Algorithms and Datastructures

Linked Lists, Binary Search Trees

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Algorithms and Datastructures, January 2018

Structure

Sorted Sequences

Linked Lists

Binary Search Trees

Sorted Sequences

Introduction

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Sorted Sequences

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- We have a set of keys mapped to values

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Sorted Sequences

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Sorted Sequences

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Sorted Sequences
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  - `next()/previous()`: Returns the element with the next bigger/smaller key. This enables iteration over all elements

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- How could we implement this?

Sorted Sequences

Implementation 1 (not good) - Static Array

Static array:

3	5	9	14	18	21	26	40	41	42	43	46

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Sorted Sequences

Implementation 2 (bad) - Hash Table

Hash map:

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  Order of the elements is independent of the order of the keys

Sorted Sequences

Implementation 3 (good?) - Linked List

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Sorted Sequences

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Sorted Sequences

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- Not yet what we want, but structure is related to binary search trees
- Let's have a closer look

Structure

Sorted Sequences

Linked Lists

Binary Search Trees

Linked Lists

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- Dynamic datastructure
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- Elements are linked through references / pointer to the predecessor / successor
- Single / doubly linked lists possible

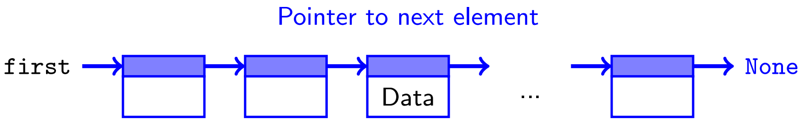

Figure: Linked list

Properties in comparison to an array:

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Linked Lists

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Properties in comparison to an array:

- Minimal extra space for storing pointer
- We do not need to copy elements on `insert` or `remove`
- The number of elements can be simply modified
- No direct access of elements
  $\Rightarrow$ We have to iterate over the list

List with head / last element pointer:

Linked Lists

Variants

List with head / last element pointer:

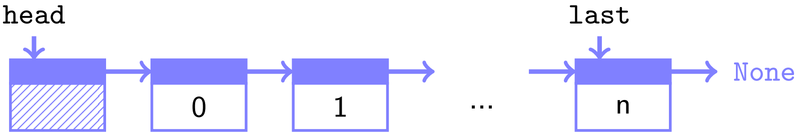

Figure: Singly linked list

Linked Lists
Variants

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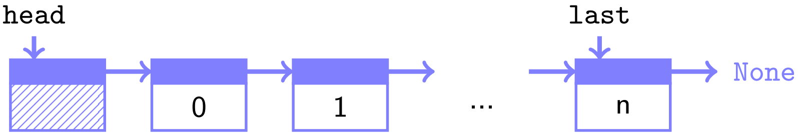

Figure: Singly linked list

- Head element has pointer to first list element

Linked Lists

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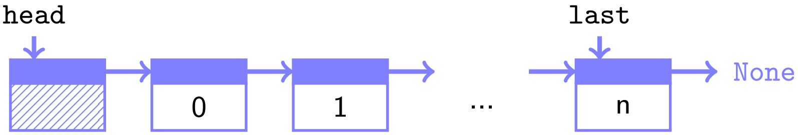

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- Head element has pointer to first list element
- May also hold additional information:

Linked Lists

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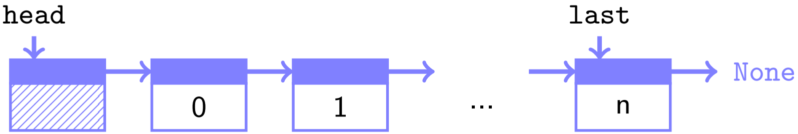

Figure: Singly linked list

- Head element has pointer to first list element
- May also hold additional information:
  - Number of elements

Doubly linked list:

Linked Lists

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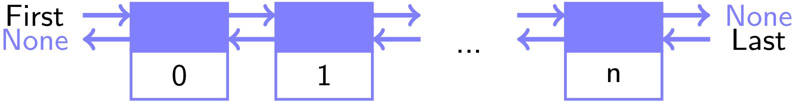

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Linked Lists

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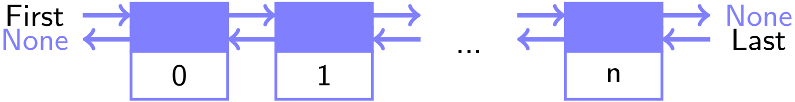

Figure: Doubly linked list

- Pointer to successor element

Linked Lists
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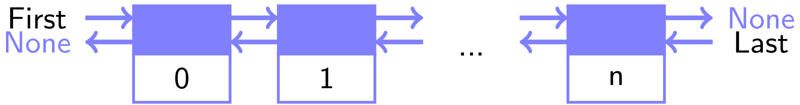

Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element

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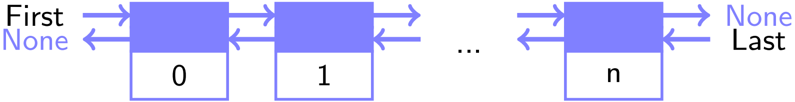

Figure: Doubly linked list

- Pointer to successor element
- Pointer to predecessor element
- Iterate forward and backward

Linked Lists

Implementation - Node/Element - Python

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```

Creating linked lists - Python:

Linked Lists

Usage examples

Creating linked lists - Python:

- `first = Node(7)`

Linked Lists

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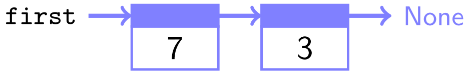
Creating linked lists - Python:

- `first = Node(7)`

- `first.nextNode = Node(3)`



Linked Lists

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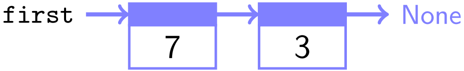
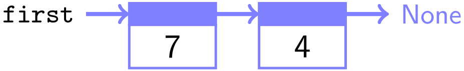


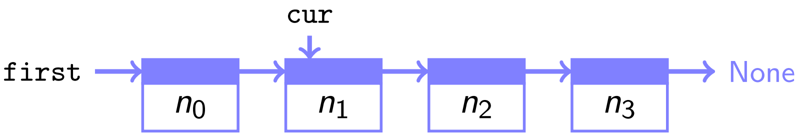
- `first.nextNode.value = 4`



Linked Lists

Implementation - Insert

Inserting a node after node `cur`:



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Linked Lists

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Linked Lists

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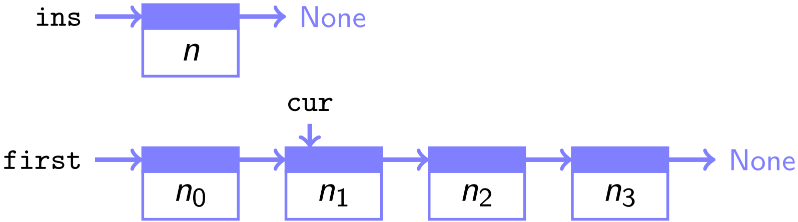

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Linked Lists

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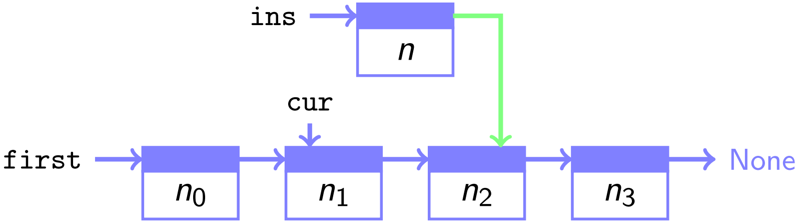

Linked Lists

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Linked Lists

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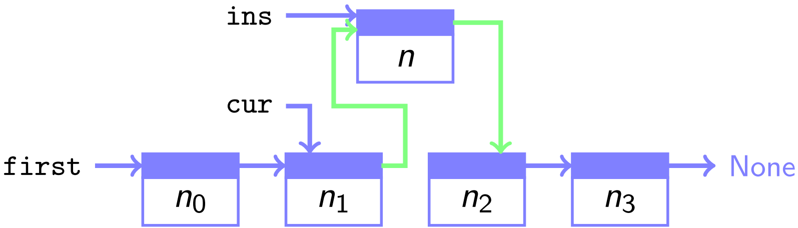

Linked Lists

Implementation - Insert

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Linked Lists

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Linked Lists

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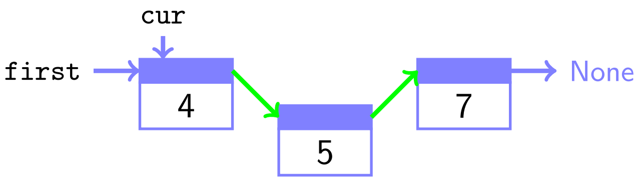
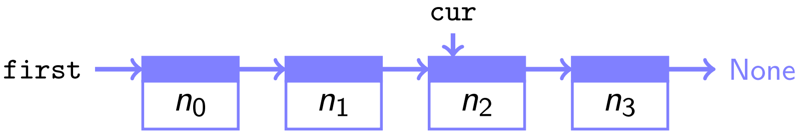


Linked Lists

Implementation - Remove

Removing a node `cur`:



