

# Algorithms and Datastructures

Levenshtein distance, Dynamic programming

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Algorithms and Datastructures, February 2018

# Structure

Introduction

Edit distance

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**Edit distance:**

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## **Edit distance:**

- ▶ Measurement for similarity of two words / strings

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## **Edit distance:**

- ▶ Measurement for similarity of two words / strings
- ▶ Algorithm for efficient calculation

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## **Edit distance:**

- ▶ Measurement for similarity of two words / strings
- ▶ Algorithm for efficient calculation
- ▶ General principle: dynamic programming

# Introduction

Motivation: Error tolerant string comparison

## BioInfSearch



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ejafjatljökuk
eyjafjallajökull
eyjafjallajökull movie
eyjafjallajälull trailer

Search!

Wikipedia.org:

"Der Eyjafjallajökull ([ˈeɪjaˌfjatlaˌjœːkʏtʃ])[3], auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárþing eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."



# Introduction

## Motivation

**A lot of applications where similar string are searched:**

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### **A lot of applications where similar string are searched:**

- ▶ Duplicates in databases:

Hein Blöd	27568	Bremerhaven
Hein Bloed	27568	Bremerhafen
Hein Doof	27478	Cuxhaven

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- ▶ Product search:

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eyjaföllajaküll

uniwersität verien 2017

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eyjaföllajaküll

uniwersität verien 2017

- ▶ Bioinformatics: Similarity of DNA-sequences

# Introduction

Example: Bioinformatics DNA-matching

**Search of similar proteins:**

# Introduction

Example: Bioinformatics DNA-matching

## Search of similar proteins:

- ▶ BLAST (**B**asic **L**ocal **A**lignment **S**earch **T**ool)

# Introduction

Example: Bioinformatics DNA-matching

## Search of similar proteins:

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- ▶ Alignment  $\hat{=}$  Edit distance



# Introduction

Example: Bioinformtics DNA-matching

## Search of similar proteins:

- ▶ BLAST (**B**asic **L**ocal **A**lignment **S**earch **T**ool)
- ▶ Alignment  $\hat{=}$  Edit distance
- ▶ Changed life-science completely

# Introduction

Example: Bioinformatics DNA-matching

## Search of similar proteins:

- ▶ BLAST (**B**asic **L**ocal **A**lignment **S**earch **T**ool)
- ▶ Alignment  $\hat{=}$  Edit distance
- ▶ Changed life-science completely
- ▶ Cited 63437 times on Google Scholar (Sep. 2017)

# Structure

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Edit distance

# Edit distance

**Definition of edit distance:** (*Levenshtein-distance*)

# Edit distance

## **Definition of edit distance:** (*Levenshtein-distance*)

- ▶ Let  $x$ ,  $y$  be two strings
- ▶ Edit distance  $ED(x, y)$  of  $x$  and  $y$ :  
The minimal number of operations to transform  $x$  into  $y$

# Edit distance

## Definition of edit distance: (*Levenshtein-distance*)

- ▶ Let  $x$ ,  $y$  be two strings
- ▶ Edit distance  $ED(x, y)$  of  $x$  and  $y$ :  
The minimal number of operations to transform  $x$  into  $y$ 
  - ▶ Insert a character

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- ▶ Let  $x$ ,  $y$  be two strings
- ▶ Edit distance  $ED(x, y)$  of  $x$  and  $y$ :  
The minimal number of operations to transform  $x$  into  $y$ 
  - ▶ Insert a character
  - ▶ Replace a character with another

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- ▶ Let  $x$ ,  $y$  be two strings
- ▶ Edit distance  $ED(x, y)$  of  $x$  and  $y$ :  
The minimal number of operations to transform  $x$  into  $y$ 
  - ▶ Insert a character
  - ▶ Replace a character with another
  - ▶ Delete a character



# Edit distance

## Example

1 2 3 4 5  
D O O F

B L O E D

# Edit distance

## Example

1 2 3 4 5

DOOF

↓

BOOF

replace(1, B)

BLOED

# Edit distance

## Example

1 2 3 4 5

DOOF

↓

replace(1, B)

BOOF

↓

replace(2, L)

BLOF

BLOED

# Edit distance

## Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
BLOED	

# Edit distance

## Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	

# Edit distance

## Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	

  
ED=4

# Edit distance

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1 2 3 4 5

DOOF

↓

replace(1, B)

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↓

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↓

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BLOED

1 2 3 4 5

BLOED

⏟  
ED=4

# Edit distance

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1 2 3 4 5

B LOED

DOOF

⏟  
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BLOED

1 2 3 4 5

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↓

replace(5, F)

