

# Algorithms and Datastructures

Linked Lists, Binary Search Trees

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Algorithms and Datastructures, January 2018

# Structure

Sorted Sequences

Linked Lists

Binary Search Trees

# Sorted Sequences

## Introduction

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# Sorted Sequences

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- ▶ We have a set of **keys** mapped to **values**

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  - ▶ **lookup(key)**: Find the element with the given **key**, if it is not available find the element with the next smallest key
  - ▶ **next()/previous()**: Returns the element with the next bigger/smaller **key**. This enables iteration over all elements

# Sorted Sequences

## Introduction

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  - ▶ This is called a **range query**

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  - ▶ We can implement this with a combination of **lookup(key)** and **next()**



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  - ▶ We can implement this with a combination of **lookup(key)** and **next()**
  - ▶ It's not essential if an apartments exists with **exactly** 400€ monthly rent
- ▶ We do not want to sort all elements every time on an **insert** operation
- ▶ How could we implement this?

# Sorted Sequences

Implementation 1 (not good) - Static Array

**Static array:**

|   |   |   |    |    |    |    |    |    |    |    |    |
|---|---|---|----|----|----|----|----|----|----|----|----|
| 3 | 5 | 9 | 14 | 18 | 21 | 26 | 40 | 41 | 42 | 43 | 46 |
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- **lookup** in time  $O(\log n)$

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- ▶ **insert** and **remove** up to  $\Theta(n)$

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- ▶ **lookup** in time  $O(\log n)$ 
  - ▶ With **binary search**
  - ▶ Example: **lookup(41)**
- ▶ **next** / **previous** in time  $O(1)$ 
  - ▶ They are next to each other
- ▶ **insert** and **remove** up to  $\Theta(n)$ 
  - ▶ We have to copy up to  $n$  elements

# Sorted Sequences

## Implementation 2 (bad) - Hash Table

**Hash map:**

# Sorted Sequences

## Implementation 2 (bad) - Hash Table

### Hash map:

- ▶ `insert` and `remove` in  $O(1)$

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If the hash table is big enough and we use a good hash function

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If the hash table is big enough and we use a good hash function

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If element with **exactly** this key exists, otherwise we get `None` as result

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If element with **exactly** this key exists, otherwise we get `None` as result

- ▶ `next` / `previous` in time up to  $\Theta(n)$

Order of the elements is independent of the order of the keys

# Sorted Sequences

Implementation 3 (good?) - Linked List

**Linked list:**

# Sorted Sequences

Implementation 3 (good?) - Linked List

## **Linked list:**

- ▶ Runtimes for doubly linked lists:

# Sorted Sequences

## Implementation 3 (good?) - Linked List

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- ▶ Not yet what we want, but structure is related to binary search trees

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## Implementation 3 (good?) - Linked List

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- ▶ Runtimes for doubly linked lists:
  - ▶ `next` / `previous` in time  $O(1)$
  - ▶ `insert` and `remove` in  $O(1)$
  - ▶ `lookup` in time  $\Theta(n)$
- ▶ Not yet what we want, but structure is related to binary search trees
- ▶ Let's have a closer look



# Structure

Sorted Sequences

Linked Lists

Binary Search Trees

# Linked Lists

## Introduction

**Linked list:**

# Linked Lists

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### **Linked list:**

- ▶ Dynamic datastructure

# Linked Lists

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- ▶ Number of elements changeable

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# Linked Lists

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- ▶ Data elements can be simple types or composed datastructures
- ▶ **Elements are linked** through references / pointer to the predecessor / successor

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- ▶ Number of elements changeable
- ▶ Data elements can be simple types or composed datastructures
- ▶ **Elements are linked** through references / pointer to the predecessor / successor
- ▶ Single / doubly linked lists possible

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### Linked list:

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- ▶ Single / doubly linked lists possible



Figure: Linked list



# Linked Lists

## Introduction

**Properties in comparison to an array:**

# Linked Lists

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### **Properties in comparison to an array:**

- ▶ Minimal extra space for storing pointer

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- ▶ The number of elements can be simply modified

# Linked Lists

## Introduction

### **Properties in comparison to an array:**

- ▶ Minimal extra space for storing pointer
- ▶ We do not need to copy elements on `insert` or `remove`
- ▶ The number of elements can be simply modified
- ▶ No direct access of elements  
⇒ We have to iterate over the list

