

Algorithms and Datastructures

Divide and Conquer, Master theorem

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Structure

Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Divide and Conquer

Introduction

Concept:

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all subsolutions to solve the overall problem

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all subsolutions to solve the overall problem
- ▶ **Recursive** application of the algorithm on smaller subproblems

Divide and Conquer

Introduction

Concept:

- ▶ **Divide** the problem into smaller subproblems
- ▶ **Conquer** the subproblems through recursive solving.
If subproblems are small enough solve them directly
- ▶ **Connect** all subsolutions to solve the overall problem
- ▶ **Recursive** application of the algorithm on smaller subproblems
- ▶ **Direct** solving of small subproblems

Structure

Divide and Conquer

Concept

Maximum Subtotal

Recursion Equations

Substitution Method

Recursion Tree Method

Master theorem

Master theorem (Simple Form)

Master theorem (General Form)

Divide and Conquer

Maximum Subtotal

Input:

Output:

Divide and Conquer

Maximum Subtotal

Input:

- ▶ Sequence X of n integers

Output:

Divide and Conquer

Maximum Subtotal

Input:

- ▶ Sequence X of n integers

Output:

- ▶ Maximum sum of an uninterrupted subsequence of X and its index boundary

Divide and Conquer

Maximum Subtotal

Input:

- ▶ Sequence X of n integers

Output:

- ▶ Maximum sum of an uninterrupted subsequence of X and its index boundary

Table: Input values

| Index | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|----|-----|----|----|-----|----|----|-----|-----|----|
| Value | 31 | -41 | 59 | 26 | -53 | 58 | 97 | -93 | -23 | 84 |

Output: Sum: 187, Start: 2, End: 6

Divide and Conquer

Maximum Subtotal

Idea:



Divide and Conquer

Maximum Subtotal

Idea:



- Solve the left / right half of the problem [recursive](#)

Divide and Conquer

Maximum Subtotal

Idea:



- ▶ Solve the left / right half of the problem [recursive](#)
- ▶ Combine both solutions into a overall solution

Divide and Conquer

Maximum Subtotal

Idea:



- ▶ Solve the left / right half of the problem **recursive**
- ▶ Combine both solutions into a overall solution
- ▶ The maximum is located in the **left half (A)** or the **right half (B)**

Divide and Conquer

Maximum Subtotal

Idea:



- ▶ Solve the left / right half of the problem **recursive**
- ▶ Combine both solutions into a overall solution
- ▶ The maximum is located in the **left half (A)** or the **right half (B)**
- ▶ The maximum interval can **overlap with the border (C)**

Divide and Conquer

Maximum Subtotal

Principle:



Divide and Conquer

Maximum Subtotal

Principle:

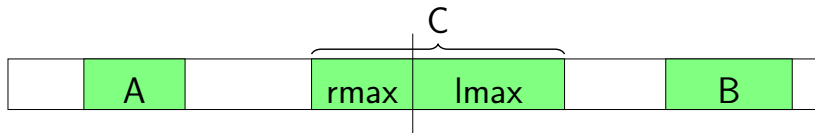


- ▶ Small problems are solved directly: $n = 1 \Rightarrow \max = X[0]$

Divide and Conquer

Maximum Subtotal

Principle:



- ▶ Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- ▶ Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned

Divide and Conquer

Maximum Subtotal

Principle:



- ▶ Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- ▶ Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned
- ▶ To solve C we have to calculate rmax and lmax

Divide and Conquer

Maximum Subtotal

Principle:



- ▶ Small problems are solved directly: $n = 1 \Rightarrow \text{max} = X[0]$
- ▶ Big problems are decomposed into two subproblems and solved recursively. Subsolutions A and B are returned
- ▶ To solve C we have to calculate rmax and lmax
- ▶ Overall solution is maximum of A , B and C

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):
```


Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2
```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)
```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subsolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for corner case C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])
```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # trivial case  
        return (X[i], i, i)  
  
    # recursive subolutions for A, B  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    # rmax and lmax for cornercase C  
    C1, C2 = rmax(X, i, m), lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    # compute solution from results A, B, C  
    return max([A, B, C], key=lambda i: i[0])
```

Divide and Conquer

Maximum Subtotal - Python

```
#Alternative trivial case  
def maxSubArray(X, i, j):
```

Divide and Conquer

Maximum Subtotal - Python

```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)
```

Divide and Conquer

Maximum Subtotal - Python

```
#Alternative trivial case
def maxSubArray(X, i, j):
    # trivial: only one element
    if i == j:
        return (X[i], i, i)

    # trivial: only two elements
    if i + 1 == j:
        return max([
            (X[i], i, i),
            (X[j], j, j),
            (X[i] + X[j], i, j)
        ], key=lambda item: item[0])

    ... # continue as before
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max  
def max(a, b, c):
```


Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation max
def max(a, b, c):
    if a > b:
        if a > c:
            return a
        else:
            return c
    else:
        if c > b:
            return c
        else:
            return b
```

Divide and Conquer

Maximum Subtotal - Python

```
#Alternative implementation max
```

Divide and Conquer

Maximum Subtotal - Python

```
#Alternative implementation max
```

```
def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b
```

Divide and Conquer

Maximum Subtotal - Python

```
#Alternative implementation max
```

```
def max(a, b):  
    if a > b:  
        return a  
    else:  
        return b
```

```
def maxTripel(a, b, c):  
    return max(max(a,b),c)
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation left maximum
def lmax(X, i, j):
    maxSum = (X[i], i)
    s = X[i]

    # sum up from the lower index going up
    # (from left to right)
    for k in range(i+1, j+1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation right maximum
def rmax(X, i, j):
    maxSum = (X[j], j)
    s = X[i]

    # sum up from the upper index going down
    # (from right to left)
    for k in range(j-1, i-1, -1):
        s += X[k]

        if s > maxSum[0]:
            maxSum = (s, k)

    return maxSum
```

Divide and Conquer

Maximum Subtotal

Table: $lmax$ example

| index | i | $i + 1$ | \dots | \dots | $j - 1$ | j |
|-------------|-----|---------|---------|---------|---------|-----|
| X | 58 | -53 | 26 | 59 | -41 | 31 |
| <i>sum</i> | 58 | 5 | 31 | 90 | 49 | 80 |
| <i>lmax</i> | 58 | 58 | 58 | 90 | 90 | 90 |