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DOOF

⏟  
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ED=4

1 2 3 4 5	
B LOED	
↓	replace(5, F)
B LOEF	
↓	delete(4)
B LOF	
↓	replace(2, O)
B OOF	
DOOF	

# Edit distance

## Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	

ED=4

1 2 3 4 5	
B LOED	
↓	replace(5, F)
B LOEF	
↓	delete(4)
B LOF	
↓	replace(2, O)
B OOF	
↓	replace(1, D)
DOOF	

# Edit distance

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ED=4	

# Edit distance

**Notation:**

# Edit distance

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- ▶  $\varepsilon$  is the empty string

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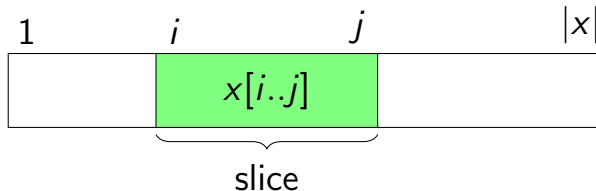
## Notation:

- ▶  $\varepsilon$  is the empty string
- ▶  $|x|$  is the length of the string  $x$  (number of characters)
- ▶  $x[i..j]$  is the slice of  $x$  from  $i$  to  $j$  where  $1 \leq i \leq j \leq |x|$

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# Edit distance

**Trivial facts:**

# Edit distance

## Trivial facts:

- ▶  $\text{ED}(x, y) = \text{ED}(y, x)$

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- ▶  $\text{ED}(x, \epsilon) = |x|$

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## Trivial facts:

- ▶  $ED(x, y) = ED(y, x)$
- ▶  $ED(x, \epsilon) = |x|$
- ▶  $ED(x, y) \geq \text{abs}(|x| - |y|)$

$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

# Edit distance

## Trivial facts:

- ▶  $ED(x, y) = ED(y, x)$
- ▶  $ED(x, \epsilon) = |x|$
- ▶  $ED(x, y) \geq \text{abs}(|x| - |y|)$
- ▶  $ED(x, y) \leq ED(x[1..n-1], y[1..m-1]) + 1$   $n = |x|, m = |y|$

$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$



# Edit distance

## Solving examples

**Solutions based on examples:**

# Edit distance

## Solving examples

### **Solutions based on examples:**

- ▶ From VERIEN to FERIE?

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- ▶ From MEXIKO to AMERIKA?

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## Solving examples

### **Solutions based on examples:**

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
- ▶ From AAEBEAABEAREEEAEBA to RBEAAEEBAAAEBBAEAE?

# Edit distance

## Solving examples

### **Solutions based on examples:**

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
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- ▶ Searching biggest substrings can yield the solution but doesn't have to

# Edit distance

## Solving examples

### **Solutions based on examples:**

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### **Recursive approach:**

# Edit distance

## Solving examples

### Solutions based on examples:

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
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- ▶ Searching biggest substrings can yield the solution but doesn't have to

### Recursive approach:

- ▶ Dividing in two halves? Not a good idea:

$ED(\textit{GRAU}, \textit{RAUM}) = 2$  but

$ED(\textit{GR}, \textit{RA}) + ED(\textit{AU}, \textit{UM}) = 4$

# Edit distance

## Solving examples

### Solutions based on examples:

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
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### Recursive approach:

- ▶ Dividing in two halves? Not a good idea:

$$\text{ED}(\text{GRAU}, \text{RAUM}) = 2 \quad \text{but} \\ \text{ED}(\text{GR}, \text{RA}) + \text{ED}(\text{AU}, \text{UM}) = 4$$

- ▶ Finding “smaller” sub problems?  
Let's try it!



# Edit distance

## **Terminology:**

# Edit distance

## Terminology:

- ▶ Let  $x$ ,  $y$  be two strings

# Edit distance

## Terminology:

- ▶ Let  $x, y$  be two strings
- ▶ Let  $\sigma_1, \dots, \sigma_k$  be a sequence of  $k$  operations where  $k = \text{ED}(x, y)$  for  $x \rightarrow y$  (transform  $x$  into  $y$ )  
(We do not know this sequence but we assume it exists)

# Edit distance

## **Terminology:**

# Edit distance

## Terminology:

- ▶ We only consider **monotonous** sequences:  
The position of  $\sigma_{i+1}$  is  $\geq$  the position of  $\sigma_i$  where we only allow the positions to be equal on a delete operation

# Edit distance

## Terminology:

- ▶ We only consider **monotonous** sequences:

The position of  $\sigma_{i+1}$  is  $\geq$  the position of  $\sigma_i$  where we only allow the positions to be equal on a delete operation

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	

1 2 3 4 5 6 7	
SAUDOOOF	
↓	delete(1)
AUDOOOF	
↓	delete(1)
UDOOOF	
↓	delete(1)
DOOF	
↓	insert(4, 0)
DOOOOF	

