

Algorithms and Datastructures

Cache Efficiency, Divide and Conquer

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Structure

Cache Efficiency

- Introduction

- Cache Organization

Divide and Conquer

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Divide and Conquer

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Cache Efficiency

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Background:

Cache Efficiency

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- ▶ Up to now we always counted **number of operations**
- ▶ Assuming this is a good measure for the runtime of a algorithm/tool

Cache Efficiency

Introduction

Background:

- ▶ Up to now we always counted **number of operations**
- ▶ Assuming this is a good measure for the runtime of a algorithm/tool
- ▶ Today we will see examples where this is not suitable

Cache Efficiency

Introduction

Example:

- ▶ We sum up all elements of a field a of size n in ...
 - ▶ natural order:

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

- ▶ random order:

$$\text{sum}(a) = a[21] + a[5] + \cdots + a[8]$$

Cache Efficiency

Linear Order - Python

Python:

```
def init(size):  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```


Cache Efficiency

Linear Order - Python

Python:

```
def run(param):  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```

Cache Efficiency

Linear Order

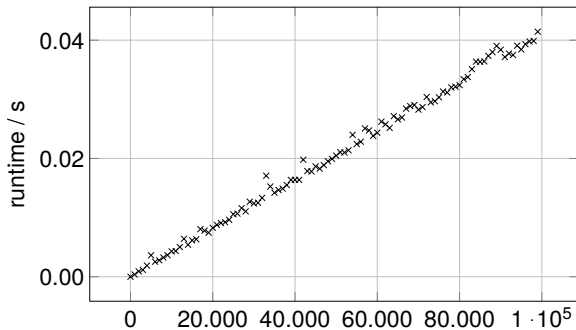


Figure: Summing elements in linear order

Cache Efficiency

Random Order - Python

Python:

```
def init(size):  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

Cache Efficiency

Random Order

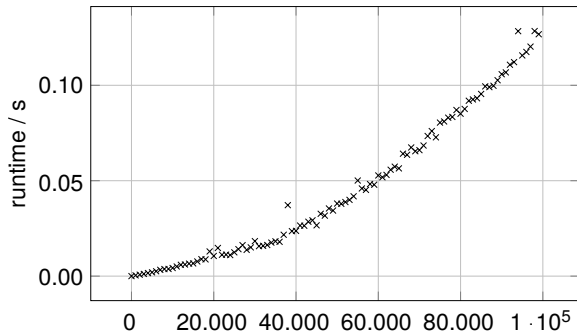


Figure: Summing elements in random order

Cache Efficiency

Algorithm Comparision

Conclusion:

Cache Efficiency

Algorithm Comparision

Conclusion:

- ▶ The number of operations are identical for both algorithms

Cache Efficiency

Algorithm Comparision

Conclusion:

- ▶ The number of operations are identical for both algorithms
- ▶ Accessing elements in random order takes a lot longer (Factor 10)

Why?

- ▶ The costs in terms of memory access are very different

Structure

Cache Efficiency

Introduction

Cache Organization

Divide and Conquer

Introduction

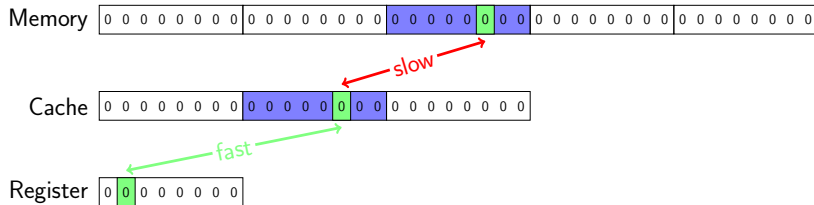
Cache Efficiency

CPU Cache



Cache Efficiency

CPU Cache



Principle / organization:

Cache Efficiency