Linked Lists

Implementation - Remove

Removing a node `cur`:

Linked Lists

Implementation - Remove

Removing a node `cur`:

- Find the predecessor of `cur`:

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pre = first
while pre.nextNode != cur:
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Linked Lists

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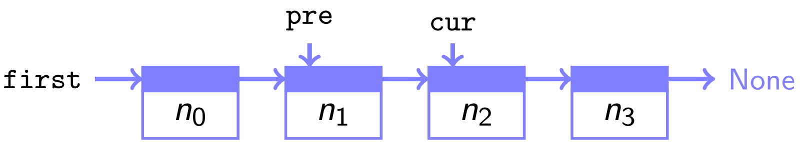

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Linked Lists

Implementation - Remove

Removing a node `cur`:

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Linked Lists

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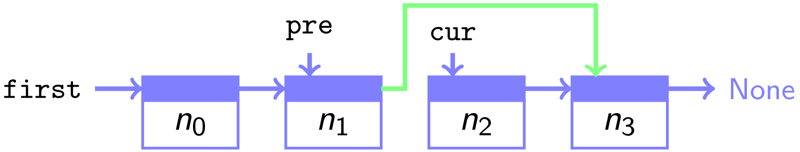

Removing the first node:

Linked Lists

Implementation - Remove

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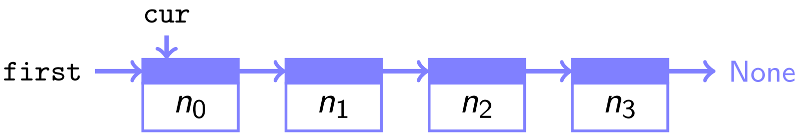

Linked Lists

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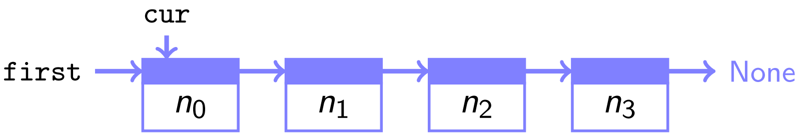

- Update the pointer to the next element:
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Linked Lists

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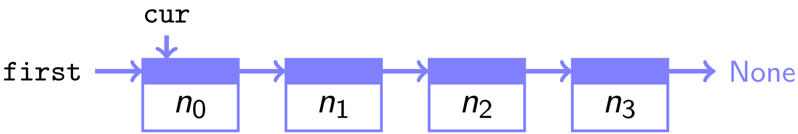

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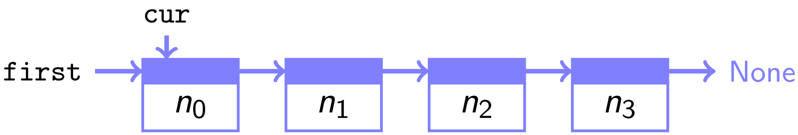

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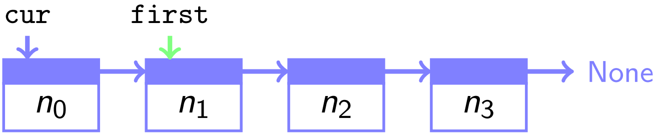

Linked Lists

Implementation - Remove

Removing a node `cur`: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```

Using a `head` node:

Linked Lists

Implementation - Head Node

Using a `head` node:

- Advantage:

Linked Lists

Implementation - Head Node

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  - Deleting the first node is no special case

Linked Lists

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Linked Lists

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  - Counting of all nodes

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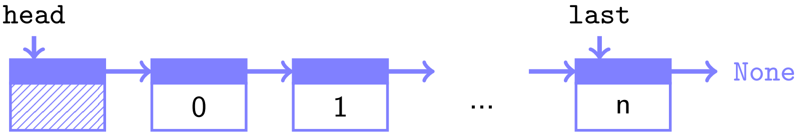


Linked Lists

Implementation - LinkedList - Python

```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```

Linked Lists

Implementation - LinkedList - Python

```
def append(self, value):
    ...

def insertAfter(self, cur, value):
    ...

def remove(self, cur):
    ...

def get(self, position):
    ...

