

Algorithms and Datastructures

Levenshtein distance, Dynamic programming

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Algorithms and Datastructures, February 2018

Structure

Introduction

Edit distance

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Edit distance:

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Edit distance:

- ▶ Measurement for similarity of two words / strings

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- ▶ Measurement for similarity of two words / strings
- ▶ Algorithm for efficient calculation

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Edit distance:

- ▶ Measurement for similarity of two words / strings
- ▶ Algorithm for efficient calculation
- ▶ General principle: dynamic programming

Introduction

Motivation: Error tolerant string comparison

BioInfSearch



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ejafjatljökull
eyjafjallajökull
eyjafjallajökull movie
eyjafjallajölull trailer

Search!

Wikipedia.org:

"Der Eyjafjallajökull ([ˈeɪjaˌfjatlaˌjœːkʏtʃ])[3], auf Deutsch Eyjafjöll-Gletscher, ist der sechstgrößte Gletscher Islands.

Er liegt an der äußersten Südküste, westlich des Gletschers Mýrdalsjökull in der Gemeinde Rangárþing eystra, die größte Höhe beträgt 1651 m. Unter dem Gletscher befindet sich der Vulkan Eyjafjöll mit eigener Magmakammer, der seit der Besiedelung von Island in den Jahren 920, 1612 (oder 1613), 1821 bis 1823 und zuletzt im Jahr 2010 aktiv war."

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Motivation

A lot of applications where similar string are searched:

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- ▶ Duplicates in databases:

Hein Blöd 27568 Bremerhaven

Hein Bloed 27568 Bremerhafen

Hein Doof 27478 Cuxhaven

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- ▶ Product search:

memory stik

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- ▶ Bioinformatics: Similarity of DNA-sequences

Introduction

Example: Bioinformatics DNA-matching

Search of similar proteins:

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Search of similar proteins:

- ▶ BLAST (**B**asic **L**ocal **A**lignment **S**earch **T**ool)

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- ▶ Alignment $\hat{=}$ Edit distance

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Example: Bioinformtics DNA-matching

Search of similar proteins:

- ▶ BLAST (**B**asic **L**ocal **A**lignment **S**earch **T**ool)
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- ▶ Changed life-science completely

Introduction

Example: Bioinformatics DNA-matching

Search of similar proteins:

- ▶ BLAST (**B**asic **L**ocal **A**lignment **S**earch **T**ool)
- ▶ Alignment $\hat{=}$ Edit distance
- ▶ Changed life-science completely
- ▶ Cited 63437 times on Google Scholar (Sep. 2017)

Structure

Introduction

Edit distance

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Definition of edit distance: (*Levenshtein-distance*)

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- ▶ Let x , y be two strings
- ▶ Edit distance $ED(x, y)$ of x and y :
The minimal number of operations to transform x into y

Edit distance

Definition of edit distance: (*Levenshtein-distance*)

- ▶ Let x , y be two strings
- ▶ Edit distance $ED(x, y)$ of x and y :
The minimal number of operations to transform x into y
 - ▶ Insert a character

Edit distance

Definition of edit distance: (*Levenshtein-distance*)

- ▶ Let x , y be two strings
- ▶ Edit distance $ED(x, y)$ of x and y :
The minimal number of operations to transform x into y
 - ▶ Insert a character
 - ▶ Replace a character with another

Edit distance

Definition of edit distance: (*Levenshtein-distance*)

- ▶ Let x , y be two strings
- ▶ Edit distance $ED(x, y)$ of x and y :
The minimal number of operations to transform x into y
 - ▶ Insert a character
 - ▶ Replace a character with another
 - ▶ Delete a character

Edit distance

Example

1 2 3 4 5
D O O F

B L O E D

Edit distance

Example

1 2 3 4 5

DOOF

↓

BOOF

replace(1, B)

BLOED

Edit distance

Example

1 2 3 4 5

DOOF

↓

replace(1, B)

BOOF

↓

replace(2, L)

BLOF

BLOED

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
BLOED	

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
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BLOED	


ED=4

Edit distance

Example

1 2 3 4 5

DOOF

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replace(1, B)

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1 2 3 4 5

BLOED

⏟
ED=4

Edit distance

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1 2 3 4 5

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replace(1, B)

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BLOED

1 2 3 4 5

B LOED

DOOF

⏟
ED=4

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1 2 3 4 5

DOOF

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replace(1, B)

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insert(4, E)

BLOEF

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BLOED

1 2 3 4 5

B LOED

↓

replace(5, F)

B LOEF

DOOF

ED=4

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	

ED=4

1 2 3 4 5	
B LOED	
↓	replace(5, F)
B LOEF	
↓	delete(4)
B LOF	
DOOF	

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	


ED=4

1 2 3 4 5	
B LOED	
↓	replace(5, F)
B LOEF	
↓	delete(4)
B LOF	
↓	replace(2, O)
B OOF	
DOOF	

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	



ED=4

1 2 3 4 5	
B LOED	
↓	replace(5, F)
B LOEF	
↓	delete(4)
B LOF	
↓	replace(2, O)
BOOF	
↓	replace(1, D)
DOOF	

Edit distance

Example

1 2 3 4 5	
DOOF	
↓	replace(1, B)
BOOF	
↓	replace(2, L)
BLOF	
↓	insert(4, E)
BLOEF	
↓	replace(5, D)
BLOED	
	
ED=4	

1 2 3 4 5	
B LOED	
↓	replace(5, F)
B LOEF	
↓	delete(4)
B LOF	
↓	replace(2, O)
B OOF	
↓	replace(1, D)
DOOF	
	
ED=4	

Edit distance

Notation:

Edit distance

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- ▶ ε is the empty string

Edit distance

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- ▶ ε is the empty string
- ▶ $|x|$ is the length of the string x (number of characters)

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- ▶ $|x|$ is the length of the string x (number of characters)
- ▶ $x[i..j]$ is the slice of x from i to j where $1 \leq i \leq j \leq |x|$

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Edit distance

Trivial facts:

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- ▶ $\text{ED}(x, y) = \text{ED}(y, x)$

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- ▶ $\text{ED}(x, \epsilon) = |x|$

Edit distance

Trivial facts:

- ▶ $ED(x, y) = ED(y, x)$
- ▶ $ED(x, \epsilon) = |x|$
- ▶ $ED(x, y) \geq \text{abs}(|x| - |y|)$

$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

Edit distance

Trivial facts:

- ▶ $ED(x, y) = ED(y, x)$
- ▶ $ED(x, \epsilon) = |x|$
- ▶ $ED(x, y) \geq \text{abs}(|x| - |y|)$
- ▶ $ED(x, y) \leq ED(x[1..n-1], y[1..m-1]) + 1$ $n = |x|, m = |y|$

$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

Edit distance

Solving examples

Solutions based on examples:

Edit distance

Solving examples

Solutions based on examples:

- ▶ From VERIEN to FERIE?

Edit distance

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- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?

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Solving examples

Solutions based on examples:

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
- ▶ From AAEBEAABEAREEEAEBA to RBEAAEEBAAAEBBAEAE?

Edit distance

Solving examples

Solutions based on examples:

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
- ▶ From AAEBEAABEAREEEAEBA to RBEAAEEBAAAEBBAEAE?
- ▶ Searching biggest substrings can yield the solution but doesn't have to

Edit distance

Solving examples

Solutions based on examples:

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
- ▶ From AAEBEAABEAREEEAEBA to RBEAAEEBAAAEBBAEAE?
- ▶ Searching biggest substrings can yield the solution but doesn't have to

Recursive approach:

Edit distance

Solving examples

Solutions based on examples:

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
- ▶ From AAEBEAABEAREEEAEBA to RBEAAEEBAAAEBBAEAE?
- ▶ Searching biggest substrings can yield the solution but doesn't have to

Recursive approach:

- ▶ Dividing in two halves? Not a good idea:

$ED(GRAU, RAUM) = 2$ but

$ED(GR, RA) + ED(AU, UM) = 4$

Edit distance

Solving examples

Solutions based on examples:

- ▶ From VERIEN to FERIEEN?
- ▶ From MEXIKO to AMERIKA?
- ▶ From AAEBEAAABEAREEEAEBA to RBEAAEEBAAAEBBAEAE?
- ▶ Searching biggest substrings can yield the solution but doesn't have to

Recursive approach:

- ▶ Dividing in two halves? Not a good idea:

$$\text{ED}(\text{GRAU}, \text{RAUM}) = 2 \quad \text{but} \\ \text{ED}(\text{GR}, \text{RA}) + \text{ED}(\text{AU}, \text{UM}) = 4$$

- ▶ Finding “smaller” sub problems?
Let's try it!

Edit distance

Terminology:

Edit distance

Terminology:

- ▶ Let x , y be two strings

Edit distance

Terminology:

- ▶ Let x, y be two strings
- ▶ Let $\sigma_1, \dots, \sigma_k$ be a sequence of k operations where $k = \text{ED}(x, y)$ for $x \rightarrow y$ (transform x into y)
(We do not know this sequence but we assume it exists)

Edit distance

Terminology:

Edit distance

Terminology:

- ▶ We only consider **monotonous** sequences:
The position of σ_{i+1} is \geq the position of σ_i where we only allow the positions to be equal on a delete operation

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The position of σ_{i+1} is \geq the position of σ_i where we only allow the positions to be equal on a delete operation

1 2 3 4 5

DOOF



replace(1, B)

BOOF



replace(2, L)

BLOF



insert(4, E)

BLOEF



replace(5, D)

BLOED

1 2 3 4 5 6 7

SAUDOOOF



delete(1)

AUDOOOF



delete(1)

UDOOOF



delete(1)

DOOF



insert(4, 0)

DOOOOF

Edit distance

Terminology:

- ▶ We only consider **monotonous** sequences:

The position of σ_{i+1} is \geq the position of σ_i where we only allow the positions to be equal on a delete operation

1 2 3 4 5

DOOF



replace(1, B)

BOOF



replace(2, L)

BLOF



insert(4, E)

BLOEF



replace(5, D)

BLOED

1 2 3 4 5 6 7

SAUDOOOF



delete(1)

AUDOOOF



delete(1)

UDOOOF



delete(1)

DOOF



insert(4, 0)

DOOOOF

Edit distance

Terminology:

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Terminology:

- **Lemma:** For any x and y with $k = \text{ED}(x, y)$ exists a **monotonous** sequence of k operations for $x \rightarrow y$

Edit distance

Terminology:

- ▶ **Lemma:** For any x and y with $k = \text{ED}(x, y)$ exists a **monotonous** sequence of k operations for $x \rightarrow y$
- ▶ **Intuition:** The order of our sequence is not relevant
(Therefore we can also sort them monotonously)

Edit distance

Terminology:

- ▶ **Lemma:** For any x and y with $k = \text{ED}(x, y)$ exists a **monotonous** sequence of k operations for $x \rightarrow y$
- ▶ **Intuition:** The order of our sequence is not relevant
(Therefore we can also sort them monotonously)

1	2	3	4	5
D	O	O	F	

B	L	O	E	D
---	---	---	---	---

1	2	3	4	5	6	7
S	A	U	D	O	O	F

D	O	O	O	F
---	---	---	---	---

Edit distance

Recursive approach

Consider the last operation:

Edit distance

Recursive approach

Consider the last operation:

- ▶ Solve **blue** part recursively

Edit distance

Recursive approach

Consider the last operation:

- Solve **blue** part recursively

DOOF

↓↓↓↓↓

BLOE

↓insert

BLOED

Figure: Case 1a

DOOF

↓↓↓↓↓↓↓

BLOEDF

↓delete

BLOED

Figure: Case 1b

DOOF

↓↓↓↓↓↓↓

BLOEF

↓replace

BLOED

Figure: Case 1c

Edit distance

Recursive approach

Consider the last operation:

Edit distance

Recursive approach

Consider the last operation:

- ▶ Solve **blue** part recursively

Edit distance

Recursive approach

Consider the last operation:

- Solve **blue** part recursively

W I N T E R
↓ ↓ ↓ ↓ ↓
S O M M E R
 ↓ nothing
S O M M E R

Figure: Case 2

Display of solution:

- Alignment
- Example:

-	-	-	B	L	O	E	D
S	A	U	B	L	O	E	D

Edit distance

Dynamic programming

Dynamic programming:

Edit distance

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Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:

Edit distance

Dynamic programming

Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
 - ▶ Shortest paths

Edit distance

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 - ▶ Edit distance



Figure: Richard Bellman
(1920 - 1984)

Edit distance

Dynamic programming

Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
 - ▶ Shortest paths
 - ▶ Edit distance



Figure: Richard Bellman
(1920 - 1984)

Edit distance

Dynamic programming

Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
 - ▶ Shortest paths
 - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions



Figure: Richard Bellman
(1920 - 1984)

Edit distance

Dynamic programming

Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
 - ▶ Shortest paths
 - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions
 - ▶ Shortest paths: Each partial path has to be optimal



Figure: Richard Bellman
(1920 - 1984)

Edit distance

Dynamic programming

Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
 - ▶ Shortest paths
 - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions
 - ▶ Shortest paths: Each partial path has to be optimal
 - ▶ Edit distance: Each partial alignment has to be optimal



Figure: Richard Bellman
(1920 - 1984)

Edit distance

Dynamic programming

Dynamic programming:

- ▶ Instances of Bellman's principle of optimality:
 - ▶ Shortest paths
 - ▶ Edit distance
- ▶ Optimal solutions consist of optimal partial solutions
 - ▶ Shortest paths: Each partial path has to be optimal
 - ▶ Edit distance: Each partial alignment has to be optimal
- ▶ Always solvable through dynamic programming (Caching of optimal partial solutions)



Figure: Richard Bellman
(1920 - 1984)

Edit distance

Case analysis:

Edit distance

Case analysis:

- ▶ We consider the last operation σ_k

Edit distance

Case analysis:

- ▶ We consider the last operation σ_k
 - ▶ $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$ and $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

Edit distance

Case analysis:

- ▶ We consider the last operation σ_k
 - ▶ $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$ and $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

- ▶ Let $n = |x|$, $m = |y|$, $m' = |z|$

Edit distance

Case analysis:

- ▶ We consider the last operation σ_k
 - ▶ $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow z$ and $\sigma_k: z \rightarrow y$

Example:

$x = \text{DOOF}, z = \text{SAUBLOEF}, y = \text{SAUBLOED}$

- ▶ Let $n = |x|$, $m = |y|$, $m' = |z|$
- ▶ We note $m' \in \{m-1, m, m+1\}$ why?

Edit distance

Case analysis:

Edit distance

Case analysis:

- ▶ Case 1: σ_k does something at the outer end:

Edit distance

Case analysis:

- ▶ Case 1: σ_k does something at the outer end:
 - ▶ Case 1a: $\sigma_k = \text{insert}(m' + 1, y[m])$ [then $m' = m - 1$]

Edit distance

Case analysis:

- ▶ Case 1: σ_k does something at the outer end:
 - ▶ Case 1a: $\sigma_k = \text{insert}(m' + 1, y[m])$ [then $m' = m - 1$]
 - ▶ Case 1b: $\sigma_k = \text{delete}(m')$ [then $m' = m + 1$]

Edit distance

Case analysis:

- ▶ Case 1: σ_k does something at the outer end:
 - ▶ Case 1a: $\sigma_k = \text{insert}(m' + 1, y[m])$ [then $m' = m - 1$]
 - ▶ Case 1b: $\sigma_k = \text{delete}(m')$ [then $m' = m + 1$]
 - ▶ Case 1c: $\sigma_k = \text{replace}(m', y[m])$ [then $m' = m$]

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Case analysis:

- ▶ Case 1: σ_k does something at the outer end:
 - ▶ Case 1a: $\sigma_k = \text{insert}(m' + 1, y[m])$ [then $m' = m - 1$]
 - ▶ Case 1b: $\sigma_k = \text{delete}(m')$ [then $m' = m + 1$]
 - ▶ Case 1c: $\sigma_k = \text{replace}(m', y[m])$ [then $m' = m$]
- ▶ Case 2: σ_k does nothing at the outer end:

Edit distance

Case analysis:

- ▶ Case 1: σ_k does something at the outer end:
 - ▶ Case 1a: $\sigma_k = \text{insert}(m' + 1, y[m])$ [then $m' = m - 1$]
 - ▶ Case 1b: $\sigma_k = \text{delete}(m')$ [then $m' = m + 1$]
 - ▶ Case 1c: $\sigma_k = \text{replace}(m', y[m])$ [then $m' = m$]
- ▶ Case 2: σ_k does nothing at the outer end:
 - ▶ Then $z[m'] = y[m]$ and $x[n'] = z[m']$ and with that
 $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$ and $x[n] = y[m]$