CPU Cache

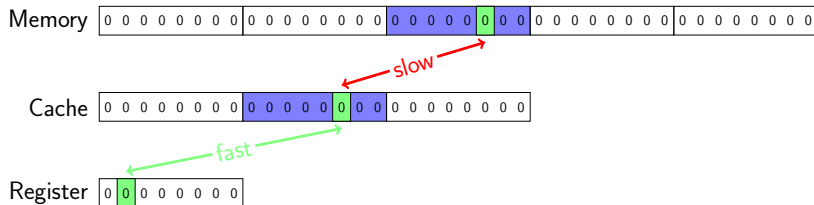


Principle / organization:

- ▶ Accessing one byte of the main memory takes ≈ 100 ns

Cache Efficiency

CPU Cache



Principle / organization:

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Cache Efficiency

CPU Cache

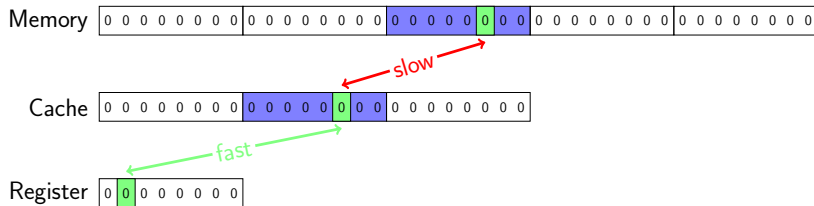


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- ▶ Accessing one or more byte/s of main memory loads a whole block ≈ 100 B into the cache

Cache Efficiency

CPU Cache

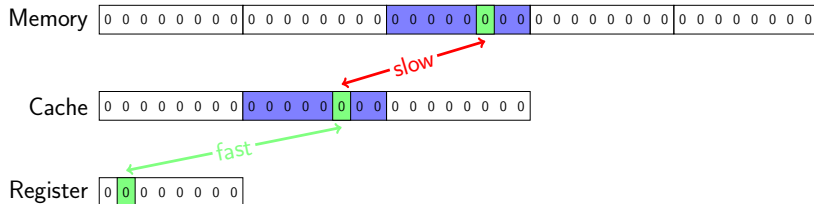


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- ▶ Accessing one byte of (L1-)cache takes ≈ 1 ns
- ▶ Accessing one or more byte/s of main memory loads a whole block ≈ 100 B into the cache
- ▶ As long as this block is in the cache, it is not necessary to access the memory for bytes of this block

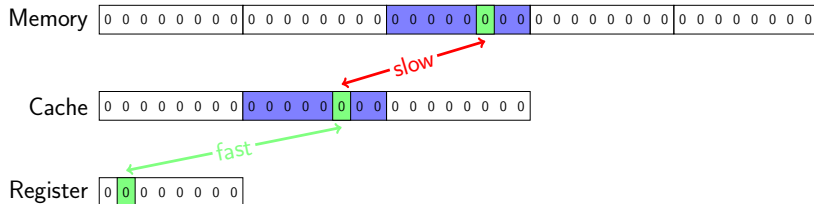
Cache Efficiency

CPU Cache



Cache Efficiency

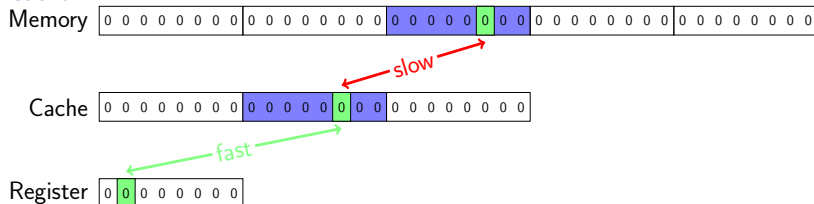
CPU Cache



Cache organization:

Cache Efficiency

CPU Cache

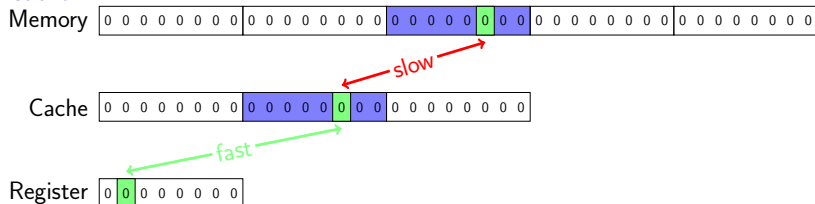


Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks (cache lines)

Cache Efficiency

CPU Cache

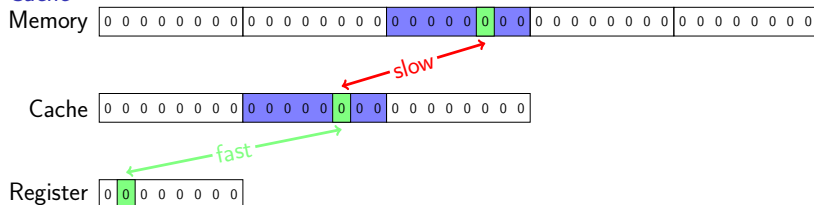


Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks (cache lines)
 - ▶ ≈ 100 kB

Cache Efficiency

CPU Cache

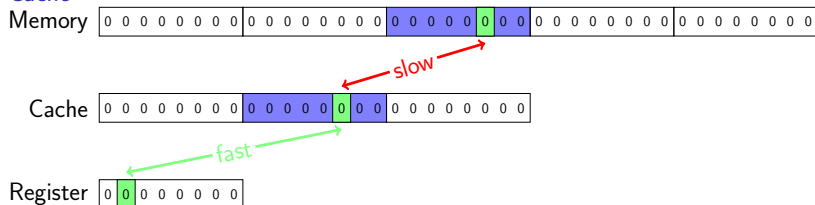


Cache organization:

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Cache Efficiency

CPU Cache

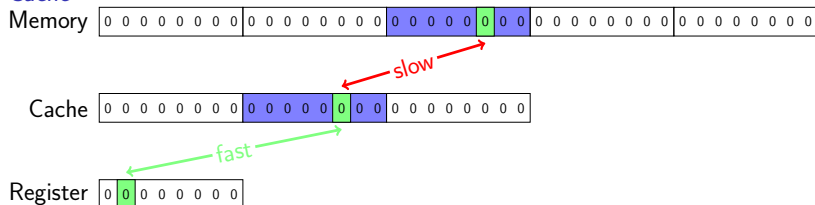


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- ▶ The (L1-)cache can hold multiple memory blocks (cache lines)
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Cache Efficiency

CPU Cache

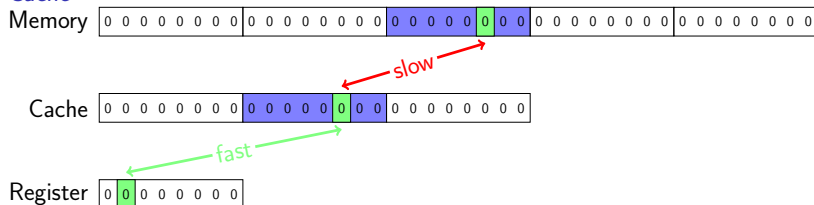


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Cache Efficiency

CPU Cache

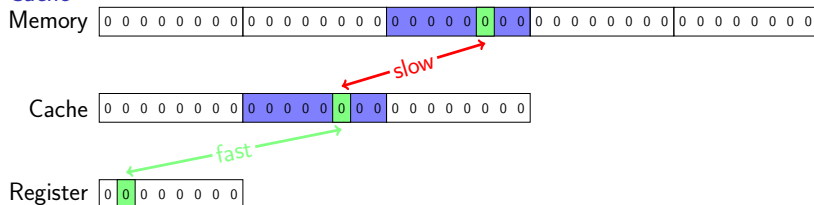


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Cache Efficiency

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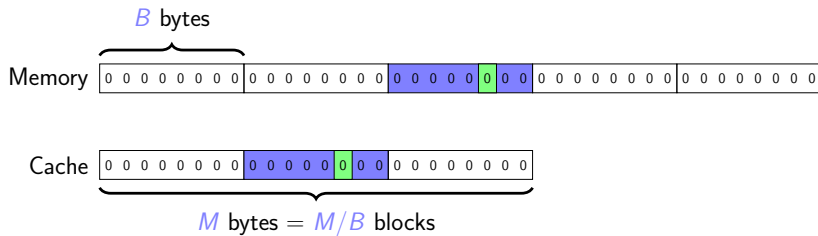


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 - ▶ ≈ 100 kB
- ▶ If the capacity is reached unused blocks are discarded
 - ▶ Least recently used (LRU)
 - ▶ Least frequently used (LFU)
 - ▶ First in first out (FIFO)
- ▶ Details of discarding are not the topic for today

Cache Efficiency

Block Operations



