

Algorithms and Datastructures

Runtime analysis Minsort / Heapsort, Induction

Prof. Dr. Rolf Backofen

Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, October 2017

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

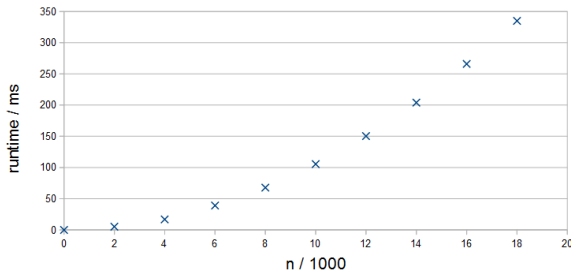
Logarithms

Runtime analysis - Minsort



How long does the program run?

Runtime analysis - Minsort



How long does the program run?

- ▶ In the last lecture we had a schematic
- ▶ **Observation:** It is going to be “disproportionally” slower the more numbers are being sorted

Runtime analysis - Minsort



How long does the program run?

- ▶ In the last lecture we had a schematic
- ▶ **Observation:** It is going to be “disproportionally” slower the more numbers are being sorted
- ▶ How can we say more precisely what is happening?

Runtime analysis - Minsort

How can we analyze the runtime?

- ▶ Ideally we have a formula which provides the runtime of the program for an specific input

Runtime analysis - Minsort

How can we analyze the runtime?

- ▶ Ideally we have a formula which provides the runtime of the program for an specific input
- ▶ **Problem:** The runtime is also depending on many other influences, especially:
 - ▶ Which kind of computer is the code executed on
 - ▶ What is running in the background
 - ▶ Which compiler is used to compile the code

Runtime analysis - Minsort

How can we analyze the runtime?

- ▶ Ideally we have a formula which provides the runtime of the program for an specific input
- ▶ **Problem:** The runtime is also depending on many other influences, especially:
 - ▶ Which kind of computer is the code executed on
 - ▶ What is running in the background
 - ▶ Which compiler is used to compile the code
- ▶ **Abstraction 1:** Analyze the number of basic operations, rather than analyzing the runtime

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Basic Operations

Incomplete list of basic operations:

- ▶ Arithmetic operation, for example: $a + b$
- ▶ Assignment of variables, for example: $x = y$
- ▶ Function call, for example: *minsort(lst)*

Basic Operations

Intuitive:

lines of code

Better:

lines of machine
code

Best:

process cycles

Important:

The actual runtime has to be roughly proportional to the number of operations.

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Runtime analysis - Minsort

How many operations does *Minsort* need?

- ▶ **Abstraction 2:** We calculate the upper (lower) bound, rather than counting the operations exactly

Reason: Runtime is approximated by number of basic operations, but we can still infer:

- ▶ Upper bound
- ▶ Lower bound

Runtime analysis - Minsort

How many operations does *Minsort* need?

- ▶ **Abstraction 2:** We calculate the upper (lower) bound, rather than counting the operations exactly

Reason: Runtime is approximated by number of basic operations, but we can still infer:

- ▶ Upper bound
 - ▶ Lower bound
-
- ▶ **Basic Assumption:**
 - ▶ n is size of the input data (i.e. array)
 - ▶ $T(n)$ number of operations for input n

Runtime analysis - Minsort

How many operations does *Minsort* need?

- ▶ **Observation:** The number of operations depends only on the size n of the array and not on the content!

Runtime analysis - Minsort

How many operations does *Minsort* need?

- ▶ **Observation:** The number of operations depends only on the size n of the array and not on the content!
- ▶ **Claim:** There are constants C_1 and C_2 such that:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

Runtime analysis - Minsort

How many operations does *Minsort* need?

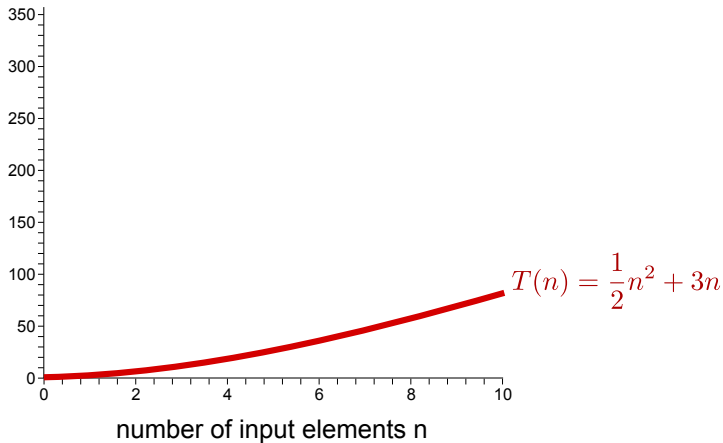
- ▶ **Observation:** The number of operations depends only on the size n of the array and not on the content!
- ▶ **Claim:** There are constants C_1 and C_2 such that:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

- ▶ This is called “quadratic runtime” (due to n^2)

Runtime Example

Number of Steps
(runtime [s])



Runtime Example

Number of Steps
(runtime [s])



Runtime Example

Number of Steps
(runtime [s])



$C_2 = 7/2$ could have
been larger or small
(exact value not
relevant)

$$T(n) = \frac{1}{2}n^2 + 3n$$

$$\frac{1}{2} \cdot n^2$$

$C_1 = 1/2$ could have been
chosen smaller (not
relevant), but not larger

number of input elements n

Runtime analysis - Minsort

We declare:

