

Algorithms and Datastructures

Cache Efficiency, Divide and Conquer

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Structure

Cache Efficiency

- Introduction

- Cache Organization

Divide and Conquer

- Introduction

Cache Efficiency

Introduction

Background:

Cache Efficiency

Introduction

Background:

- ▶ Up to now we always counted the **number of operations**
- ▶ Assuming this is a good measure for the runtime of a algorithm/tool

Cache Efficiency

Introduction

Background:

- ▶ Up to now we always counted the **number of operations**
- ▶ Assuming this is a good measure for the runtime of a algorithm/tool
- ▶ Today we will see examples where this is not suitable

Cache Efficiency

Introduction

Example:

- ▶ We sum up all elements of a field a of size n in ...
 - ▶ natural order:

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

- ▶ random order:

$$\text{sum}(a) = a[21] + a[5] + \cdots + a[8]$$

Cache Efficiency

Linear Order - Python

Python:

```
def init(size):  
    """Creates the dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

Cache Efficiency

Linear Order - Python

Python:

```
def run(param):  
    """Processes the dataset."""  
  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```


Cache Efficiency

Linear Order



Figure: Summing elements in linear order

Cache Efficiency

Random Order - Python

```
def init(size):  
    """Creates a randomly ordered dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

Cache Efficiency

Random Order



Figure: Summing elements in random order

Cache Efficiency

Algorithm Comparision

Conclusion:

Cache Efficiency

Algorithm Comparision

Conclusion:

- ▶ The number of operations are identical for both algorithms

Cache Efficiency

Algorithm Comparision

Conclusion:

- ▶ The number of operations are identical for both algorithms
- ▶ Accessing elements in random order takes a lot longer (Factor 10) Why?
- ▶ The costs in terms of memory access are very different

Structure

Cache Efficiency

Introduction

Cache Organization

Divide and Conquer

Introduction

Cache Efficiency

CPU Cache



Cache Efficiency

CPU Cache



Principle / organization:

Cache Efficiency

CPU Cache



Principle / organization:

- ▶ Accessing one byte of the main memory takes ≈ 100 ns

Cache Efficiency

CPU Cache



Principle / organization:

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- ▶ Accessing one byte of (L1-)cache takes ≈ 1 ns

Cache Efficiency

CPU Cache



Principle / organization:

- ▶ Accessing one byte of the main memory takes ≈ 100 ns
- ▶ Accessing one byte of (L1-)cache takes ≈ 1 ns
- ▶ Accessing one or more byte/s of main memory loads a whole block ≈ 100 B into the cache

Cache Efficiency

CPU Cache

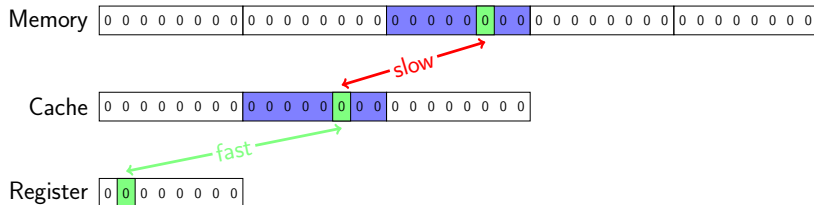


Principle / organization:

- ▶ Accessing one byte of the main memory takes ≈ 100 ns
- ▶ Accessing one byte of (L1-)cache takes ≈ 1 ns
- ▶ Accessing one or more byte/s of main memory loads a whole block ≈ 100 B into the cache
- ▶ As long as this block is in the cache, it is not necessary to access the memory for bytes of this block

Cache Efficiency

CPU Cache



Cache Efficiency

CPU Cache



Cache organization:

Cache Efficiency

CPU Cache



Cache organization:

- The (L1-)cache can hold multiple memory blocks

Cache Efficiency

CPU Cache



Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks
 - ▶ Cache lines ≈ 100 kB

Cache Efficiency

CPU Cache



Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks
 - ▶ Cache lines ≈ 100 kB
- ▶ If the capacity is reached unused blocks are discarded

Cache Efficiency

CPU Cache



Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks
 - ▶ Cache lines ≈ 100 kB
- ▶ If the capacity is reached unused blocks are discarded
 - ▶ Least recently used (LRU)

Cache Efficiency

CPU Cache



Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks
 - ▶ Cache lines \approx 100 kB
