

Algorithms and Datastructures

Shortest Path, Dijkstra Algorithm

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Bioinformatics Group / Department of Computer Science

Algorithms and Datastructures, March 2018

Structure

Graphs

Dijkstra Algorithm

Structure

Graphs

Dijkstra Algorithm

Graphs

Paths

For a graph $G = (V, E)$:

- ▶ A path of G is a sequence of edges $u_1, u_2, \dots, u_i \in V$ with
 - ▶ Undirected graph: $\{u_1, u_2\}, \{u_2, u_3\}, \dots, \{u_{i-1}, u_i\} \in E$
 - ▶ Directed graph: $(u_1, u_2), (u_2, u_3), \dots, (u_{i-1}, u_i) \in E$
- ▶ The length of a path is
 - ▶ Without weights: number of edges taken
 - ▶ With weights: sum of weights of edges taken

Graphs

Paths

For a graph $G = (V, E)$:

- ▶ The **shortest path** between two vertices u, v is the path $P = (u, \dots, v)$ with the shortest length $d(u, v)$ or lowest costs
- ▶ The **diameter** of a graph is the **longest shortest path**

Structure

Graphs

Dijkstra Algorithm

Dijkstra Algorithm

Shortest Path without Computer

- ▶ Wanted: Shortest path from M to all other points
- ▶ Place pearls on crossings and clamp strings between them



Dijkstra Algorithm

Shortest Path without Computer



Figure: Based on OpenStreetMaps;
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- ▶ Take the net and pull it slowly upwards until fully lifted



- ▶ Each node (pearl) now has a specific height
- ▶ The distance to M is exactly the **shortest path**

Dijkstra Algorithm

Shortest Path without Computer



Figure: Based on OpenStreetMaps;
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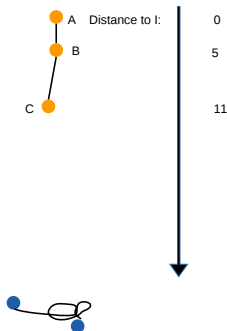
Dijkstra Algorithm

Shortest Path without Computer



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Dijkstra Algorithm

Shortest Path without Computer

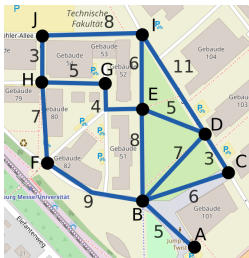
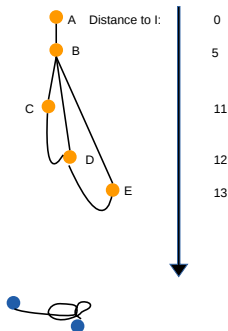


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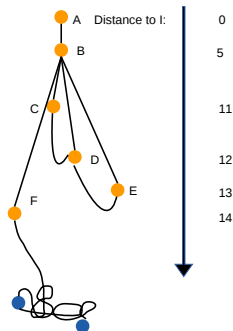
Dijkstra Algorithm

Shortest Path without Computer



Figure: Based on OpenStreetMaps;
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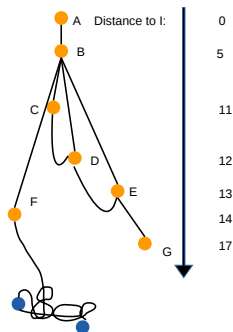
Dijkstra Algorithm

Shortest Path without Computer



Figure: Based on OpenStreetMaps;
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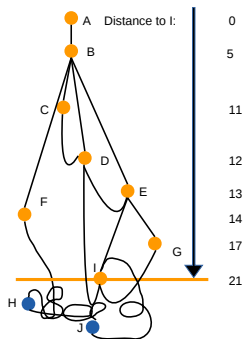
Dijkstra Algorithm

Shortest Path without Computer



Figure: Based on OpenStreetMaps;
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- Take the net and pull it slowly upwards until fully lifted



- Each node (pearl) now has a specific height
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Dijkstra Algorithm

Shortest Path

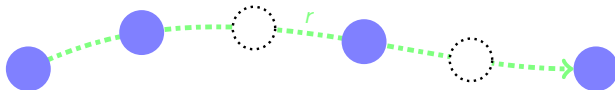


Figure: Shortest path from s to t

- ▶ Let r be the shortest path from s to t
- ▶ For each node u on path r the path from u to t is the shortest path

Proof:

- ▶ If there was a shorter path from s to u then we could choose this path to get faster to t
- ▶ Then r would not be the shortest path

Dijkstra Algorithm

Shortest Path

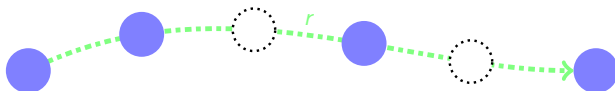


Figure: Shortest path from s to t

- ▶ This is also correct for all sub paths on r
- ▶ If the shortest path from s to t passes u_1 and u_2 then the sub path (u_1, u_2) is the shortest path from u_1 to u_2

Dijkstra Algorithm

Shortest Path

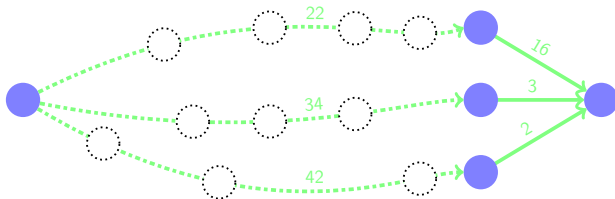


Figure: Shortest paths from s to t

- If we know the shortest path from s to the preceding nodes of t (v_1, v_2, v_3) we can determine the shortest path to t

Dijkstra Algorithm

Shortest Path

Idea:

- ▶ Attach the cost of the shortest path to each node
- ▶ Let the information travel over the edges (message passing)
- ▶ In which order should we process the nodes?

