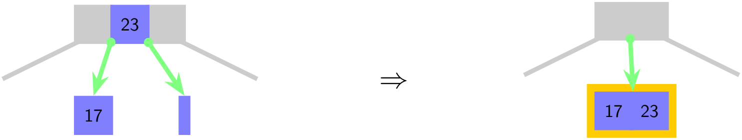

Figure: Merging two nodes

- The potential rises by m

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

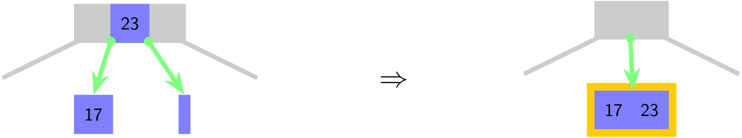

Figure: Merging two nodes

- The potential rises by m
- If the “stop-node” is of degree 2 then the potential eventually goes down by one

(a,b)-Trees
Runtime Complexity - (2,4)-Tree

Case 2: *i-th* operation is an `remove` operation

- **Case 2.2:** Merging two node

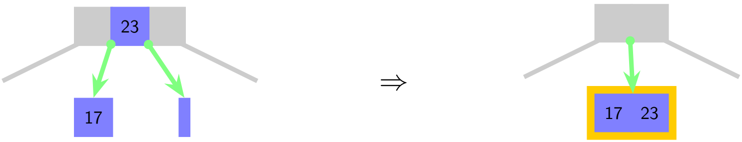

Figure: Merging two nodes

- The potential rises by m
- If the "stop-node" is of degree 2 then the potential eventually goes down by one
- Same costs as `insert`

Lemma:

Lemma:

- We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

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- We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \qquad A > 0 \text{ and } B > A$$

- With that we can conclude:

$$\sum_{i=0}^{n} c_i \in O(n)$$

(a,b)-Trees

Runtime Complexity - (2,4)-Tree - Lemma - Proof

Proof:

$$\sum_{i=0}^{n} c_i \leq \underbrace{A \cdot (\Phi_1 - \Phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\Phi_2 - \Phi_1) + B}_{\leq c_2} + \cdots + \underbrace{A \cdot (\Phi_n - \Phi_{n-1})}_{\leq c_n}$$

$$= A \cdot (\Phi_n - \Phi_0) + B \cdot n \qquad | \text{ telescope sum}$$

$$= A \cdot \Phi_n + B \cdot n \qquad | \text{ we start with an empty tree}$$

$$< A \cdot n + B \cdot n \in O(n) \qquad | \text{ number of degree 3 nodes}$$

¡ number of nodes

Structure

Red-Black Tree:

Red-Black-Trees

Introduction

Red-Black Tree:

- Binary tree with red and black nodes

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Red-Black Tree:

- Binary tree with red and black nodes
- Number of black nodes on path to leaves is equal
- Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- Each (2,4)-tree-node is a small red-black-tree with a black root node

Red-Black-Trees

Introduction

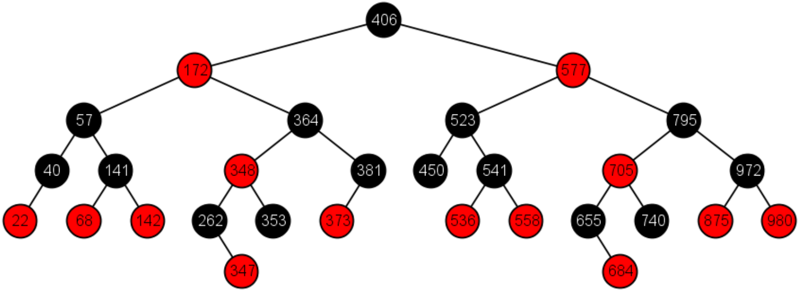

Figure: Example of an red-black-tree [Gna]

- **General**

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[Wikb] (a,b)-tree
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- **Red-Black-Tree**

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