- ▶ Runtime of operations: $T(n)$
- ▶ Number of Elements: n
- ▶ Constants: C_1 (lower bound), C_2 (upper bound)
$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$
- ▶ Number of operations in round i : T_i



Figure: *Minsort* at the iteration $i = 4$. We have to check $n - 3$ elements

Runtime analysis - Minsort

Compares at each iteration:



Figure: *Minsort* with start data

Runtime analysis - Minsort

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$



Figure: *Minsort* at iteration $i = 1$

Runtime analysis - Minsort

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

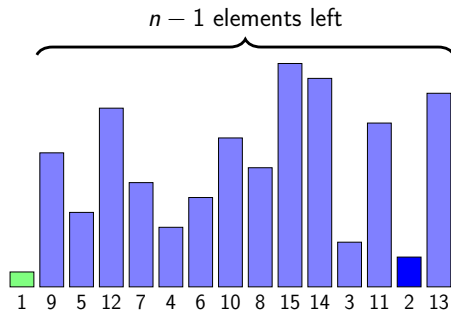


Figure: *Minsort* at iteration $i = 2$

Runtime analysis - Minsort

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

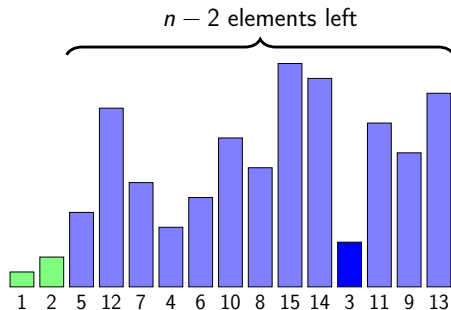


Figure: *Minsort* at iteration $i = 3$

Runtime analysis - Minsort

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

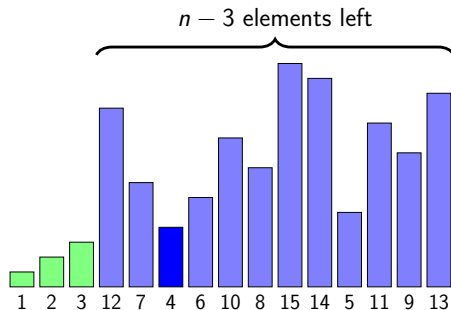


Figure: *Minsort* at iteration $i = 4$

Runtime analysis - Minsort

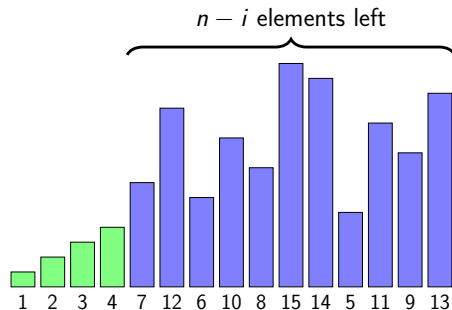


Figure: Minsort at iteration i

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

\vdots

$$T_{n-1} \leq C'_2 \cdot 2$$

$$T_n \leq C'_2 \cdot 1$$

Runtime analysis - Minsort



Figure: Minsort at iteration

Compares at each iteration:

$$T_1 \leq C'_2 \cdot (n - 0)$$

$$T_2 \leq C'_2 \cdot (n - 1)$$

$$T_3 \leq C'_2 \cdot (n - 2)$$

$$T_4 \leq C'_2 \cdot (n - 3)$$

\vdots

$$T_{n-1} \leq C'_2 \cdot 2$$

$$T_n \leq C'_2 \cdot 1$$

$$T(n) = C'_2 \cdot (T_1 + \cdots + T_n) \leq \sum_{i=1}^n (C'_2 \cdot i)$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j  
  
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
  
    return elements
```

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j  
        } const.  
        } runtime  
  
    if minimum != i:  
        elements[i], elements[minimum] = \  
            elements[minimum], elements[i]  
  
    return elements
```

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):  
    for i in range(0, len(elements)-1):  
        minimum = i  
  
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j  
  
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]  
  
    return elements
```

} const. runtime } n-i-1 times

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
```

```
    for i in range(0, len(elements)-1):  
        minimum = i
```

```
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j
```

```
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]
```

```
    return elements
```

} const.
runtime } n-i-1
times } n-1
times

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
```

```
    for i in range(0, len(elements)-1):  
        minimum = i
```

```
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j
```

```
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]
```

```
    return elements
```

} const.
runtime } n-i-1
times } n-1
times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C_2'$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
```

```
    for i in range(0, len(elements)-1):  
        minimum = i
```

```
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j
```

```
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]
```

```
    return elements
```

} const. runtime } n-i-1 times } n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
```

```
    for i in range(0, len(elements)-1):  
        minimum = i
```

```
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j
```

```
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]
```

```
    return elements
```

} const.
runtime } n-i-1
times } n-1
times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2 = \sum_{i=1}^{n-1} (n-i) \cdot C'_2$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
```

```
    for i in range(0, len(elements)-1):  
        minimum = i
```

```
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j
```

```
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]
```

```
    return elements
```

} const. runtime } n-i-1 times } n-1 times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2 = \sum_{i=1}^{n-1} (n-i) \cdot C'_2 \leq \sum_{i=1}^n i \cdot C'_2$$

