

# Algorithms and Datastructures

## Shortest Path, Dijkstra Algorithm

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# Structure

Graphs

Dijkstra Algorithm

# Graphs

## Paths

For a graph  $G = (V, E)$ :

- ▶ A path of  $G$  is a sequence of edges  $u_1, u_2, \dots, u_i \in V$  with
  - ▶ Undirected graph:  $\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{i-1}, u_i\} \in E$
  - ▶ Directed graph:  $(u_1, u_2), (u_2, u_3), \dots, (u_{i-1}, u_i) \in E$
- ▶ The length of a path is
  - ▶ Without weights: number of edges taken
  - ▶ With weights: sum of weights of edges taken

# Graphs

## Paths

For a graph  $G = (V, E)$ :

- ▶ The **shortest path** between two vertices  $u, v$  is the path  $P = (u, \dots, v)$  with the shortest length  $d(u, v)$  or lowest costs
- ▶ The **diameter** of a graph is the **longest shortest path**

# Dijkstra Algorithm

## Shortest Path without Computer

- ▶ Wanted: Shortest path from M to all other points
- ▶ Place pearls on crossings and clamp strings between them



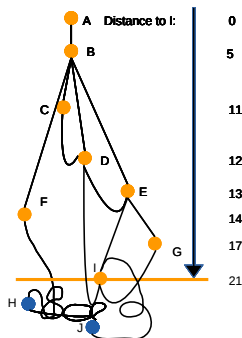
# Dijkstra Algorithm

## Shortest Path without Computer



Figure: Based on OpenStreetMaps;  
CC BY-SA 2.0

- Take the net and pull it slowly upwards until fully lifted



- Each node (pearl) now has a specific height
- The distance to M is exactly the **shortest path**

# Dijkstra Algorithm

## Shortest Path

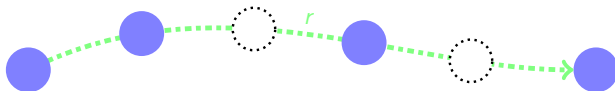


Figure: Shortest path from  $s$  to  $t$

- ▶ Let  $r$  be the shortest path from  $s$  to  $t$
- ▶ For each node  $u$  on path  $r$  the path from  $u$  to  $t$  is the shortest path

### Proof:

- ▶ If there was a shorter path from  $s$  to  $u$  then we could choose this path to get faster to  $t$
- ▶ Then  $r$  would not be the shortest path

# Dijkstra Algorithm

## Shortest Path

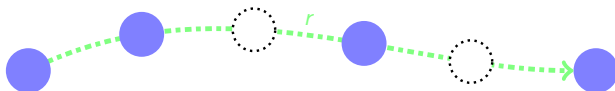


Figure: Shortest path from  $s$  to  $t$

- ▶ This is also correct for all sub paths on  $r$
- ▶ If the shortest path from  $s$  to  $t$  passes  $u_1$  and  $u_2$  then the sub path  $(u_1, u_2)$  is the shortest path from  $u_1$  to  $u_2$



# Dijkstra Algorithm

## Shortest Path

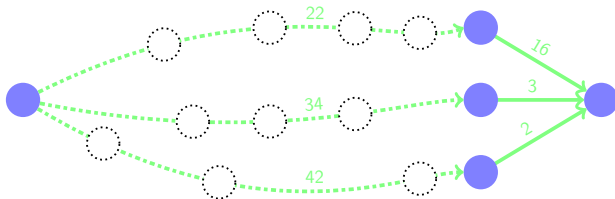


Figure: Shortest paths from  $s$  to  $t$

- If we know the shortest path from  $s$  to the preceding nodes of  $t$  ( $v_1, v_2, v_3$ ) we can determine the shortest path to  $t$

# Dijkstra Algorithm

## Shortest Path

### Idea:

- ▶ Attach the cost of the shortest path to each node
- ▶ Let the information travel over the edges (message passing)
- ▶ In which order should we process the nodes?

