

Algorithms and Datastructures

Cache Efficiency, Divide and Conquer

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Structure

Cache Efficiency

- Introduction

- Cache Organization

Divide and Conquer

- Introduction

Cache Efficiency

Introduction

Background:

- ▶ Up to now we always counted the **number of operations**
- ▶ Assuming this is a good measure for the runtime of a algorithm/tool
- ▶ Today we will see examples where this is not suitable

Cache Efficiency

Introduction

Example:

- ▶ We sum up all elements of a field a of size n in ...
 - ▶ natural order:

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

- ▶ random order:

$$\text{sum}(a) = a[21] + a[5] + \cdots + a[8]$$

Cache Efficiency

Linear Order - Python

Python:

```
def init(size):  
    """Creates the dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set linear order as accessor  
    order = [a for a in range(0, size)]  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```

Cache Efficiency

Linear Order - Python

Python:

```
def run(param):  
    """Processes the dataset."""  
  
    # unpack data  
    (order, data) = param  
  
    # init the sum value  
    s = 0  
  
    for index in order:  
        s += data[index]  
  
    return s
```

Cache Efficiency

Linear Order



Figure: Summing elements in linear order

Cache Efficiency

Random Order - Python

```
def init(size):  
    """Creates a randomly ordered dataset."""  
  