Runtime analysis - Minsort

Alternative: Analyse the Code:

```
def minsort(elements):
```

```
    for i in range(0, len(elements)-1):  
        minimum = i
```

```
        for j in range(i+1, len(elements)):  
            if elements[j] < elements[minimum]:  
                minimum = j
```

```
        if minimum != i:  
            elements[i], elements[minimum] = \  
                elements[minimum], elements[i]
```

```
    return elements
```

} const.
runtime } n-i-1
times } n-1
times

$$T(n) \leq \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} C'_2 = \sum_{i=0}^{n-2} (n-i-1) \cdot C'_2 = \sum_{i=1}^{n-1} (n-i) \cdot C'_2 \leq \sum_{i=1}^n i \cdot C'_2$$

Remark: C'_2 is cost of comparison \Rightarrow assumed constant

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$T(n) \leq \sum_{i=1}^n C'_2 \cdot i$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \end{aligned}$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \\ &\quad \downarrow \text{Small Gauss sum} \\ &= C'_2 \cdot \frac{n(n+1)}{2} \end{aligned}$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \\ &\quad \downarrow \text{Small Gauss sum} \\ &= C'_2 \cdot \frac{n(n+1)}{2} \\ &\leq C'_2 \cdot \frac{n(n+n)}{2}, \quad 1 \leq n \end{aligned}$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \\ &\quad \downarrow \text{Small Gauss sum} \\ &= C'_2 \cdot \frac{n(n+1)}{2} \\ &\leq C'_2 \cdot \frac{n(n+n)}{2}, \quad 1 \leq n \\ &= C'_2 \cdot \frac{2 \cdot n^2}{2} \end{aligned}$$

Runtime analysis - Minsort

Proof of upper bound: $T(n) \leq C_2 \cdot n^2$

$$\begin{aligned} T(n) &\leq \sum_{i=1}^n C'_2 \cdot i \\ &= C'_2 \cdot \sum_{i=1}^n i \\ &\quad \downarrow \text{Small Gauss sum} \\ &= C'_2 \cdot \frac{n(n+1)}{2} \\ &\leq C'_2 \cdot \frac{n(n+n)}{2}, \quad 1 \leq n \\ &= C'_2 \cdot \frac{2 \cdot n^2}{2} = C'_2 \cdot n^2 \end{aligned}$$

Excursion - Small Gauss Formula

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1 . Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C'_1 \cdot (n - i)$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1 . Summation analysis is the same

$$T(n) \geq \sum_{i=1}^{n-1} C'_1 \cdot (n - i) = C'_1 \sum_{i=1}^{n-1} i$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1 . Summation analysis is the same

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n - i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \end{aligned}$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1 . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n - i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \end{aligned}$$

How do we get to n^2 ?

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1 . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n - i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \quad \text{How do we get to } n^2? \\ &\quad \Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \end{aligned}$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1 . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n - i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \quad \text{How do we get to } n^2? \\ &\quad \Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \\ &\geq C'_1 \cdot \frac{n \cdot n}{2 \cdot 2} \end{aligned}$$

Runtime analysis - Minsort

Proof of lower bound: $C_1 \cdot n^2 \leq T(n)$

Like for the upper boundary there exists a C_1 . Summation analysis is the same, **only final approximation differs**

$$\begin{aligned} T(n) &\geq \sum_{i=1}^{n-1} C'_1 \cdot (n - i) = C'_1 \sum_{i=1}^{n-1} i \\ &\geq C'_1 \cdot \frac{(n-1) \cdot n}{2} \quad \text{How do we get to } n^2? \\ &\quad \Downarrow \quad n-1 \geq \frac{n}{2} \text{ for } n \geq 2 \\ &\geq C'_1 \cdot \frac{n \cdot n}{2 \cdot 2} = \frac{C'_1}{4} \cdot n^2 \end{aligned}$$

Runtime analysis - Minsort

Runtime Analysis:

- ▶ Upper bound: $T(n) \leq C'_2 \cdot n^2$

Runtime analysis - Minsort

Runtime Analysis:

- ▶ Upper bound: $T(n) \leq C'_2 \cdot n^2$
- ▶ Lower bound: $\frac{C'_1}{4} \cdot n^2 \leq T(n)$

Runtime analysis - Minsort

Runtime Analysis:

- ▶ Upper bound: $T(n) \leq C'_2 \cdot n^2$
- ▶ Lower bound: $\frac{C'_1}{4} \cdot n^2 \leq T(n)$

Summarized:

$$\frac{C'_1}{4} \cdot n^2 \leq T(n) \leq C'_2 \cdot n^2$$

Quadratic runtime proven:

$$C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$$

Runtime Example

- ▶ The runtime is growing quadratic with the number of elements n in the list

Runtime Example

- ▶ The runtime is growing quadratic with the number of elements n in the list
- ▶ Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$

Runtime Example

- ▶ The runtime is growing quadratic with the number of elements n in the list
- ▶ Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- ▶ $2 \times$ elements $\Rightarrow 4 \times$ runtime

Runtime Example

- ▶ The runtime is growing quadratic with the number of elements n in the list
- ▶ Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- ▶ $2 \times$ elements $\Rightarrow 4 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)

Runtime Example

- ▶ The runtime is growing quadratic with the number of elements n in the list
- ▶ Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- ▶ $2 \times$ elements $\Rightarrow 4 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)
 - ▶ $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
 - ▶ $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{12} = 10^3 \text{ s} = 16.7 \text{ min}$