Edit distance

Case analysis:

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Case analysis:

- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$

Edit distance

Case analysis:

- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- ▶ Case 1b (delete): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y$

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Case analysis:

- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- ▶ Case 1b (delete): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y$
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- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
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- ▶ Case 1c (replace): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$
- ▶ Case 2 (nothing): $\sigma_1, \dots, \sigma_k: x[1..n-1] \rightarrow y[1..m-1]$

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Case analysis:

- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
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- ▶ Case 1c (replace): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$
- ▶ Case 2 (nothing): $\sigma_1, \dots, \sigma_k: x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

Edit distance

Case analysis:

- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- ▶ Case 1b (delete): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y$
- ▶ Case 1c (replace): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$
- ▶ Case 2 (nothing): $\sigma_1, \dots, \sigma_k: x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

- ▶ For $|x| > 0$ and $|y| > 0$ is $\text{ED}(x, y)$ the minimum of

Edit distance

Case analysis:

- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
- ▶ Case 1b (delete): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y$
- ▶ Case 1c (replace): $\sigma_1, \dots, \sigma_{k-1}: x[1..n-1] \rightarrow y[1..m-1]$
- ▶ Case 2 (nothing): $\sigma_1, \dots, \sigma_k: x[1..n-1] \rightarrow y[1..m-1]$

This results in the recursive formula:

- ▶ For $|x| > 0$ and $|y| > 0$ is $\text{ED}(x, y)$ the minimum of
 - ▶ $\text{ED}(x, y[1..m-1]) + 1$ and

Edit distance

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- ▶ Case 1a (insert): $\sigma_1, \dots, \sigma_{k-1}: x \rightarrow y[1..m-1]$
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- ▶ For $|x| = 0$ is $ED(x, y) = |y|$
- ▶ For $|y| = 0$ is $ED(x, y) = |x|$

Edit distance

Implementation - Python

```
def edit_distance(x, y):  
    if len(x) == 0:  
        return len(y)  
    if len(y) == 0:  
        return len(x)  
  
    ed1 = edit_distance(x, y[:-1]) + 1  
    ed2 = edit_distance(x[:-1], y) + 1  
    ed3 = edit_distance(x[:-1], y[:-1])  
    if x[-1] != y[-1]:  
        ed3 += 1  
  
    return min(ed1, ed2, ed3)
```

Edit distance

Runtime analysis

Recursive program:

Edit distance

Runtime analysis

Recursive program:

- ▶ The algorithm results in the following recursive formular:

$$\begin{aligned}T(n, m) &= T(n-1, m) + T(n, m-1) + T(n-1, m-1) + 1 \\&\geq T(n-1, m-1) + T(n-1, m-1) + T(n-1, m-1) \\&= 3 \cdot T(n-1, m-1)\end{aligned}$$

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- ▶ This results in $T(n, n) \geq 3^n$
- ⇒ The runtime is at least exponential

Edit distance

Dynamic programming:

Edit distance

Dynamic programming:

- ▶ We create a table with all possible combination of substrings and save calculated entries
- ▶ This results in a runtime and space consumption of $O(n \cdot m)$

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Edit distance

Dynamic programming:

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- ▶ This results in a runtime and space consumption of $O(n \cdot m)$

Visualization on the next slide:

- ▶ Operations always refer to the last position (indices are omitted)
- ▶ We also display the replaced character on a replace operation to visualize operations without costs
 $\Rightarrow \text{repl}(\text{A}, \text{A})$

Edit Distance

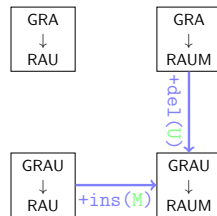
Edit Distance



Edit Distance



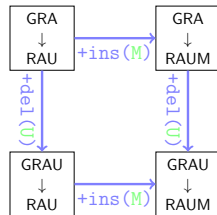
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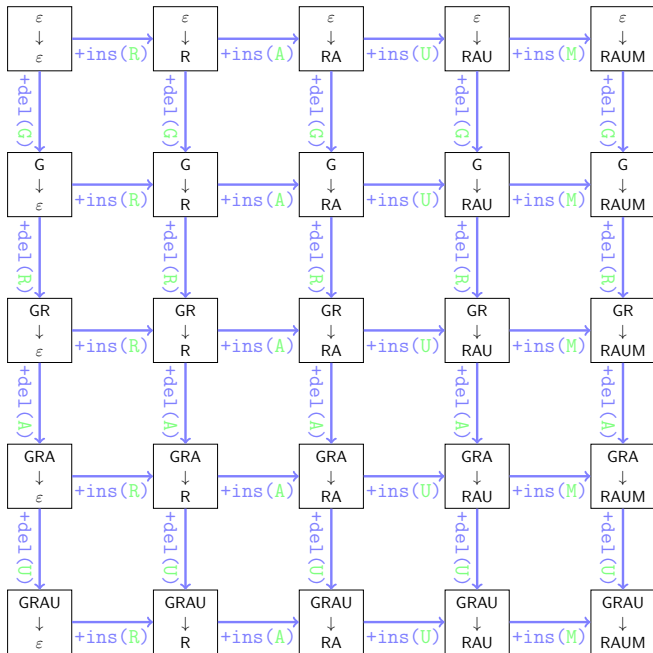
Edit Distance



Edit Distance



Edit Distance



Edit distance

Fast algorithm

Fast algorithm:

We can determine the **edit distance** for all combination of partial strings from the top left to bottom right.