Divide and Conquer

Maximum Subtotal

Table: *lmax* example

| index | i | $i + 1$ | \dots | \dots | $j - 1$ | j |
|-------------|-----|---------|---------|---------|---------|-----|
| X | 58 | -53 | 26 | 59 | -41 | 31 |
| <i>sum</i> | 58 | 5 | 31 | 90 | 49 | 80 |
| <i>lmax</i> | 58 | 58 | 58 | 90 | 90 | 90 |

- The *sum* and *lmax* are initialized with $X[i]$

Divide and Conquer

Maximum Subtotal

Table: $lmax$ example

| index | i | $i + 1$ | \dots | \dots | $j - 1$ | j |
|--------|-----|---------|---------|---------|---------|-----|
| X | 58 | -53 | 26 | 59 | -41 | 31 |
| sum | 58 | 5 | 31 | 90 | 49 | 80 |
| $lmax$ | 58 | 58 | 58 | 90 | 90 | 90 |

- ▶ The sum and $lmax$ are initialized with $X[i]$
- ▶ We iterate over X from $i + 1$ to j and update sum

Divide and Conquer

Maximum Subtotal

Table: $lmax$ example

| index | i | $i + 1$ | \dots | \dots | $j - 1$ | j |
|--------|-----|---------|---------|---------|---------|-----|
| X | 58 | -53 | 26 | 59 | -41 | 31 |
| sum | 58 | 5 | 31 | 90 | 49 | 80 |
| $lmax$ | 58 | 58 | 58 | 90 | 90 | 90 |

- ▶ The sum and $lmax$ are initialized with $X[i]$
- ▶ We iterate over X from $i + 1$ to j and update sum
- ▶ If $sum > lmax$ then $lmax$ gets updated

Divide and Conquer

Maximum Subtotal

Call with array of four elements

`maxSubArray(-3,9,-4,7)`

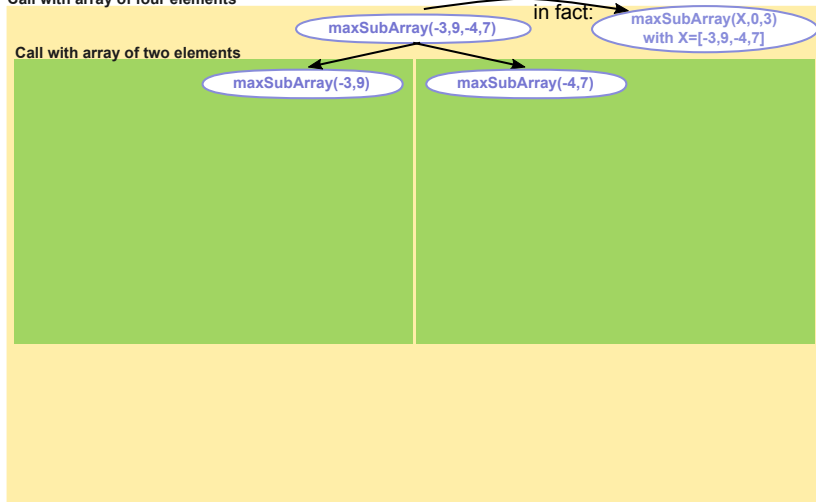
in fact:

`maxSubArray(X,0,3)`
with `X=[-3,9,-4,7]`

Divide and Conquer

Maximum Subtotal

Call with array of four elements



Divide and Conquer

Maximum Subtotal

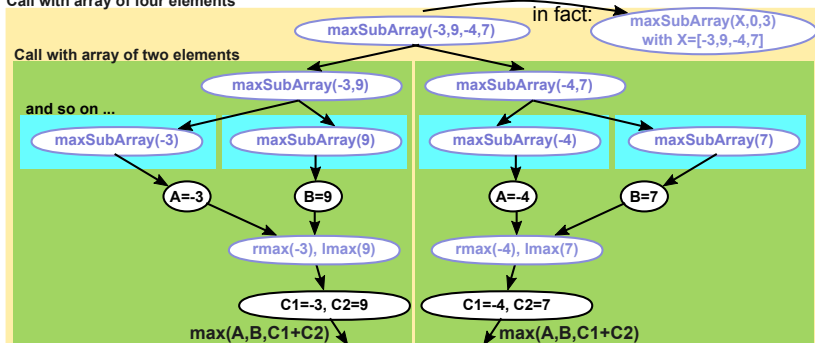
Call with array of four elements



Divide and Conquer

Maximum Subtotal

Call with array of four elements



Divide and Conquer

Maximum Subtotal

Call with array of four elements



Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j:  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
               key=lambda item: item[0])
```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j:  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
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    return max([A, B, C], \  
               key=lambda item: item[0])
```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j: # 0(1)  
        return (X[i], i, i)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
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```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
        return (X[i], i, i)                       # 0(1)  
  
    m = (i + j) / 2  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
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Divide and Conquer

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def maxSubArray(X, i, j):  
    if i == j:                                     # 0(1)  
        return (X[i], i, i)                       # 0(1)  
  
    m = (i + j) / 2                                # 0(1)  
    A = maxSubArray(X, i, m)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
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```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j:                                # O(1)  
        return (X[i], i, i)                  # O(1)  
  
    m = (i + j) / 2                           # O(1)  
    A = maxSubArray(X, i, m)                  # T(n/2)  
    B = maxSubArray(X, m + 1, j)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
    return max([A, B, C], \  
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```

Divide and Conquer

Maximum Subtotal - Python

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    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
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    m = (i + j) / 2                           # O(1)  
    A = maxSubArray(X, i, m)                  # T(n/2)  
    B = maxSubArray(X, m + 1, j)              # T(n/2)  
  
    C1 = rmax(X, i, m)                        # O(n)  
    C2 = lmax(X, m + 1, j)  
    C = (C1[0] + C2[0], C1[1], C2[1])  
  
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Divide and Conquer

Maximum Subtotal - Python

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    if i == j:                                # O(1)  
        return (X[i], i, i)                  # O(1)  
  
    m = (i + j) / 2                            # O(1)  
    A = maxSubArray(X, i, m)                  # T(n/2)  
    B = maxSubArray(X, m + 1, j)              # T(n/2)  
  
    C1 = rmax(X, i, m)                        # O(n)  
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Divide and Conquer