Dijkstra Algorithm

Inventor:

- ▶ Edsger Dijkstra (1930 - 2002)
- ▶ Computer scientist from Netherlands
- ▶ Won Turing-Award as one of few Europeans for his studies of structured programming
- ▶ Invented the Dijkstra-Algorithm in 1959

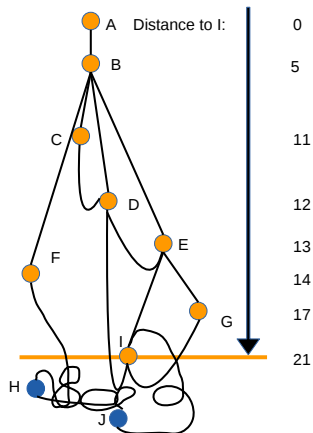


Figure: Portrait © Hamilton Richards - manuscripts of Edsger W. Dijkstra, University Texas at Austin

Dijkstra Algorithm

Example:

- ▶ Lift pearl **A** a little bit
 - ▶ Connection to pearl **B** is hanging in the air
 - ▶ Lift further until pearl **B** starts to lift at 5 m
 - ▶ The shortest path to **B** is now known
 - ▶ Lift further: The wires from **C**, **D**, **E** and **F** are now in the air
 - ▶ One of the pearls **C**, **D**, **E** or **F** is the next one
- Which one?



Dijkstra Algorithm

Example:

- ▶ At 11 m pearl **C** gets lifted
- ▶ The wire to **D** is now in the air
- ▶ One of the pearls **D**, **E** and **F** is the next one
Which one?
- ▶ At 12 m pearl **D** gets lifted
- ...
- ▶ How to translate this into an computer algorithm?



Dijkstra Algorithm

High level description: Three types of nodes

- ▶ **Settled:** For node u we know $\text{dist}(s, u)$
(Pearl example: This pearl is hanging in the air)
- ▶ **Active:** For node u we know a tentative distance $\text{td}(u) \geq \text{dist}(s, u)$ (Can be optimal but doesn't have to)
(Pearl example: This pearl is laying on the table but one connected wire is already in the air)
- ▶ **Unreached:** We have not reached the node yet
(Pearl example: This preal is hanging in the air)



Dijkstra Algorithm

High level description:

- ▶ Each iteration take the **active** node u with the **smallest** $td(u)$
(The pearl getting lifted next)
- ▶ We update the state of the node u to **settled**
(The pearl gets lifted)
- ▶ We check for each **neighbor** v of node u if we can reach v faster than currently possible
(Check all outgoing wires from this pearl: Activate all connected pearls, update tentative distance if smaller)
- ▶ Iterate until no active nodes exist anymore

Dijkstra Algorithm



Figure: Start at u_1

Dijkstra Algorithm



Figure: Iteration 1

Dijkstra Algorithm



Figure: Iteration 2

Dijkstra Algorithm



Figure: Iteration 3

Dijkstra Algorithm

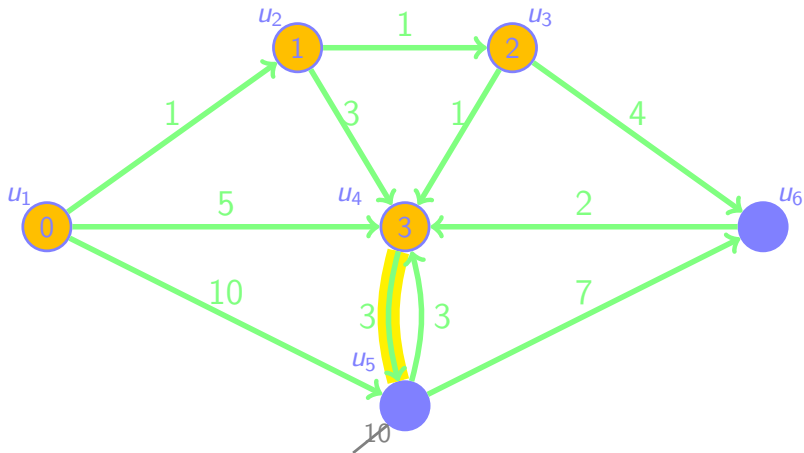


Figure: Iteration 4

Dijkstra Algorithm

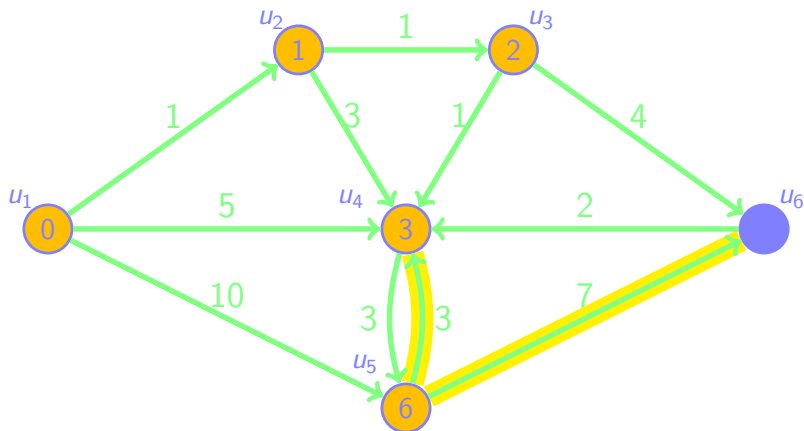


Figure: Iteration 5

Dijkstra Algorithm



Figure: Iteration 6

Dijkstra Algorithm

Proof

Proof:

- ▶ **Assumption 1:** All edges have a positive length
- ▶ **Assumption 2:** Each node has a unique distance $\text{dist}(s, u)$ to the start s
(This was not the case on the previous slides)

This results in an easy and intuitive proof.