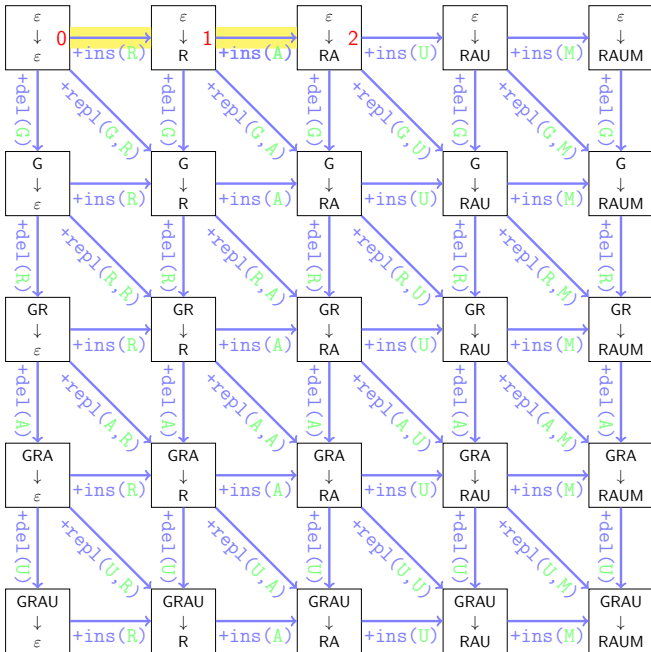
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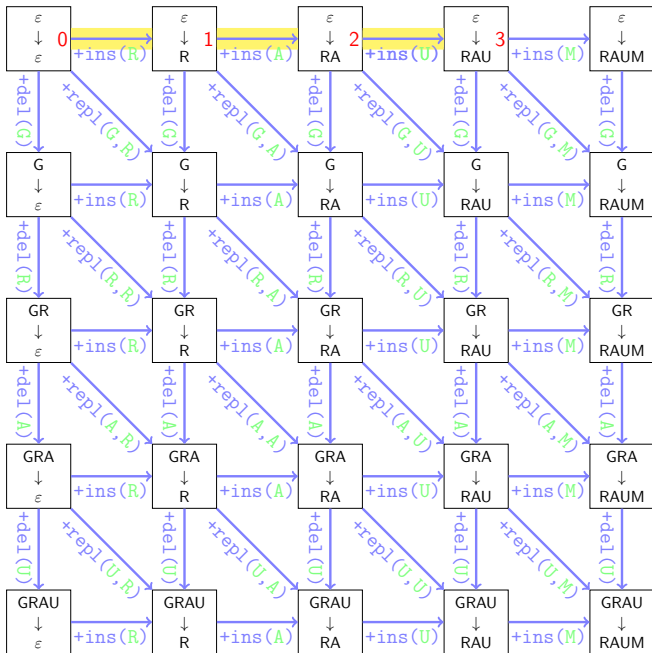
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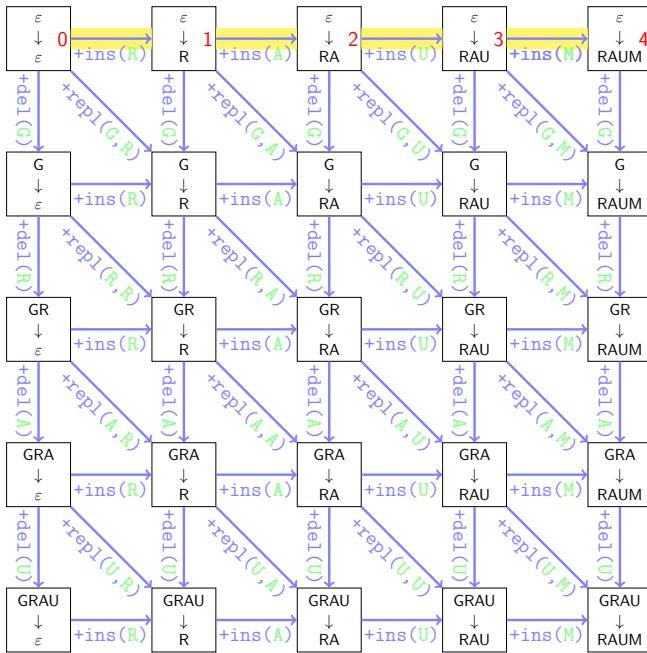
Edit Distance