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Cache Efficiency

CPU Cache



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 - ▶ First in first out (FIFO)

Cache Efficiency

CPU Cache



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- ▶ If the capacity is reached unused blocks are discarded
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 - ▶ Least frequently used (LFU)
 - ▶ First in first out (FIFO)
- ▶ Details of discarding are not the topic for today

Cache Efficiency

Block Operations



Terminology:

Cache Efficiency

Block Operations



Terminology:

- The system consists of slow and fast memory

Cache Efficiency

Block Operations



Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size B**

Cache Efficiency

Block Operations



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- ▶ The **fast cache** has size M and can store M/B blocks

Cache Efficiency

Block Operations



Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size B**
- ▶ The **fast cache** has size M and can store M/B blocks
- ▶ If data is not in fast memory, the corresponding block is loaded into the **cache**

Cache Efficiency

Block Operations



Terminology:

Cache Efficiency

Block Operations



Terminology:

- ▶ The program defines which blocks are held in the **cache**

Cache Efficiency

Block Operations



Terminology:

- ▶ The program defines which blocks are held in the **cache**
- ▶ We use the number of **block operations** as runtime estimation

Cache Efficiency

Block Operations



Terminology:

- ▶ The program defines which blocks are held in the **cache**
- ▶ We use the number of **block operations** as runtime estimation
- ▶ We ignore runtime costs of cache accesses / management

Cache Efficiency



Figure: Comparison good / bad locality

Accessing the cache B times:

- ▶ **Best case:** 1 block operation \rightarrow good locality
- ▶ **Worst case:** B block operations \rightarrow bad locality

Cache Efficiency

Block Operations

Additional factors:

Cache Efficiency

Block Operations

Additional factors:

- ▶ The following settings change only a small constant factor in number of block operations

Cache Efficiency

Block Operations

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Cache Efficiency

Block Operations

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Cache Efficiency

Block Operations

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Note:

Cache Efficiency

Block Operations

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Note:

- ▶ If the input size is smaller than M we load the complete data chunk directly into the cache

Cache Efficiency

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 - ▶ Partitioning of the slow memory into blocks
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Note:

- ▶ If the input size is smaller than M we load the complete data chunk directly into the cache
- ▶ Cache handling is only interesting when the input size is greater than M

Cache Efficiency

Block Operations

Typical values: (Intel® i7-4770 Haswell, WD® Blue 2 TB)

Cache Efficiency

Block Operations

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Cache Efficiency

Block Operations

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- ▶ CPU L1 Cache: $B = 64 \text{ B}$, $M = 4 \times (32 \text{ kB} + 32 \text{ kB})$
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Cache Efficiency

Block Operations

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- ▶ CPU L3 Cache: $B = 64 \text{ B}$, $M = 8 \text{ MB}$
- ▶ Disk Cache: $B = 64 \text{ kB}$, $M = 64 \text{ MB}$

Cache Efficiency

Block Operations

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- ▶ CPU L1 Cache: $B = 64 \text{ B}$, $M = 4 \times (32 \text{ kB} + 32 \text{ kB})$
- ▶ CPU L2 Cache: $B = 64 \text{ B}$, $M = 4 \times 256 \text{ kB}$
- ▶ CPU L3 Cache: $B = 64 \text{ B}$, $M = 8 \text{ MB}$
