

# Algorithms and Datastructures

## Hash Map, Universal Hashing

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# Structure

## Associative Arrays

- Introduction

- Hash Map

## Universal Hashing

- Introduction

- Probability Calculation

- Proof

- Examples

# Associative Arrays

How do we build a Map?

## Reminder:

- ▶ An associative array is like a normal array, only that the indices are not  $0, 1, 2, \dots$ , but different, e.g. telephone numbers

## Problem:

- ▶ Quickly find a element with a specific key
- ▶ Naive solution: Store pairs of key and value in a normal field
- ▶ For  $n$  keys searching requires  $\Theta(n)$  time
- ▶ With a **hash map** this just requires  $\Theta(1)$  in the best case, ... regardless how many elements are in the map!

# Associative Arrays

## The Hash Map

### Idea:

- ▶ Mapping the keys onto indices with a **hash function**
- ▶ Store the values at the calculated indices in a normal array

### Example:

- ▶ Key set:  $x = \{3904433, 312692, 5148949\}$
- ▶ Hash function:  $h(x) = x \bmod 5$ , in the range  $[0, \dots, 4]$
- ▶ We need an array **T** with **5** elements.  
A “hashtable” with 5 “buckets”
- ▶ The element with the key **x** is stored in  $T[h(x)]$

# Associative Arrays

## The Hash Map

### Storage:

- ▶ `insert(3904433, "A")`:  $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$
- ▶ `insert(312692, "B")`:  $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$
- ▶ `insert(5148949, "C")`:  $h(5148949) = 4 \Rightarrow T[4] = (5148949, "C")$

Figure: Hashtable T



# Associative Arrays

## The Hash Map

### Searching:

- ▶ `search(3904433):  $h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$`
- ▶ `search(123459):  $h(123459) = 4 \Rightarrow T[4]$`   
 $\Rightarrow$  Value with key 123459 does not exist
- ▶ Search time for this example:  $\mathcal{O}(1)$

Figure: Hashtable T

0	
1	
2	← 312692   B
3	← 3904433   A
4	← 5148949   C

# Associative Arrays

## Hash Collisions

### Further inserting:

- ▶ `insert(876543, "D")`:  $h(876543) = 3$   
 $\Rightarrow T[3] = (876543, "D") \Rightarrow$  Collision
- ▶ This happens more often than expected
  - ▶ **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

Figure: Hashtable T

0	
1	
2	← 312692   B
3	← 3904433   A
4	← 5148949   C

# Associative Arrays

## Hash Collisions

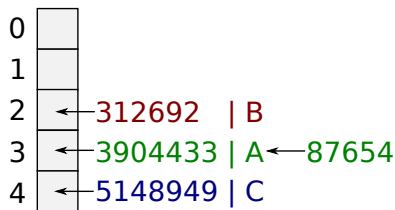
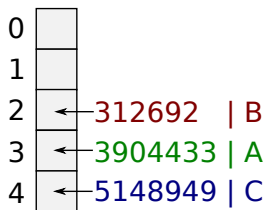
### Problem:

- ▶ Two keys are equal  $h(x) = h(y)$  but not the values  $x \neq y$

### Easiest Solution:

- ▶ Represent each bucket as list of key value pairs
- ▶ Append new values to the end of the list

Figure: Hashtable T





# Associative Arrays

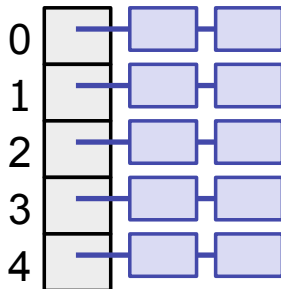
## Expected Runtime

### Best case:

- ▶ We have  $n$  keys which are equally distributed over  $m$  buckets
- ▶ We have  $\approx \frac{n}{m}$  pairs per bucket
- ▶ The runtime for searching is nearly  $\mathcal{O}(1)$  when **not**  $n \gg m$

### Best case

$(m = 5, n = 10)$



# Associative Arrays

Expected Runtime

Worst case:

- ▶ All  $n$  keys are mapped onto the same bucket
- ▶ The runtime is  $\Theta(n)$  for searching