def contains(self, value):
    ...
```

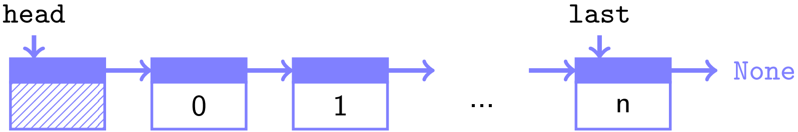
Linked Lists
Implementation

Head, last:

Linked Lists

Implementation

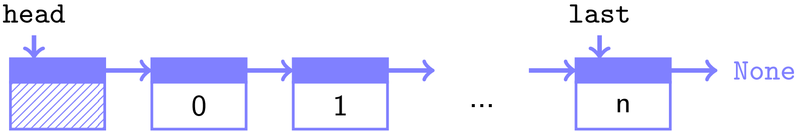
Head, last:



Linked Lists

Implementation

Head, last:



- `Head` points to the first node, `last` to the last node

Linked Lists

Implementation

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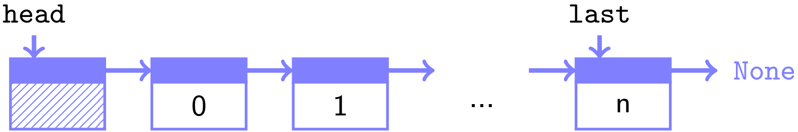

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Linked Lists

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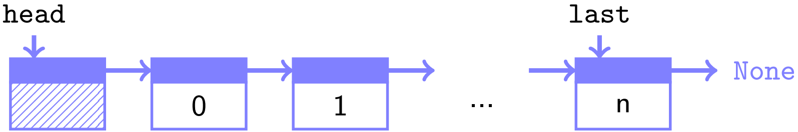

- `Head` points to the first node, `last` to the last node
- We can append elements to the end of the list in $O(1)$ through the `last` node
- We have to keep the pointer to `last` updated after all operations

Linked Lists

Implementation - Append

Appending an element:

Linked Lists

Implementation - Append

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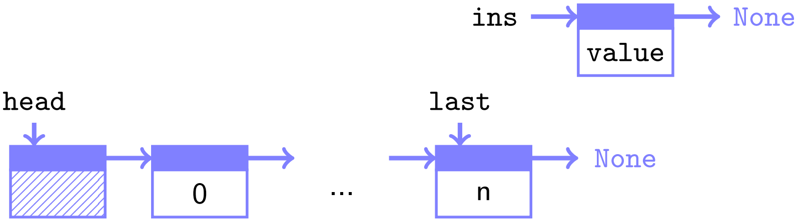

Linked Lists

Implementation - Append

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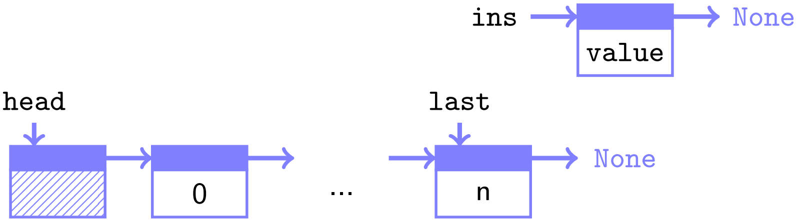


```
def append(self, value):
    last.nextNode = Node(value)
    last = last.NextNode
    itemCount += 1
```

Linked Lists

Implementation - Append

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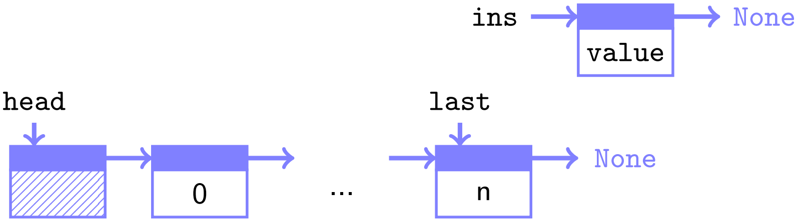


```python
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```
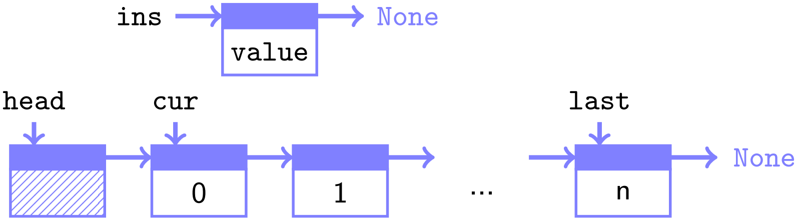

- The pointer to `last` avoids the iteration of the whole list

Inserting after node `cur`:



Linked Lists

Implementation - Insert After

Inserting after node `cur`:

- The pointer to `head` is not modified

Linked Lists

Implementation - Insert After

Inserting after node `cur`:

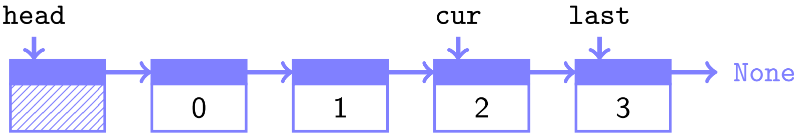
- The pointer to `head` is not modified

```
def insertAfter(self, cur, value):
    if cur == last:
        # also update last node
        append(value)
    else:
        # last node is not modified
        cur.nextNode = Node(value, \
            cur.nextNode)
        itemCount += 1
```

Linked Lists

Implementation - Remove

Remove node `cur`:



Linked Lists

Implementation - Remove

Remove node `cur`:

- Searching the predecessor in $O(n)$

Remove node `cur`**:**

- Searching the predecessor in $O(n)$

```
def remove(self, cur):
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
    itemCount -= 1

    if pre.nextNode == None:
        last = pre
```

Getting a reference to node at `pos`:

- Iterate the entries of the list until at position in $O(n)$

Linked Lists

Implementation - Get

Getting a reference to node at `pos`:

- Iterate the entries of the list until at position in $O(n)$

```
def get(self, pos):
    if pos < 0 or pos >= itemCount:
        return None

    cur = head
    for i in range(0, pos):
        cur = cur.nextNode

    return cur
```

Searching a `value`:

Linked Lists

Implementation - Contains

Searching a `value`:

- First element is head without an assigned value

Linked Lists

Implementation - Contains

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Linked Lists

Implementation - Contains

Searching a `value`:

- First element is head without an assigned value
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```
def contains(self, value):
    cur = head

    for i in range(0, itemCount):
        cur = cur.nextNode
        if cur.value == value:
            return True