- ▶ Disk Cache: $B = 64 \text{ kB}$, $M = 64 \text{ MB}$
 - ▶ Many operating systems use free system memory as disk cache

Cache Efficiency

Block Operations

Terminology:

Cache Efficiency

Block Operations

Terminology:

- ▶ Block loads on CPU-cache are called **cache misses**

Cache Efficiency

Block Operations

Terminology:

- ▶ Block loads on CPU-cache are called **cache misses**
- ▶ Block operations on disk-cache are called **IOs**
(input / output operations)

Cache Efficiency

Block Operations

Terminology:

- ▶ Block loads on CPU-cache are called **cache misses**
- ▶ Block operations on disk-cache are called **IOs**
(input / output operations)
- ▶ These also fall under the term **cache efficiency** or **IO efficiency**

Cache Efficiency

Block Operations - Linear Order

Example 1 - Linear order:

Cache Efficiency

Block Operations - Linear Order

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- ▶ We sum up all elements in **natural order**

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Cache Efficiency

Block Operations - Linear Order

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- ▶ The number of block operations is $\text{ceil}\left(\frac{n}{B}\right)$

Cache Efficiency

Block Operations - Linear Order

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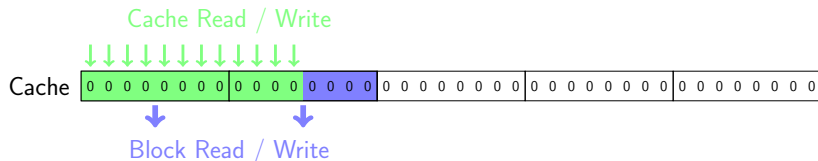


Figure: Good locality of sum operation

Cache Efficiency

Block Operations - Random Order

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Cache Efficiency

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Cache Efficiency

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Cache Efficiency

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Cache Efficiency

Block Operations - Random Order

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Figure: Bad locality of sum operation

Cache Efficiency

Block Operations

Generally the factor is substantially $< B$

Cache Efficiency

Block Operations

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- ▶ Even with a **random order** we access 4 neighboring bytes at once per int (int32_t)

Cache Efficiency

Block Operations

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- ▶ Even with a **random order** we access 4 neighboring bytes at once per `int` (`int32_t`)
- ▶ The next element might already be loaded in the cache

Cache Efficiency

Block Operations

Generally the factor is substantially $< B$

- ▶ Even with a **random order** we access 4 neighboring bytes at once per `int` (`int32_t`)
- ▶ The next element might already be loaded in the cache
- ▶ If **not** $n \gg M$ this might occur with a high probability

Cache Efficiency

Block Operations - QuickSort

QuickSort:

Cache Efficiency

Block Operations - QuickSort

QuickSort:

- ▶ **Strategy:** Divide and conquer

Cache Efficiency

Block Operations - QuickSort

QuickSort:

- ▶ **Strategy:** Divide and conquer
- ▶ Divide the data into two parts where the “left” part contains all values \leq those in the right part

Cache Efficiency

Block Operations - QuickSort

QuickSort:

- ▶ **Strategy:** Divide and conquer
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Cache Efficiency

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Cache Efficiency

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Cache Efficiency

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Figure: QuickSort with pivot-element

Idea of Quicksort

- ▶ **At start:** Pivot in first position, first re-arrange list such that left part contains small, right part larger elements
- ▶ Do required changes *in place*



- ▶ **End point:** k is left to left-most element greater than pivot
swap position 0 (pivot) with k (smaller than pivot)

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Cache Efficiency

Block Operations - QuickSort - Python

Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[0]  
  
    ...
```