# Dijkstra Algorithm

## Inventor:

- ▶ Edsger Dijkstra (1930 - 2002)
- ▶ Computer scientist from Netherlands
- ▶ Won Turing-Award as one of few Europeans for his studies of structured programming
- ▶ Invented the Dijkstra-Algorithm in 1959

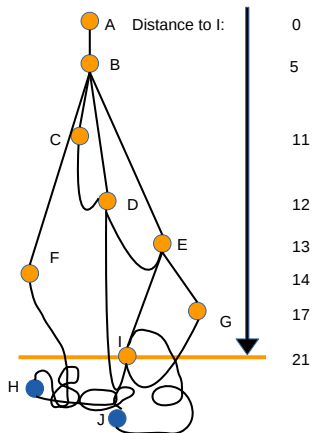


**Figure:** Portrait © Hamilton Richards - manuscripts of Edsger W. Dijkstra, University Texas at Austin

# Dijkstra Algorithm

## Example:

- ▶ Lift pearl **A** a little bit
  - ▶ Connection to pearl **B** is hanging in the air
  - ▶ Lift further until pearl **B** starts to lift at 5 m
  - ▶ The shortest path to **B** is now known
  - ▶ Lift further: The wires from **C**, **D**, **E** and **F** are now in the air
  - ▶ One of the pearls **C**, **D**, **E** or **F** is the next one
- Which one?



# Dijkstra Algorithm

## Example:

- ▶ At 11 m pearl **C** gets lifted
- ▶ The wire to **D** is now in the air
- ▶ One of the pearls **D**, **E** and **F** is the next one  
Which one?
- ▶ At 12 m pearl **D** gets lifted
- ...
- ▶ How to translate this into an computer algorithm?



# Dijkstra Algorithm

**High level description:** Three types of nodes

- ▶ **Settled:** For node  $u$  we know  $\text{dist}(s, u)$   
(Pearl example: This pearl is hanging in the air)
- ▶ **Active:** For node  $u$  we know a tentative distance  $\text{td}(u) \geq \text{dist}(s, u)$  (Can be optimal but doesn't have to)  
(Pearl example: This pearl is laying on the table but one connected wire is already in the air)
- ▶ **Unreached:** We have not reached the node yet  
(Pearl example: This preal is hanging in the air)



# Dijkstra Algorithm

## High level description:

- ▶ Each iteration take the **active** node  $u$  with the **smallest**  $td(u)$   
(The pearl getting lifted next)
- ▶ We update the state of the node  $u$  to **settled**  
(The pearl gets lifted)
- ▶ We check for each **neighbor**  $v$  of node  $u$  if we can reach  $v$  faster than currently possible  
(Check all outgoing wires from this pearl: Activate all connected pearls, update tentative distance if smaller)
- ▶ Iterate until no active nodes exist anymore

# Dijkstra Algorithm

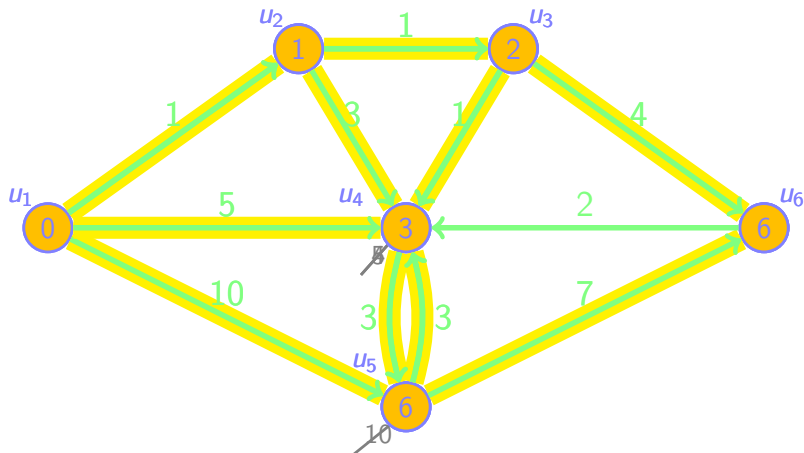


Figure: Start at  $u_1$  Iteration 1 Iteration 2 Iteration 3 Iteration 4 Iteration 5 Iteration 6



# Dijkstra Algorithm

## Proof

### Proof:

- ▶ **Assumption 1:** All edges have a positive length
- ▶ **Assumption 2:** Each node has a unique distance  $\text{dist}(s, u)$  to the start  $s$   
(This was not the case on the previous slides)

This results in an easy and intuitive proof.