# Edit distance

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- ▶ We only consider **monotonous** sequences:

The position of  $\sigma_{i+1}$  is  $\geq$  the position of  $\sigma_i$  where we only allow the positions to be equal on a delete operation

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DOOOOF	

# Edit distance

## **Terminology:**



# Edit distance

## Terminology:

- **Lemma:** For any  $x$  and  $y$  with  $k = \text{ED}(x, y)$  exists a **monotonous** sequence of  $k$  operations for  $x \rightarrow y$

# Edit distance

## Terminology:

- ▶ **Lemma:** For any  $x$  and  $y$  with  $k = \text{ED}(x, y)$  exists a **monotonous** sequence of  $k$  operations for  $x \rightarrow y$
- ▶ **Intuition:** The order of our sequence is not relevant  
(Therefore we can also sort them monotonously)

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## Terminology:

- ▶ **Lemma:** For any  $x$  and  $y$  with  $k = \text{ED}(x, y)$  exists a **monotonous** sequence of  $k$  operations for  $x \rightarrow y$
- ▶ **Intuition:** The order of our sequence is not relevant  
(Therefore we can also sort them monotonously)

1	2	3	4	5
D	O	O	F	

B	L	O	E	D
---	---	---	---	---

1	2	3	4	5	6	7
S	A	U	D	O	O	F

D	O	O	O	F
---	---	---	---	---

# Edit distance

## Recursive approach

**Consider the last operation:**

# Edit distance

## Recursive approach

### **Consider the last operation:**

- ▶ Solve **blue** part recursively

# Edit distance

## Recursive approach

### Consider the last operation:

- Solve **blue** part recursively

DOOF

↓↓↓↓↓

BLOE

↓insert

BLOED

Figure: Case 1a

DOOF

↓↓↓↓↓↓↓

BLOEDF

↓delete

BLOED

Figure: Case 1b

DOOF

↓↓↓↓↓↓↓

BLOEF

↓replace

BLOED

Figure: Case 1c

# Edit distance

## Recursive approach

**Consider the last operation:**

# Edit distance

## Recursive approach

### **Consider the last operation:**

- ▶ Solve **blue** part recursively



# Edit distance

## Recursive approach

### Consider the last operation:

- Solve **blue** part recursively

W I N T E R  
↓ ↓ ↓ ↓ ↓  
S O M M E R  
          ↓ nothing  
S O M M E R

Figure: Case 2

### Display of solution:

- Alignment
- Example:

-	-	-	B	L	O	E	D
S	A	U	B	L	O	E	D

# Edit distance

Dynamic programming

**Dynamic programming:**

# Edit distance

## Dynamic programming

### **Dynamic programming:**

- ▶ Instances of Bellman's principle of optimality:

# Edit distance

## Dynamic programming

### **Dynamic programming:**

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths

# Edit distance

## Dynamic programming

### **Dynamic programming:**

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths
  - ▶ Edit distance

# Edit distance

## Dynamic programming

### Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths
  - ▶ Edit distance



**Figure:** Richard Bellman  
(1920 - 1984)

# Edit distance

## Dynamic programming

### Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths
  - ▶ Edit distance



**Figure:** Richard Bellman  
(1920 - 1984)

# Edit distance

## Dynamic programming

### Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths
  - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions



**Figure:** Richard Bellman  
(1920 - 1984)



# Edit distance

## Dynamic programming

### Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths
  - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions
  - ▶ Shortest paths: Each partial path has to be optimal



**Figure:** Richard Bellman  
(1920 - 1984)

# Edit distance

## Dynamic programming

### Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths
  - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions
  - ▶ Shortest paths: Each partial path has to be optimal
  - ▶ Edit distance: Each partial alignment has to be optimal



**Figure:** Richard Bellman  
(1920 - 1984)

# Edit distance

## Dynamic programming

### Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
  - ▶ Shortest paths
  - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions
  - ▶ Shortest paths: Each partial path has to be optimal
  - ▶ Edit distance: Each partial alignment has to be optimal
- ▶ Always solvable through dynamic programming (Caching of optimal partial solutions)



Figure: Richard Bellman  
(1920 - 1984)

# Edit distance

**Case analysis:**

# Edit distance

## Case analysis:

- ▶ We consider the last operation  $\sigma_k$

# Edit distance

## Case analysis:

- ▶ We consider the last operation  $\sigma_k$ 
  - ▶  $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$  and  $\sigma_k: z \rightarrow y$

Example:

$$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$$

# Edit distance

## Case analysis:

- ▶ We consider the last operation  $\sigma_k$ 
  - ▶  $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$  and  $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

- ▶ Let  $n = |x|$ ,  $m = |y|$ ,  $m' = |z|$

# Edit distance

## Case analysis:

- ▶ We consider the last operation  $\sigma_k$ 
  - ▶  $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$  and  $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

- ▶ Let  $n = |x|$ ,  $m = |y|$ ,  $m' = |z|$
- ▶ We note  $m' \in \{m-1, m, m+1\}$       why?