Edit Distance



Edit Distance



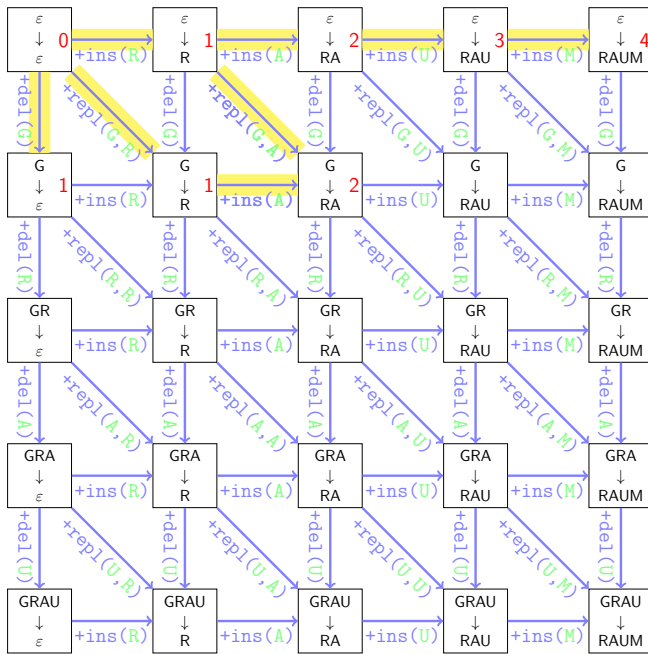
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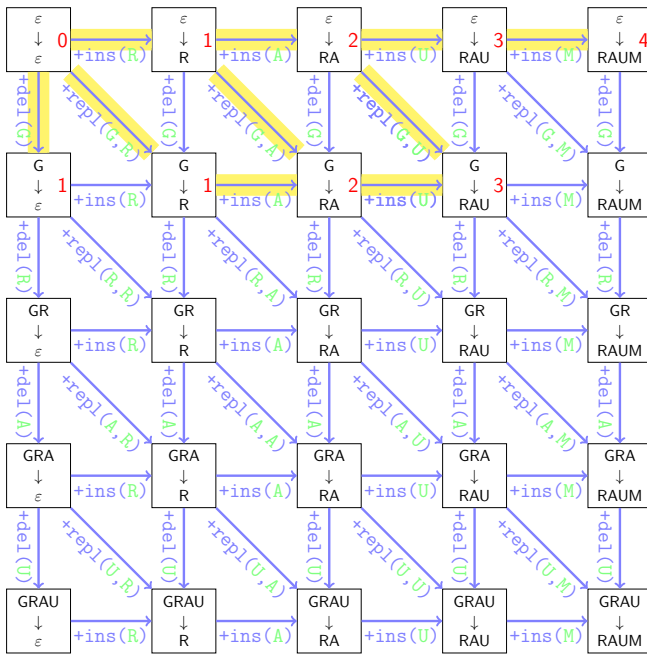
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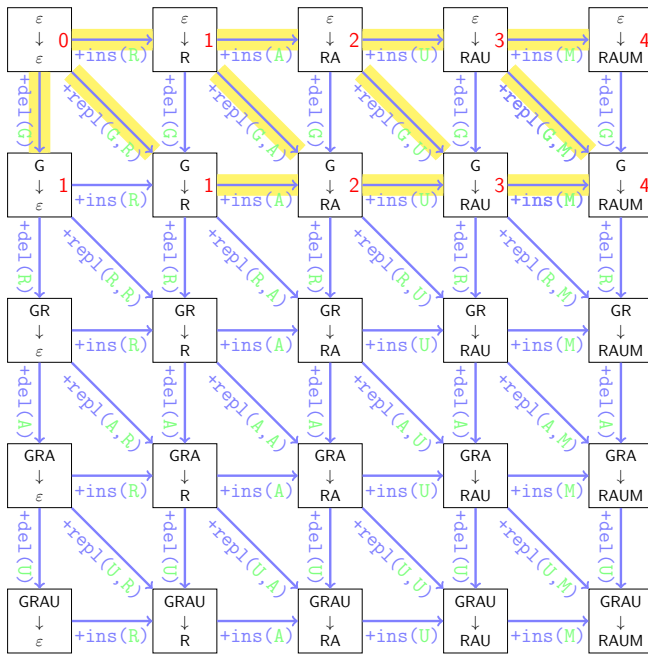
Edit Distance



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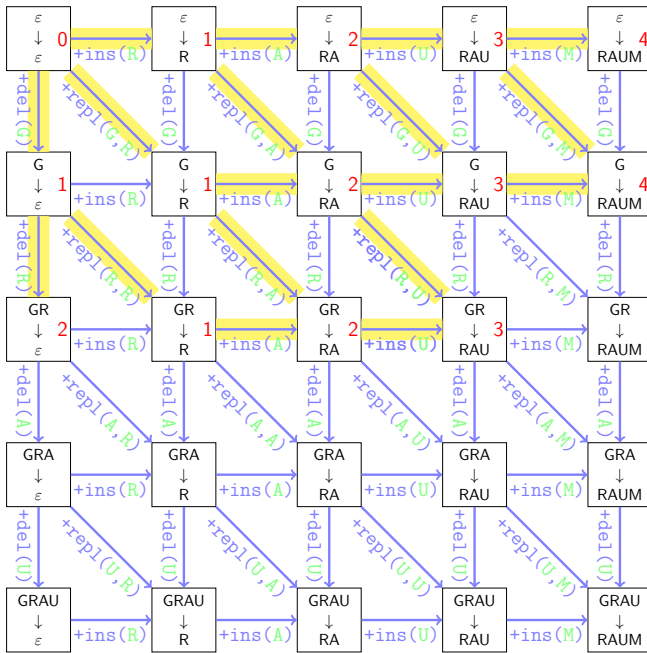
Edit Distance