It is possible to show this without assumption 2. See references if interested

- ▶ With assumption 2 there exists a sorting u_1, u_2, \dots with that:

$$\text{dist}(s, u_1) < \text{dist}(s, u_2) < \text{dist}(s, u_3) < \dots$$

Dijkstra Algorithm

Proof

Proof:

- ▶ With **assumption 2** there exists a sorting u_1, u_2, \dots with that:

$$\text{dist}(s, u_1) < \text{dist}(s, u_2) < \text{dist}(s, u_3) < \dots$$

- ▶ We want to show that the *Dijkstra* algorithm finds the shortest path for each node u_i so that $\text{td}(u_i) = \text{dist}(s, u_i)$ holds
- ▶ Additionally we show that each node gets solved in order of the distance: Node u_i gets solved in iteration i

$$u_1, u_2, u_3, \dots$$

Dijkstra Algorithm

Proof

To show: Node u_i gets solved in round i

1. Node u_i contains the correct distance ($td(u_i) = \text{dist}(s, u_i)$) and is active
2. Node u_i has the smallest value for $td(u_i)$ and gets selected by the algorithm

Induction start:

1.
 - ▶ Only the start node $s = u_1$ is active and $td(s) = 0$
 - ▶ Node u_1 gets solved and $td(u_1) = \text{dist}(s, u_1) = 0$
2. Only the start node u_1 is active

Dijkstra Algorithm

Proof

Induction step: $i = i + 1$

1. **To show:** Node u_{i+1} contains the correct distance ($\text{td}(u_{i+1}) = \text{dist}(s, u_{i+1})$) and is active
 - ▶ On the shortest path from s to u_{i+1} is a preceding node that:

$$\text{dist}(s, u_{i+1}) = \text{dist}(s, v) + c(v, u_{i+1})$$

($c(v, u_{i+1})$ are the costs of the edge)



- ▶ Hence $\text{dist}(s, v) < \text{dist}(s, u_{i+1})$ because $c > 0$ (c =cost of edge)
- ▶ Because u_{i+1} is currently settled, the node v is one of the preceding nodes u_1, \dots, u_i , hence $v = u_j$ with $0 \leq j \leq i$

Dijkstra Algorithm

Proof - Example of Iteration 6



- ▶ Preceding node of u_6 is $v = u_3$
- ▶ In round 3 $td(u_6) = 2 + 4 = 6$ was already solved

Dijkstra Algorithm

Proof



1. **To show:** Node u_i contains the correct distance $\text{td}(u_i) = \text{dist}(s, u_i)$ and is active
 - ▶ With **induction assumption:** v already contains the correct distance which was evaluated in round j (edge from v to u_{i+1}) and is stored in $\text{td}(u_{i+1})$
 - ▶ u_{i+1} is active because the preceding node was solved

Dijkstra Algorithm

Proof



2. **To show:** Node u_{i+1} has the smallest value for $\text{td}(u_{i+1})$ and gets selected by the algorithm
- ▶ All nodes with smaller dist are already solved
 - ▶ All other nodes u_k with $k > i + 1$ have a greater $\text{dist}(s, u_k)$ and with that the $\text{td}(u_k)$ is greater or equal
- $\Rightarrow u_{i+1}$ is the node with the smallest td and gets selected by the algorithm

Dijkstra Algorithm

Implementation

Implementation:

- ▶ We have to manage a set of **active nodes**
- ▶ We start with only the **start node** in our set
- ▶ At the start of each iteration we need the node u with the smallest $td(u)$

How to implement this?

Dijkstra Algorithm

Implementation

Implementation:

- ▶ Using a **priority queue** with $td(u)$ as keys
- ▶ The following problem occurs:
 - ▶ The **tentative distance** of an active node might change multiple times before it is settled
 - ▶ We have to change the key in our **priority queue** without removing the entry

Limitations:

- ▶ Often only `insert`, `getMin` and `deleteMin` are implemented
- ⇒ We only have access to the first element and not any desired one

Dijkstra Algorithm

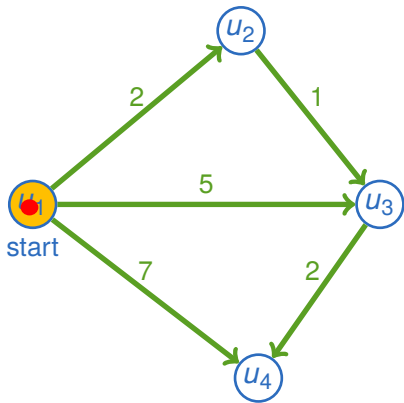
Implementation

Alternative:

- ▶ If a node reoccurs with a smaller **dist** we insert the element one more time into the **priority queue**
(We do nothing if the distance is greater or equal)
- ▶ We do not remove the old entry
- ▶ The node always gets solved with the smallest distance because of the **smaller key**
- ▶ If a settled node reoccurs with a higher **dist** we remove it and do simply **nothing**