Maximum Subtotal - Python

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def maxSubArray(X, i, j):  
    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)                             # O(n)  
    C2 = lmax(X, m + 1, j)                         # O(n)  
    C = (C1[0] + C2[0], C1[1], C2[1])              # O(1)  
  
    return max([A, B, C], \  
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```

Divide and Conquer

Maximum Subtotal - Python

```
def maxSubArray(X, i, j):  
    if i == j:                                     # O(1)  
        return (X[i], i, i)                       # O(1)  
  
    m = (i + j) / 2                                # O(1)  
    A = maxSubArray(X, i, m)                       # T(n/2)  
    B = maxSubArray(X, m + 1, j)                   # T(n/2)  
  
    C1 = rmax(X, i, m)                             # O(n)  
    C2 = lmax(X, m + 1, j)                         # O(n)  
    C = (C1[0] + C2[0], C1[1], C2[1])              # O(1)  
  
    return max([A, B, C], \                        # O(1)  
               key=lambda item: item[0])
```

Divide and Conquer

Maximum Subtotal - Number of steps $T(n)$

Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

$\underbrace{\Theta(1)}_{\text{trivial case}}$

Divide and Conquer

Maximum Subtotal - Number of steps $T(n)$

Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

trivial case

- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

Divide and Conquer

Maximum Subtotal - Number of steps $T(n)$

Recursion equation:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \underbrace{2 \cdot T\left(\frac{n}{2}\right)}_{\text{solving of subproblems}} + \underbrace{\Theta(n)}_{\text{combination of solutions}} & n > 1 \end{cases}$$

trivial case

- There exist two constants a and b with:

$$T(n) \leq \begin{cases} a & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + b \cdot n & n > 1 \end{cases}$$

- We define $c := \max(a, b)$:

$$T(n) \leq \begin{cases} c & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n & n > 1 \end{cases}$$

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

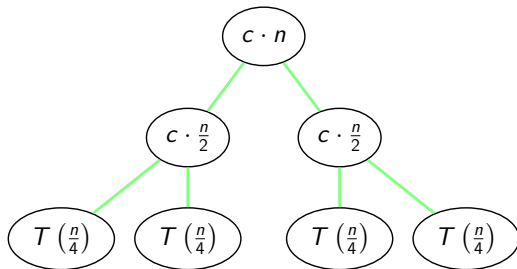


$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

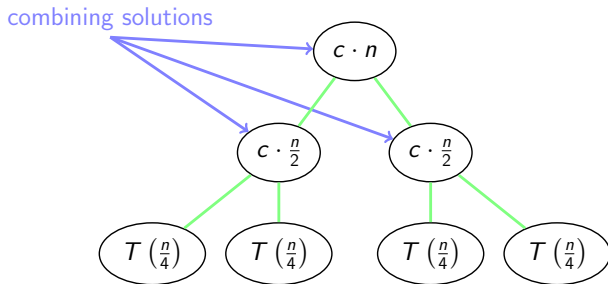


$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



$$T\left(\frac{n}{2}\right) = 2 \cdot T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}$$

Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



Figure: Illustration of the runtime

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$


$$c \cdot n$$

1 node processing n elements
 $\Rightarrow c \cdot n$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements

$$\Rightarrow c \cdot n$$

2 nodes processing $\frac{n}{2}$ elements

$$\Rightarrow 2c \cdot \frac{n}{2} = c \cdot n$$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements

$$\Rightarrow c \cdot n$$

2 nodes processing $\frac{n}{2}$ elements

$$\Rightarrow 2 c \cdot \frac{n}{2} = c \cdot n$$

4 nodes processing $\frac{n}{4}$ elements

$$\Rightarrow 4 c \cdot \frac{n}{4} = c \cdot n$$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements

$$\Rightarrow c \cdot n$$

2 nodes processing $\frac{n}{2}$ elements

$$\Rightarrow 2 c \cdot \frac{n}{2} = c \cdot n$$

4 nodes processing $\frac{n}{4}$ elements

$$\Rightarrow 4 c \cdot \frac{n}{4} = c \cdot n$$

2^i nodes processing $\frac{n}{2^i}$ elements

$$\Rightarrow 2^i c \cdot \frac{n}{2^i} = c \cdot n$$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$



1 node processing n elements

$$\Rightarrow c \cdot n$$

2 nodes processing $\frac{n}{2}$ elements

$$\Rightarrow 2 c \cdot \frac{n}{2} = c \cdot n$$

4 nodes processing $\frac{n}{4}$ elements

$$\Rightarrow 4 c \cdot \frac{n}{4} = c \cdot n$$

2^i nodes processing $\frac{n}{2^i}$ elements

$$\Rightarrow 2^i c \cdot \frac{n}{2^i} = c \cdot n$$

n nodes processing 1 element

$$\Rightarrow c \cdot n$$

Figure: Recursion tree method

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

- ▶ Top level with depth $i = 0$

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

- ▶ Top level with depth $i = 0$
- ▶ Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

- ▶ Top level with depth $i = 0$
- ▶ Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

Depth:

- ▶ Top level with depth $i = 0$
- ▶ Lowest level with $2^i = n$ elements

$$\Rightarrow i = \log_2 n$$

Runtime:

- ▶ A total of $\log_2 n + 1$ levels with each cost of $c \cdot n$
The costs of merging the solutions and solving of the trivial problems are the same here

Divide and Conquer

Maximum Subtotal - Illustration of $T(n)$

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The costs of merging the solutions and solving of the trivial problems are the same here

$$T(n) = c \cdot n \log_2 n + c \cdot n \in \Theta(n \log n)$$

Divide and Conquer

Maximum Subtotal - Summary

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Divide and Conquer

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Divide and Conquer

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Divide and Conquer

Maximum Subtotal - Summary

Summary:

- ▶ Direct solution is slow with $\mathcal{O}(n^3)$
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- ▶ Divide and conquer approach results in $\mathcal{O}(n \log n)$

Divide and Conquer

Maximum Subtotal - Summary

Summary:

- ▶ Direct solution is slow with $\mathcal{O}(n^3)$
- ▶ Better solution with incremental update of sum was $\mathcal{O}(n^2)$
- ▶ Divide and conquer approach results in $\mathcal{O}(n \log n)$
- ▶ There is an approach running in $\mathcal{O}(n)$ if you assume that all subtotals are positive