    return False
```

Runtime:

Runtime:

- Singly linked list:

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  - `next` in $O(1)$

Runtime:

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Linked Lists

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Linked Lists

Runtime

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  - `next` in $O(1)$
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  - `insert` in $O(1)$
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  - `lookup` in $\Theta(n)$
- Better with doubly linked lists

Doubly linked list:

Doubly linked list:

- Each node has a reference to its successor and its predecessor

Linked Lists

Doubly Linked List

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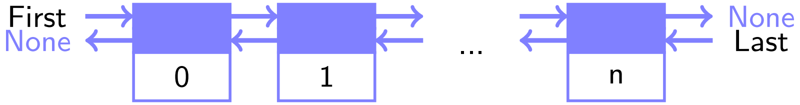

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Linked Lists

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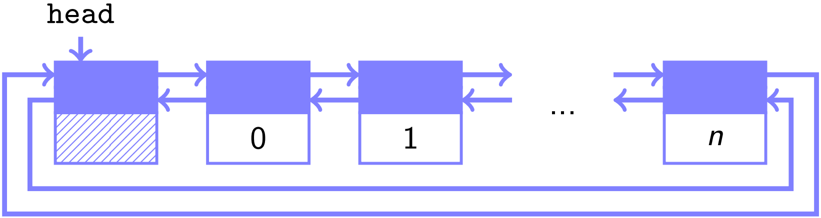

Runtime of doubly linked list:

Linked Lists

Runtime

Runtime of doubly linked list:

- `next` and `previous` in $O(1)$

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Linked Lists

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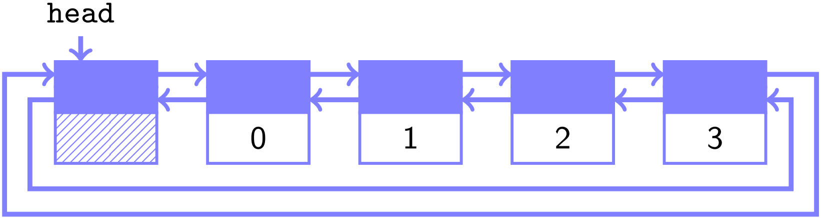
  Even if the elements are sorted we can only retrieve them in $\Theta(n)$ Why?

Linked Lists

List in real program

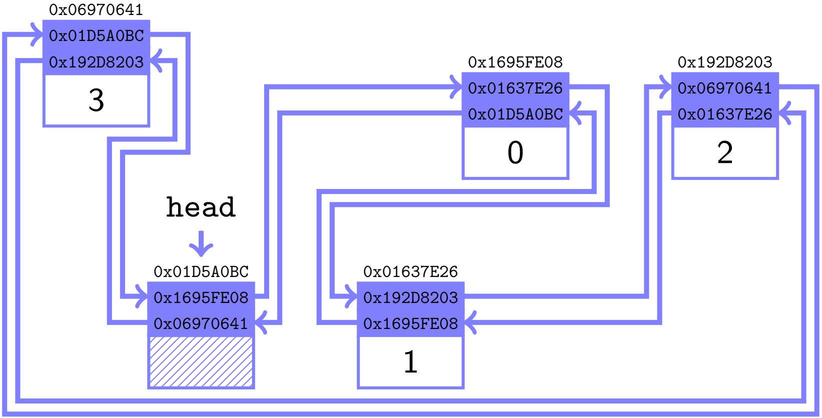
Linked list in book:



Linked Lists

List in real program

Linked list in memory:



Structure

Sorted Sequences

Linked Lists

Binary Search Trees

Runtime of a search tree:

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- `next` and `previous` in $O(1)$

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  The structure helps searching efficiently

Binary Search Trees

Introduction

Idea:

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- We define a total order for the search tree

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Binary Search Trees

Introduction

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Binary Search Trees

Introduction

- Edge direction indicates ordering

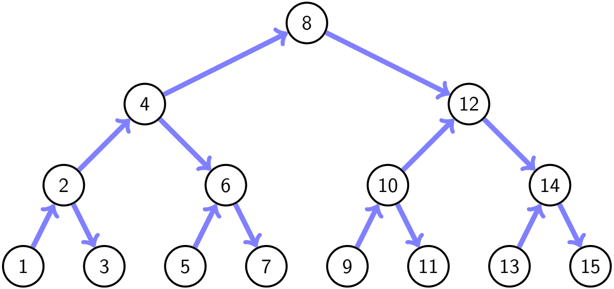

Figure: A binary search tree

Binary Search Trees

Introduction

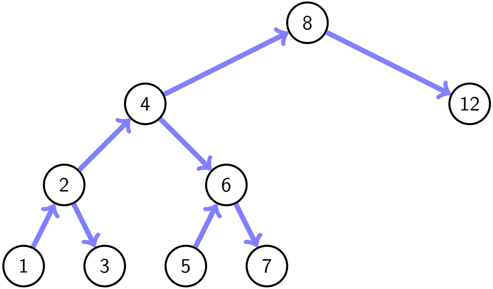

Figure: Another binary search tree

Binary Search Trees

Introduction

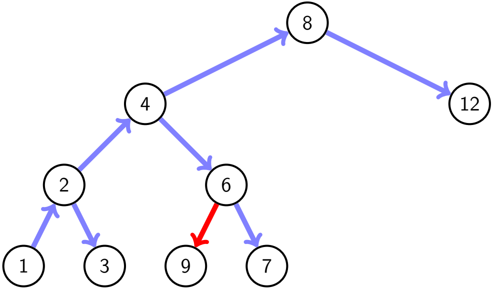

Figure: **Not** a binary search tree

Binary Search Trees

Implementation

Implementation:

Implementation:

- For the heap we had all elements stored in an array
- Here we link all nodes through pointer / references, like linked lists

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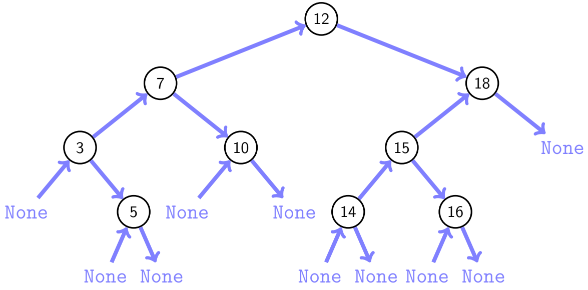