# Edit distance

**Case analysis:**

# Edit distance

## Case analysis:

- ▶ Case 1:  $\sigma_k$  does something at the outer end:

# Edit distance

## Case analysis:

- ▶ Case 1:  $\sigma_k$  does something at the outer end:
  - ▶ Case 1a:  $\sigma_k = \text{insert}(m' + 1, y[m])$  [then  $m' = m - 1$ ]

# Edit distance

## Case analysis:

- ▶ Case 1:  $\sigma_k$  does something at the outer end:
  - ▶ Case 1a:  $\sigma_k = \text{insert}(m' + 1, y[m])$  [then  $m' = m - 1$ ]
  - ▶ Case 1b:  $\sigma_k = \text{delete}(m')$  [then  $m' = m + 1$ ]

# Edit distance

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- ▶ Case 1:  $\sigma_k$  does something at the outer end:
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  - ▶ Case 1b:  $\sigma_k = \text{delete}(m')$  [then  $m' = m + 1$ ]
  - ▶ Case 1c:  $\sigma_k = \text{replace}(m', y[m])$  [then  $m' = m$ ]

# Edit distance

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- ▶ Case 1:  $\sigma_k$  does something at the outer end:
  - ▶ Case 1a:  $\sigma_k = \text{insert}(m' + 1, y[m])$  [then  $m' = m - 1$ ]
  - ▶ Case 1b:  $\sigma_k = \text{delete}(m')$  [then  $m' = m + 1$ ]
  - ▶ Case 1c:  $\sigma_k = \text{replace}(m', y[m])$  [then  $m' = m$ ]
- ▶ Case 2:  $\sigma_k$  does nothing at the outer end:

# Edit distance

## Case analysis:

- ▶ Case 1:  $\sigma_k$  does something at the outer end:
  - ▶ Case 1a:  $\sigma_k = \text{insert}(m' + 1, y[m])$  [then  $m' = m - 1$ ]
  - ▶ Case 1b:  $\sigma_k = \text{delete}(m')$  [then  $m' = m + 1$ ]
  - ▶ Case 1c:  $\sigma_k = \text{replace}(m', y[m])$  [then  $m' = m$ ]
- ▶ Case 2:  $\sigma_k$  does nothing at the outer end:
  - ▶ Then  $z[m'] = y[m]$  and  $x[n'] = z[m']$  and with that  
 $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$  and  $x[n] = y[m]$

# Edit distance

## Case analysis:



# Edit distance

## Case analysis:

- ▶ Case 1a (insert):  $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$

# Edit distance

## Case analysis:

- ▶ Case 1a (insert):  $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
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- ▶ For  $|y| = 0$  is  $ED(x, y) = |x|$

# Edit distance

## Implementation - Python

```
def edit_distance(x, y):  
    if len(x) == 0:  
        return len(y)  
    if len(y) == 0:  
        return len(x)  
  
    ed1 = edit_distance(x, y[:-1]) + 1  
    ed2 = edit_distance(x[:-1], y) + 1  
    ed3 = edit_distance(x[:-1], y[:-1])  
    if x[-1] != y[-1]:  
        ed3 += 1  
  
    return min(ed1, ed2, ed3)
```

# Edit distance

## Runtime analysis

**Recursive program:**

# Edit distance

## Runtime analysis

### Recursive program:

- ▶ The algorithm results in the following recursive formular:

$$\begin{aligned}T(n, m) &= T(n-1, m) + T(n, m-1) + T(n-1, m-1) + 1 \\&\geq T(n-1, m-1) + T(n-1, m-1) + T(n-1, m-1) \\&= 3 \cdot T(n-1, m-1)\end{aligned}$$

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- ▶ This results in  $T(n, n) \geq 3^n$
- ⇒ The runtime is at least exponential

# Edit distance

**Dynamic programming:**

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## Dynamic programming:

- ▶ We create a table with all possible combination of substrings and save calculated entries
- ▶ This results in a runtime and space consumption of  $O(n \cdot m)$

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## Dynamic programming:

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## Visualization on the next slide:

- ▶ Operations always refer to the last position (indices are omitted)
- ▶ We also display the replaced character on a replace operation to visualize operations without costs  
 $\Rightarrow \text{repl}(A, A)$

# Edit Distance

# Edit Distance





# Edit Distance



# Edit Distance



# Edit Distance



# Edit Distance



# Edit Distance



# Edit distance

## Fast algorithm

### **Fast algorithm:**

We can determine the **edit distance** for all combination of partial strings from the top left to bottom right.