Divide and Conquer

Maximum Subtotal



Figure: Scanning the array in linear time

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation - linear runtime  
def maxSubArray(X):
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation - linear runtime
def maxSubArray(X):
    # sum, start index
    rMax, irMax = 0, 0 # current maximum
    tMax, itMax = 0, 0 # total maximum
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation - linear runtime
def maxSubArray(X):
    # sum, start index
    rMax, irMax = 0, 0 # current maximum
    tMax, itMax = 0, 0 # total maximum

    for i in range(len(X)):
        if rMax == 0:
            irMax = i
        rMax = max(0, rMax + X[i])
```

Divide and Conquer

Maximum Subtotal - Python

```
#Implementation - linear runtime
def maxSubArray(X):
    # sum, start index
    rMax, irMax = 0, 0 # current maximum
    tMax, itMax = 0, 0 # total maximum

    for i in range(len(X)):
        if rMax == 0:
            irMax = i
            rMax = max(0, rMax + X[i])

        if rMax > tMax:
            tMax, itMax = rMax, irMax

    return (tMax, itMax)
```


Structure

Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Recursion Equations

Recursion Equation

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} \overbrace{f_0(n)}^{\text{trivial case for } n_0} & n = n_0 \\ \underbrace{a \cdot T\left(\frac{n}{b}\right)}_{\substack{\text{solving of } a \\ \text{subproblems} \\ \text{with reduced} \\ \text{input size } \frac{n}{b}}} + \underbrace{f(n)}_{\substack{\text{slicing and} \\ \text{splicing of} \\ \text{subsolutions}}} & n > n_0 \end{cases}$$

Recursion Equations

Recursion Equation

Recursion equation:

Recursion Equations

Recursion Equation

Recursion equation:

- Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

Recursion Equations

Recursion Equation

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Recursion Equations

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Recursion Equations

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Recursion Equations

Recursion Equation

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- ▶ Describes the runtime for recursive functions:

$$T(n) = \begin{cases} f_0(n) & n = n_0 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > n_0 \end{cases}$$

- ▶ n_0 is normally small, $f_0(n_0) \in \Theta(1)$
- ▶ Normally $a > 1$ and $b > 1$
- ▶ Dependent on the strategy of solving $T(n)$ f_0 is ignored
- ▶ $T(n)$ is only defined for integers of $\frac{n}{b}$ which is often ignored in benefit of a simpler solution

Structure

Divide and Conquer

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Recursion Equations

Substitution Method

Substitution Method:

Recursion Equations

Substitution Method

Substitution Method:

- ▶ Guess the solution and prove it with induction

Recursion Equations

Substitution Method

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- ▶ Guess the solution and prove it with induction
- ▶ Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

Recursion Equations

Substitution Method

Substitution Method:

- ▶ Guess the solution and prove it with induction
- ▶ Example:

$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

- ▶ Assumption: $T(n) = n + n \cdot \log_2 n$

Recursion Equations

Substitution Method

Induction:

Recursion Equations

Substitution Method

Induction:

- ▶ Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$

Recursion Equations

Substitution Method

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- ▶ Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
- ▶ Induction step (from $\frac{n}{2}$ to n):

Recursion Equations

Substitution Method

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$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$$

Recursion Equations

Substitution Method

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- ▶ Induction basis (for $n = 1$): $T(1) = 1 + 1 \cdot \log_2 1 = 1$
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$$\begin{aligned} T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\ &\stackrel{IA}{=} 2 \cdot \left(\frac{n}{2} + \frac{n}{2} \cdot \log_2 \frac{n}{2}\right) + n \end{aligned}$$

Recursion Equations

Substitution Method

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Recursion Equations

Substitution Method

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Recursion Equations

Substitution Method

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Recursion Equations

Substitution Method

Substitution Method:

Recursion Equations

Substitution Method

Substitution Method:

- ▶ Alternative assumption

Recursion Equations

Substitution Method

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$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 1 \end{cases}$$

Recursion Equations

Substitution Method

Substitution Method:

- ▶ Alternative assumption
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$$T(n) = \begin{cases} 1 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + n & n > 0 \end{cases}$$

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Recursion Equations

Substitution Method

Substitution Method:

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- ▶ Assumption: $T(n) \in O(n \log n)$
- ▶ Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$

Recursion Equations

Substitution Method

Induction:

Recursion Equations

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- ▶ Solution: Find $c > 0$ with $T(n) \leq c \cdot n \log_2 n$

Recursion Equations

Substitution Method

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Recursion Equations

Substitution Method

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Recursion Equations

Substitution Method

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Recursion Equations

Substitution Method

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Recursion Equations

Substitution Method

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Recursion Equations

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$$\begin{aligned}T(n) &= 2 \cdot T\left(\frac{n}{2}\right) + n \\&\leq 2 \cdot \left(c \cdot \frac{n}{2} \log_2 \frac{n}{2}\right) + n \\&= c \cdot n \log_2 n - c \cdot n \log_2 2 + n \\&= c \cdot n \log_2 n - c \cdot n + n \\&\leq c \cdot n \log_2 n, \quad c \geq 1\end{aligned}$$

Structure

Divide and Conquer

- Concept

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Recursion Equations