# Universal Hashing

## Thought Experiment

### Thought Experiment:

- ▶ A **hash function** is defined for a given **key set**
- ▶ Find a **set of keys** resulting in a degenerated **hash table**
  - ▶ *The **hash function** stays fixed*
  - ▶ *For table size of 100: Try  $100 \times (99 + 1)$  different numbers*
  - ▶ *Worst case: All 100 **key sets** map to one bucket*
- ▶ **Now:** Find a solution to avoid that problem

# Universal Hashing

## Idea

### Solution: universal hashing

- ▶ Out of a set of hash functions we randomly choose one
- ▶ The **expected result** of the hash function is an equal distribution over the buckets
- ▶ This hash function stays fixed for the lifetime of table  
Optional: copy table with new hash when degenerated



Figure: Hash func. 1



Figure: Hash func. 2



Figure: Hash func. coll.

# Universal Hashing

## Definition

### Definition:

- ▶ We call  $\mathbb{U}$  the set (universum) of possible keys
- ▶ The size  $m$  of the hash table  $T$
- ▶ Set of hash functions  $\mathbb{H} = \{h_1, h_2, \dots, h_n\}$  with  $h_i : \mathbb{U} \rightarrow \{0, \dots, m-1\}$
- ▶ Idea: runtime should be  $O(1 + \frac{|\mathbb{S}|}{m})$ , where  $\frac{|\mathbb{S}|}{m}$  is the table load



Figure: Hash function  $h_1$

# Universal Hashing

## Definition

- ▶ We choose two random keys  $x, y \in \mathbb{U} \mid x \neq y$
- ▶ An average of 3 out of 15 functions produce collisions



Figure: Set of hash functions  $H$

# Universal Hashing

## Definition

**Definition:**  $\mathbb{H}$  is  $c$ -universal if  $\forall x, y \in \mathbb{U} \mid x \neq y :$

Number of hash functions that create collisions

$$\frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}, \quad c \in \mathbb{R}$$

Number of hash functions

- ▶ With other words, given a arbitrary but fixed pair  $x, y$ .  
If  $h \in \mathbb{H}$  is chosen randomly then

$$\text{Prob}(h(x) = h(y)) \leq c \cdot \frac{1}{m}$$

Note: If the hash function assigns each key  $x$  and  $y$  randomly to buckets then:

$$\text{Prob}(\text{Collision}) = \frac{1}{m} \Leftrightarrow c = 1$$

# Universal Hashing

## Definition

- ▶  $\mathcal{U}$ : Key universe
- ▶  $\mathcal{S}$ : Used Keys
- ▶  $\mathcal{S}_i \subseteq \mathcal{S}$ : Keys mapping to Bucket  $i$  (“synonyms”)
- ▶ Ideal would be  $|\mathcal{S}_i| = \frac{|\mathcal{S}|}{m}$



Figure: Hash function  $h \in \mathbb{H}$



# Universal Hashing

## Definition

- ▶ Let  $\mathbb{H}$  be a  $c$ -universal class of hash functions
- ▶ Let  $\mathbb{S}$  be a set of keys and  $h \in \mathbb{H}$  selected randomly
- ▶ Let  $\mathbb{S}_i$  be the key  $x$  for which  $h(x) = i$
- ▶ The expected average number of elements to search through per bucket is

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m}$$

- ▶ Particular: If  $(m = \Omega(|\mathbb{S}|))$  then  $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

# Universal Hashing

## Probability Calculation

- ▶ We just discuss the discrete case
- ▶ Probability space  $\Omega$  with elementary (simple) events
- ▶ Events  $e$  have probabilities ...

$$\sum_{e \in \Omega} P(e) = 1$$

- ▶ The probability for a subset of events  $E \subseteq \Omega$  is

$$P(E) = \sum_{e \in E} P(e) \mid e \in E$$

Table: Throwing a dice

$e$	$P(e)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

# Universal Hashing

## Probability Calculation

### Example:

- ▶ Rolling a dice twice ( $\Omega = \{1, \dots, 6\}^2$ )
- ▶ Each event  $e \in \Omega$  has the probability  
 $P(e) = 1/36$
- ▶  $E =$  if both results are even, then  
 $P(E) =$