# Edit Distance



# Edit Distance

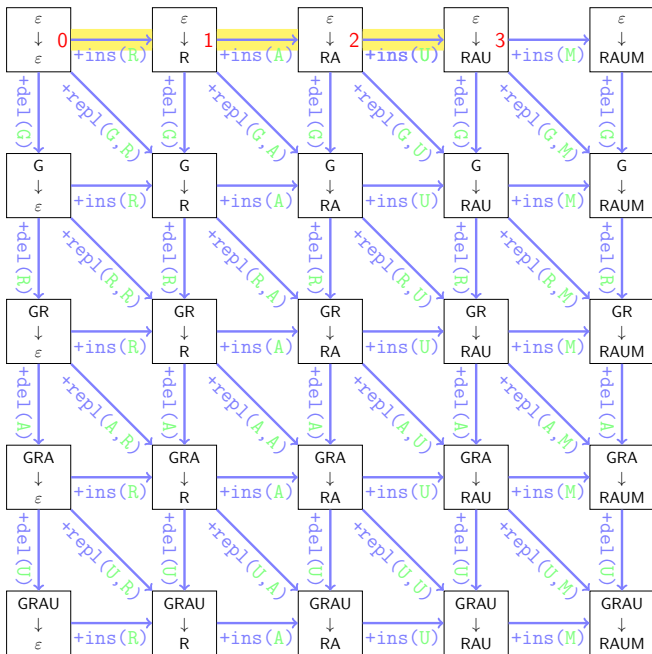




# Edit Distance



# Edit Distance



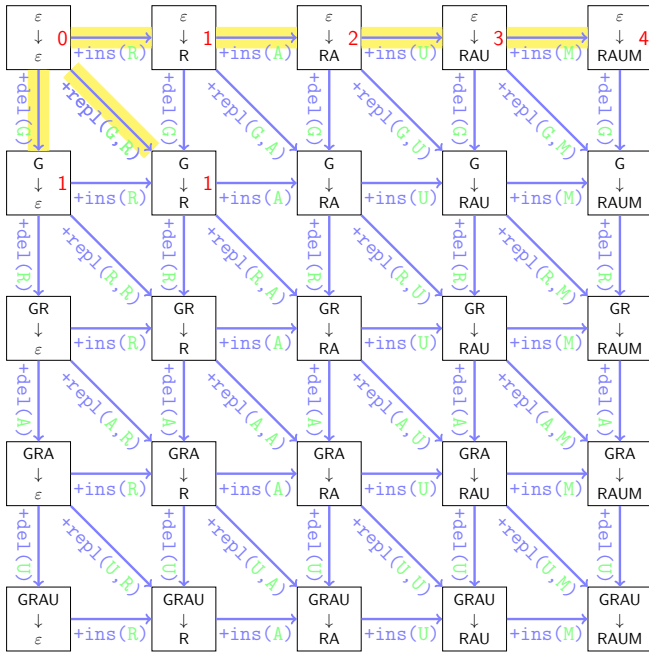
# Edit Distance



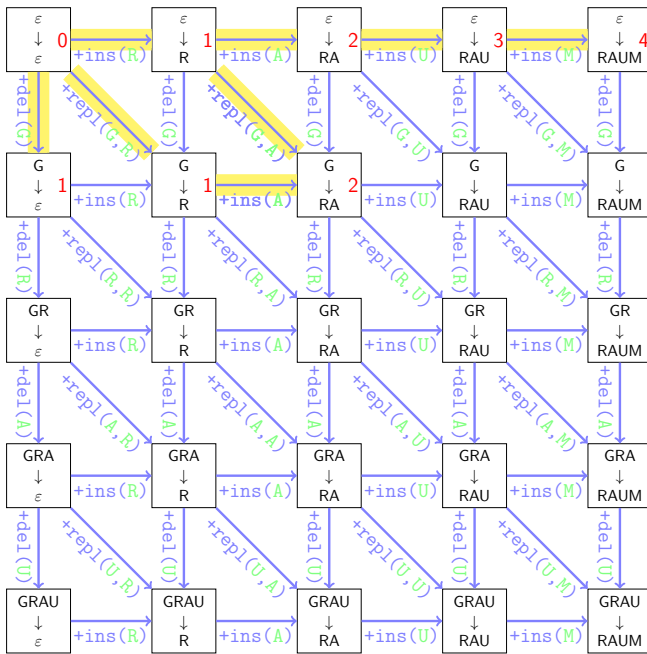
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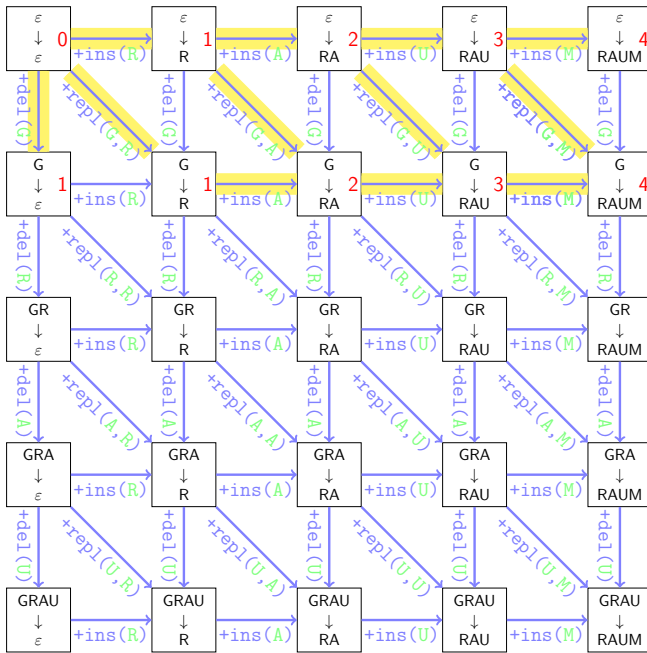
# Edit Distance