Recursion Tree Method

Recursion tree method:

Recursion Equations

Recursion Tree Method

Recursion tree method:

- ▶ Can be used to make assumptions about the runtime

Recursion Equations

Recursion Tree Method

Recursion tree method:

- ▶ Can be used to make assumptions about the runtime
- ▶ Example:

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + \Theta(n^2) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

Recursion Equations

Recursion Tree Method

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

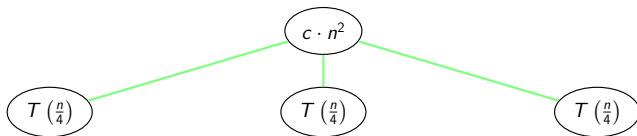


Figure: Recursion tree of example

Recursion Equations

Recursion Tree Method

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

Figure: Recursion tree of example

Recursion Equations

Recursion Tree Method

$$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$



$$T(n) = 12 \cdot T\left(\frac{n}{16}\right) + 3c \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2$$

Figure: Recursion tree of example

Recursion Equations

Recursion Tree Method



Figure: Levels of the recursion tree

Recursion Equations

Recursion Tree Method Costs

Costs of connecting the partial solutions:
(excludes the last layer)

Recursion Equations

Recursion Tree Method Costs

Costs of connecting the partial solutions:

(excludes the last layer)

- ▶ Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$

Recursion Equations

Recursion Tree Method Costs

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- ▶ Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$
- ▶ Costs of partial problem on level i :

$$T_{i_p}(n) = c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n\right)^2$$

Recursion Equations

Recursion Tree Method Costs

Costs of connecting the partial solutions:

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- ▶ Size of partial problems on level i : $s_i(n) = \left(\frac{1}{4}\right)^i \cdot n$
- ▶ Costs of partial problem on level i :

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- ▶ Number of partial problems on level i : $n_i = 3^i$

Recursion Equations

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- ▶ Number of partial problems on level i : $n_i = 3^i$
- ▶ Costs on level i :

$$T_i(n) = 3^i \cdot c \cdot \left(\left(\frac{1}{4}\right)^i \cdot n\right)^2 = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$$

Recursion Equations

Recursion Tree Method Costs

Costs of solving partial solutions: (only the last layer)

Recursion Equations

Recursion Tree Method Costs

Costs of solving partial solutions: (only the last layer)

- ▶ Size of partial problems on the last level: $s_{i+1}(n) = 1$

Recursion Equations

Recursion Tree Method Costs

Costs of solving partial solutions: (only the last layer)

- ▶ Size of partial problems on the last level: $s_{i+1}(n) = 1$
- ▶ Costs of partial problem on the last level: $T_{i+1_p}(n) = d$

Recursion Equations

Recursion Tree Method Costs

Costs of solving partial solutions: (only the last layer)

- ▶ Size of partial problems on the last level: $s_{i+1}(n) = 1$
- ▶ Costs of partial problem on the last level: $T_{i+1_p}(n) = d$
- ▶ With this the depth of the tree is:

$$\left(\frac{1}{4}\right)^i \cdot n = 1 \quad \Rightarrow \quad n = 4^i \quad \Rightarrow \quad i = \log_4 n$$

Recursion Equations

Recursion Tree Method Costs

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- ▶ Size of partial problems on the last level: $s_{i+1}(n) = 1$
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- ▶ Number of partial problems on the last level:

$$n_{i+1} = 3^{\log_4 n}$$

Recursion Equations

Recursion Tree Method Costs

Costs of solving partial solutions: (only the last layer)

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- ▶ Number of partial problems on the **last level**:

$$n_{i+1} = 3^{\log_4 n} = n^{\log_4 3} \quad \leftarrow \text{next slide}$$

Recursion Equations

Recursion Tree Method Costs

Costs of solving partial solutions: (only the last layer)

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- ▶ Number of partial problems on the last level:

$$n_{i+1} = 3^{\log_4 n} = n^{\log_4 3} \quad \leftarrow \text{next slide}$$

- ▶ Costs on the last level: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

Fun with logarithm

Logarithm

- ▶ Transforming $3^{\log_4 n}$ uses general log rules

$$\log_4 n = \log_4 \left(3^{\log_3 n} \right) \quad \text{uses } n = 3^{\log_3 n}$$

Fun with logarithm

Logarithm

- ▶ Transforming $3^{\log_4 n}$ uses general log rules

$$\begin{aligned}\log_4 n &= \log_4 \left(3^{\log_3 n} \right) \\ &= \log_3 n \cdot \log_4 3\end{aligned}$$

uses $n = 3^{\log_3 n}$

uses $\log a^b = b \cdot \log a$

Fun with logarithm

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- ▶ This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$

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Logarithm

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- ▶ This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- ▶ Now the whole expression:

$$3^{\log_4 n} = 3^{\log_3 n \cdot \log_4 3}$$

uses reformulation above

Fun with logarithm

Logarithm

- ▶ Transforming $3^{\log_4 n}$ uses general log rules

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uses $n = 3^{\log_3 n}$

uses $\log a^b = b \cdot \log a$

- ▶ This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- ▶ Now the whole expression:

$$\begin{aligned}3^{\log_4 n} &= 3^{\log_3 n \cdot \log_4 3} \\ &= \left(3^{\log_3 n} \right)^{\log_4 3}\end{aligned}$$

uses reformulation above

uses $x^{a \cdot b} = (x^a)^b$

Fun with logarithm

Logarithm

- ▶ Transforming $3^{\log_4 n}$ uses general log rules

$$\begin{aligned}\log_4 n &= \log_4 \left(3^{\log_3 n} \right) \\ &= \log_3 n \cdot \log_4 3\end{aligned}$$

$$\text{uses } n = 3^{\log_3 n}$$

$$\text{uses } \log a^b = b \cdot \log a$$

- ▶ This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
- ▶ Now the whole expression:

$$\begin{aligned}3^{\log_4 n} &= 3^{\log_3 n \cdot \log_4 3} \\ &= \left(3^{\log_3 n} \right)^{\log_4 3} \\ &= n^{\log_4 3}\end{aligned}$$

uses reformulation above

$$\text{uses } x^{a \cdot b} = (x^a)^b$$

Fun with logarithm

Logarithm

- ▶ Transforming $3^{\log_4 n}$ uses general log rules

$$\begin{aligned}\log_4 n &= \log_4 \left(3^{\log_3 n} \right) \\ &= \log_3 n \cdot \log_4 3\end{aligned}$$

$$\text{uses } n = 3^{\log_3 n}$$

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- ▶ This proves the general log rule $\log_b c = \log_a c \cdot \log_b a$