Table: Throwing a dice twice

$e$	$P(e)$
(1, 1)	1/36
(1, 2)	1/36
(1, 3)	1/36
...	...
(6, 5)	1/36
(6, 6)	1/36

# Universal Hashing

## Probability Calculation

### Example:

- ▶ Random variable
  - ▶ Assigns a number to the result of an experiment
  - ▶ For example:  $X$  = Sum of results for rolling twice
  - ▶  $X = 12$  and  $X \geq 7$  are regarded as events
  - ▶ Example 1:  $P(X = 2) =$
  - ▶ Example 2:  $P(X = 4) =$

Table: Throwing a dice twice

$e$	$P(e)$	$X$
(1, 1)	$1/36$	2
(1, 2)	$1/36$	3
(1, 3)	$1/36$	4
...	...	...
(6, 5)	$1/36$	11
(6, 6)	$1/36$	12

# Universal Hashing

## Probability Calculation

**Expected value** is defined as  $\mathbb{E}(X) = \sum (k \cdot P(X = k))$

- Intuitive: The weighted average of possible values of  $X$ , where the weights are the probabilities of the values

Table: Throwing a dice once

$X$	$P(X)$
1	$1/6$
2	$1/6$
3	$1/6$
4	$1/6$
5	$1/6$
6	$1/6$

Table: Throwing a dice twice

$X$	$P(X)$
2	$1/36$
3	$2/36$
4	$3/36$
...	...
11	$2/36$
12	$1/36$

- Example rolling once:  $\mathbb{E}(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = 3.5$
- Example rolling twice:  $\mathbb{E}(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \dots + 12 \cdot \frac{1}{36} = 7$

# Universal Hashing

## Probability Calculation

**Sum of expected values:** For arbitrary discrete random variables  $X_1, \dots, X_n$  we can write:

$$\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n)$$

**Example:** Throwing two dice

- ▶  $X_1$ : Expected result of dice 1:  $\mathbb{E}(X_1) = 3.5$
- ▶  $X_2$ : Expected result of dice 2:  $\mathbb{E}(X_2) = 3.5$
- ▶  $X = X_1 + X_2$ : Expected total number:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

# Universal Hashing

## Probability Calculation

### Corollary:

The probability of the event  $E$  is  $p = P(E)$ . Let  $X$  be the occurrences of the event  $E$  and  $n$  be the number of executions of the experiment. Then  $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurrences of } 6) = \frac{1}{6} \cdot 60 = 10$$

# Universal Hashing

## Probability Calculation

### Proof Corollary:

Indicator variable:  $X_i$

$$X_i = \begin{cases} 1, & \text{if event occurs} \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow X = \sum_{i=1}^n X_i$$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbb{E}(X_i) \stackrel{\text{def. } \mathbb{E}\text{-value}}{=} \sum_{i=1}^n p = n \cdot p$$

□

Def.  $\mathbb{E}$ -value:  $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$



# Universal Hashing

## Proof

### Given:

- ▶ We pick two random keys  $x, y \in \mathbb{S} \mid x \neq y$  and a random hash function  $h \in \mathbb{H}$
- ▶ We know the probability of a collision:

$$P(h(x) = h(y)) \leq c \cdot \frac{1}{m}$$

### To proof:

$$\mathbb{E}[|S_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \quad \forall i$$

# Universal Hashing

Proof

**We know:**

$$\mathbb{S}_i = \{x \in \mathbb{S} : h(x) = i\}$$

If  $\mathbb{S}_i = \emptyset \Rightarrow |\mathbb{S}_i| = 0$  otherwise, let  $x \in \mathbb{S}_i$  be any key

We define an indicator variable:

$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \quad y \in \mathbb{S} \setminus \{x\}$$

$$\Rightarrow |\mathbb{S}_i| = 1 + \sum_{y \in \mathbb{S} \setminus x} I_y$$

$$\Rightarrow \mathbb{E}(|\mathbb{S}_i|) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