Runtime Example

- ▶ The runtime is growing quadratic with the number of elements n in the list
- ▶ Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- ▶ $2 \times$ elements $\Rightarrow 4 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)
 - ▶ $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
 - ▶ $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{12} = 10^3 \text{ s} = 16.7 \text{ min}$
 - ▶ $n = 10^9$ (1 billion numbers = 4 GB)
 - ▶ $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{18} = 10^9 \text{ s} = 31.7 \text{ years}$

Runtime Example

- ▶ The runtime is growing quadratic with the number of elements n in the list
- ▶ Let constants C_1 and C_2 for which $C_1 \cdot n^2 \leq T(n) \leq C_2 \cdot n^2$
- ▶ $2 \times$ elements $\Rightarrow 4 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)
 - ▶ $n = 10^6$ (1 million numbers = 4 MB with 4 B/number)
 - ▶ $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{12} = 10^3 \text{ s} = 16.7 \text{ min}$
 - ▶ $n = 10^9$ (1 billion numbers = 4 GB)
 - ▶ $C \cdot n^2 = 10^{-9} \text{ s} \cdot 10^{18} = 10^9 \text{ s} = 31.7 \text{ years}$
- ▶ **Quadratic runtime = “big” problems unsolvable**

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Runtime - Heapsort

Intuitive to extract minimum:

- ▶ **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.

Runtime - Heapsort

Intuitive to extract minimum:

- ▶ **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.
- ▶ **Heapsort:** The root node is always the smallest (minheap). We only need to repair a part of the full tree after delete operation.

Runtime - Heapsort

Intuitive to extract minimum:

- ▶ **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.
- ▶ **Heapsort:** The root node is always the smallest (minheap). We only need to repair a part of the full tree after delete operation.

Formal:

Runtime - Heapsort

Intuitive to extract minimum:

- ▶ **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.
- ▶ **Heapsort:** The root node is always the smallest (minheap). We only need to repair a part of the full tree after delete operation.

Formal:

- ▶ Let $T(n)$ be the runtime for the *Heapsort* algorithm with n elements

Runtime - Heapsort

Intuitive to extract minimum:

- ▶ **Minsort:** To determine the minimum value we have to iterate through all the unsorted elements.
- ▶ **Heapsort:** The root node is always the smallest (minheap). We only need to repair a part of the full tree after delete operation.

Formal:

- ▶ Let $T(n)$ be the runtime for the *Heapsort* algorithm with n elements
- ▶ On the next pages we will proof $T(n) \leq C \cdot n \log_2 n$

Runtime - Heapsort

Depth of a binary tree:

- ▶ **Depth d :** longest path through the tree
- ▶ Complete binary tree has $n = 2^d - 1$ nodes
- ▶ Example: $d = 4$
 $\Rightarrow n = 2^4 - 1 = 15$

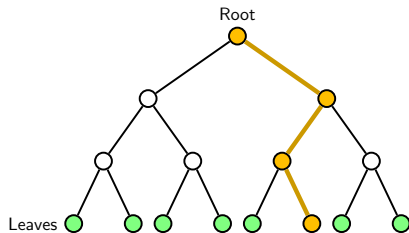


Figure: Binary tree with 15 nodes

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Induction

Basics:

Induction

Basics:

- ▶ You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$

Induction

Basics:

- ▶ You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- ▶ We show induction in two steps:

Induction

Basics:

- ▶ You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- ▶ We show induction in two steps:
 1. **Induction basis:** we show that our assumption is valid at one point (for example: $n = 1, A(1)$).

Induction

Basics:

- ▶ You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- ▶ We show induction in two steps:
 1. **Induction basis:** we show that our assumption is valid at one point (for example: $n = 1, A(1)$).
 2. **Induction step:** we show that the assumption is valid for all n (normally one step forward: $n = n + 1, A(1), \dots, A(n)$).

Induction

Basics:

- ▶ You want to show assumption $A(n)$ is valid $\forall n \in \mathbb{N}$
- ▶ We show induction in two steps:
 1. **Induction basis:** we show that our assumption is valid at one point (for example: $n = 1, A(1)$).
 2. **Induction step:** we show that the assumption is valid for all n (normally one step forward: $n = n + 1, A(1), \dots, A(n)$).
- ▶ If both has been proven, then $A(n)$ holds for all natural numbers n by **induction**

Induction - Example 1

Claim:

A **complete** binary tree of depth d has $n(d) = 2^d - 1$ nodes

Induction - Example 1

Claim:

A **complete** binary tree of depth d has $n(d) = 2^d - 1$ nodes

- **Induction basis:** Assumption holds for $d = 1$

Root



$$n(1) = 2^1 - 1 = 1$$

Figure: Tree of depth 1 has 1 node

Induction - Example 1

Claim:

A **complete** binary tree of depth d has $n(d) = 2^d - 1$ nodes

- **Induction basis:** Assumption holds for $d = 1$

Root



$$n(1) = 2^1 - 1 = 1$$

\Rightarrow correct ✓

Figure: Tree of depth 1 has 1 node

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$
- ▶ **Induction basis:** $n(1) = 2^1 - 1 = 2^1 - 1 = 1$ ✓