Cache Efficiency

Block Operations - QuickSort - Python

```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```

Cache Efficiency

Block Operations - QuickSort

Number of operations for Quicksort:

Cache Efficiency

Block Operations - QuickSort

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- ▶ Let $T(n)$ be the runtime for the input size n

Cache Efficiency

Block Operations - QuickSort

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Assumptions:

Cache Efficiency

Block Operations - QuickSort

Number of operations for Quicksort:

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- ▶ Fields are always separated perfectly in the middle

Cache Efficiency

Block Operations - QuickSort

Number of operations for Quicksort:

- ▶ Let $T(n)$ be the runtime for the input size n

Assumptions:

- ▶ Fields are always separated perfectly in the middle
- ▶ n is a power of two and recursion depth is $k = \log_2 n$

Cache Efficiency

Block Operations - QuickSort

$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left(A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$

Cache Efficiency

Block Operations - QuickSort



Figure: Locality of quicksort

Cache Efficiency

Block Operations - QuickSort

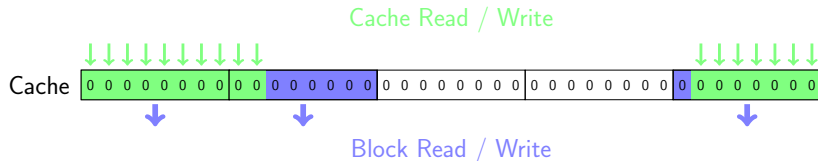


Figure: Locality of quicksort

- Let $IO(n)$ be the number of **block operations** for input size n

Cache Efficiency

Block Operations - QuickSort



Figure: Locality of quicksort

- ▶ Let $IO(n)$ be the number of **block operations** for input size n
- ▶ Assumptions as before but recursion depth is $k = \log_2 \frac{n}{B}$
Why?

Cache Efficiency

Block Operations - QuickSort

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in \mathcal{O}\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

Structure

Cache Efficiency

Introduction

Cache Organization

Divide and Conquer

Introduction

Divide and Conquer

Introduction

Concept:

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly

Divide and Conquer

Introduction

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- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
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- ▶ **Connect** all solutions of the subproblems to a solution of the full problem

Divide and Conquer

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Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
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- ▶ **Recursive** application of the algorithm to ever smaller subproblems

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all solutions of the subproblems to a solution of the full problem
- ▶ **Recursive** application of the algorithm to ever smaller subproblems
- ▶ **Direct** solving of sufficiently small subproblems

Divide and Conquer

Introduction - Python

Divide and Conquer

Introduction - Python

- ▶ Function `solve` for solving a `problem` of size `n`

Divide and Conquer

Introduction - Python

- Function `solve` for solving a `problem` of size `n`