    # use system time as seed  
    random.seed(None)  
  
    # set random order as accessor  
    order = [a for a in range(0, size)]  
    random.shuffle(order)  
  
    # init array with random data  
    data = [random.random() for a in order]  
  
    return (order, data)
```


Cache Efficiency

Random Order



Figure: Summing elements in random order

Cache Efficiency

Algorithm Comparision

Conclusion:

- ▶ The number of operations are identical for both algorithms
- ▶ Accessing elements in random order takes a lot longer (Factor 10)
- ▶ The costs in terms of memory access are very different

Cache Efficiency

CPU Cache



Principle / organization:

- ▶ Accessing one byte of the main memory takes ≈ 100 ns
- ▶ Accessing one byte of (L1-)cache takes ≈ 1 ns
- ▶ Accessing one or more byte/s of main memory loads a whole block ≈ 100 B into the cache
- ▶ As long as this block is in the cache, it is not necessary to access the memory for bytes of this block

Cache Efficiency

CPU Cache



Cache organization:

- ▶ The (L1-)cache can hold multiple memory blocks
 - ▶ Cache lines ≈ 100 kB
- ▶ If the capacity is reached unused blocks are discarded
 - ▶ Least recently used (LRU)
 - ▶ Least frequently used (LFU)
 - ▶ First in first out (FIFO)
- ▶ Details of discarding are not the topic for today

Cache Efficiency

Block Operations



Terminology:

- ▶ The system consists of slow and fast memory
- ▶ The **slow memory** is divided in **blocks of size B**
- ▶ The **fast cache** has size M and can store M/B blocks
- ▶ If data is not in fast memory, the corresponding block is loaded into the **cache**

Cache Efficiency

Block Operations



Terminology:

- ▶ The program defines which blocks are held in the **cache**
- ▶ We use the number of **block operations** as runtime estimation
- ▶ We ignore runtime costs of cache accesses / management

Cache Efficiency



Figure: Comparison good / bad locality

Accessing the cache B times:

- ▶ **Best case:** 1 block operation \rightarrow good locality
- ▶ **Worst case:** B block operations \rightarrow bad locality

Cache Efficiency

Block Operations

Additional factors:

- ▶ The following settings change only a small constant factor in number of block operations
 - ▶ Partitioning of the slow memory into blocks
 - ▶ Regardless of the block size: 1 Bytes or 4 Bytes or 8 Bytes

Note:

- ▶ If the input size is smaller than M we load the complete data chunk directly into the cache
- ▶ Cache handling is only interesting when the input size is greater than M

Cache Efficiency

Block Operations

Typical values: (Intel® i7-4770 Haswell, WD® Blue 2 TB)

- ▶ CPU L1 Cache: $B = 64 \text{ B}$, $M = 4 \times (32 \text{ kB} + 32 \text{ kB})$
- ▶ CPU L2 Cache: $B = 64 \text{ B}$, $M = 4 \times 256 \text{ kB}$
- ▶ CPU L3 Cache: $B = 64 \text{ B}$, $M = 8 \text{ MB}$
- ▶ Disk Cache: $B = 64 \text{ kB}$, $M = 64 \text{ MB}$
 - ▶ Many operating systems use free system memory as disk cache

Cache Efficiency

Block Operations

Terminology:

- ▶ Block loads on CPU-cache are called **cache misses**
- ▶ Block operations on disk-cache are called **IOs**
(input / output operations)
- ▶ These also fall under the term **cache efficiency** or **IO efficiency**

Cache Efficiency

Block Operations - Linear Order

Example 1 - Linear order:

- We sum up all elements in **natural order**

$$\text{sum}(a) = a[1] + a[2] + \cdots + a[n]$$

- The number of block operations is $\text{ceil}(\frac{n}{B})$



Figure: Good locality of sum operation

Cache Efficiency

Block Operations - Random Order

Example 2 - Random order:

- ▶ We sum up all elements in **random order**

$$\text{sum}(a) = a[21] + a[5] + \dots + a[8]$$

- ▶ The number of block operations is n in the **worst case**
- ▶ This leads to a runtime factor difference of B



Figure: Bad locality of sum operation

Cache Efficiency

Block Operations

Generally the factor is substantially $< B$

- ▶ Even with a **random order** we access 4 neighboring bytes at once per `int` (`int32_t`)
- ▶ The next element might already be loaded in the cache
- ▶ If **not** $n \gg M$ this might occur with a high probability

Cache Efficiency

Block Operations - QuickSort

QuickSort:

- ▶ **Strategy:** Divide and conquer
- ▶ Divide the data into two parts where the “left” part contains all values \leq those in the right part
- ▶ Choose one element (e.g the first one) as “pivot”-element
- ▶ Ideally both parts are the same size
- ▶ Both parts are sorted recursively



Figure: QuickSort with pivot-element

Idea of Quicksort

- ▶ **At start:** Pivot in first position, first re-arrange list such that left part contains small, right part larger elements
- ▶ Do required changes *in place*



- ▶ **End point:** k is left to left-most element greater than pivot
swap position 0 (pivot) with k (smaller than pivot)

Cache Efficiency

Block Operations - QuickSort - Python

Python:

```
def quicksort(l, start, end):  
    if (end - start) < 1:  
        return  
  
    i = start  
    k = end  
    piv = l[0]  
  
    ...
```


Cache Efficiency

Block Operations - QuickSort - Python

```
def quicksort(l, start, end):  
    ...  
  
    while k > i:  
        while l[i] <= piv and i <= end and k > i:  
            i += 1  
        while l[k] > piv and k >= start and k >= i:  
            k -= 1  
  
        if k > i: # swap elements  
            (l[i], l[k]) = (l[k], l[i])  
  