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$
- ▶ **Induction basis:** $n(1) = 2^1 - 1 = 2^1 - 1 = 1$ ✓
- ▶ **Induction step:** to show for $d := d + 1$



Figure: Binary tree with subtrees

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$
- ▶ **Induction basis:** $n(1) = 2^1 - 1 = 2^1 - 1 = 1$ ✓
- ▶ **Induction step:** to show for $d := d + 1$



$$n(d+1) = 2 \cdot n(d) + 1$$

Figure: Binary tree with subtrees

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$
- ▶ **Induction basis:** $n(1) = 2^1 - 1 = 2^1 - 1 = 1$ ✓
- ▶ **Induction step:** to show for $d := d + 1$



Figure: Binary tree with subtrees

$$\begin{aligned}n(d+1) &= 2 \cdot n(d) + 1 \\&= 2 \cdot (2^d - 1) + 1\end{aligned}$$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$
- ▶ **Induction basis:** $n(1) = 2^1 - 1 = 2^1 - 1 = 1$ ✓
- ▶ **Induction step:** to show for $d := d + 1$

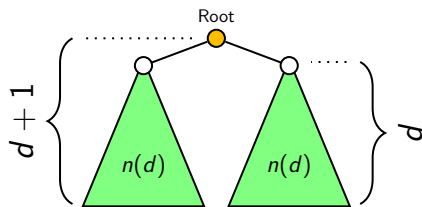


Figure: Binary tree with subtrees

$$\begin{aligned}n(d+1) &= 2 \cdot n(d) + 1 \\&= 2 \cdot (2^d - 1) + 1 \\&= 2^{d+1} - 2 + 1\end{aligned}$$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$
- ▶ **Induction basis:** $n(1) = 2^1 - 1 = 2^1 - 1 = 1$ ✓
- ▶ **Induction step:** to show for $d := d + 1$



Figure: Binary tree with subtrees

$$\begin{aligned}n(d+1) &= 2 \cdot n(d) + 1 \\&= 2 \cdot (2^d - 1) + 1 \\&= 2^{d+1} - 2 + 1 \\&= 2^{d+1} - 1 \quad \checkmark\end{aligned}$$

Induction - Example 1

Number of nodes $n(d)$ in a binary tree with depth d :

- ▶ **Induction assumption:** $n(d) = 2^d - 1$
- ▶ **Induction basis:** $n(1) = 2^1 - 1 = 2^1 - 1 = 1$ ✓
- ▶ **Induction step:** to show for $d := d + 1$



Figure: Binary tree with subtrees

⇒ **By induction:**

$$\begin{aligned}n(d+1) &= 2 \cdot n(d) + 1 \\&= 2 \cdot (2^d - 1) + 1 \\&= 2^{d+1} - 2 + 1 \\&= 2^{d+1} - 1 \quad \checkmark\end{aligned}$$

$$n(d) = 2^d - 1 \quad \forall n \in \mathbb{N} \quad \square$$

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Runtime - Heapsort

Heapsort has the following steps:

- ▶ **Initially:** heapify list of n elements

Runtime - Heapsort

Heapsort has the following steps:

- ▶ **Initially:** heapify list of n elements
- ▶ **Then:** until all n elements are sorted

Runtime - Heapsort

Heapsort has the following steps:

- ▶ **Initially:** heapify list of n elements
- ▶ **Then:** until all n elements are sorted
 - ▶ Remove root as minimal element

Runtime - Heapsort

Heapsort has the following steps:

- ▶ **Initially:** heapify list of n elements
- ▶ **Then:** until all n elements are sorted
 - ▶ Remove root as minimal element
 - ▶ Move last leaf to root position

Runtime - Heapsort

Heapsort has the following steps:

- ▶ **Initially:** heapify list of n elements
- ▶ **Then:** until all n elements are sorted
 - ▶ Remove root as minimal element
 - ▶ Move last leaf to root position
 - ▶ Repair heap by sifting

Runtime - Heapsort

Heapify

Runtime of heapify depends on depth d :



Runtime - Heapsort

Heapify

Runtime of heapify depends on depth d :



Runtime of heapify with depth of d :

- No costs at depth d with 2^{d-1} (or less) nodes

Runtime - Heapsort

Heapify

Runtime of heapify depends on depth d :



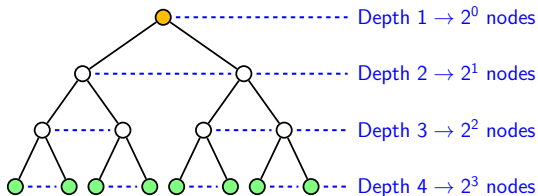
Runtime of heapify with depth of d :

- ▶ No costs at depth d with 2^{d-1} (or less) nodes
- ▶ The cost for sifting with depth 1 is at most $1C$ per node

Runtime - Heapsort

Heapify

Runtime of heapify depends on depth d :



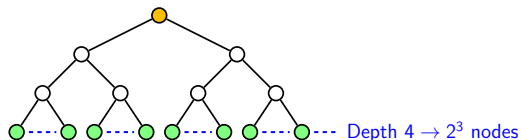
Runtime of heapify with depth of d :

- ▶ No costs at depth d with 2^{d-1} (or less) nodes
- ▶ The cost for sifting with depth 1 is at most $1C$ per node
- ▶ In general: Sifting costs are linear with path length **and** number of nodes

Runtime - Heapsort

Heapify

Heapify total runtime:



Depth	Nodes	Path length	Costs per node
d	2^{d-1}	0	$\leq C \cdot 0$

Runtime - Heapsort

Heapify

Heapify total runtime:



Depth	Nodes	Path length	Costs per node
d	2^{d-1}	0	$\leq C \cdot 0$
$d - 1$	2^{d-2}	1	$\leq C \cdot 1$

Runtime - Heapsort

Heapify

Heapify total runtime:



Depth	Nodes	Path length	Costs per node
d	2^{d-1}	0	$\leq C \cdot 0$
$d - 1$	2^{d-2}	1	$\leq C \cdot 1$
$d - 2$	2^{d-3}	2	$\leq C \cdot 2$

Runtime - Heapsort

Heapify

Heapify total runtime:



Generally: Depth $d \rightarrow 2^{d-1}$ nodes

Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d - 1$	2^{d-2}	1	$\leq C \cdot 1$	
$d - 2$	2^{d-3}	2	$\leq C \cdot 2$	
$d - 3$	2^{d-4}	3	$\leq C \cdot 3$	

Runtime - Heapsort

Heapify

Heapify total runtime:



Generally: Depth $d \rightarrow 2^{d-1}$ nodes

Depth	Nodes	Path length	Costs per node	
d	2^{d-1}	0	$\leq C \cdot 0$	
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

In total:
$$T(d) \leq \sum_{i=1}^d \left(C \cdot (i-1) \cdot 2^{d-i} \right)$$

Runtime - Heapsort

Heapify

Heapify total runtime:



Generally: Depth $d \rightarrow 2^{d-1}$ nodes

Depth	Nodes	Path length	Costs per node	Upper bound
d	2^{d-1}	0	$\leq C \cdot 0$	Standard Equation
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	

In total:
$$T(d) \leq \sum_{i=1}^d \left(C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^d \left(C \cdot i \cdot 2^{d-i} \right)$$

Runtime - Heapsort

Heapify

Heapify total runtime:



Generally: Depth $d \rightarrow 2^{d-1}$ nodes

Depth	Nodes	Path length	Costs per node	Upper bound
d	2^{d-1}	0	$\leq C \cdot 0$	$\leq C \cdot 1$
$d-1$	2^{d-2}	1	$\leq C \cdot 1$	$\leq C \cdot 2$
$d-2$	2^{d-3}	2	$\leq C \cdot 2$	$\leq C \cdot 3$
$d-3$	2^{d-4}	3	$\leq C \cdot 3$	$\leq C \cdot 4$

In total:
$$T(d) \leq \sum_{i=1}^d \left(C \cdot (i-1) \cdot 2^{d-i} \right) \leq \sum_{i=1}^d \left(C \cdot i \cdot 2^{d-i} \right)$$

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot \sum_{i=1}^d \left(i \cdot 2^{d-i} \right) \leq C \cdot 2^{d+1}$$

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot \sum_{i=1}^d \left(i \cdot 2^{d-i} \right) \leq C \cdot 2^{d+1}$$

- **Hence:** Resulting costs for heapify:

$$T(d) \leq C \cdot 2^{d+1}$$

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot \sum_{i=1}^d \left(i \cdot 2^{d-i} \right) \leq C \cdot 2^{d+1}$$

- **Hence:** Resulting costs for heapify:

$$T(d) \leq C \cdot 2^{d+1}$$

- **However:** We want costs in relation to n

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- ▶ A binary tree of depth d has $2^{d+1} - 1 \leq n$ nodes

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- ▶ A binary tree of depth d has $2^{d+1} - 1 \leq n$ nodes Why?

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- ▶ A binary tree of depth d has $2^{d-1} \leq n$ nodes **Why?**



Figure: Partial binary tree

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- ▶ A binary tree of depth d has $2^{d-1} \leq n$ nodes **Why?**
- ▶ $2^{d-1} - 1$ nodes in full tree
till layer $d - 1$



Figure: Partial binary tree

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- ▶ A binary tree of depth d has $2^{d+1} - 1$ nodes **Why?**
- ▶ $2^{d-1} - 1$ nodes in full tree till layer $d - 1$
- ▶ At least 1 node in layer d



Figure: Partial binary tree

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- ▶ A binary tree of depth d has $2^{d-1} \leq n$ nodes **Why?**
- ▶ $2^{d-1} - 1$ nodes in full tree till layer $d - 1$
- ▶ At least 1 node in layer d
- ▶ Equation multiplied by 2^2
 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$



Figure: Partial binary tree

Runtime - Heapsort

Heapify

Heapify total runtime:

$$T(d) \leq C \cdot 2^{d+1}$$

- ▶ A binary tree of depth d has $2^{d-1} \leq n$ nodes **Why?**
- ▶ $2^{d-1} - 1$ nodes in full tree till layer $d - 1$
- ▶ At least 1 node in layer d
- ▶ Equation multiplied by 2^2
 $\Rightarrow 2^{d-1} \cdot 2^2 \leq 2^2 \cdot n$
- ▶ Cost for heapify:
 $\Rightarrow T(n) \leq C \cdot 4 \cdot n$