# Linked Lists

## Variants

**List with head / last element pointer:**

# Linked Lists

## Variants

**List with head / last element pointer:**

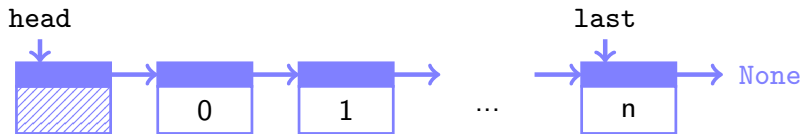


Figure: Singly linked list

# Linked Lists

## Variants

### List with head / last element pointer:



Figure: Singly linked list

- ▶ Head element has pointer to first list element



# Linked Lists

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### List with head / last element pointer:



Figure: Singly linked list

- ▶ Head element has pointer to first list element
- ▶ May also hold additional information:

# Linked Lists

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### List with head / last element pointer:



Figure: Singly linked list

- ▶ Head element has pointer to first list element
- ▶ May also hold additional information:
  - ▶ Number of elements

# Linked Lists

## Variants

**Doubly linked list:**

# Linked Lists

## Variants

### Doubly linked list:



Figure: Doubly linked list

# Linked Lists

## Variants

### Doubly linked list:



Figure: Doubly linked list

- Pointer to successor element

# Linked Lists

## Variants

### Doubly linked list:



Figure: Doubly linked list

- ▶ Pointer to successor element
- ▶ Pointer to predecessor element

# Linked Lists

## Variants

### Doubly linked list:



Figure: Doubly linked list

- ▶ Pointer to successor element
- ▶ Pointer to predecessor element
- ▶ Iterate forward and backward

# Linked Lists

## Implementation - Node/Element - Python

```
class Node:
    """ Defines a node of a singly linked
        list.
    """

    def __init__(self, value, nextNode):
        self.value = value
        self.nextNode = nextNode

    def __init__(self, value):
        self.value = value;
        self.nextNode = None
```



# Linked Lists

Usage examples

**Creating linked lists - Python:**

# Linked Lists

## Usage examples

### Creating linked lists - Python:

► `first = Node(7)`



# Linked Lists

## Usage examples

### Creating linked lists - Python:

► `first = Node(7)`



► `first.nextNode = Node(3)`



# Linked Lists

## Usage examples

### Creating linked lists - Python:

- ▶ `first = Node(7)`



- ▶ `first.nextNode = Node(3)`



- ▶ `first.nextNode.value = 4`



# Linked Lists

## Implementation - Insert

**Inserting a node after node cur:**



# Linked Lists

## Implementation - Insert

**Inserting a node after node `cur`:**

# Linked Lists

## Implementation - Insert

**Inserting a node after node cur:**

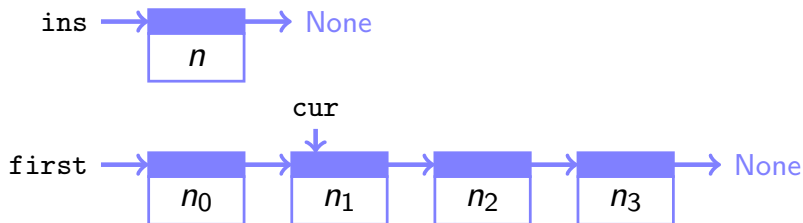
- ▶ `ins = Node(n)`

# Linked Lists

## Implementation - Insert

**Inserting a node after node cur:**

► `ins = Node(n)`





# Linked Lists

## Implementation - Insert

**Inserting a node after node cur:**

- ▶ `ins.nextNode = cur.nextNode`

# Linked Lists

## Implementation - Insert

**Inserting a node after node cur:**

► `ins.nextNode = cur.nextNode`



# Linked Lists

## Implementation - Insert

**Inserting a node after node cur:**

- ▶ `cur.nextNode = ins`

# Linked Lists

## Implementation - Insert

**Inserting a node after node `cur`:**

► `cur.nextNode = ins`



# Linked Lists

## Implementation - Insert

**Inserting a node after node `cur` - single line of code:**

# Linked Lists

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## Implementation - Insert

**Inserting a node after node `cur` - single line of code:**



► `cur.nextNode = Node(value, cur.nextNode)`

# Linked Lists

## Implementation - Insert

**Inserting a node after node `cur` - single line of code:**



► `cur.nextNode = Node(value, cur.nextNode)`

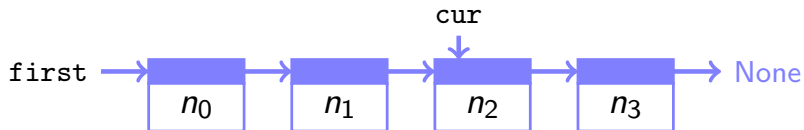




# Linked Lists

## Implementation - Remove

**Removing a node** `cur`:



# Linked Lists

## Implementation - Remove

**Removing a node** `cur:`

# Linked Lists

## Implementation - Remove

### Removing a node cur:

- Find the predecessor of cur:

```
pre = first
while pre.nextNode != cur:
    pre = pre.nextNode
```