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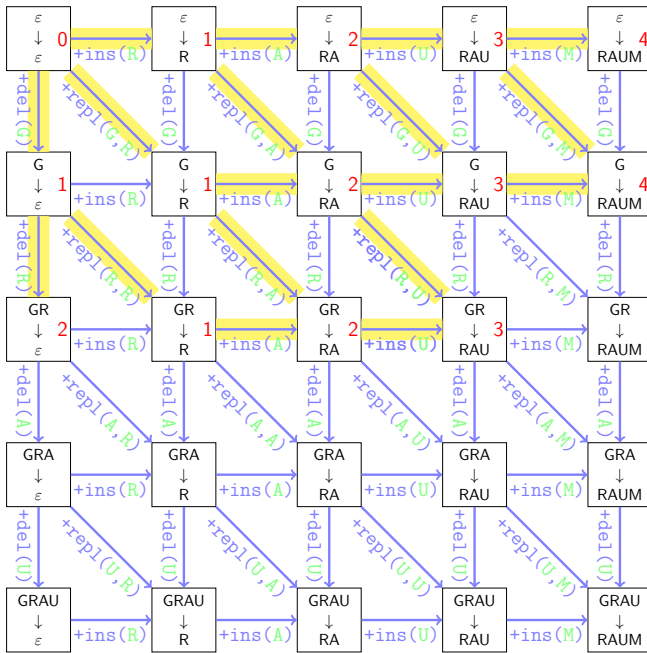
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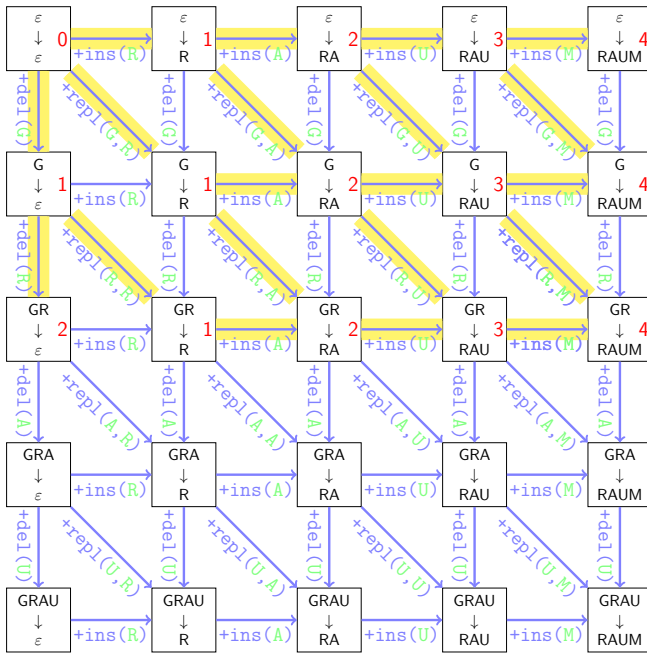
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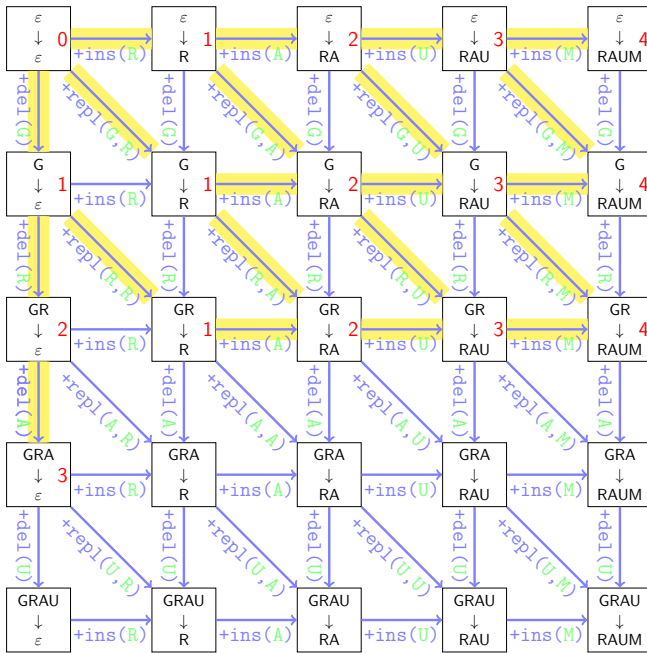
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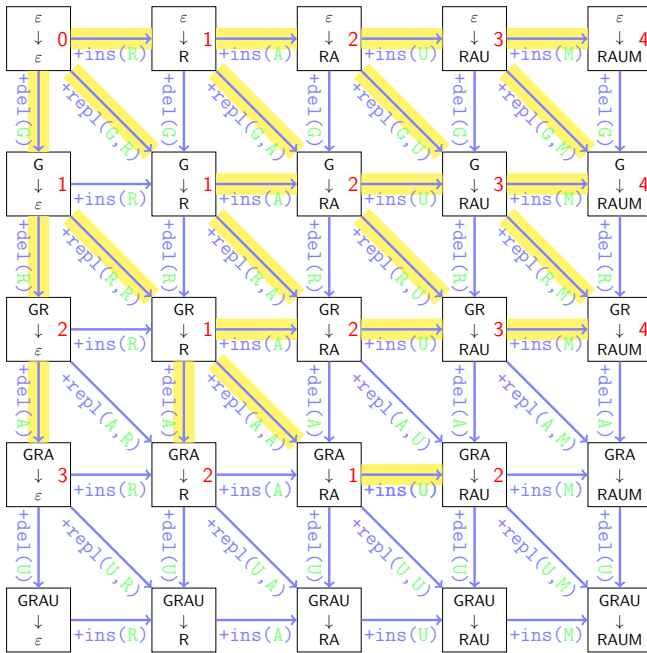