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uses reformulation above

$$\text{uses } x^{a \cdot b} = (x^a)^b$$

- ▶ This term will recur in the master theorem

Recursion Equations

Total costs

Total costs:

Recursion Equations

Total costs

Total costs:

- ▶ Costs of level i : $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$

Recursion Equations

Total costs

Total costs:

- ▶ Costs of level i : $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
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Recursion Equations

Total costs

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- ▶ Costs of **level i** : $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
- ▶ Costs of **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \left(\begin{array}{c} \text{even with} \\ \text{infinite elements} \end{array} \right)}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

Recursion Equations

Total costs

Total costs:

- ▶ Costs of **level i**: $T_i(n) = \left(\frac{3}{16}\right)^i \cdot c \cdot n^2$
- ▶ Costs of **last level**: $T_{i+1}(n) = d \cdot n^{\log_4 3}$

$$T(n) = \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i \cdot c \cdot n^2}_{\substack{\text{geometric series,} \\ \text{constant} \\ \left(\begin{array}{c} \text{even with} \\ \text{infinite elements} \end{array} \right)}} + \underbrace{d \cdot n^{\log_4 3}}_{\substack{\log_4 3 < 1, \\ \text{grows a lot} \\ \text{slower than } n^2}} \in \mathcal{O}(n^2)$$

- ▶ Here: The costs of connecting the partial problems dominate

Recursion Equations

Geometric Series

- ▶ **Geometric progression:**

Quotient of two neighboring sequence parts is constant

$$2^0, 2^1, 2^2, \dots, 2^k$$

Recursion Equations

Geometric Series

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The series (cumulative sum) of a geometric sequence

Recursion Equations

Geometric Series

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- ▶ **Geometric series:**

The series (cumulative sum) of a geometric sequence

- ▶ For $|q| < 1$:

$$\sum_{k=0}^{\infty} a_0 \cdot q^k = \frac{a_0}{1 - q} \Rightarrow \text{constant}$$

Recursion Equations

Proof of $O(n^2)$

Proof of $O(n^2)$:

Recursion Equations

Proof of $O(n^2)$

Proof of $O(n^2)$:

► We know:

$$\begin{aligned} T(n) &= 3T\left(\frac{n}{4}\right) + \Theta(n^2) \\ &\leq 3T\left(\frac{n}{4}\right) + c \cdot n^2 \end{aligned}$$

Recursion Equations

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► Assumption: $T(n) \in O(n^2)$, so there exists a $k > 0$ with

$$T(n) \leq k \cdot n^2$$

Recursion Equations

Proof of $O(n^2)$

Proof of $\mathcal{O}(n^2)$:

Recursion Equations

Proof of $O(n^2)$

Proof of $O(n^2)$:

- Presumption: $T(n) \in O(n^2)$, so there exists a $k > 0$ with

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Recursion Equations

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Recursion Equations

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$$T(n) \leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2$$

Recursion Equations

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Recursion Equations

Proof of $O(n^2)$

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- Presumption: $T(n) \in O(n^2)$, so there exists a $k > 0$ with

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- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \\ &= \frac{3}{16} k \cdot n^2 + c \cdot n^2 \end{aligned}$$

Recursion Equations

Proof of $O(n^2)$

Proof of $O(n^2)$:

- Presumption: $T(n) \in O(n^2)$, so there exists a $k > 0$ with

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- Substitution method:

$$\begin{aligned} T(n) &\leq 3 \cdot T\left(\frac{n}{4}\right) + c \cdot n^2 \\ &\leq 3k \cdot \left(\frac{n}{4}\right)^2 + c \cdot n^2 \\ &= \frac{3}{16} k \cdot n^2 + c \cdot n^2 \\ &\leq k \cdot n^2 \qquad \text{for } k \geq \frac{16}{13} c \end{aligned}$$

Structure

Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem**

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Recursion Equations

Master theorem

Master theorem:

Recursion Equations

Master theorem

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

Recursion Equations

Master theorem

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- $T(n)$ is the runtime of an algorithm ...

Recursion Equations

Master theorem

Master theorem:

- ▶ Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

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 - ▶ ... which divides a problem of size n in a partial problems

Recursion Equations

Master theorem

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- ▶ Approach to solve for a recursion equation of the form:

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 - ▶ ... which divides a problem of size n in a partial problems
 - ▶ ... which solves each partial problem recursively with a runtime of $T\left(\frac{n}{b}\right)$

Recursion Equations

Master theorem

Master theorem:

- Approach to solve for a recursion equation of the form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- $T(n)$ is the runtime of an algorithm ...
 - ... which divides a **problem of size n** in **a partial problems**
 - ... which solves each partial problem recursively with a **runtime of $T\left(\frac{n}{b}\right)$**
 - ... which takes **$f(n)$** steps to merge all partial solutions

Recursion Equations

Master theorem (Simple Form)

Master theorem:

Recursion Equations

Master theorem (Simple Form)

Master theorem:

- ▶ In the examples we have seen that ...

Recursion Equations

Master theorem (Simple Form)

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- ▶ In the examples we have seen that ...
 - ▶ Either the runtime of **connecting the solutions** dominates

Recursion Equations

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- ▶ In the examples we have seen that ...
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 - ▶ Or the runtime of **solving the problems** dominates

Recursion Equations

Master theorem (Simple Form)

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- ▶ In the examples we have seen that ...
 - ▶ Either the runtime of **connecting the solutions** dominates
 - ▶ Or the runtime of **solving the problems** dominates
 - ▶ Or both have **equal influence on runtime**

Recursion Equations

Master theorem (Simple Form)

Master theorem:

- ▶ In the examples we have seen that ...