# Linked Lists

## Implementation - Remove

### Removing a node `cur`:

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- Runtime of  $O(n)$

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- ▶ Runtime of  $O(n)$
- ▶ Does not work for first node!

# Linked Lists

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while pre.nextNode != cur:  
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# Linked Lists

## Implementation - Remove

**Removing a node** `cur:`

# Linked Lists

## Implementation - Remove

### **Removing a node** `cur`:

- ▶ Update the pointer to the next element:  
`pre.nextNode = cur.nextNode`



# Linked Lists

## Implementation - Remove

### Removing a node `cur`:

- ▶ Update the pointer to the next element:  
`pre.nextNode = cur.nextNode`
- ▶ `cur` will get automatically destroyed if no more references exist (`cur=None`)

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- ▶ Update the pointer to the next element:  
`pre.nextNode = cur.nextNode`
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# Linked Lists

## Implementation - Remove

**Removing the first node:**

# Linked Lists

## Implementation - Remove

**Removing the first node:**



# Linked Lists

## Implementation - Remove

### Removing the first node:



- Update the pointer to the next element:  
`first = first.nextNode`

# Linked Lists

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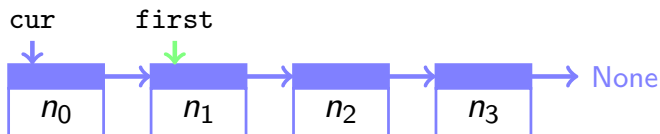
# Linked Lists

## Implementation - Remove

### Removing the first node:



- Update the pointer to the next element:  
`first = first.nextNode`
- `cur` will get automatically destroyed if no more references exist  
(`cur=None`)



# Linked Lists

## Implementation - Remove

**Removing a node** cur: (General case)

```
if cur == first:
    first = first.nextNode
else:
    pre = first
    while pre.nextNode != cur:
        pre = pre.nextNode

    pre.nextNode = cur.nextNode
```



# Linked Lists

## Implementation - Head Node

**Using a head node:**

# Linked Lists

## Implementation - Head Node

### **Using a head node:**

- ▶ Advantage:

# Linked Lists

## Implementation - Head Node

### **Using a head node:**

- ▶ Advantage:
  - ▶ Deleting the first node is no special case

# Linked Lists

## Implementation - Head Node

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  - ▶ We have to consider the first node at other operations

# Linked Lists

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  - ▶ We have to consider the first node at other operations
  - ▶ Iterating all nodes
  - ▶ Counting of all nodes

# Linked Lists

## Implementation - Head Node

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  - ▶ Iterating all nodes
  - ▶ Counting of all nodes
  - ▶ ...



# Linked Lists

## Implementation - LinkedList - Python

```
class LinkedList:
    def __init__(self):
        self.itemCount = 0
        self.head = Node()
        self.last = self.head

    def size(self):
        return self.itemCount

    def isEmpty(self):
        return self.itemCount == 0
```



# Linked Lists

## Implementation - LinkedList - Python

```
def append(self, value):
```

```
...
```

```
def insertAfter(self, cur, value):
```

```
...
```

```
def remove(self, cur):
```

```
...
```

```
def get(self, position):
```

```
...
```

```
def contains(self, value):
```

```
...
```