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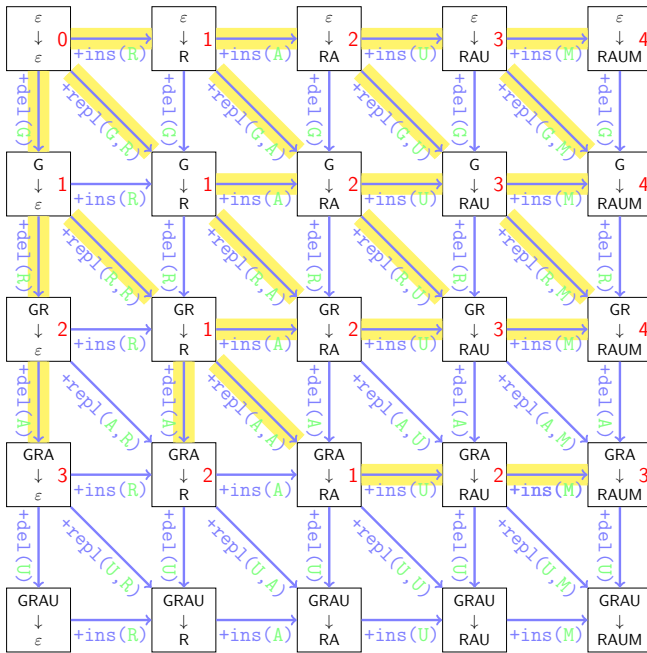