Implementation:

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Binary Search Trees

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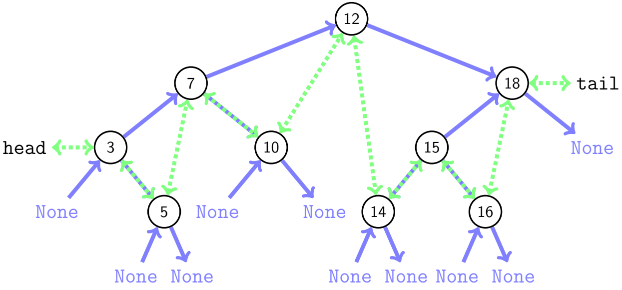

Figure: Binary search tree with links

Lookup:

Lookup:

- Definition:
  " Search the element with the given key. If no element is found return the element with the next (bigger) key. "

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Binary Search Trees

Implementation - Lookup

For each node applies the total order:

Binary Search Trees

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For each node applies the total order:

keys of left subtree ¡ `node.key` ¡ keys of right subtree

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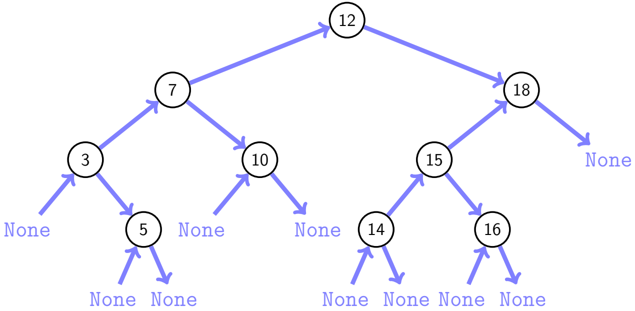

Examples:

Figure: Binary search tree with total order "¡"

Binary Search Trees

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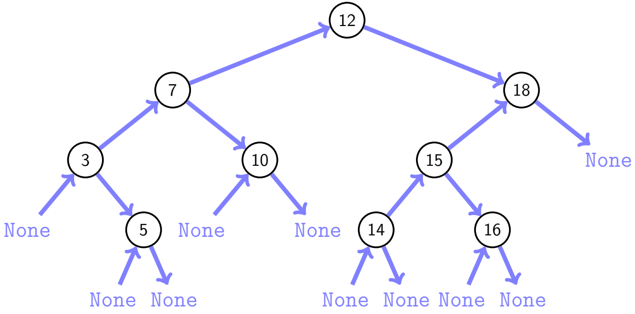

Examples:

`lookup(14)`

Figure: Binary search tree with total order "¡"

Binary Search Trees

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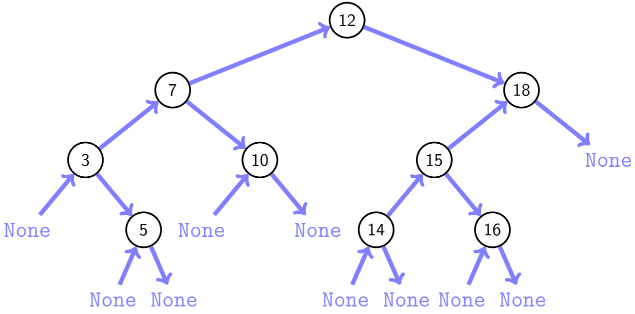

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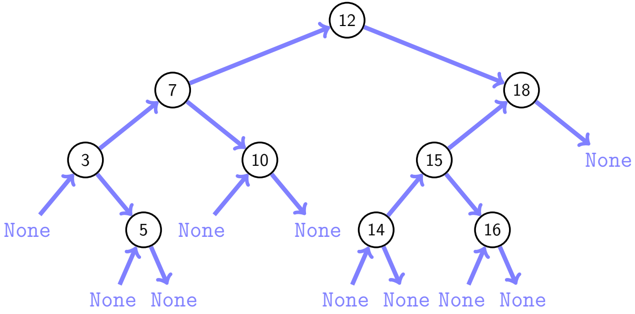

Figure: Binary search tree with total order "¡"

Examples:

`lookup(14)`

`lookup(6)`

`lookup(19)`

Binary Search Trees

Implementation - Insert

Insert:

Binary Search Trees

Implementation - Insert

Insert:

- We search for the key in our search tree

Binary Search Trees

Implementation - Insert

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Binary Search Trees

Implementation - Insert

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Binary Search Trees

Implementation - Insert

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Binary Search Trees

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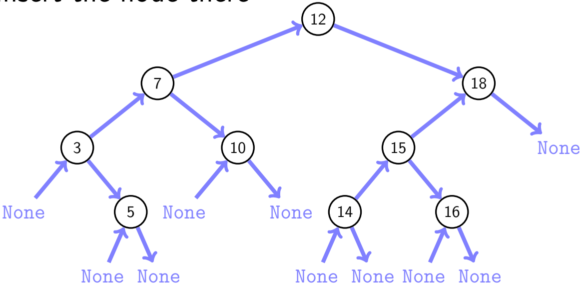

Figure: Binary search tree with total order "¡"

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node “5” has no children

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node “5” has no children

- Find parent of node “5” (“6”)

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node "5" has no children

- Find parent of node "5" ("6")
- Set left / right child of node "6" to `None` depending on position of node "5"

Binary Search Trees

Implementation - Remove

Remove: Case 1: The node “5” has no children

- Find parent of node “5” (“6”)
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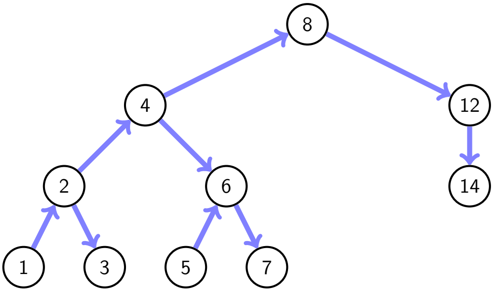

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Binary Search Trees