# Linked Lists

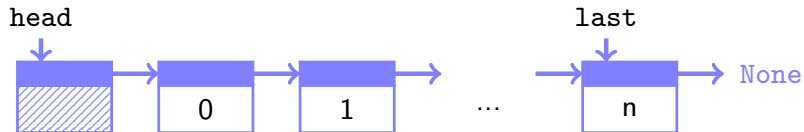
## Implementation

**Head, last:**

# Linked Lists

## Implementation

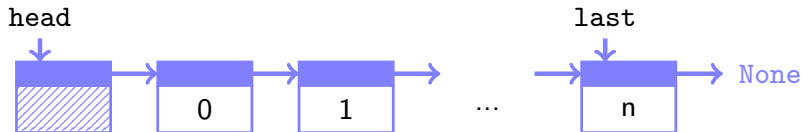
**Head, last:**



# Linked Lists

## Implementation

**Head, last:**



- Head points to the first node, last to the last node

# Linked Lists

## Implementation

### Head, last:



- ▶ Head points to the first node, last to the last node
- ▶ We can append elements to the end of the list in  $O(1)$  through the last node

# Linked Lists

## Implementation

### Head, last:



- ▶ Head points to the first node, last to the last node
- ▶ We can append elements to the end of the list in  $O(1)$  through the last node
- ▶ We have to keep the pointer to last updated after all operations

# Linked Lists

## Implementation - Append

**Appending an element:**

# Linked Lists

## Implementation - Append

### Appending an element:





# Linked Lists

## Implementation - Append

### Appending an element:



```
def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```

# Linked Lists

## Implementation - Append

### Appending an element:



```
def append(self, value):  
    last.nextNode = Node(value)  
    last = last.NextNode  
    itemCount += 1
```

- ▶ The pointer to last avoids the iteration of the whole list

# Linked Lists

## Implementation - Insert After

**Inserting after node cur:**



# Linked Lists

## Implementation - Insert After

**Inserting after node** `cur`:

- ▶ The pointer to head is not modified

# Linked Lists

## Implementation - Insert After

### Inserting after node cur:

- The pointer to head is not modified

```
def insertAfter(self, cur, value):  
    if cur == last:  
        # also update last node  
        append(value)  
    else:  
        # last node is not modified  
        cur.nextNode = Node(value, \  
                             cur.nextNode)  
        itemCount += 1
```

# Linked Lists

## Implementation - Remove

**Remove node** `cur`:



# Linked Lists

## Implementation - Remove

**Remove node** `cur`:

- ▶ Searching the predecessor in  $O(n)$

# Linked Lists

## Implementation - Remove

### Remove node cur:

- ▶ Searching the predecessor in  $O(n)$

```
def remove(self, cur):  
    pre = first  
    while pre.nextNode != cur:  
        pre = pre.nextNode  
  
    pre.nextNode = cur.nextNode  
    itemCount -= 1  
  
    if pre.nextNode == None:  
        last = pre
```



# Linked Lists

## Implementation - Get

**Getting a reference to node at pos:**

- ▶ Iterate the entries of the list until at position in  $O(n)$

# Linked Lists

## Implementation - Get

### Getting a reference to node at pos:

- Iterate the entries of the list until at position in  $O(n)$

```
def get(self, pos):  
    if pos < 0 or pos >= itemCount:  
        return None  
  
    cur = head  
    for i in range(0, pos):  
        cur = cur.nextNode  
  
    return cur
```

# Linked Lists

Implementation - Contains

**Searching a value:**

# Linked Lists

## Implementation - Contains

### **Searching a value:**

- ▶ First element is head without an assigned value

# Linked Lists

## Implementation - Contains

### **Searching a value:**

- ▶ First element is head without an assigned value
- ▶ Iterate the entries of the list until value found in  $O(n)$

# Linked Lists

## Implementation - Contains

### Searching a value:

- ▶ First element is head without an assigned value
- ▶ Iterate the entries of the list until value found in  $O(n)$

```
def contains(self, value):  
    cur = head  
  
    for i in range(0, itemCount):  
        cur = cur.nextNode  
        if cur.value == value:  
            return True  
  