Dijkstra Algorithm

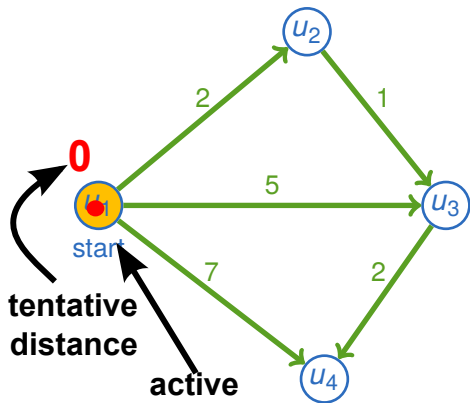
Implementation - Example



priority queue

Dijkstra Algorithm

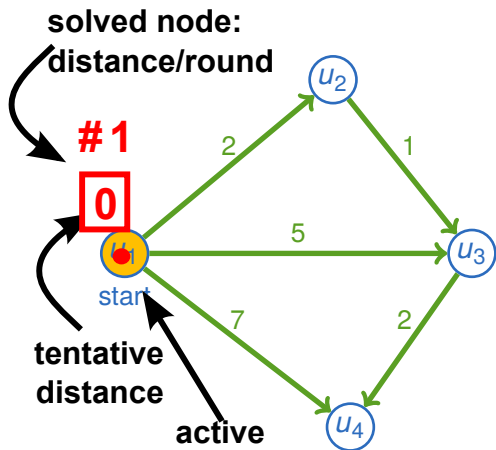
Implementation - Example



priority queue

Dijkstra Algorithm

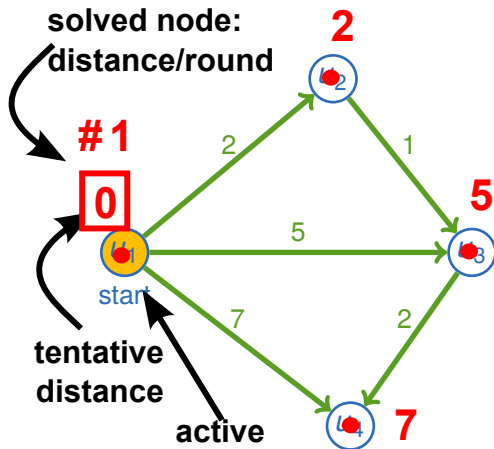
Implementation - Example



priority queue

Dijkstra Algorithm

Implementation - Example



priority queue

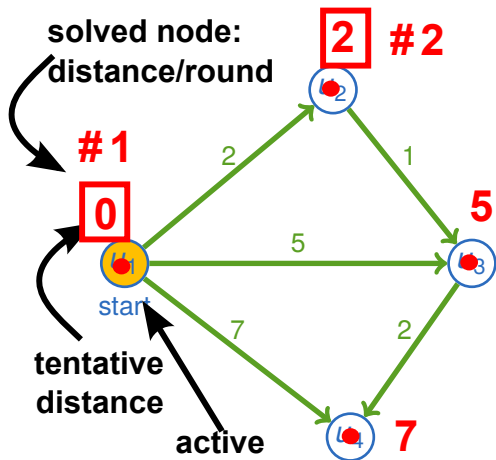
(u2, 2)

(u3, 5)

(u4, 7)

Dijkstra Algorithm

Implementation - Example



priority queue

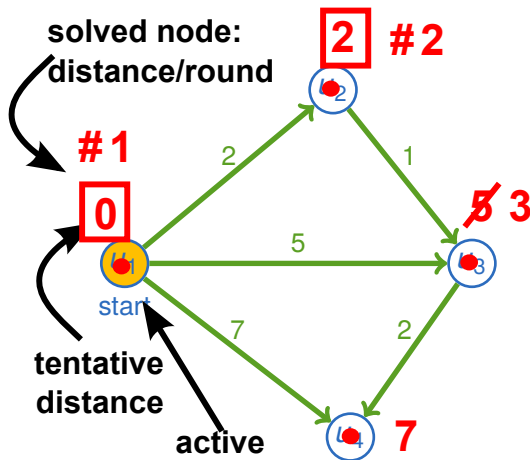
(u2, 2) → solved #2

(u3, 5)

(u4, 7)

Dijkstra Algorithm

Implementation - Example



priority queue

(u2, 2) → solved #2

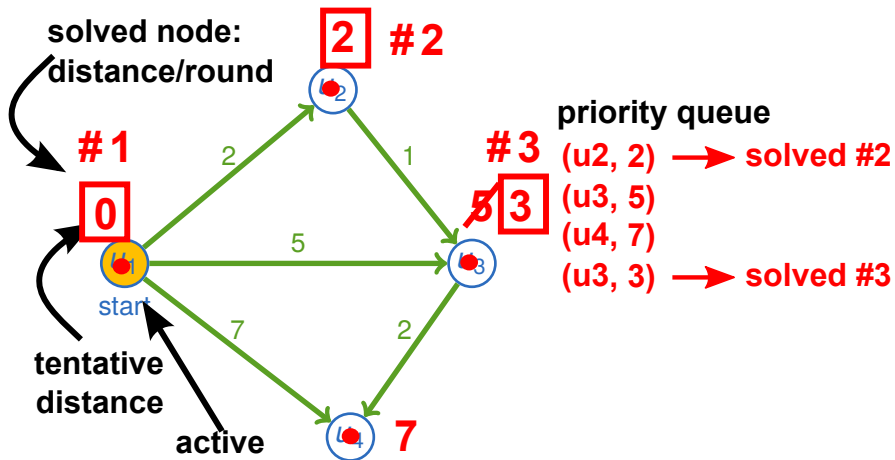
(u3, 5)

(u4, 7)

(u3, 3)