    (l[start], l[k]) = (l[k], l[start])  
    quicksort(l, start, k - 1)  
    quicksort(l, k + 1, end)
```

Cache Efficiency

Block Operations - QuickSort

Number of operations for Quicksort:

- ▶ Let $T(n)$ be the runtime for the input size n

Assumptions:

- ▶ Fields are always separated perfectly in the middle
- ▶ n is a power of two and recursion depth is $k = \log_2 n$

Cache Efficiency

Block Operations - QuickSort

$$\begin{aligned} T(n) &\leq \underbrace{A \cdot n}_{\text{splitting in two parts}} + \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{recursive sort}} \\ &\leq A \cdot n + 2 \left(A \cdot \frac{n}{2} + 2 \cdot T\left(\frac{n}{4}\right) \right) \\ &= 2A \cdot n + 4 \cdot T\left(\frac{n}{4}\right) \\ &\leq 3A \cdot n + 8 \cdot T\left(\frac{n}{8}\right) \\ &\leq \dots \\ &\leq k \cdot A \cdot n + 2^k \cdot T(1) \\ &= \log_2 n \cdot A \cdot n + n \cdot T(1) \\ &\leq \log_2 n \cdot A \cdot n + n \cdot A \in \mathcal{O}(n \log_2 n) \end{aligned}$$

Cache Efficiency

Block Operations - QuickSort



Figure: Locality of quicksort

- ▶ Let $IO(n)$ be the number of **block operations** for input size n
- ▶ Assumptions as before but recursion depth is $k = \log_2 \frac{n}{B}$

Cache Efficiency

Block Operations - QuickSort

$$\begin{aligned} IO(n) &\leq \underbrace{A \cdot n/B}_{\text{splitting in two parts}} + \underbrace{2 \cdot IO(n/2)}_{\text{recursive sort}} \\ &\leq A \cdot n/B + 2(A \cdot n/2B + 2 \cdot IO(n/4)) \\ &\leq 2 \cdot A \cdot n/B + 4 \cdot IO(n/4) \\ &\leq 3 \cdot A \cdot n/B + 8 \cdot IO(n/8) \\ &\leq \dots \\ &\leq k \cdot A \cdot n/B + 2^k \cdot IO(n/2^k) \\ &= \log_2(n/B) \cdot A \cdot (n/B) + n/B \cdot IO(B) \\ &\leq \log_2(n/B) \cdot A \cdot (n/B) + A \cdot n/B \in \mathcal{O}\left(\frac{n}{B} \cdot \log_2\left(\frac{n}{B}\right)\right) \end{aligned}$$

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all solutions of the subproblems to a solution of the full problem
- ▶ **Recursive** application of the algorithm to ever smaller subproblems
- ▶ **Direct** solving of sufficiently small subproblems

Divide and Conquer

Introduction - Python

- Function `solve` for solving a `problem` of size `n`

```
def solve(problem):  
    if n < threshold:  
        return solution # solve directly  
    else:  
        # divide problem into subproblems  
        # P1, P2, ..., Pk with k>=2  
        S1 = solve(P1)  
        S2 = solve(P2)  
        ...  
        Sk = solve(Pk)  
  
        # combine solutions  
        return S1 + S2 + ... + Sk
```

Divide and Conquer

Features

Divide and Conquer:

- ▶ Can help with conceptual hard problems
- ▶ **Solution** of the trivial problems has to be known
- ▶ **Dividing** in subproblems has to be possible
- ▶ **Combination** of solutions has to be possible

Divide and Conquer

Features

Features:

- ▶ Realization of **efficient solutions**
 - ▶ If trivial solution is $\in O(1)$
 - ▶ And separation / combination of subproblems is $\in O(n)$
 - ▶ And the number of subproblems is limited
 - ▶ The runtime is $\in O(n \cdot \log n)$
- ▶ Suitable for parallel processing
 - ▶ Subproblems are **independent** of each other
 - ▶ Only needed input for each subproblem has to be known

Divide and Conquer

Implementation

Definition of the trivial case:

- ▶ Smaller subproblems are elegant and simple
- ▶ Otherwise the efficiency will be improved if relative big subproblems can be solved directly
- ▶ Recursion depth should not get too big (stack / memory overhead)

Divide and Conquer

Implementation

Division in subproblems:

- ▶ Choosing the number of subproblems and the concrete allocation can be demanding

Combination of solutions:

- ▶ Typically conceptual demanding

Divide and Conquer

Example - Maximum Subtotal

Example - Maximum Subtotal Input:

- ▶ Sequence X of n integers

Output:

- ▶ Maximum sum of related subsequence and its index boundary

Index	0	1	2	3	4	5	6	7	8	9
Value	31	-41	59	26	-53	58	97	-93	-23	84

Output: Sum: 187, Start: 2, End: 6

Divide and Conquer

Example - Maximum Subtotal

Application:

- ▶ Maximum profit of buying and selling shares



Figure: Stock value over time

Divide and Conquer

Example - Maximum Subtotal - Python

Naive solution (brute force)