Terminology:

Cache Efficiency

Block Operations



Terminology:

- The system consists of slow and fast memory

Cache Efficiency

Block Operations



Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size B**

Cache Efficiency

Block Operations

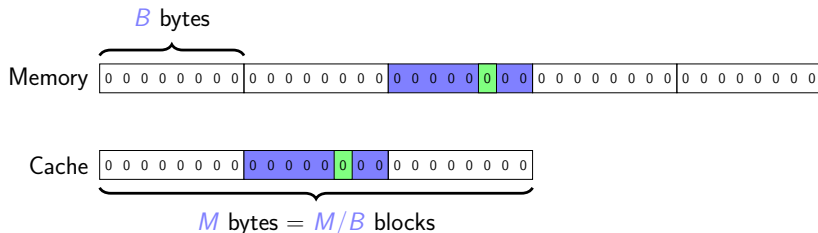


Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size B**
- ▶ The **fast cache** has size M and can store M/B blocks

Cache Efficiency

Block Operations



Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size B**
- ▶ The **fast cache** has size M and can store M/B blocks
- ▶ If data is not in fast memory, the corresponding block is loaded into the **cache**

Cache Efficiency

Block Operations



Terminology:

Cache Efficiency

Block Operations



Terminology:

- ▶ The program defines which blocks are held in the **cache**

Cache Efficiency

Block Operations

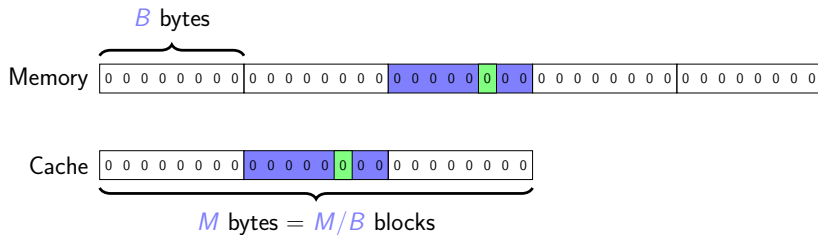


Terminology:

- ▶ The program defines which blocks are held in the **cache**
- ▶ We use the number of **block operations** as runtime estimation

Cache Efficiency

Block Operations



Terminology:

- ▶ The program defines which blocks are held in the **cache**
- ▶ We use the number of **block operations** as runtime estimation
- ▶ We ignore runtime costs of cache accesses / management

Cache Efficiency

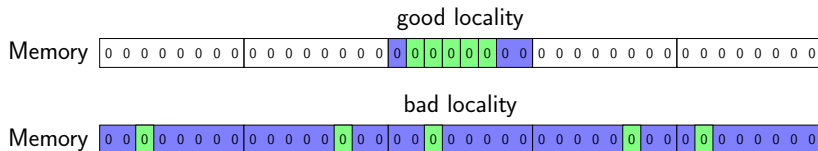


Figure: Comparison good / bad locality

Accessing the cache B times:

- ▶ **Best case:** 1 block operation \rightarrow good locality
- ▶ **Worst case:** B block operations \rightarrow bad locality

Cache Efficiency

Block Operations

Additional factors:

Cache Efficiency

Block Operations

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- ▶ The following settings change only a small constant factor in number of block operations

Cache Efficiency

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Cache Efficiency

Block Operations

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 - ▶ If the block is 1 Bytes or 4 Bytes or 8 Bytes

Cache Efficiency

Block Operations

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Note:

Cache Efficiency

Block Operations

Additional factors:

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Note:

- ▶ If the input size is smaller than M we load the complete data chunk directly into the cache

Cache Efficiency

Block Operations

Additional factors:

- ▶ The following settings change only a small constant factor in number of block operations
 - ▶ The partitioning of the slow memory into blocks
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Note:

- ▶ If the input size is smaller than M we load the complete data chunk directly into the cache
- ▶ Cache handling is only interesting when the input size is greater than M

Cache Efficiency

Block Operations

Typical values: (Intel® i7-4770 Haswell, WD® Blue 2 TB)

Cache Efficiency

Block Operations

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- ▶ CPU L1 Cache: $B = 64 \text{ B}$, $M = 4 \times (32 \text{ kB} + 32 \text{ kB})$

Cache Efficiency

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Cache Efficiency

Block Operations

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- ▶ CPU L3 Cache: $B = 64 \text{ B}$, $M = 8 \text{ MB}$
- ▶ Disk Cache: $B = 64 \text{ kB}$, $M = 64 \text{ MB}$

Cache Efficiency

Block Operations

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- ▶ CPU L3 Cache: $B = 64 \text{ B}$, $M = 8 \text{ MB}$
- ▶ Disk Cache: $B = 64 \text{ kB}$, $M = 64 \text{ MB}$
 - ▶ Many operating systems use free system memory as disk cache

Cache Efficiency

Block Operations

Terminology:

Cache Efficiency

Block Operations

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- ▶ Block loads on CPU-cache are called **cache misses**

Cache Efficiency

Block Operations

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- ▶ Block operations on disk-cache are called **IOs**
(input / output operations)

Cache Efficiency

Block Operations

Terminology:

- ▶ Block loads on CPU-cache are called **cache misses**
- ▶ Block operations on disk-cache are called **IOs**
(input / output operations)
- ▶ These also fall under the term **cache efficiency** or **IO efficiency**

Cache Efficiency

Block Operations - Linear Order

Example 1 - Linear order:

Cache Efficiency

Block Operations - Linear Order

Example 1 - Linear order:

- ▶ We sum up all elements in **natural order**

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

Cache Efficiency

Block Operations - Linear Order

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Cache Efficiency

Block Operations - Linear Order

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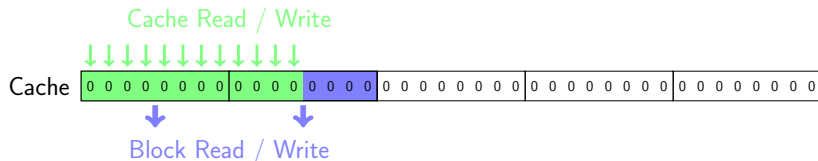


Figure: Good locality of sum operation