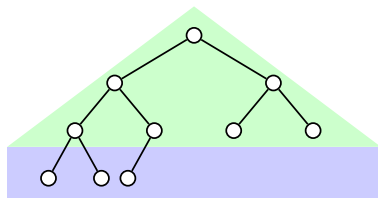


Figure: Partial binary tree

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Induction - Example 2

- ▶ We want to proof (induction assumption):

$$\underbrace{\sum_{i=1}^d (i \cdot 2^{d-i})}_{A(d)} \leq \underbrace{2^{d+1}}_{B(d)}$$

- ▶ We denote the left side with A , the right side with B

Induction - Example 2

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

Induction - Example 2

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

$$\sum_{i=1}^d \left(i \cdot 2^{d-i} \right) \leq 2^{d+1}$$

Induction - Example 2

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

$$\sum_{i=1}^d (i \cdot 2^{d-i}) \leq 2^{d+1}$$

$$\sum_{i=1}^1 (i \cdot 2^{1-i}) \leq 2^{1+1}$$

Induction - Example 2

- **Induction basis:** $d := 1$:

$$A(d) \leq B(d)$$

$$\sum_{i=1}^d (i \cdot 2^{d-i}) \leq 2^{d+1}$$

$$\sum_{i=1}^1 (i \cdot 2^{1-i}) \leq 2^{1+1}$$

$$2^0 \leq 2^2 \quad \checkmark$$

Induction - Example 2

Induction step: ($d := d + 1$):

- ▶ **Idea:** Write down right hand formula and try to get $A(d)$ and $B(d)$ out of it

$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

Induction - Example 2

Induction step: ($d := d + 1$):

- **Idea:** Write down right hand formula and try to get $A(d)$ and $B(d)$ out of it

$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

$$\sum_{i=1}^{d+1} \left(i \cdot 2^{d+1-i} \right) \leq 2^{d+1+1}$$

Induction - Example 2

Induction step: ($d := d + 1$):

- **Idea:** Write down right hand formula and try to get $A(d)$ and $B(d)$ out of it

$$A(d) \leq B(d) \quad \Rightarrow \quad A(d+1) \leq B(d+1)$$

$$\sum_{i=1}^{d+1} (i \cdot 2^{d+1-i}) \leq 2^{d+1+1}$$
$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

\vdots

Induction - Example 2

Induction step: ($d := d + 1$):

\vdots

$$2 \cdot \sum_{i=1}^{d+1} \left(i \cdot 2^{d-i} \right) \leq 2 \cdot 2^{d+1}$$

Induction - Example 2

Induction step: ($d := d + 1$):

\vdots

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot B(d)$$

Induction - Example 2

Induction step: ($d := d + 1$):

\vdots

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot B(d)$$

$$2 \cdot \sum_{i=1}^d (i \cdot 2^{d-i}) + 2 \cdot (d+1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

Induction - Example 2

Induction step: ($d := d + 1$):

\vdots

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot B(d)$$

$$2 \cdot \sum_{i=1}^d (i \cdot 2^{d-i}) + 2 \cdot (d+1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

$$2 \cdot A(d) + (d+1) \leq 2 \cdot B(d)$$

Induction - Example 2

Induction step: ($d := d + 1$):

\vdots

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot 2^{d+1}$$

$$2 \cdot \sum_{i=1}^{d+1} (i \cdot 2^{d-i}) \leq 2 \cdot B(d)$$

$$2 \cdot \sum_{i=1}^d (i \cdot 2^{d-i}) + 2 \cdot (d+1) \cdot 2^{d-(d+1)} \leq 2 \cdot B(d)$$

$$2 \cdot A(d) + (d+1) \leq 2 \cdot B(d)$$

► **Problem:** Does not work but claim still holds

Induction - Example 2

Working proof:

- Show a **little bit stronger** claim

$$\sum_{i=1}^d \left(i \cdot 2^{d-i} \right) \leq 2^{d+1} - d - 2 \leq 2^{d+1}$$

Induction - Example 2

Working proof:

- ▶ Show a **little bit stronger** claim

$$\sum_{i=1}^d \left(i \cdot 2^{d-i} \right) \leq 2^{d+1} - d - 2 \leq 2^{d+1}$$

- ▶ **Advantage:** Results in a stronger induction assumption
 \Rightarrow **exercise**

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Runtime of the other operations:

Runtime of the other operations:

- ▶ Constant costs for taking out $n \times$ maximum

Runtime of the other operations:

- ▶ Constant costs for taking out $n \times$ maximum
- ▶ Maximum of d steps repairing the heap n times

Runtime of the other operations:

- ▶ Constant costs for taking out $n \times$ maximum
- ▶ Maximum of d steps repairing the heap n times
- ▶ Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \Rightarrow d - 1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

Runtime of the other operations:

- ▶ Constant costs for taking out $n \times$ maximum
- ▶ Maximum of d steps repairing the heap n times
- ▶ Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \Rightarrow d - 1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

- ▶ **Recall:** The depth and number of elements is decreasing

Runtime of the other operations:

- ▶ Constant costs for taking out $n \times$ maximum
- ▶ Maximum of d steps repairing the heap n times
- ▶ Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \Rightarrow d - 1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

- ▶ **Recall:** The depth and number of elements is decreasing
 - ▶ **Hence:** $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$

Runtime of the other operations:

- ▶ Constant costs for taking out $n \times$ maximum
- ▶ Maximum of d steps repairing the heap n times
- ▶ Depth of heap at the start is $d \leq 1 + \log_2 n$ Why?