 - ▶ Either the runtime of **connecting the solutions** dominates
 - ▶ Or the runtime of **solving the problems** dominates
 - ▶ Or both have **equal influence on runtime**
- ▶ **Simple form:** Special case with runtime of connecting the solutions $f(n) \in O(n)$

Recursion Equations

Master theorem (Simple Form)

Simple form:

Recursion Equations

Master theorem (Simple Form)

Simple form:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \underbrace{c \cdot n}_{\text{Is any } f(n) \text{ in general form}}, \quad a \geq 1, b > 1, c > 0$$

Recursion Equations

Master theorem (Simple Form)

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- This yields a runtime of:

Recursion Equations

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- This yields a runtime of:

$$T(n) = \begin{cases} \Theta(\overbrace{n^{\log_b a}}^{\text{Number of leaves}}) & \text{if } a > b \\ \Theta(n \log n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

Recursion Equations

Master theorem (Simple Form)

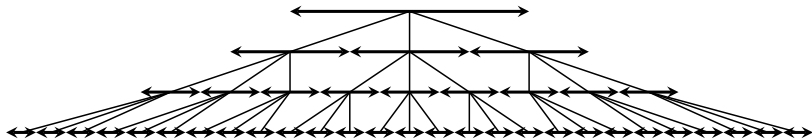


Figure: Simple recursion equation with $a = 3, b = 2$

Recursion Equations

Master theorem (Simple Form)

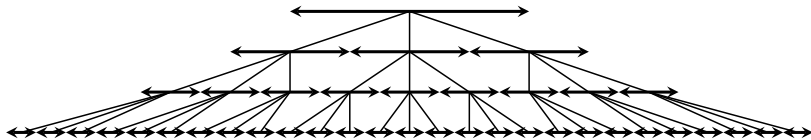


Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- ▶ Three partial problems with $\frac{1}{2}$ the size

Recursion Equations

Master theorem (Simple Form)

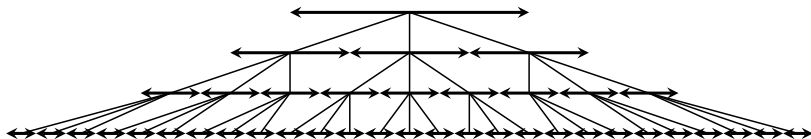


Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- ▶ Three partial problems with $\frac{1}{2}$ the size
- ▶ Solving the partial problems dominates (last layer, leaves)

Recursion Equations

Master theorem (Simple Form)

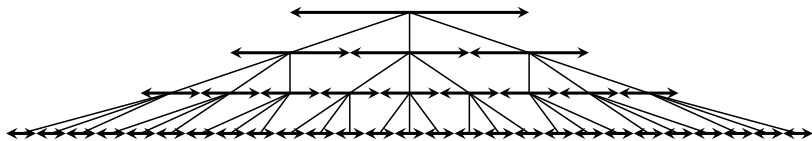


Figure: Simple recursion equation with $a = 3, b = 2$

Case 1: $a > b$

- ▶ Three partial problems with $\frac{1}{2}$ the size
- ▶ Solving the partial problems dominates (last layer, leaves)
- ▶ Runtime of $\Theta(n^{\log_b a})$

Recursion Equations

Master theorem (Simple Form)

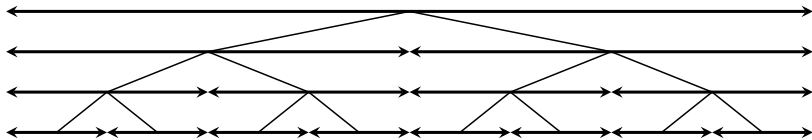


Figure: Simple recursion equation with $a = 2, b = 2$

Recursion Equations

Master theorem (Simple Form)

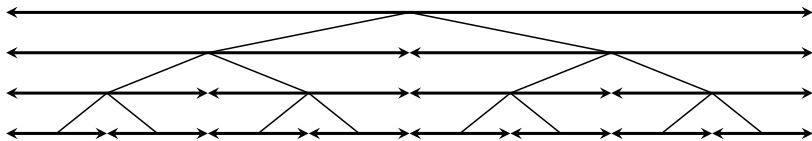


Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

Recursion Equations

Master theorem (Simple Form)



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Case 2: $a = b$

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Recursion Equations

Master theorem (Simple Form)

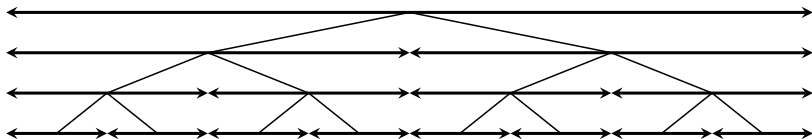


Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

- ▶ Two partial problems with $\frac{1}{2}$ the size
- ▶ Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 2$

Case 2: $a = b$

- ▶ Two partial problems with $\frac{1}{2}$ the size
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- ▶ Runtime of $\Theta(n \log n)$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

Recursion Equations

Master theorem (Simple Form)

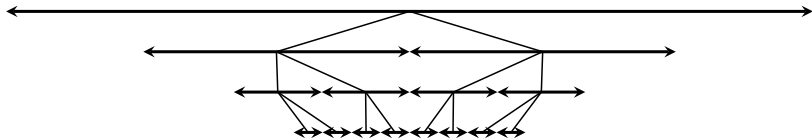


Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

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Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

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- ▶ Connecting all partial solutions dominates (first layer, root)

Recursion Equations

Master theorem (Simple Form)



Figure: Simple recursion equation with $a = 2, b = 3$

Case 3: $a < b$

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- ▶ Connecting all partial solutions dominates (first layer, root)
- ▶ Runtime of $\Theta(n)$

Recursion Equations

Master theorem (Simple Form)

For a recursion equation like

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + c \cdot n, \quad a \geq 1, b > 1, c > 0$$

Recursion Equations

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Recursion Equations

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► ... yields to a runtime of:

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & \text{if } a > b \\ \Theta(n \log_b n) & \text{if } a = b \\ \Theta(n) & \text{if } a < b \end{cases}$$

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- Proof with *geometric series*: Number of operations per layer grows / shrinks by constant factor $\frac{a}{b}$

Structure

Divide and Conquer

- Concept

- Maximum Subtotal

Recursion Equations

- Substitution Method

- Recursion Tree Method

- Master theorem**

 - Master theorem (Simple Form)

 - Master theorem (General Form)

Recursion Equations

Master theorem (General Form)

Master theorem (general form):

Recursion Equations

Master theorem (General Form)

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Recursion Equations

Master theorem (General Form)

Master theorem (general form):

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n), \quad a \geq 1, b > 1$$

- **Case 1:** $T(n) \in \Theta(n^{\log_b a})$ if $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$, $\varepsilon > 0$

Solving the partial problems dominates
(last layer, leaves)

Recursion Equations

Master theorem (General Form)

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Solving the partial problems dominates
(last layer, leaves)

- **Case 2:** $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log_b n$ layers

Recursion Equations

Master theorem (General Form)

Master theorem (general form):

- **Case 3:** $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$
Connecting all partial solutions in first layer (root) dominates

Regularity condition:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n), \quad 0 \leq c \leq 1, \\ n > n_0$$

Recursion Equations

Master theorem (General Form) - Case 1

Case 1 - Example:

$$f(n) \in O(n^{\log_b a - \varepsilon}), \varepsilon > 0$$

Solving the partial problems dominates (last layer, leaves)

if

Recursion Equations

Master theorem (General Form) - Case 1

Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$

$f(n) \in O(n^{\log_b a - \varepsilon}), \varepsilon > 0$

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Solving the partial problems dominates (last layer, leaves)

Recursion Equations

Master theorem (General Form) - Case 1

Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$

if

$f(n) \in O(n^{\log_b a - \epsilon}), \epsilon > 0$

Solving the partial problems dominates (last layer, leaves)

$$\blacktriangleright T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$$

$$a = 8, b = 2, f(n) = 1000 \cdot n^2, \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in O(n^{3-\epsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