It is possible to show this without assumption 2. See references if interested

- ▶ With assumption 2 there exists a sorting  $u_1, u_2, \dots$  with that:

$$\text{dist}(s, u_1) < \text{dist}(s, u_2) < \text{dist}(s, u_3) < \dots$$

# Dijkstra Algorithm

## Proof

### Proof:

- ▶ With **assumption 2** there exists a sorting  $u_1, u_2, \dots$  with that:

$$\text{dist}(s, u_1) < \text{dist}(s, u_2) < \text{dist}(s, u_3) < \dots$$

- ▶ We want to show that the *Dijkstra* algorithm finds the shortest path for each node  $u_i$  so that  $\text{td}(u_i) = \text{dist}(s, u_i)$  holds
- ▶ Additionally we show that each node gets solved in order of the distance: Node  $u_i$  gets solved in iteration  $i$

$$u_1, u_2, u_3, \dots$$

# Dijkstra Algorithm

## Proof

**To show:** Node  $u_i$  gets solved in round  $i$

1. Node  $u_i$  contains the correct distance ( $\text{td}(u_i) = \text{dist}(s, u_i)$ ) and is active
2. Node  $u_i$  has the smallest value for  $\text{td}(u_i)$  and gets selected by the algorithm

**Induction start:**

1.
  - ▶ Only the start node  $s = u_1$  is active and  $\text{td}(s) = 0$
  - ▶ Node  $u_1$  gets solved and  $\text{td}(u_1) = \text{dist}(s, u_1) = 0$
2. Only the start node  $u_1$  is active

# Dijkstra Algorithm

## Proof

**Induction step:**  $i = i + 1$

1. **To show:** Node  $u_{i+1}$  contains the correct distance ( $\text{td}(u_{i+1}) = \text{dist}(s, u_{i+1})$ ) and is active
  - ▶ On the shortest path from  $s$  to  $u_{i+1}$  is a preceding node that:

$$\text{dist}(s, u_{i+1}) = \text{dist}(s, v) + c(v, u_{i+1})$$

( $c(v, u_{i+1})$  are the costs of the edge)



- ▶ Hence  $\text{dist}(s, v) < \text{dist}(s, u_{i+1})$  because  $c > 0$  ( $c$ =cost of edge)
- ▶ Because  $u_{i+1}$  is currently settled, the node  $v$  is one of the preceding nodes  $u_1, \dots, u_i$ , hence  $v = u_j$  with  $0 \leq j \leq i$

# Dijkstra Algorithm

Proof - Example of Iteration 6



- ▶ Preceding node of  $u_6$  is  $v = u_3$
- ▶ In round 3  $\text{td}(u_6) = 2 + 4 = 6$  was already solved

# Dijkstra Algorithm

## Proof



1. **To show:** Node  $u_i$  contains the correct distance  $\text{td}(u_i) = \text{dist}(s, u_i)$  and is active
  - ▶ With **induction assumption:**  $v$  already contains the correct distance which was evaluated in round  $j$  (edge from  $v$  to  $u_{i+1}$ ) and is stored in  $\text{td}(u_{i+1})$
  - ▶  $u_{i+1}$  is active because the preceding node was solved

# Dijkstra Algorithm

## Proof



2. **To show:** Node  $u_{i+1}$  has the smallest value for  $\text{td}(u_{i+1})$  and gets selected by the algorithm
- ▶ All nodes with smaller  $\text{dist}$  are already solved
  - ▶ All other nodes  $u_k$  with  $k > i + 1$  have a greater  $\text{dist}(s, u_k)$  and with that the  $\text{td}(u_k)$  is greater or equal
- $\Rightarrow u_{i+1}$  is the node with the smallest  $\text{td}$  and gets selected by the algorithm

# Dijkstra Algorithm

## Implementation

### Implementation:

- ▶ We have to manage a set of **active nodes**
- ▶ We start with only the **start node** in our set
- ▶ At the start of each iteration we need the node  $u$  with the smallest  $td(u)$

How to implement this?



# Dijkstra Algorithm

## Implementation

### Implementation:

- ▶ Using a **priority queue** with  $td(u)$  as keys
- ▶ The following problem occurs:
  - ▶ The **tentative distance** of an active node might change multiple times before it is settled
  - ▶ We have to change the key in our **priority queue** without removing the entry

### Limitations:

- ▶ Often only `insert`, `getMin` and `deleteMin` are implemented
- ⇒ We only have access to the first element and not any desired one

# Dijkstra Algorithm

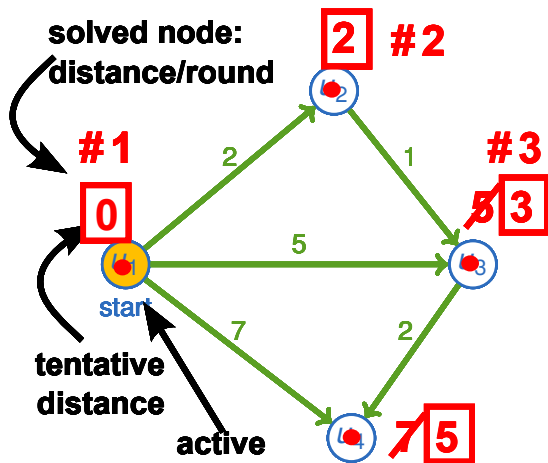
## Implementation

### Alternative:

- ▶ If a node reoccurs with a smaller **dist** we insert the element one more time into the **priority queue**  
(We do nothing if the distance is greater or equal)
- ▶ We do not remove the old entry
- ▶ The node always gets solved with the smallest distance because of the **smaller key**
- ▶ If a settled node reoccurs with a higher **dist** we remove it and do simply **nothing**