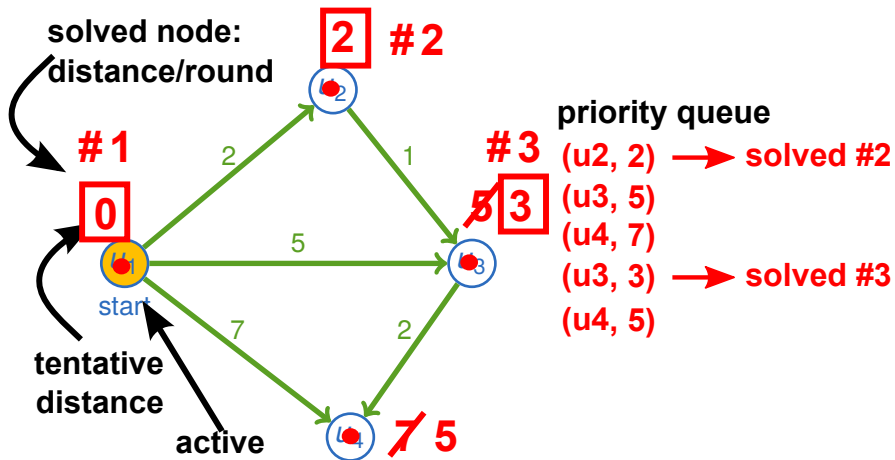
Dijkstra Algorithm

Implementation - Example



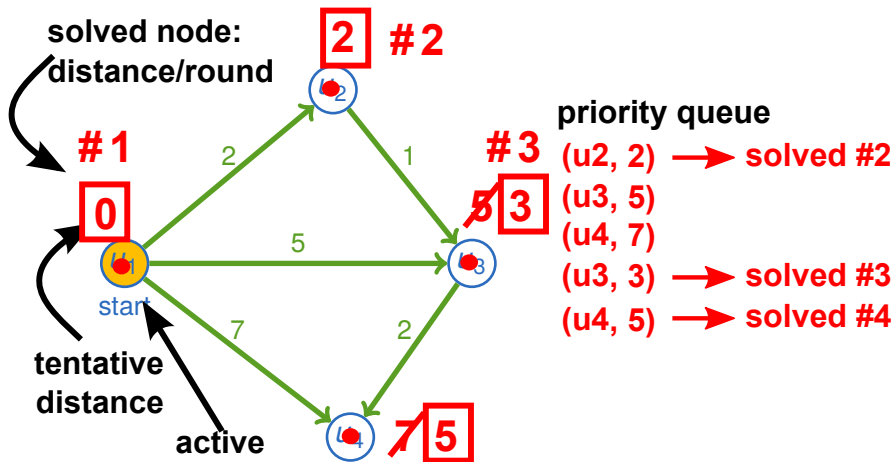
Dijkstra Algorithm

Implementation - Example



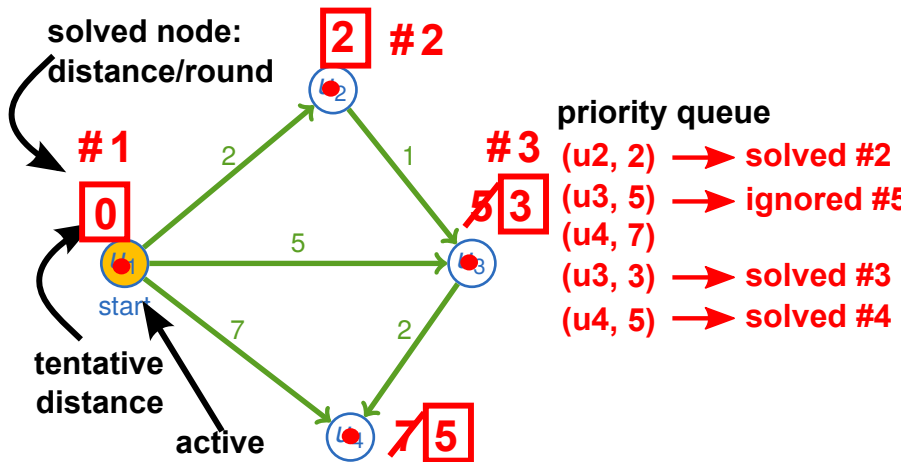
Dijkstra Algorithm

Implementation - Example



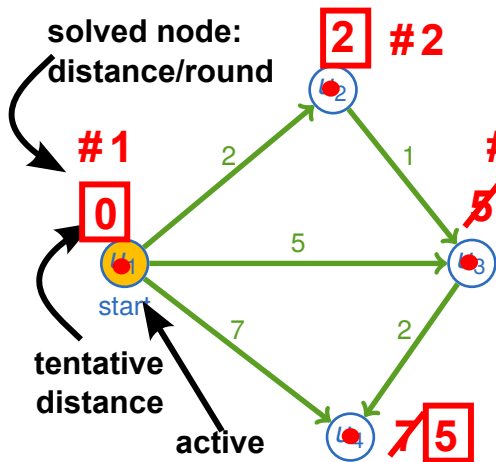
Dijkstra Algorithm

Implementation - Example



Dijkstra Algorithm

Implementation - Example



priority queue

- (u2, 2) → solved #2
- (u3, 5) → ignored #5
- (u4, 7) → ignored #6
- (u3, 3) → solved #3
- (u4, 5) → solved #4

Dijkstra Algorithm

Runtime analysis

Graph with n nodes and m edges: ($m \geq n$)

- ▶ Each node gets solved exactly **one time**
- ▶ When solving a node its outgoing edges are taken into account
- ▶ Each edge triggers at maximum one insert operation
- ▶ The number of operations on the **priority queue** is at maximum $O(m)$
- ▶ This results in a runtime of $O(m \cdot \log m)$
($\log m$ because of at max. m elements in the priority queue)

Dijkstra Algorithm

Runtime analysis

Runtime of $O(m \cdot \log m)$:

- ▶ Because of $m \leq n^2$ we have a maximum runtime of $O(m \cdot \log n)$, because $\log n^2 = 2 \log n$
 - ▶ With a complex **priority queue** the runtime can be reduced to $O(m + n \log n)$
 - ▶ For example with a **Fibonacci heap**
 - ▶ This results in a better runtime for complex graphs $m \sim n^2$
 - ▶ Complex heaps create a management overhead
- ⇒ In practice $m \in O(n)$ with a **binary heap** being faster
(See lecture 6)