Implementation - Remove

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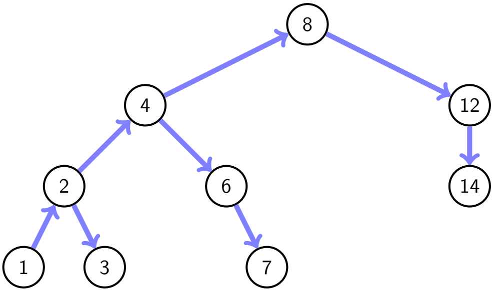

Figure: Binary search tree after deleting node “5”

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node “12” has one child

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node “12” has one child

- Find the child of node “12” (“14”)

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node “12” has one child

- Find the child of node “12” (“14”)
- Find the parent of node “12” (“8”)

Binary Search Trees

Implementation - Remove

Remove: Case 2: The node “12” has one child

- Find the child of node “12” (“14”)
- Find the parent of node “12” (“8”)
- Set left / right child of node “8” to “14” depending on position of node “12” (skip node “14”)

Binary Search Trees

Implementation - Remove

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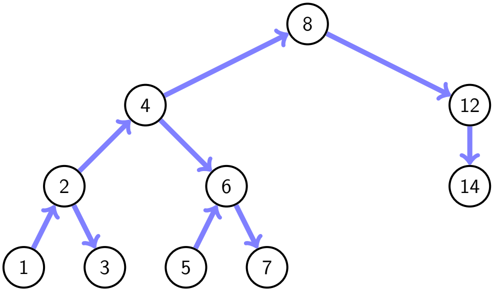

Figure: Binary search tree with total order "¡"

Binary Search Trees

Implementation - Remove

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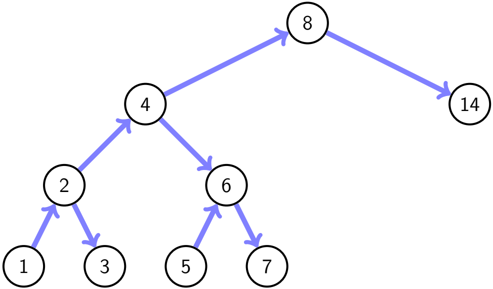

Figure: Binary search tree after delting node "12"

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node "4" has two children

- Find the successor of node "4" ("5")

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

- Find the successor of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”

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Binary Search Trees

Implementation - Remove

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- Delete node “5” (the successor of node “4”) with remove-case 1 or 2
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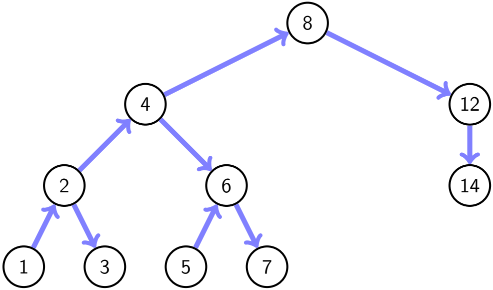

Binary Search Trees

Implementation - Remove

Remove: Case 3: The node “4” has two children

- Find the successor of node “4” (“5”)
- Replace the value of node “4” with the value of node “5”
- Delete node “5” (the successor of node “4”) with remove-case 1 or 2
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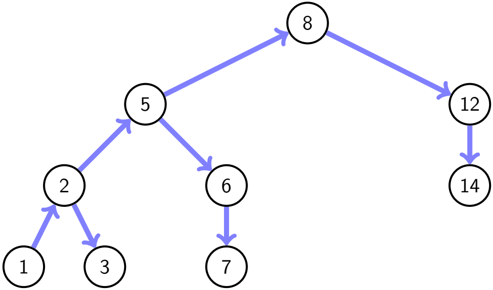

Binary Search Trees

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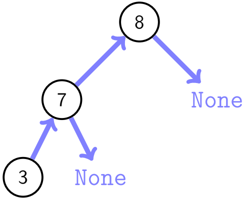

Figure: Degenerated binary tree $d = n$

Binary Search Trees

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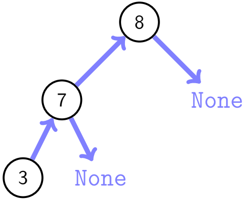

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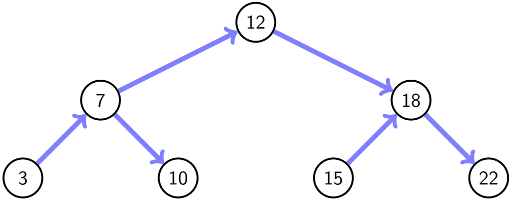

Figure: Complete binary tree $d = \log n$

- **General**

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- **Binary Search Tree**

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