# Dijkstra Algorithm

## Implementation - Example



### priority queue

- $(u_2, 2) \rightarrow$  **solved #2**
- $(u_3, 5) \rightarrow$  **ignored #5**
- $(u_4, 7) \rightarrow$  **ignored #6**
- $(u_3, 3) \rightarrow$  **solved #3**
- $(u_4, 5) \rightarrow$  **solved #4**

# Dijkstra Algorithm

## Runtime analysis

Graph with  $n$  nodes and  $m$  edges: ( $m \geq n$ )

- ▶ Each node gets solved exactly **one time**
- ▶ When solving a node its outgoing edges are taken into account
- ▶ Each edge triggers at maximum one insert operation
- ▶ The number of operations on the **priority queue** is at maximum  $O(m)$
- ▶ This results in a runtime of  $O(m \cdot \log m)$   
( $\log m$  because of at max.  $m$  elements in the priority queue)

# Dijkstra Algorithm

## Runtime analysis

Runtime of  $O(m \cdot \log m)$ :

- ▶ Because of  $m \leq n^2$  we have a maximum runtime of  $O(m \cdot \log n)$ , because  $\log n^2 = 2 \log n$
  - ▶ With a complex **priority queue** the runtime can be reduced to  $O(m + n \log n)$ 
    - ▶ For example with a **Fibonacci heap**
    - ▶ This results in a better runtime for complex graphs  $m \sim n^2$
    - ▶ Complex heaps create a management overhead
- ⇒ In practice  $m \in O(n)$  with a **binary heap** being faster  
(See lecture 6)

# Dijkstra Algorithm

## Additional comments

### Termination criteria:

- ▶ Terminate as soon as the target node  $t$  is settled  
... never before because tentative distance might change:

$$td(t) \geq \text{dist}(s, t)$$

- ▶ Before the node  $t$  is solved all nodes  $u$  with  $\text{dist}(s, u) \leq \text{dist}(s, t)$  are settled

# Dijkstra Algorithm

## Additional comments

### Termination criteria:

- ▶ Not only the **single source single target** shortest path problem is solved by the Dijkstra algorithm but also the **single source all targets** problem
- ▶ This sounds wasteful but there is not a (much) better method for general graphs

**Intuitive:** We only know that there is no shorter path if all nodes in the distance of  $\text{dist}(s, t)$  are evaluated

# Dijkstra Algorithm

## Additional comments

### Calculate the shortest path:

- ▶ With the current implementation of the Dijkstra algorithm we only get the **length** of the path  
How to get the path itself too?
- ▶ If we save the preceding node of the current shortest path on **settling** of each node we can reconstruct the **path**



# Dijkstra Algorithm



Figure: Start at  $u_1$  Iteration 1 Iteration 2 Iteration 3 Iteration 4 Iteration 5 Iteration 6

# Dijkstra Algorithm

## Additional comments

### Enhancement:

- ▶ In our proof we used the assumption that all costs are **not negative** (even  $> 0$ )
- ▶ With **negative costs** there might be **negative cycles**:

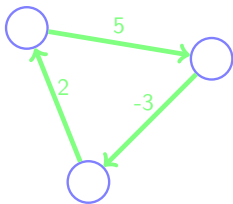


Figure: Here no problem ...

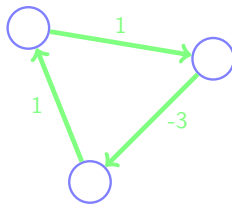
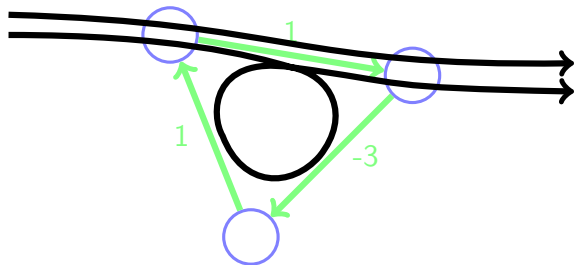


Figure: ... but here

# Dijkstra Algorithm

## Additional comments

### Negative cycles:



- ▶ No cycle:  
cost of 1
- ▶ 1 cycle:  
cost of 0
- ▶ 2 cycles:  
cost of -1
- ▶ 3 cycles:  
cost of -2
- ▶ ...