Dijkstra Algorithm

Additional comments

Termination criteria:

- ▶ Terminate as soon as the target node t is settled
... never before because tentative distance might change:

$$td(t) \geq \text{dist}(s, t)$$

- ▶ Before the node t is solved all nodes u with $\text{dist}(s, u) \leq \text{dist}(s, t)$ are settled

Dijkstra Algorithm

Additional comments

Termination criteria:

- ▶ Not only the **single source single target** shortest path problem is solved by the Dijkstra algorithm but also the **single source all targets** problem
- ▶ This sounds wasteful but there is not a (much) better method for general graphs

Intuitive: We only know that there is no shorter path if all nodes in the distance of $\text{dist}(s, t)$ are evaluated

Dijkstra Algorithm

Additional comments

Calculate the shortest path:

- ▶ With the current implementation of the Dijkstra algorithm we only get the **length** of the path
How to get the path itself too?
- ▶ If we save the preceding node of the current shortest path on **settling** of each node we can reconstruct the **path**

Dijkstra Algorithm

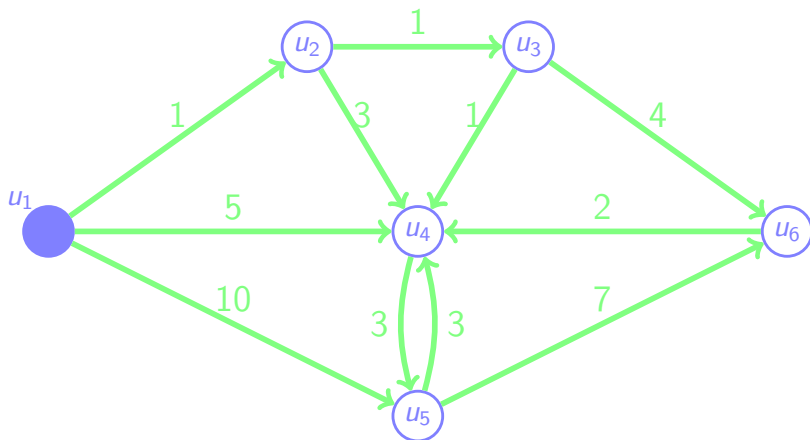


Figure: Start at u_1

Dijkstra Algorithm

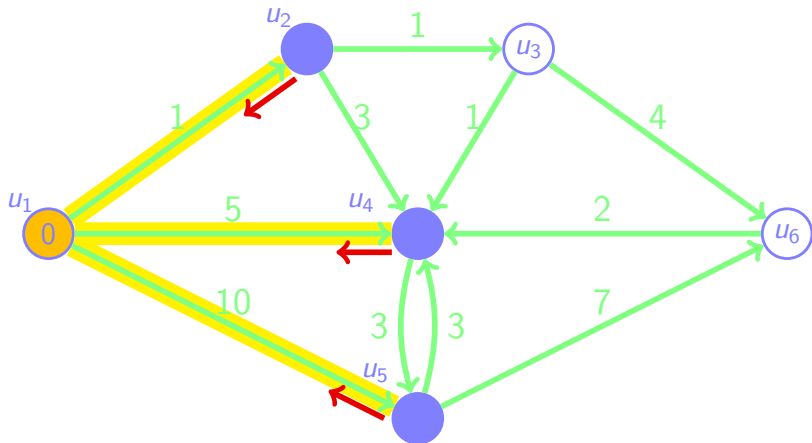


Figure: Iteration 1

Dijkstra Algorithm

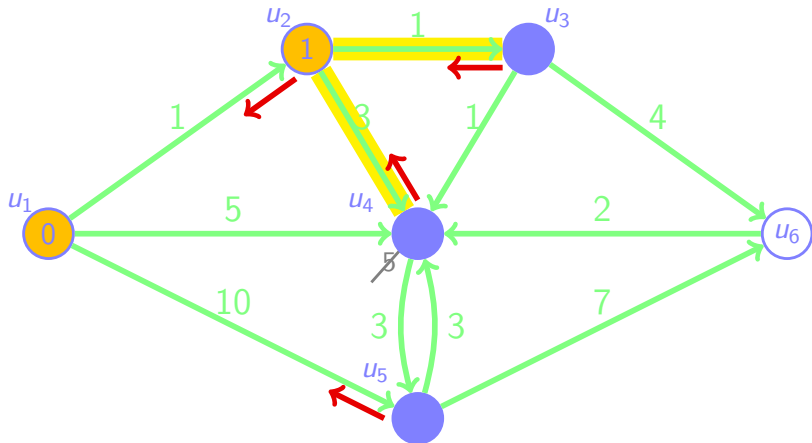


Figure: Iteration 2

Dijkstra Algorithm

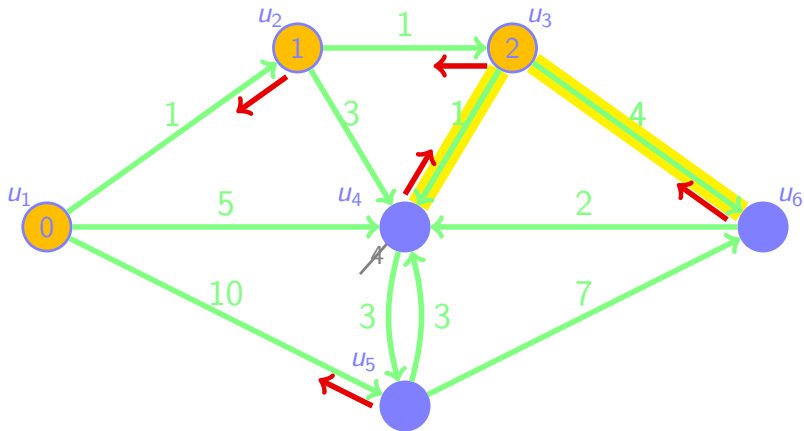


Figure: Iteration 3

Dijkstra Algorithm

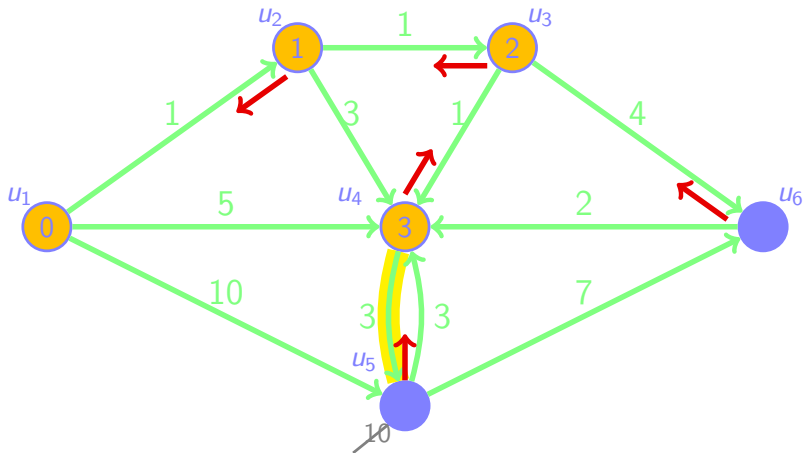


Figure: Iteration 4

Dijkstra Algorithm

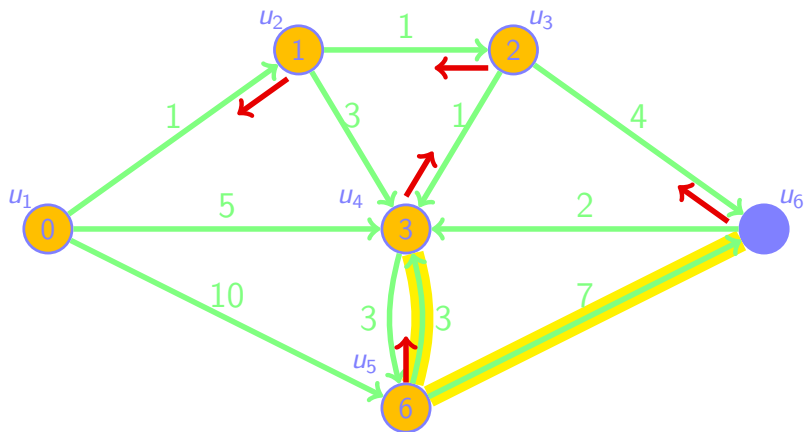


Figure: Iteration 5

Dijkstra Algorithm

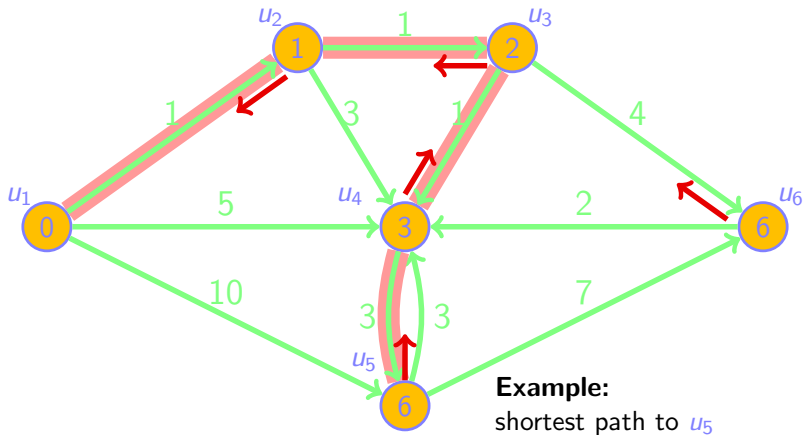


Figure: Iteration 6

Dijkstra Algorithm

Additional comments

Enhancement:

- ▶ In our proof we used the assumption that all costs are **not negative** (even > 0)
- ▶ With **negative costs** there might be **negative cycles**:

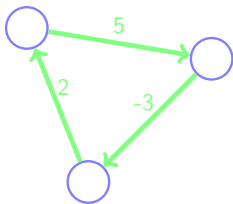


Figure: Here no problem ...

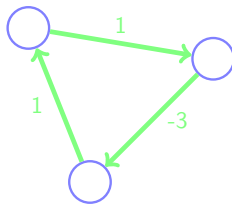
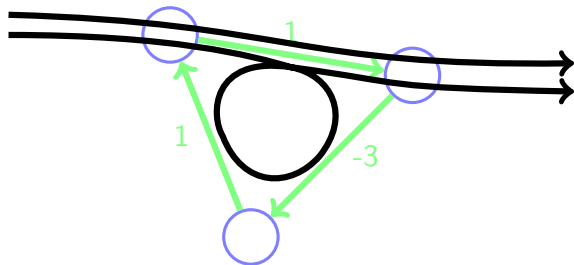


Figure: ... but here

Dijkstra Algorithm

Additional comments

Negative cycles:



- ▶ No cycle:
cost of 1
- ▶ 1 cycle:
cost of 0
- ▶ 2 cycles:
cost of -1
- ▶ 3 cycles:
cost of -2
- ▶ ...