    return False
```

# Linked Lists

Runtime

**Runtime:**

# Linked Lists

## Runtime

### **Runtime:**

- ▶ Singly linked list:



# Linked Lists

## Runtime

### Runtime:

- ▶ Singly linked list:
  - ▶ `next` in  $O(1)$

# Linked Lists

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### Runtime:

- ▶ Singly linked list:
  - ▶ `next` in  $O(1)$
  - ▶ `previous` in  $\Theta(n)$

# Linked Lists

## Runtime

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- ▶ Singly linked list:
  - ▶ `next` in  $O(1)$
  - ▶ `previous` in  $\Theta(n)$
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# Linked Lists

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  - ▶ `insert` in  $O(1)$
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  - ▶ `lookup` in  $\Theta(n)$

# Linked Lists

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- ▶ Singly linked list:
  - ▶ `next` in  $O(1)$
  - ▶ `previous` in  $\Theta(n)$
  - ▶ `insert` in  $O(1)$
  - ▶ `remove` in  $\Theta(n)$
  - ▶ `lookup` in  $\Theta(n)$
- ▶ Better with `doubly linked lists`

# Linked Lists

## Doubly Linked List

**Doubly linked list:**

# Linked Lists

## Doubly Linked List

### **Doubly linked list:**

- ▶ Each node has a reference to its successor and its predecessor



# Linked Lists

## Doubly Linked List

### **Doubly linked list:**

- ▶ Each node has a reference to its successor and its predecessor
- ▶ We can iterate the list forward and backward

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**Doubly linked list:**

# Linked Lists

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### **Doubly linked list:**

- ▶ It is helpful to have a **head** node

# Linked Lists

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### **Doubly linked list:**

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- ▶ We only need **one head** node if we connect the list cyclic

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# Linked Lists

## Runtime

**Runtime of doubly linked list:**

# Linked Lists

## Runtime

### **Runtime of doubly linked list:**

- ▶ `next` and `previous` in  $O(1)$



# Linked Lists

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Each element has a pointer to pred-/sucessor

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A constant number of pointers needs to be modified

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Even if the elements are sorted we can only retrieve them in  $\Theta(n)$

# Linked Lists

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Even if the elements are sorted we can only retrieve them in  $\Theta(n)$  Why?

# Linked Lists

List in real program

**Linked list in book:**



# Linked Lists

List in real program

## Linked list in memory:





# Structure

Sorted Sequences

Linked Lists

Binary Search Trees

# Binary Search Trees

## Introduction

**Runtime of a search tree:**

# Binary Search Trees

## Introduction

### Runtime of a search tree:

- ▶ `next` and `previous` in  $O(1)$

# Binary Search Trees

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Pointers corresponding to linked list

# Binary Search Trees

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### Runtime of a search tree:

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- ▶ `insert` and `remove` in  $O(\log n)$

# Binary Search Trees

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- ▶ `next` and `previous` in  $O(1)$

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### Runtime of a search tree:

- ▶ `next` and `previous` in  $O(1)$

Pointers corresponding to linked list

- ▶ `insert` and `remove` in  $O(\log n)$

- ▶ `lookup` in  $O(\log n)$

The structure helps searching efficiently

# Binary Search Trees

## Introduction

**Idea:**



# Binary Search Trees

## Introduction

### **Idea:**

- ▶ We define a total order for the search tree