Cache Efficiency

Block Operations - Random Order

Example 2 - Random order:

Cache Efficiency

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Cache Efficiency

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Cache Efficiency

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Cache Efficiency

Block Operations - Random Order

Example 2 - Random order:

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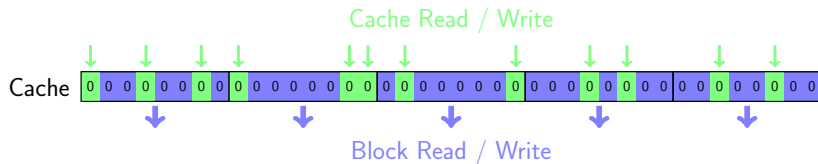


Figure: Bad locality of sum operation

Cache Efficiency

Block Operations

Generally the factor is substantially $< B$

Cache Efficiency

Block Operations

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- ▶ Even with a **random order** we access per element 4 (int) / neighboring bytes at once

Cache Efficiency

Block Operations

Generally the factor is substantially $< B$

- ▶ Even with a **random order** we access per element 4 (int) / neighboring bytes at once
- ▶ If **not $n \gg M$** the next element might already with a high probability loaded in cache

Cache Efficiency

Block Operations - QuickSort

QuickSort:

Cache Efficiency

Block Operations - QuickSort

QuickSort:

- ▶ **Strategy:** Divide and conquer

Cache Efficiency

Block Operations - QuickSort

QuickSort:

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- ▶ Divide the data into two parts where the “left” part contains all values \leq those in the right part

Cache Efficiency

Block Operations - QuickSort

QuickSort:

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Cache Efficiency

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Cache Efficiency

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Figure: *QuickSort* with pivot-element

Idea of Quicksort

- ▶ **at start:** pivot in first position, first re-arrange list such that left part contains small, right part larger elements
- ▶ do required changes *in place*



- ▶ **end point:** k is left to left-most element greater than pivot
swap position 0 (pivot) with k (smaller than pivot)

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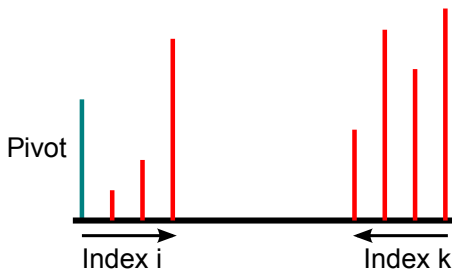
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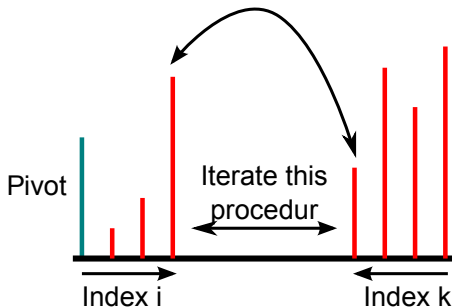
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Cache Efficiency

Block Operations - QuickSort - Python

Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[0]  
  
    ...
```