```
def maxSubArray(X):  
    # Store sum, start, end  
    result = (X[0], 0, 0)  
    for i in range(0, len(X)):  
        for j in range(i, len(X)):  
            subSum = 0  
            for k in range(i, j + 1):  
                subSum += X[k]  
            if result[0] < subSum:  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal - Python

Runtime - Upper bound

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            # max n loops -> O(n)  
            subSum = sum(X[i:j+1])  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

Divide and Conquer

Example - Maximum Subtotal

Upper bound:

- ▶ Three interleaved loops
- ▶ Each loop with runtime $O(n)$
- ▶ Algorithm runtime of $O(n^3)$

Divide and Conquer

Example - Maximum Subtotal - Runtime

Lower bound:

Table: Operations

i	Additions	j
$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$	$\frac{n}{3} \in O(n)$

- ▶ We iterate at least $\frac{n}{3}$ values for i
- ▶ For each i we iterate at least $\frac{n}{3}$ values for j
- ▶ For each j we have at least $\frac{n}{3}$ additions
- ▶ We need at least $T(n) = (\frac{n}{3})^3 \in \Omega(n^3)$ steps

Divide and Conquer

Example - Maximum Subtotal - Runtime

Runtime:

- ▶ With $T(n) \in O(n^3)$ and $T(n) \in \Omega(n^3)$ we know:

$$T(n) \in \Theta(n^3)$$

- ▶ It is hard to solve the problem in a worse way ...

Divide and Conquer

Example - Maximum Subtotal - Runtime

Current approach:

- ▶ Calculating the sum for range from i to j with loop

$$S_{i,j} = X[i] + X[i + 1] + \cdots + X[j]$$

Better approach:

- ▶ Incremental sum instead of loop

$$S_{i,j+1} = X[i] + X[i + 1] + \cdots + X[j] + X[j + 1]$$

$$S_{i,j+1} = S_{i,j} + X[j + 1] \in O(1) \quad \text{instead of} \quad \in O(n)$$

Divide and Conquer

Example - Maximum Subtotal - Python

Better solution:

```
def maxSubArray(X):  
    result = (X[0], 0, 0)  
    # n loops -> O(n)  
    for i in range(0, len(X)):  
        subSum = 0  
        # max n loops -> O(n)  
        for j in range(i, len(X)):  
            subSum += X[j] # O(1)  
            if result[0] < subSum: # O(1)  
                result = (subSum, i, j)  
    return result
```

► Runtime $\in O(n^2)$

Divide and Conquer

Example - Maximum Subtotal

Divide and Conquer:



Divide and Conquer Idea to solve:

- ▶ Split the sequence in the middle
- ▶ Solve left half of the problem
- ▶ Solve right half and combine both solutions into one
- ▶ Maximum might be located in left half (*A*) or right half (*B*)
- ▶ Problem: Maximum can overlap the split
- ▶ To solve this case we have to calculate *rmax* and *lmax*
- ▶ The overall solution is the maximum of *A*, *B* and *C*

Divide and Conquer

Example - Maximum Subtotal

Principle - Divide and Conquer:

- ▶ Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- ▶ Bigger problems are partitioned into two subproblems and recursively solved. Subsolutions A and B are returned
- ▶ To determine subsolution C , rmax and lmax for the subproblems are computed
- ▶ The overall solution is the maximum of A , B and C

Divide and Conquer

Example - Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

Further Literature

► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

- ▶ **Caching**

[Wik] [Cache](https://en.wikipedia.org/wiki/Cache)

`https://en.wikipedia.org/wiki/Cache`