# Binary Search Trees

## Introduction

### Idea:

- ▶ We define a total order for the search tree
- ▶ All nodes of the left subtree have **smaller keys** than the current node

# Binary Search Trees

## Introduction

### Idea:

- ▶ We define a total order for the search tree
- ▶ All nodes of the left subtree have **smaller keys** than the current node
- ▶ All nodes of the right subtree have **bigger keys** than the current node

# Binary Search Trees

## Introduction

- Edge direction indicates ordering

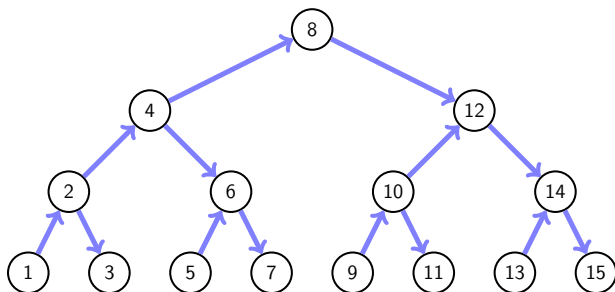


Figure: A binary search tree

# Binary Search Trees

## Introduction

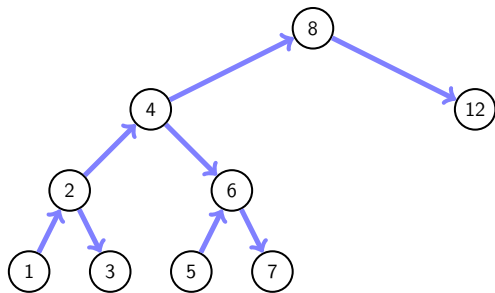


Figure: Another binary search tree

# Binary Search Trees

## Introduction

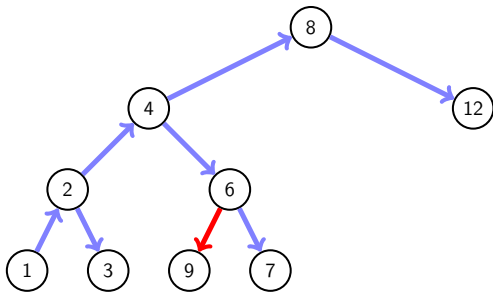


Figure: **Not** a binary search tree

# Binary Search Trees

## Implementation

### **Implementation:**

# Binary Search Trees

## Implementation

### **Implementation:**

- ▶ For the heap we had all elements stored in an array
- ▶ Here we link all nodes through pointer / references, like linked lists



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- ▶ `None` for missing children

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# Binary Search Trees

## Implementation

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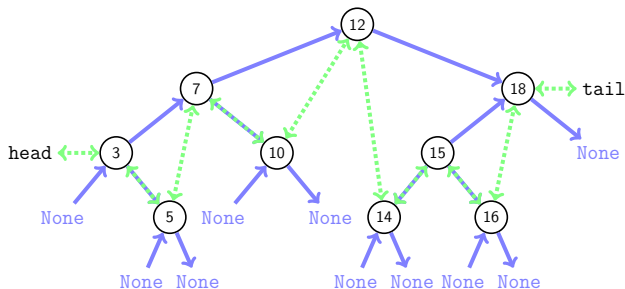


Figure: Binary search tree with links

# Binary Search Trees

## Implementation - Lookup

**Lookup:**



# Binary Search Trees

## Implementation - Lookup

### **Lookup:**

- ▶ Definition:  
“ Search the element with the given key. If no element is found return the element with the next (bigger) key. ”

# Binary Search Trees

## Implementation - Lookup

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## Implementation - Lookup

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  - ▶ Go to the left / right until the child is **None** or the key is found

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  - ▶ If the key is not found return the next bigger one

# Binary Search Trees

## Implementation - Lookup

**For each node applies the total order:**

# Binary Search Trees

## Implementation - Lookup

**For each node applies the total order:**

keys of left subtree | `node.key` | keys of right subtree

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**Examples:**

Figure: Binary search tree with total order “i”



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## Implementation - Lookup

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**Examples:**

`lookup(14)`

Figure: Binary search tree with total order “i”

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**Examples:**

`lookup(14)`

`lookup(6)`

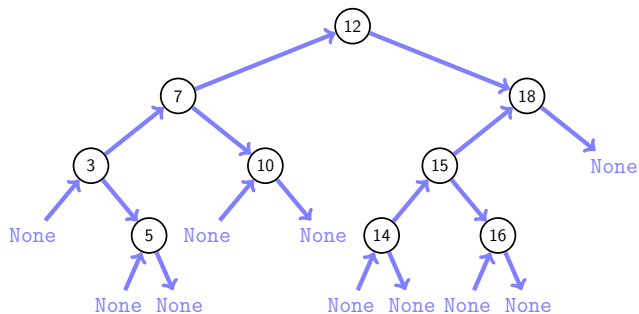
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## Implementation - Lookup

**For each node applies the total order:**

keys of left subtree | `node.key` | keys of right subtree



**Examples:**

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`lookup(6)`

`lookup(19)`

Figure: Binary search tree with total order “`i`”

# Binary Search Trees

Implementation - Insert

**Insert:**

# Binary Search Trees

## Implementation - Insert

### **Insert:**

- ▶ We search for the key in our search tree

# Binary Search Trees

## Implementation - Insert

### **Insert:**

- ▶ We search for the key in our search tree
- ▶ If a node is found we replace the value with the new one

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## Implementation - Insert

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- ▶ If a node is found we replace the value with the new one
- ▶ Else we insert a new node

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# Binary Search Trees

## Implementation - Insert

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# Binary Search Trees