Cache Efficiency

Block Operations - QuickSort - Python

```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```

Cache Efficiency

Block Operations - QuickSort

Number of operations for Quicksort:

Cache Efficiency

Block Operations - QuickSort

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- ▶ Let $T(n)$ be the runtime for the input size n

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Block Operations - QuickSort

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Block Operations - QuickSort

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 - ▶ Fields are always separated perfectly in the middle

Cache Efficiency

Block Operations - QuickSort

Number of operations for Quicksort:

- ▶ Let $T(n)$ be the runtime for the input size n
- ▶ Assumptions:
 - ▶ Fields are always separated perfectly in the middle
 - ▶ n is a power of two and recursion depth is $k = \log_2 n$

Cache Efficiency

Block Operations - QuickSort

$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left(A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$

Cache Efficiency

Block Operations - QuickSort



Figure: Locality of quicksort

Cache Efficiency

Block Operations - QuickSort

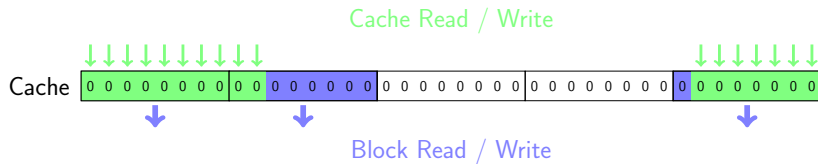


Figure: Locality of quicksort

- Let $IO(n)$ be the number of block operations for input size n

Cache Efficiency

Block Operations - QuickSort



Figure: Locality of quicksort

- ▶ Let $IO(n)$ be the number of block operations for input size n
- ▶ Assumptions as before but recursion depth is $k = \log_2 \frac{n}{B}$
Why?

Cache Efficiency

Block Operations - QuickSort

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in O\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

Structure

Cache Efficiency

Introduction

Cache Organization

Divide and Conquer

Introduction

Divide and Conquer

Introduction

Concept:

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all solutions of the subproblems to a solution of the full problem

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all solutions of the subproblems to a solution of the full problem
- ▶ **Recursive** application of the algorithm to ever smaller subproblems

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all solutions of the subproblems to a solution of the full problem
- ▶ **Recursive** application of the algorithm to ever smaller subproblems
- ▶ **Direct** solving of sufficiently small subproblems

Divide and Conquer

Introduction - Python

Divide and Conquer

Introduction - Python

- ▶ Function `solve` for solving a `problem` of size `n`

Divide and Conquer

Introduction - Python

- Function `solve` for solving a `problem` of size `n`