# Dijkstra Algorithm

## Additional comments

### Enhancement:

- ▶ We need a different algorithm to deal with negative edges
  - ▶ For example the **Bellman-Ford** algorithm
  - ▶ If the graph is **acyclic** we can simply use a topological sorting (with DFS) and settling the nodes in order of this sorting
- ▶ Another (not only) in artificial intelligence used variant of the Dijkstra algorithm is the **A\* algorithm**

Additional information given:

$h(u)$  = estimated value for  $\text{dist}(u, t)$

# Dijkstra Algorithm

Example - Negative costs (e-car consumption)

## Dijkstra algorithm:

Message passing only from solved nodes

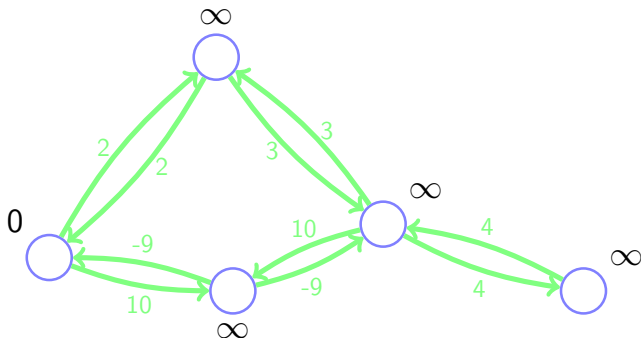


# Dijkstra Algorithm

Example - Negative costs (e-car consumption)

## Bellman-Ford algorithm:

Message passing from all nodes until the path lengths are stable



# Dijkstra Algorithm

## Application

### **Application example:**

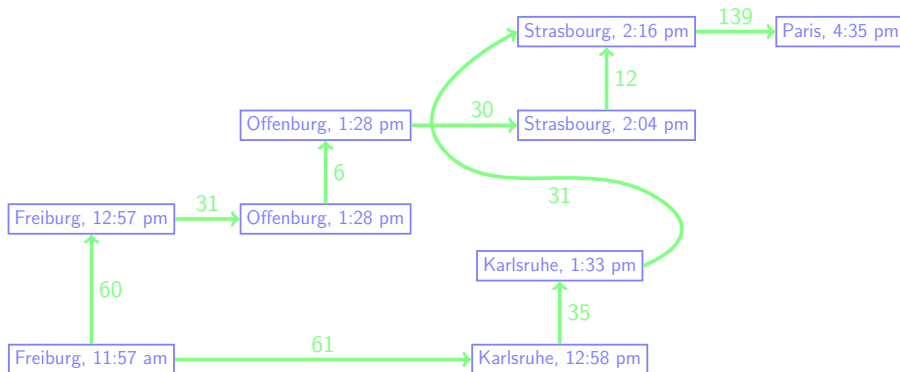
- ▶ Route planner for car trips (exercise sheet)
- ▶ Route planner for bus / train connections

What could the graph look like?

# Dijkstra Algorithm

## Application

### Space-time graph:





# Dijkstra Algorithm

Application in image processing

-6em-6em



Figure: Neurons under fluorescence microscope

- ▶ **Task:** Measure length of axons (connections of neurons)
- ▶ Demo with ImageJ plugin NeuronJ  
<http://www.imagescience.org/meijering/software/neuronj/>

# Dijkstra Algorithm

Application: Trace axons



- ▶ Image as graph: Each pixel is a node
- ▶ Implicit edges: Each pixel has an edge to its 8 neighbours (no need to save the edges)
- ▶ Costs for nodes (not edges): bright pixels are cheap, dark pixels are costly

# Further Literature

## ► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

*Introduction to Algorithms.*

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

# Further Literature

- ▶ **Dijkstra's algorithm**

[Wik] [Dijkstra's algorithm](#)

https:

[//en.wikipedia.org/wiki/Dijkstra's\\_algorithm](https://en.wikipedia.org/wiki/Dijkstra's_algorithm)

- ▶ **Shortest path problem**

[Wik] [Shortest path problem](#)

[https://en.wikipedia.org/wiki/Shortest\\_path\\_problem](https://en.wikipedia.org/wiki/Shortest_path_problem)