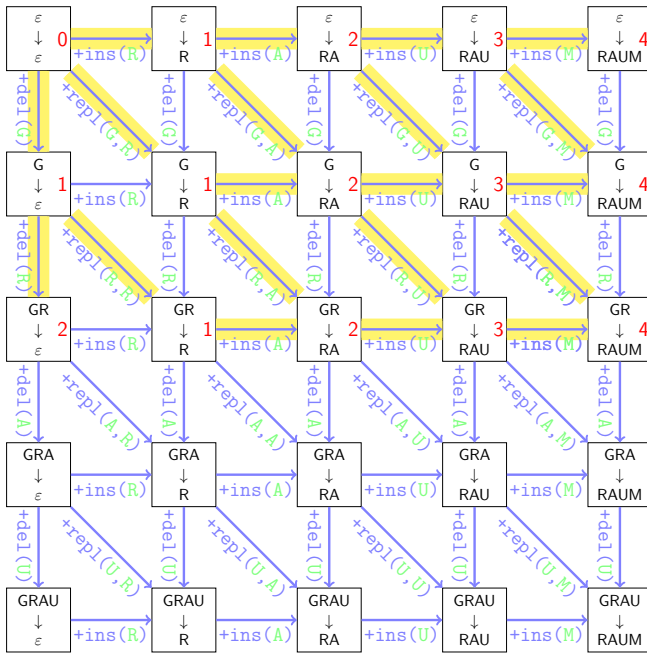
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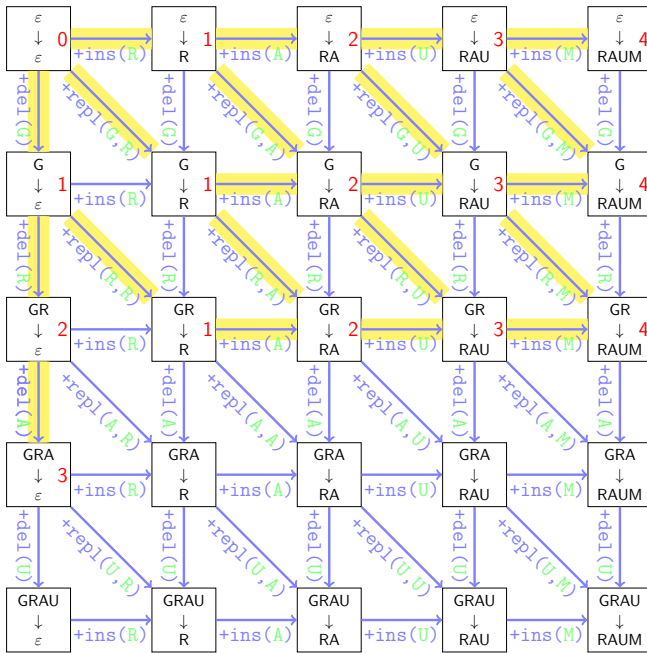
Edit Distance



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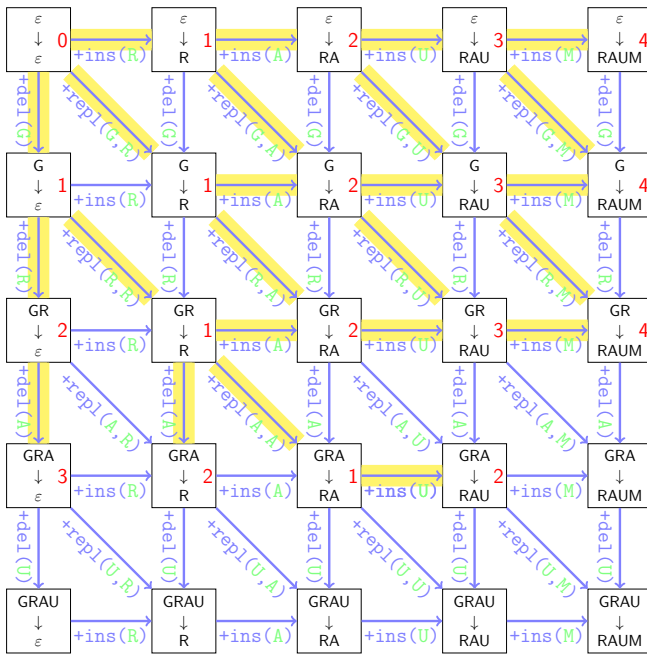
Edit Distance