```
def solve(problem):  
    if n < threshold:  
        # solve directly  
        return solution  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
        return S1 + S2 + ... + Sk
```

Divide and Conquer

Features

Divide and Conquer

Features

- ▶ Can help with conceptual hard problems

Divide and Conquer

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 - ▶ **Solution** of the trivial problems has to be known

Divide and Conquer

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Divide and Conquer

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Divide and Conquer

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 - ▶ And separation / combination of subproblems is $\in O(n)$

Divide and Conquer

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Divide and Conquer

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Divide and Conquer

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Divide and Conquer

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Divide and Conquer

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 - ▶ **Combination** of solutions has to be possible
- ▶ Realization of **efficient solutions**
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 - ▶ And separation / combination of subproblems is $\in O(n)$
 - ▶ And the number of subproblems is limited
 - ▶ The runtime is $\in O(n \cdot \log n)$
- ▶ Suitable for parallel processing
 - ▶ Subproblems are **independent** of each other
 - ▶ Only needed input for each subproblem has to be known

Divide and Conquer

Implementation

Definition of the trivial case:

Divide and Conquer

Implementation

Definition of the trivial case:

- ▶ Smaller subproblems are elegant and simple

Divide and Conquer

Implementation

Definition of the trivial case:

- ▶ Smaller subproblems are elegant and simple
- ▶ Otherwise the efficiency will be improved if relative big subproblems can be solved directly

Divide and Conquer

Implementation

Definition of the trivial case:

- ▶ Smaller subproblems are elegant and simple
- ▶ Otherwise the efficiency will be improved if relative big subproblems can be solved directly
- ▶ Recursion depth should not get too big (stack / memory overhead)

Divide and Conquer

Implementation

Division in subproblems:

Divide and Conquer

Implementation

Division in subproblems:

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

Divide and Conquer

Implementation

Division in subproblems:

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Combination of solutions:

Divide and Conquer

Implementation

Division in subproblems:

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

Combination of solutions:

- ▶ Typically conceptual demanding

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

Divide and Conquer

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- ▶ Progression X of n integers

Divide and Conquer

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Example - Maximum Subtotal Input:

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Output:

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

- ▶ Progression X of n integers

Output:

- ▶ Maximum sum of related subsequence and its index boundary

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

- ▶ Progression X of n integers

Output:

- ▶ Maximum sum of related subsequence and its index boundary

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

- ▶ Progression X of n integers

Output:

- ▶ Maximum sum of related subsequence and its index boundary

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

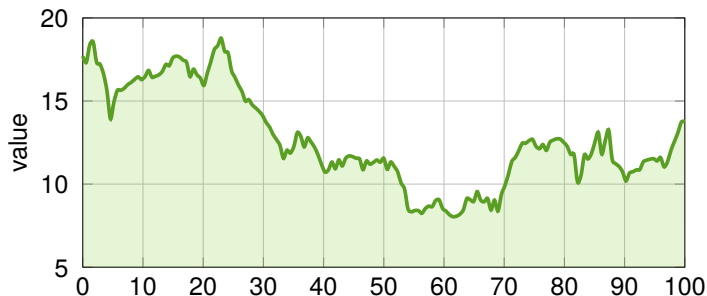
Output: Sum: 187, Start: 2, End: 6

Divide and Conquer

Example - Maximum Subtotal

Application:

- ▶ Maximum profit of buying and selling shares



Divide and Conquer

Example - Maximum Subtotal - Python

Naive solution (brute force)

Divide and Conquer

Example - Maximum Subtotal - Python

Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal - Python

Runtime - Upper bound

Divide and Conquer

Example - Maximum Subtotal - Python

Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

- ▶ Three interleaved loops

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

- ▶ Three interleaved loops
- ▶ Each loop with runtime $O(n)$

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

- ▶ Three interleaved loops
- ▶ Each loop with runtime $O(n)$
- ▶ Algorithm runtime of $O(n^3)$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- We iterate at least $\frac{n}{3}$ values for i

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least $\frac{n}{3}$ values for i
- ▶ For each i we iterate at least $\frac{n}{3}$ values for j

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least $\frac{n}{3}$ values for i
- ▶ For each i we iterate at least $\frac{n}{3}$ values for j
- ▶ For each j we have at least $\frac{n}{3}$ additions

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least $\frac{n}{3}$ values for i
- ▶ For each i we iterate at least $\frac{n}{3}$ values for j
- ▶ For each j we have at least $\frac{n}{3}$ additions
- ▶ We need at least $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$ steps

Divide and Conquer

Example - Maximum Subtotal - Runtime

Runtime:

Divide and Conquer

Example - Maximum Subtotal - Runtime

Runtime:

- ▶ With $T(n) \in O(n^3)$ and $T(n) \in \Omega(n^3)$ we know:

$$T(n) \in \Theta(n^3)$$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Runtime:

- ▶ With $T(n) \in O(n^3)$ and $T(n) \in \Omega(n^3)$ we know:

$$T(n) \in \Theta(n^3)$$

- ▶ It is hard to solve the problem in a worse way ...

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

- ▶ Calculating the sum for range from i to j with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

- ▶ Calculating the sum for range from i to j with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

Better approach:

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

- ▶ Calculating the sum for range from i to j with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

Better approach:

- ▶ Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i + 1] + \cdots + X[j] + X[j + 1]$$

$$S_{i,j+1} = S_{i,j} + X[j + 1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

Divide and Conquer

Example - Maximum Subtotal - Python

Better solution:

Divide and Conquer

Example - Maximum Subtotal - Python

Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal - Python

Better solution:

```
def maxSubArray(X):  
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        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
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    return result
```

► Runtime $\in O(n^2)$

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ split the sequence in the middle

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ split the sequence in the middle
- ▶ Solve the left half of the problem

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ split the sequence in the middle
- ▶ Solve the left half of the problem
- ▶ Solve right half and combine both solutions into a total solution

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ split the sequence in the middle
- ▶ Solve the left half of the problem
- ▶ Solve right half and combine both solutions into a total solution
- ▶ OK if maximum is located in left half (*A*) or right half (*B*)

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



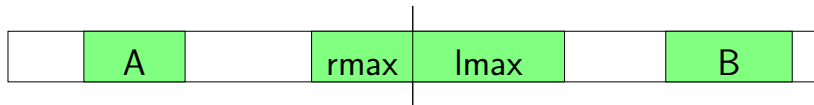
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Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



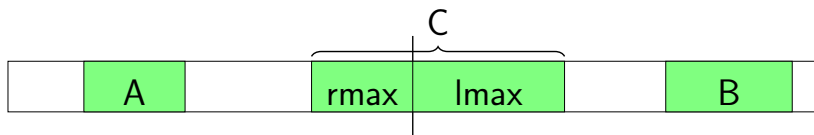
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- ▶ To solve this case we have to calculate *rmax* and *lmax*

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

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- ▶ Solve the left half of the problem
- ▶ Solve right half and combine both solutions into a total solution
- ▶ OK if maximum is located in left half (*A*) or right half (*B*)
- ▶ Problem: Maximum can overlap split
- ▶ To solve this case we have to calculate *rmax* and *lmax*
- ▶ The overall solution is the maximum of *A*, *B* and *C*

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions A and B are returned.

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions A and B are returned.
- ▶ To determine subsolution C, r_{\max} and l_{\max} for the subproblems are computed.

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions A and B are returned.
- ▶ The overall solution is the maximum of A, B and C

Divide and Conquer

Example - Maximum Subtotal - Python

Divide and conquer solution

```
def maxSubArray(X, i, j):  
    if i == j: #trivial case  
        return (X[i], i, i)  
    m = (i + j) / 2  
        #recursive Subsolutions for A,B  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
    #rmax and lmax for bordercase C  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
    #Solution results from A,B,C  
    return max([A, B, C], \  
        key=lambda item: item[0])
```

Further Literature

► General

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MIT Press, Cambridge, Mass, 2001.

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<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

- ▶ **Caching**

[Wik] [Cache](#)

`https://en.wikipedia.org/wiki/Cache`