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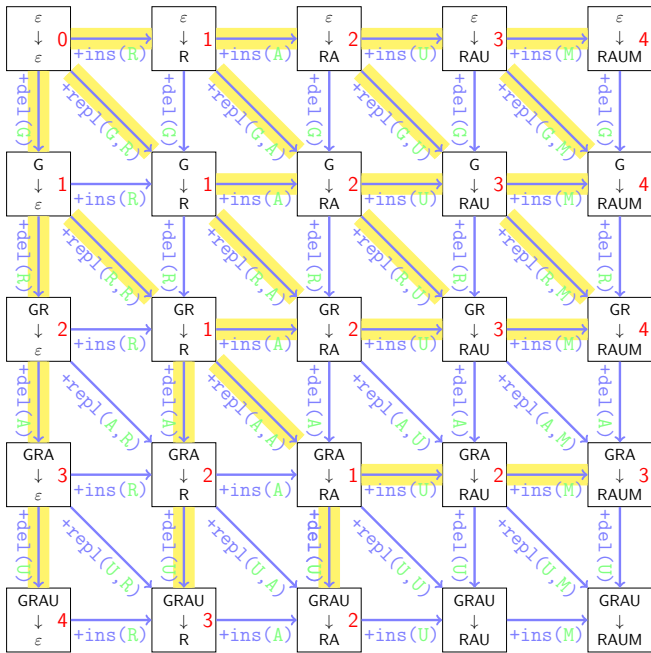
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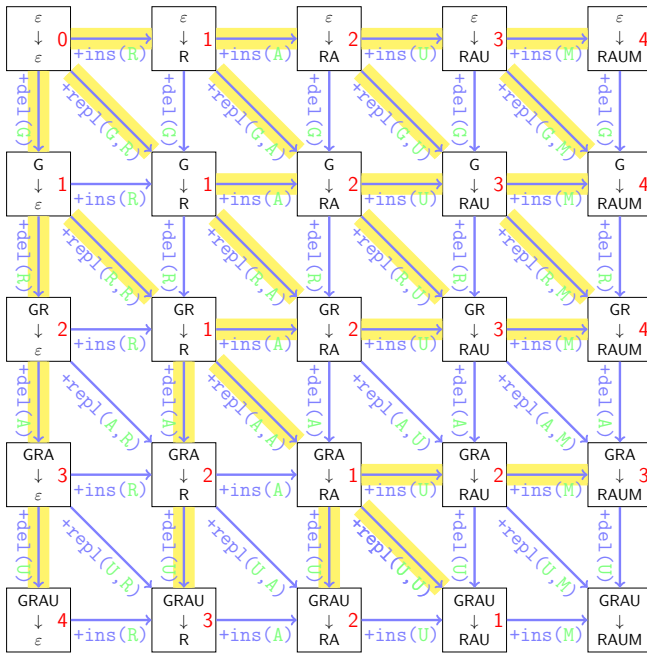
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## Edit distance

**How to get the sequence of operations?**



# Edit distance

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- ▶ We save at each recursion the most efficient previous entry (the **highlighted arrows** in our image)

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- ▶ If we follow the highlighted path from  $(n, m)$  to  $(1, 1)$  we get the optimum operations to transform  $x$  into  $y$ 
  - ▶ If we can follow more than one path there exist more than one ideal sequence

# Edit Distance



# Edit distance

**General principle:**

# Edit distance

## General principle:

- ▶ Recursive computation of ...
  - ... the same reoccurring partial problems
  - ... a limited number of partial problems

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## General principle:

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- ▶ Computation of the solutions for all partial problems
- ▶ In a order that unsolved partial problems consist of already solved partial problems
- ▶ The “path” to our solution normally gets computed while searching the best solution
- ▶ Dijkstra algorithm is basically dynamic programming!

# Edit distance

## Additional applications (I)

**Additional applications:**

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### **Additional applications:**

- ▶ *Edit distance*: global alignment with  $O(n^2)$  space and time consumption

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- ▶ *Edit distance*: global alignment with  $O(n^2)$  space and time consumption
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- ▶ Solution in  $O(n^3)$  time or  $O(n^2)$  affine



# Edit distance

## Additional applications (II)

$O(n^2)$  space consumption might be problematic:

**Hirschberg algorithm:**

# Edit distance

## Additional applications (II)

$O(n^2)$  space consumption might be problematic:

### **Hirschberg algorithm:**

- ▶ Divide-and-conquer approach

# Edit distance

## Additional applications (II)

$O(n^2)$  space consumption might be problematic:

### **Hirschberg algorithm:**

- ▶ Divide-and-conquer approach
- ▶  $O(n)$  space and  $O(n^2)$  time consumption

# Edit distance

## Additional applications (III)



# Edit distance

## Additional applications (III)



- Sequencing:  $O(n^2)$  is too much

# Edit distance

## Additional applications (III)



- ▶ Sequencing:  $O(n^2)$  is too much
- ▶ Index: suffixtree, suffixarray, burrow-wheeler-transform

# Further Literature

## ► General

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# Further Literature

- ▶ **Dynamic programming**

[Wik] [Dynamic programming](#)

https:

[//en.wikipedia.org/wiki/Dynamic\\_programming](https://en.wikipedia.org/wiki/Dynamic_programming)

- ▶ **Edit distance**

[Wik] [Levenshtein distance](#)

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