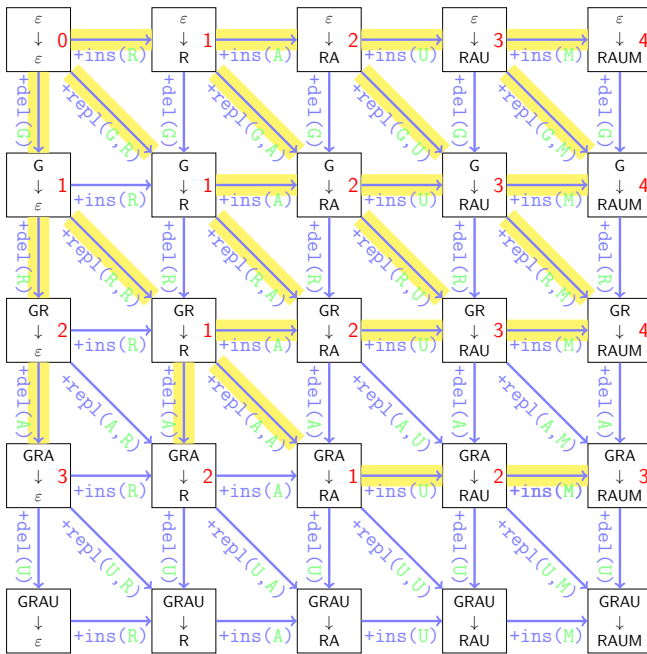
Edit Distance



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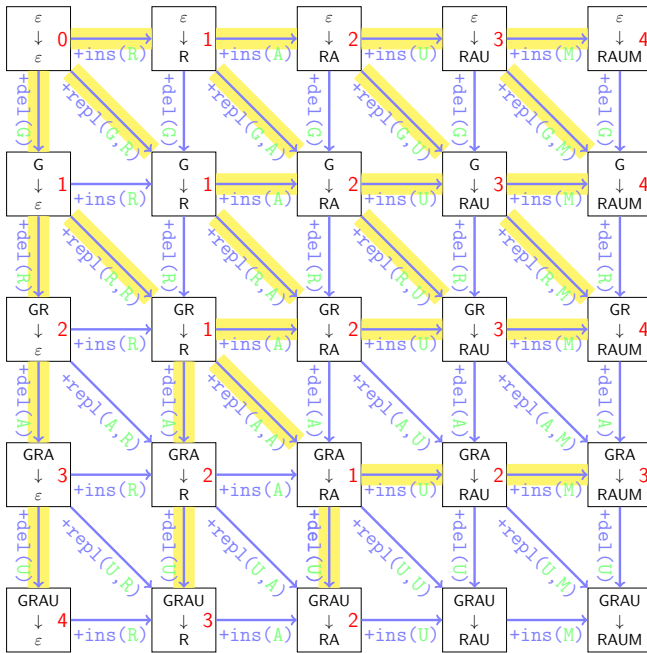
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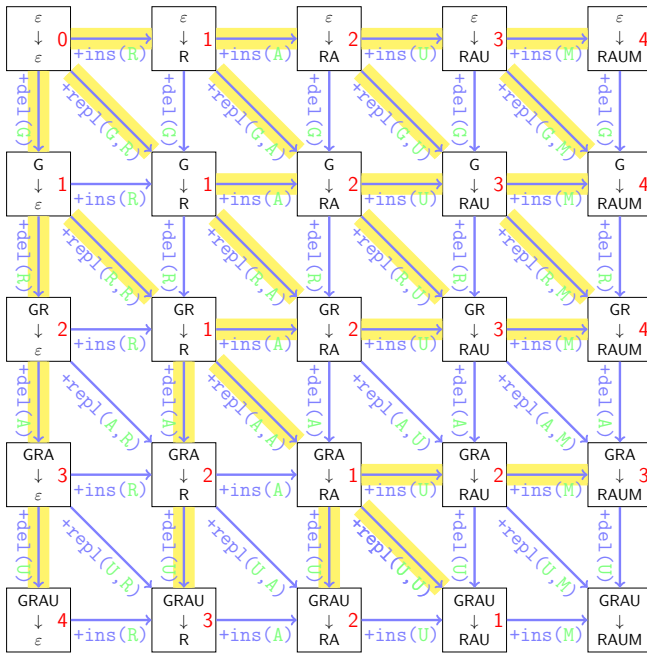
Edit Distance



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Edit distance

How to get the sequence of operations?

Edit distance

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- ▶ We save at each recursion the most efficient previous entry (the **highlighted arrows** in our image)

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- ▶ If we follow the highlighted path from (n, m) to $(1, 1)$ we get the optimum operations to transform x into y

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- ▶ There can be more than one arrows to the three previous entries
- ▶ If we follow the highlighted path from (n, m) to $(1, 1)$ we get the optimum operations to transform x into y
 - ▶ If we can follow more than one path there exist more than one ideal sequence

Edit Distance



Edit distance

General principle:

Edit distance

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- ▶ Recursive computation of ...
 - ... the same reoccurring partial problems
 - ... a limited number of partial problems

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General principle:

- ▶ Recursive computation of ...
 - ... the same reoccurring partial problems
 - ... a limited number of partial problems
- ▶ Computation of the solutions for all partial problems
- ▶ In a order that unsolved partial problems consist of already solved partial problems
- ▶ The “path” to our solution normally gets computed while searching the best solution
- ▶ Dijkstra algorithm is basically dynamic programming!

Edit distance

Additional applications (I)

Additional applications:

Edit distance

Additional applications (I)

Additional applications:

- ▶ *Edit distance*: global alignment with $O(n^2)$ space and time consumption

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-	-	-	B	L	O	E	D
S	A	U	B	L	O	E	D

Edit distance

Additional applications (I)

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-	-	-	B	L	O	E	D
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- ▶ Solution in $O(n^3)$ time or $O(n^2)$ affine

Edit distance

Additional applications (II)

$O(n^2)$ space consumption might be problematic:

Hirschberg algorithm:

Edit distance

Additional applications (II)

$O(n^2)$ space consumption might be problematic:

Hirschberg algorithm:

- ▶ Divide-and-conquer approach

Edit distance

Additional applications (II)

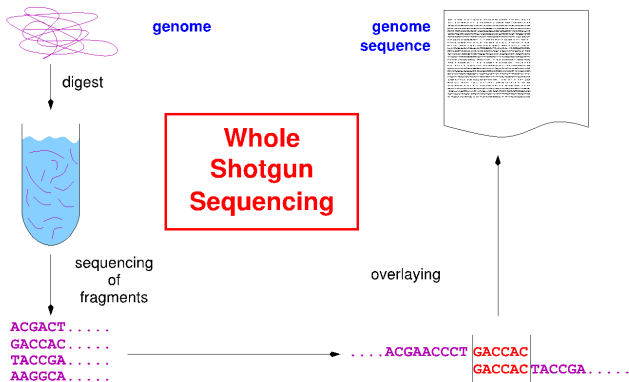
$O(n^2)$ space consumption might be problematic:

Hirschberg algorithm:

- ▶ Divide-and-conquer approach
- ▶ $O(n)$ space and $O(n^2)$ time consumption

Edit distance

Additional applications (III)



Edit distance

Additional applications (III)



- Sequencing: $O(n^2)$ is too much

Edit distance

Additional applications (III)



- ▶ Sequencing: $O(n^2)$ is too much
- ▶ Index: suffixtree, suffixarray, burrow-wheeler-transform

Further Literature

► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

- ▶ **Dynamic programming**

[Wik] [Dynamic programming](#)

https:

[//en.wikipedia.org/wiki/Dynamic_programming](https://en.wikipedia.org/wiki/Dynamic_programming)

- ▶ **Edit distance**

[Wik] [Levenshtein distance](#)

https:

[//en.wikipedia.org/wiki/Levenshtein_distance](https://en.wikipedia.org/wiki/Levenshtein_distance)