$$2^{d-1} \leq n \Rightarrow d - 1 \leq \log_2 n \Rightarrow d \leq 1 + \log_2 n$$

- ▶ **Recall:** The depth and number of elements is decreasing
 - ▶ **Hence:** $T(n) \leq n \cdot (1 + \log_2 n) \cdot C$
 - ▶ We can reduce this to:

$$T(n) \leq 2 \cdot n \log_2 n \cdot C \quad (\text{holds for } n > 2)$$

Runtime - Heapsort

Runtime costs:

- ▶ Heapify: $T(n) \leq 4 \cdot n \cdot C$

Runtime - Heapsort

Runtime costs:

- ▶ Heapify: $T(n) \leq 4 \cdot n \cdot C$
- ▶ Remove: $T(n) \leq 2 \cdot n \log_2 n \cdot C$

Runtime - Heapsort

Runtime costs:

- ▶ Heapify: $T(n) \leq 4 \cdot n \cdot C$
- ▶ Remove: $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- ▶ Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$

Runtime - Heapsort

Runtime costs:

- ▶ Heapify: $T(n) \leq 4 \cdot n \cdot C$
- ▶ Remove: $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- ▶ Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- ▶ Constraints:
 - ▶ **Upper bound:** $C_2 \cdot n \log_2 n \geq T(n)$ (for $n \geq 2$)
 - ▶ **Lower bound:** $C_1 \cdot n \log_2 n \leq T(n)$ (for $n \geq 2$)

Runtime - Heapsort

Runtime costs:

- ▶ Heapify: $T(n) \leq 4 \cdot n \cdot C$
- ▶ Remove: $T(n) \leq 2 \cdot n \log_2 n \cdot C$
- ▶ Total runtime: $T(n) \leq 6 \cdot n \log_2 n \cdot C$
- ▶ Constraints:
 - ▶ **Upper bound:** $C_2 \cdot n \log_2 n \geq T(n)$ (for $n \geq 2$)
 - ▶ **Lower bound:** $C_1 \cdot n \log_2 n \leq T(n)$ (for $n \geq 2$)
 - ▶ $\Rightarrow C_1$ and C_2 are constant

Structure

Runtime Example

Minsort

Basic Operations

Runtime analysis

Minsort

Heapsort

Introduction to Induction

Logarithms

Base of Logarithms

Logarithm to different bases:

$$\log_a n = \frac{\log_b n}{\log_b a} = \log_b n \cdot \frac{1}{\log_b a}$$

The only difference is a constant coefficient $\frac{1}{\log_b a}$

Examples:

- ▶ $\log_2 4 = \log_{10} 4 \cdot \frac{1}{\log_2 10} = 0.602 \dots \cdot 3.322 \dots = 2$ ✓
- ▶ $\log_{10} 1000 = \log_e 1000 \cdot \frac{1}{\log_e 10} = \ln 1000 \cdot \frac{1}{\ln 10} = 3$ ✓

Runtime Example

Runtime of $n \log_2 n$:

- ▶ Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

Runtime Example

Runtime of $n \log_2 n$:

- ▶ Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- ▶ $2 \times$ elements \Rightarrow only slightly larger than $2 \times$ runtime

Runtime Example

Runtime of $n \log_2 n$:

- ▶ Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- ▶ $2 \times$ elements \Rightarrow only slightly larger than $2 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)

Runtime Example

Runtime of $n \log_2 n$:

- ▶ Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- ▶ $2 \times$ elements \Rightarrow only slightly larger than $2 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)
 - ▶ $n = 2^{20}$ (1 million numbers = 4 MB with 4 B/number)
 - ▶ $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{20} \cdot 20 = 21.0 \text{ ms}$

Runtime Example

Runtime of $n \log_2 n$:

- ▶ Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- ▶ $2 \times$ elements \Rightarrow only slightly larger than $2 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)
 - ▶ $n = 2^{20}$ (1 million numbers = 4 MB with 4 B/number)
 - ▶ $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{20} \cdot 20 = 21.0 \text{ ms}$
 - ▶ $n = 2^{30}$ (1 billion numbers = 4 GB)
 - ▶ $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{30} \cdot 30 = 32 \text{ s}$

Runtime Example

Runtime of $n \log_2 n$:

- ▶ Assume we have constants C_1 and C_2 with

$$C_1 \cdot n \cdot \log_2 n \leq T(n) \leq C_2 \cdot n \cdot \log_2 n \quad \text{for } n \geq 2$$

- ▶ $2 \times$ elements \Rightarrow only slightly larger than $2 \times$ runtime
 - ▶ $C = 1 \text{ ns}$ (1 simple instruction $\approx 1 \text{ ns}$)
 - ▶ $n = 2^{20}$ (1 million numbers = 4 MB with 4 B/number)
 - ▶ $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{20} \cdot 20 = 21.0 \text{ ms}$
 - ▶ $n = 2^{30}$ (1 billion numbers = 4 GB)
 - ▶ $C \cdot n \cdot \log_2 n = 10^{-9} \text{ s} \cdot 2^{30} \cdot 30 = 32 \text{ s}$
- ▶ **Runtime $n \log_2 n$ is nearly as good as linear!**

Further Literature

► General for this Lecture

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

- ▶ **Mathematical Induction**

[Wik] [Mathematical induction](https://en.wikipedia.org/wiki/Mathematical_induction)

`https://en.wikipedia.org/wiki/Mathematical_induction`