Dijkstra Algorithm

Additional comments

Enhancement:

- ▶ We need a different algorithm to deal with negative edges
 - ▶ For example the **Bellman-Ford** algorithm
 - ▶ If the graph is **acyclic** we can simply use a topological sorting (with DFS) and settling the nodes in order of this sorting
- ▶ Another (not only) in artificial intelligence used variant of the Dijkstra algorithm is the **A* algorithm**

Additional information given:

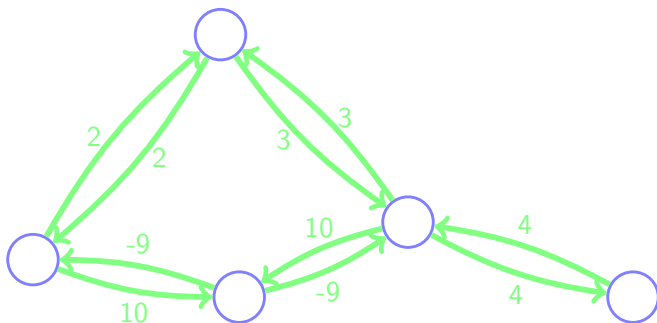
$h(u)$ = estimated value for $\text{dist}(u, t)$

Dijkstra Algorithm

Example - Negative costs (e-car consumption)

Dijkstra algorithm:

Message passing only from solved nodes

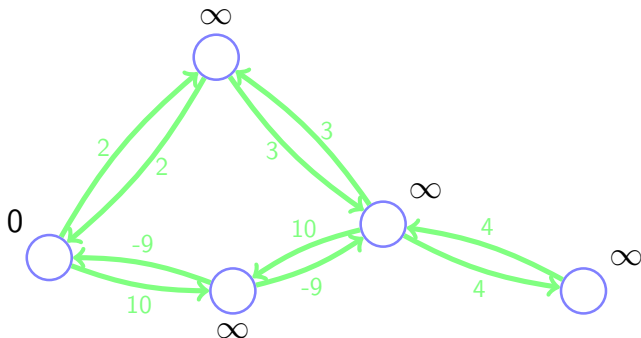


Dijkstra Algorithm

Example - Negative costs (e-car consumption)

Bellman-Ford algorithm:

Message passing from all nodes until the path lengths are stable



Dijkstra Algorithm

Application

Application example:

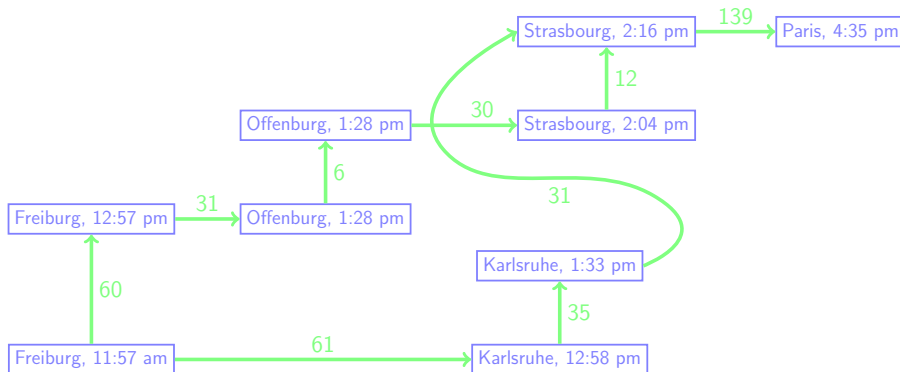
- ▶ Route planner for car trips (exercise sheet)
- ▶ Route planner for bus / train connections

What could the graph look like?

Dijkstra Algorithm

Application

Space-time graph:



Dijkstra Algorithm

Application in image processing

-6em-6em

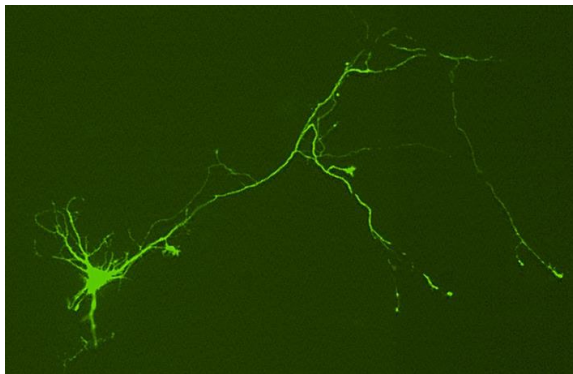
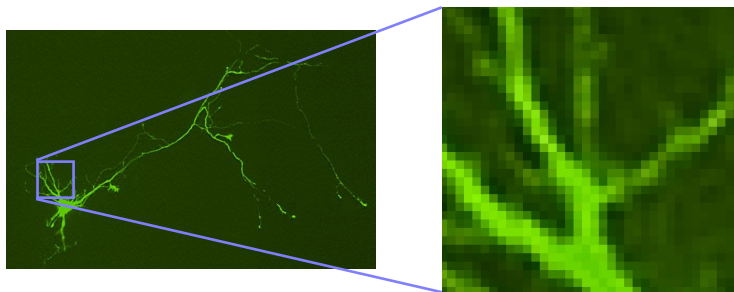


Figure: Neurons under fluorescence microscope

- ▶ **Task:** Measure length of axons (connections of neurons)
- ▶ Demo with ImageJ plugin NeuronJ
<http://www.imagescience.org/meijering/software/neuronj/>

Dijkstra Algorithm

Application: Trace axons



- ▶ Image as graph: Each pixel is a node
- ▶ Implicit edges: Each pixel has an edge to its 8 neighbours (no need to save the edges)
- ▶ Costs for nodes (not edges): bright pixels are cheap, dark pixels are costly

Further Literature

► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

- ▶ **Dijkstra's algorithm**

[Wik] [Dijkstra's algorithm](#)

`https:`

`//en.wikipedia.org/wiki/Dijkstra's_algorithm`

- ▶ **Shortest path problem**

[Wik] [Shortest path problem](#)

`https://en.wikipedia.org/wiki/Shortest_path_
problem`