## Implementation - Remove

**Remove:** Case 1: The node “5” has no children

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 1: The node “5” has no children

- ▶ Find **parent** of node “5” (“6”)

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 1: The node "5" has no children

- ▶ Find **parent** of node "5" ("6")
- ▶ Set left / right child of node "6" to **None** depending on position of node "5"

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 1: The node "5" has no children

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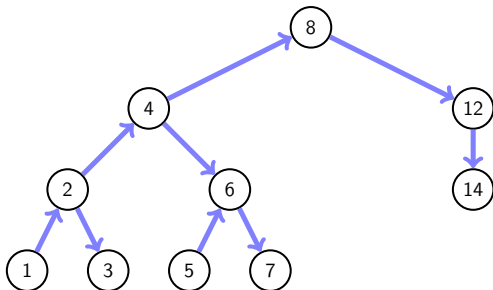


Figure: Binary search tree with total order "i"

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 1: The node "5" has no children

- ▶ Find **parent** of node "5" ("6")
- ▶ Set left / right child of node "6" to **None** depending on position of node "5"



Figure: Binary search tree after deleting node "5"

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 2: The node “12” has one child



# Binary Search Trees

## Implementation - Remove

**Remove:** Case 2: The node “12” has one child

- Find the **child** of node “12” (“14”)

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 2: The node “12” has one child

- ▶ Find the **child** of node “12” (“14”)
- ▶ Find the **parent** of node “12” (“8”)

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 2: The node "12" has one child

- ▶ Find the **child** of node "12" ("14")
- ▶ Find the **parent** of node "12" ("8")
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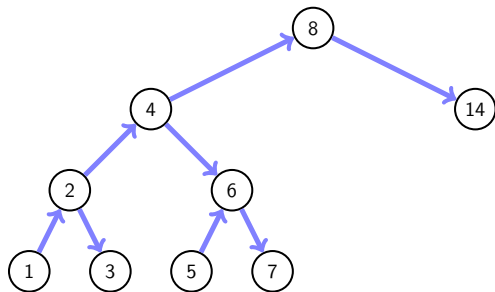


Figure: Binary search tree after deleting node "12"

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 3: The node “4” has two children

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 3: The node “4” has two children

- Find the **successor** of node “4” (“5”)

# Binary Search Trees

## Implementation - Remove

**Remove:** Case 3: The node “4” has two children

- ▶ Find the **successor** of node “4” (“5”)
- ▶ Replace the value of node “4” with the value of node “5”



# Binary Search Trees

## Implementation - Remove

**Remove:** Case 3: The node “4” has two children

- ▶ Find the **successor** of node “4” (“5”)
- ▶ Replace the value of node “4” with the value of node “5”
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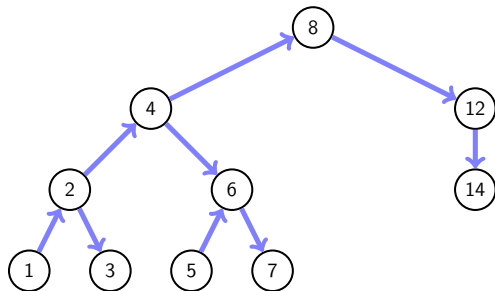
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- ▶ There is no left node because we are deleting the **predecessor**

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# Binary Search Trees

Runtime Complexity

**How long takes `insert` and `lookup`?**

# Binary Search Trees

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- ▶ Up to  $\Theta(d)$ , with  $d$  being the depth of the tree  
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**Figure:** Degenerated binary tree  $d = n$

# Binary Search Trees

## Runtime Complexity

### How long takes *insert* and *lookup*?

- ▶ Up to  $\Theta(d)$ , with  $d$  being the *depth of the tree* (The longest path from the root to a leaf)
- ▶ *Best case* with  $d = \log n$  the runtime is  $\Theta(\log n)$
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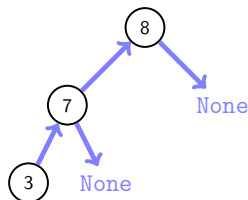


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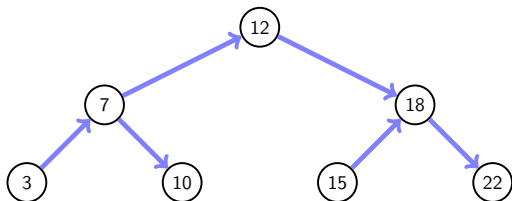


Figure: Complete binary tree  $d = \log n$

► **General**

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*Introduction to Algorithms.*

MIT Press, Cambridge, Mass, 2001.

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## ► **Linked List**

[Wik] [Linked list](https://en.wikipedia.org/wiki/Linked_list)

`https://en.wikipedia.org/wiki/Linked\_list`

## ► **Binary Search Tree**

[Wik] [Binary search tree](https://en.wikipedia.org/wiki/Binary_search_tree)

`https:`

`//en.wikipedia.org/wiki/Binary\_search\_tree`