# Universal Hashing

Proof

**Auxiliary calculation:**

$$\begin{aligned}\mathbb{E}[I_y] &= P(I_y = 1) \\ &= P(h(y) = i) \\ &= P(h(y) = h(x)) \\ &\leq c \cdot \frac{1}{m}\end{aligned}$$

$$\begin{aligned}\textbf{Hence: } \mathbb{E}[|S_i|] &= 1 + \sum_{y \in S \setminus x} \mathbb{E}[I_y] \leq 1 + \sum_{y \in S \setminus x} c \cdot \frac{1}{m} \\ &= 1 + (|S| - 1) \cdot c \cdot \frac{1}{m} \\ &\leq 1 + |S| \cdot c \cdot \frac{1}{m} \\ &= 1 + c \cdot \frac{|S|}{m} \quad \square\end{aligned}$$

# Universal Hashing

## Examples

### Negative example:

- ▶ The set of all  $h$  for which  $h_a(x) = (a \cdot x) \bmod m$ , for a  $a \in \mathbb{U}$
- ▶ Is not  $c$ -universal.
- ▶ If universal:

$$\forall x, y \quad x \neq y: \frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}$$

- ▶ Which  $x, y$  lead to a relative collision count bigger than  $\frac{c}{m}$ ?

# Universal Hashing

## Examples

### Positive example:

- ▶ Let  $p$  be a big prime number,  $p > m$  and  $p \geq |\mathbb{U}|$
- ▶ Let  $\mathbb{H}$  be the set of all  $h$  for which:

$$h_{a,b}(x) = ((a \cdot x + b) \bmod p) \bmod m,$$

where  $1 \leq a < p$ ,  $0 \leq b < p$

- ▶ This is  $\approx 1$ -universal, see Exercise 4.11 in Mehlhorn/Sanders
- ▶ E.g.:  $U = \{0, \dots, 99\}$ ,  $p = 101$ ,  $a = 47$ ,  $b = 5$
- ▶ Then  $h(x) = ((47 \cdot x + 5) \bmod 101) \bmod m$
- ▶ Easy to implement but hard to proof
- ▶ Exercise: show empirically that it is 2-universal

# Universal Hashing

## Examples

### Positive example:

- ▶ The set of hash functions is  $c$ -universal:

$$h_a(x) = a \bullet x \mod m, \quad a \in \mathbb{U}$$

- ▶ We define:

$$a = \sum_{0, \dots, k-1} a_i \cdot m^i, \quad k = \text{ceil}(\log_m |\mathbb{U}|)$$

$$x = \sum_{0, \dots, k-1} x_i \cdot m^i$$

- ▶ **Intuitive:** Scalar product with base  $m$

$$a \bullet x = \sum_{0, \dots, k-1} a_i \cdot x_i$$

# Universal Hashing

## Examples

Example ( $\mathbb{U} = \{0, \dots, 999\}$ ,  $m = 10$ ,  $a = 348$ )

With  $a = 348$ :  $a_2 = 3$ ,  $a_1 = 4$ ,  $a_0 = 8$

$$\begin{aligned}h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\&= (3x_2 + 4x_1 + 8x_0) \mod 10\end{aligned}$$

With  $x = 127$ :  $x_2 = 1$ ,  $x_1 = 2$ ,  $x_0 = 7$

$$\begin{aligned}h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\&= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\&= 7\end{aligned}$$

# Further Literature

## ► General for this Lecture

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

*Introduction to Algorithms.*

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.



# Further Literature

## ► Hash Map - Theory

[Wik] [Hash table](#)

[https://en.wikipedia.org/wiki/Hash\\_table](https://en.wikipedia.org/wiki/Hash_table)

## ► Hash Map - Implementations / API

[Cpp] [C++ - hash\\_map](#)

[http://www.sgi.com/tech/stl/hash\\_map.html](http://www.sgi.com/tech/stl/hash_map.html)

[Jav] [Java - HashMap](#)

<https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html>

[Pyt] [Python - Dictionaries \(Hash table\)](#)

[https://en.wikipedia.org/wiki/Hash\\_table](https://en.wikipedia.org/wiki/Hash_table)