Recursion Equations

Master theorem (General Form) - Case 1

Case 1 - Example: $T(n) \in \Theta(n^{\log_b a})$

if

$f(n) \in \mathcal{O}(n^{\log_b a - \epsilon})$, $\epsilon > 0$

Solving the partial problems dominates (last layer, leaves)

► $T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 1000 \cdot n^2$

$$a = 8, \quad b = 2, \quad f(n) = 1000 \cdot n^2, \quad \underbrace{\log_b a = \log_2 8 = 3}_{n^3 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{3-\epsilon}) \Rightarrow T(n) \in \Theta(n^3)$$

► $T(n) = 9 \cdot T\left(\frac{n}{3}\right) + 17 \cdot n$

$$a = 9, \quad b = 3, \quad f(n) = 17 \cdot n, \quad \underbrace{\log_b a = \log_3 9 = 2}_{n^2 \text{ leaves}}$$

$$f(n) \in \mathcal{O}(n^{2-\epsilon}) \Rightarrow T(n) \in \Theta(n^2)$$

Recursion Equations

Master theorem (General Form) - Case 2

Case 2: if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (General Form) - Case 2

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

Recursion Equations

Master theorem (General Form) - Case 2

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

► $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 10 \cdot n$

$$a = 2, \quad b = 2, \quad f(n) = 10 \cdot n, \quad \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

Recursion Equations

Master theorem (General Form) - Case 2

Case 2: $T(n) \in \Theta(n^{\log_b a} \log n)$ if $f(n) \in \Theta(n^{\log_b a})$

Each layer has equal costs, $\log n$ layers

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$$f(n) \in \Theta(n^{\log_2 2}) \Rightarrow T(n) \in \Theta(n \log n)$$

► $T(n) = T\left(\frac{2n}{3}\right) + 1$

$$a = 1, \quad b = \frac{2}{3}, \quad f(n) = 1, \quad \underbrace{\log_b a = \log_{3/2} 1 = 0}_{n^0 \text{ leaves} = 1 \text{ leaf}}$$

$$f(n) \in \Theta(n^{\log_{3/2} 1}) \Rightarrow T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$$

Recursion Equations

Master theorem (General Form) - Case 3

Case 3: if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

Recursion Equations

Master theorem (General Form) - Case 3

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

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Recursion Equations

Master theorem (General Form) - Case 3

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Connecting all partial solutions in first layer (root) dominates

► $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$

$$a = 2, b = 2, f(n) = n^2, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

$$f(n) \in \Omega(n^{1+\varepsilon})$$

Recursion Equations

Master theorem (General Form) - Case 3

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Recursion Equations

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Recursion Equations

Master theorem (General Form) - Case 3

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Connecting all partial solutions in first layer (root) dominates

- ▶ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$
- ▶ $f(n) \in \Omega(n^{1+\varepsilon})$

Recursion Equations

Master theorem (General Form) - Case 3

Case 3: $T(n) \in \Theta(f(n))$ if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

Connecting all partial solutions in first layer (root) dominates

- ▶ $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n^2$
- ▶ $f(n) \in \Omega(n^{1+\varepsilon})$
- ▶ Check if **regularity condition** also holds:

$$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$$

$$2 \cdot \left(\frac{n}{2}\right)^2 \leq c \cdot n^2 \quad \Rightarrow \quad \frac{1}{2} \cdot n^2 \leq c \cdot n^2 \quad \Rightarrow \quad c \geq \frac{1}{2}$$

$$\Rightarrow T(n) \in \Theta(n^2)$$

Recursion Equations

Master theorem (General Form)

Master theorem:

Recursion Equations

Master theorem (General Form)

Master theorem:

- ▶ Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

Recursion Equations

Master theorem (General Form)

Master theorem:

- ▶ Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, \quad b = 2, \quad f(n) = n \log n, \quad \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

Recursion Equations

Master theorem (General Form)

Master theorem:

- ▶ Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, b = 2, f(n) = n \log n, \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- ▶ **Case 1:** $f(n) \notin O(n^{1-\epsilon})$

Recursion Equations

Master theorem (General Form)

Master theorem:

- ▶ Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

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- ▶ **Case 1:** $f(n) \notin O(n^{1-\epsilon})$
- ▶ **Case 2:** $f(n) \notin \Theta(n^1)$

Recursion Equations

Master theorem (General Form)

Master theorem:

- ▶ Not always applicable: $T(n) = 2 \cdot T(\frac{n}{2}) + n \log n$

$$a = 2, \quad b = 2, \quad f(n) = n \log n, \quad \underbrace{\log_b a = \log_2 2 = 1}_{n^1 \text{ leaves}}$$

- ▶ **Case 1:** $f(n) \notin O(n^{1-\varepsilon})$
- ▶ **Case 2:** $f(n) \notin \Theta(n^1)$
- ▶ **Case 3:** $f(n) \notin \Omega(n^{1+\varepsilon})$

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$n \log n$ is *asymptotically* larger than n ,
but not *polynomial* larger

Recursion Equations

Master theorem - Summary

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- ▶ **Case 1:** Solving the partial problems is *polynomial* bigger than merging all solutions

$$T(n) \in \Theta(n^{\log_b a}), \quad T(n) \in \Theta(\text{number of leaves})$$

- ▶ **Case 2:** Each layer has equal costs

$$T(n) \in \Theta(n^{\log_b a} \log n), \quad \log n \text{ layers}$$

Recursion Equations

Master theorem - Summary

Master theorem:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- ▶ Three cases depending on the dominance of the terms
- ▶ **Case 1:** Solving the partial problems is *polynomial* bigger than merging all solutions
 $T(n) \in \Theta(n^{\log_b a})$, $T(n) \in \Theta(\text{number of leaves})$
- ▶ **Case 2:** Each layer has equal costs
 $T(n) \in \Theta(n^{\log_b a} \log n)$, $\log n$ layers
- ▶ **Case 3:** Connecting all partial solutions is *polynomial* bigger than solving all partial problems
 $T(n) \in \Theta(f(n))$

Further Literature

► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

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MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

- ▶ **Master theorem**

[Wik] [Master theorem](https://en.wikipedia.org/wiki/Master_theorem)

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