```
def solve(problem):  
    if n < threshold:  
        return solution # solve directly  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
        return S1 + S2 + ... + Sk
```

Divide and Conquer

Features

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- ▶ Can help with conceptual hard problems

Divide and Conquer

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Divide and Conquer

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Divide and Conquer:

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- ▶ **Solution** of the trivial problems has to be known
- ▶ **Dividing** in subproblems has to be possible
- ▶ **Combination** of solutions has to be possible

Divide and Conquer

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Features:

- ▶ Realization of efficient solutions

Divide and Conquer

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 - ▶ If trivial solution is $\in O(1)$

Divide and Conquer

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- ▶ Realization of efficient solutions
 - ▶ If trivial solution is $\in O(1)$
 - ▶ And separation / combination of subproblems is $\in O(n)$

Divide and Conquer

Features

Features:

- ▶ Realization of efficient solutions
 - ▶ If trivial solution is $\in O(1)$
 - ▶ And separation / combination of subproblems is $\in O(n)$
 - ▶ And the number of subproblems is limited

Divide and Conquer

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Features:

- ▶ Realization of efficient solutions
 - ▶ If trivial solution is $\in O(1)$
 - ▶ And separation / combination of subproblems is $\in O(n)$
 - ▶ And the number of subproblems is limited
 - ▶ The runtime is $\in O(n \cdot \log n)$

Divide and Conquer

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Features:

- ▶ Realization of efficient solutions
 - ▶ If trivial solution is $\in O(1)$
 - ▶ And separation / combination of subproblems is $\in O(n)$
 - ▶ And the number of subproblems is limited
 - ▶ The runtime is $\in O(n \cdot \log n)$
- ▶ Suitable for parallel processing

Divide and Conquer

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 - ▶ And separation / combination of subproblems is $\in O(n)$
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 - ▶ The runtime is $\in O(n \cdot \log n)$
- ▶ Suitable for parallel processing
 - ▶ Subproblems are **independent** of each other

Divide and Conquer

Features

Features:

- ▶ Realization of **efficient solutions**
 - ▶ If trivial solution is $\in O(1)$
 - ▶ And separation / combination of subproblems is $\in O(n)$
 - ▶ And the number of subproblems is limited
 - ▶ The runtime is $\in O(n \cdot \log n)$
- ▶ Suitable for parallel processing
 - ▶ Subproblems are **independent** of each other
 - ▶ Only needed input for each subproblem has to be known

Divide and Conquer

Implementation

Definition of the trivial case:

Divide and Conquer

Implementation

Definition of the trivial case:

- ▶ Smaller subproblems are elegant and simple

Divide and Conquer

Implementation

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- ▶ Smaller subproblems are elegant and simple
- ▶ Otherwise the efficiency will be improved if relative big subproblems can be solved directly

Divide and Conquer

Implementation

Definition of the trivial case:

- ▶ Smaller subproblems are elegant and simple
- ▶ Otherwise the efficiency will be improved if relative big subproblems can be solved directly
- ▶ Recursion depth should not get too big (stack / memory overhead)

Divide and Conquer

Implementation

Division in subproblems:

Divide and Conquer

Implementation

Division in subproblems:

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

Divide and Conquer

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Combination of solutions:

Divide and Conquer

Implementation

Division in subproblems:

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

Combination of solutions:

- ▶ Typically conceptual demanding

Divide and Conquer

Example - Maximum Subtotal

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Divide and Conquer

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Example - Maximum Subtotal Input:

Divide and Conquer

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- ▶ Sequence X of n integers

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Output:

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

- ▶ Sequence X of n integers

Output:

- ▶ Maximum sum of related subsequence and its index boundary

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

- ▶ Sequence X of n integers

Output:

- ▶ Maximum sum of related subsequence and its index boundary

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

- ▶ Sequence X of n integers

Output:

- ▶ Maximum sum of related subsequence and its index boundary

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

Divide and Conquer

Example - Maximum Subtotal

Application:

- ▶ Maximum profit of buying and selling shares

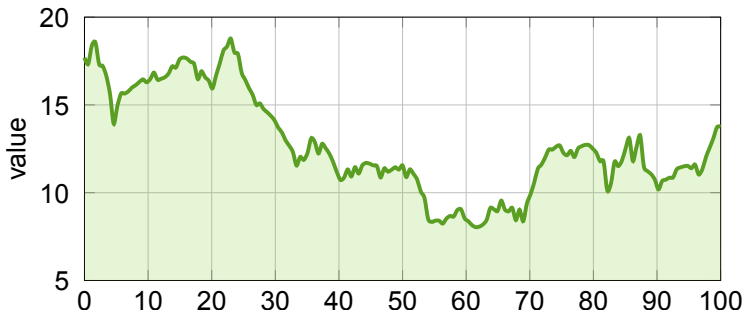


Figure: Stock value over time

Divide and Conquer

Example - Maximum Subtotal - Python

Naive solution (brute force)

Divide and Conquer

Example - Maximum Subtotal - Python

Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal - Python

Runtime - Upper bound

Divide and Conquer

Example - Maximum Subtotal - Python

Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

- ▶ Three interleaved loops

Divide and Conquer

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Upper bound:

- ▶ Three interleaved loops
- ▶ Each loop with runtime $O(n)$

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

- ▶ Three interleaved loops
- ▶ Each loop with runtime $O(n)$
- ▶ Algorithm runtime of $O(n^3)$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- We iterate at least $\frac{n}{3}$ values for i

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least $\frac{n}{3}$ values for i
- ▶ For each i we iterate at least $\frac{n}{3}$ values for j

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least $\frac{n}{3}$ values for i
- ▶ For each i we iterate at least $\frac{n}{3}$ values for j
- ▶ For each j we have at least $\frac{n}{3}$ additions

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least $\frac{n}{3}$ values for i
- ▶ For each i we iterate at least $\frac{n}{3}$ values for j
- ▶ For each j we have at least $\frac{n}{3}$ additions
- ▶ We need at least $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$ steps

Divide and Conquer

Example - Maximum Subtotal - Runtime

Runtime:

Divide and Conquer

Example - Maximum Subtotal - Runtime

Runtime:

- ▶ With $T(n) \in O(n^3)$ and $T(n) \in \Omega(n^3)$ we know:

$$T(n) \in \Theta(n^3)$$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Runtime:

- ▶ With $T(n) \in O(n^3)$ and $T(n) \in \Omega(n^3)$ we know:

$$T(n) \in \Theta(n^3)$$

- ▶ It is hard to solve the problem in a worse way ...

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

- ▶ Calculating the sum for range from i to j with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

- ▶ Calculating the sum for range from i to j with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

Better approach:

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

- ▶ Calculating the sum for range from i to j with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

Better approach:

- ▶ Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i + 1] + \cdots + X[j] + X[j + 1]$$

$$S_{i,j+1} = S_{i,j} + X[j + 1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

Divide and Conquer

Example - Maximum Subtotal - Python

Better solution:

Divide and Conquer

Example - Maximum Subtotal - Python

Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal - Python

Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
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        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
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```

► Runtime $\in O(n^2)$

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ Split the sequence in the middle

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

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- ▶ Solve left half of the problem

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ Split the sequence in the middle
- ▶ Solve left half of the problem
- ▶ Solve right half and combine both solutions into one

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ Split the sequence in the middle
- ▶ Solve left half of the problem
- ▶ Solve right half and combine both solutions into one
- ▶ Maximum might be located in **left half (A)** or **right half (B)**

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



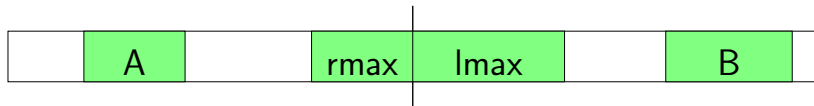
Divide and Conquer Idea to solve:

- ▶ Split the sequence in the middle
- ▶ Solve left half of the problem
- ▶ Solve right half and combine both solutions into one
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- ▶ Problem: Maximum can overlap the split

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



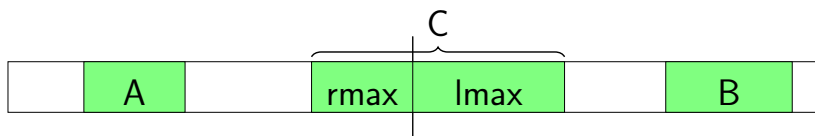
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- ▶ To solve this case we have to calculate *rmax* and *lmax*

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ Split the sequence in the middle
- ▶ Solve left half of the problem
- ▶ Solve right half and combine both solutions into one
- ▶ Maximum might be located in left half (*A*) or right half (*B*)
- ▶ Problem: Maximum can overlap the split
- ▶ To solve this case we have to calculate *rmax* and *lmax*
- ▶ The overall solution is the maximum of *A*, *B* and *C*

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions A and B are returned

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions A and B are returned
- ▶ To determine subsolution C , rmax and lmax for the subproblems are computed

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions A and B are returned
- ▶ To determine subsolution C , $rmax$ and $lmax$ for the subproblems are computed
- ▶ The overall solution is the maximum of A , B and C

Divide and Conquer

Example - Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

Further Literature

► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

- ▶ **Caching**

[Wik] [Cache](https://en.wikipedia.org/wiki/Cache)

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