

# Algorithms and Datastructures

Balanced Trees (AVL-Trees, (a,b)-Trees, Red-Black-Trees)

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# Structure

## Balanced Trees

- Motivation

- AVL-Trees

- (a,b)-Trees

  - Introduction

  - Runtime Complexity

- Red-Black Trees

# Balanced Trees

## Motivation

### Binary search tree:

- ▶ With `BinarySearchTree` we could perform an `lookup` or `insert` in  $O(d)$ , with  $d$  being the `depth` of the tree
- ▶ Best case:  $d \in O(\log n)$ , keys are inserted randomly
- ▶ Worst case:  $d \in O(n)$ , keys are inserted in ascending / descending order  $(20, 19, 18, \dots)$

# Balanced Trees

## Motivation



### Gnarley trees:

► <http://people.ksp.sk/~kuko/bak>



Figure: Binary search tree with random insert [Gna]



Figure: Binary search tree with descending insert [Gna]

# Balanced Trees

## Motivation

### Balanced trees:

- ▶ We do not want to rely on certain properties of our **key set**
- ▶ We explicitly want a depth of  $O(\log n)$
- ▶ We **rebalance** the tree from time to time

# Balanced Trees

## Motivation

How do we get a depth of  $O(\log n)$ ?

▶ **AVL-Tree:**

- ▶ Binary tree with 2 children per node
- ▶ Balancing via “rotation”

▶ **(a,b)-Tree or B-Tree:**

- ▶ Node has between  $a$  and  $b$  children
- ▶ Balancing through **splitting** and **merging** nodes
- ▶ Used in databases and file systems

▶ **Red-Black-Tree:**

- ▶ Binary tree with “black” and “red” nodes
- ▶ Balancing through “rotation” and “recoloring”
- ▶ Can be interpreted as (2, 4)-tree
- ▶ Used in C++ `std::map` and Java `SortedMap`

# Balanced Trees

## AVL-Tree

### AVL-Tree:

- ▶ Gregory Maximovich **A**delson-**V**elskii, Yevgeniy Mikhailovlovich **L**andis (1963)
- ▶ Search tree with modified **insert** and **remove** operations while satisfying a **depth** condition
- ▶ Prevents degeneration of the search tree
- ▶ Height difference of left and right subtree is at maximum one
- ▶ With that the height of the search tree is always  $O(\log n)$
- ▶ We can perform all basic operations in  $O(\log n)$

# Balanced Trees

## AVL-Tree



Figure: Example of an AVL-Tree



# Balanced Trees

## AVL-Tree

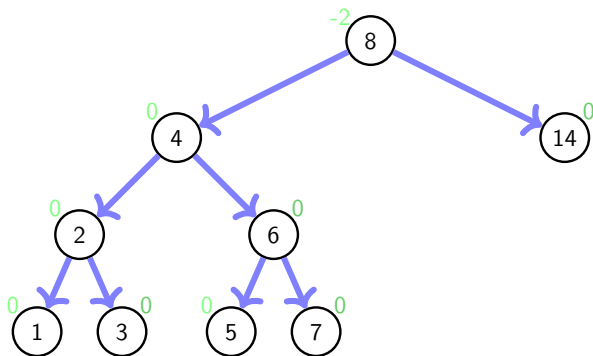


Figure: **Not** an AVL-Tree

# Balanced Trees

## AVL-Tree



Figure: Another example of an AVL-Tree

# Balanced Trees

## AVL-Tree - Rebalancing

**Rotation:**



Figure: Before rotating



Figure: After rotating

- ▶ Central operation of **rebalancing**
- ▶ After rotation to the right:
  - ▶ Subtree **A** is a layer higher and subtree **C** a layer lower
  - ▶ The parent child relations between nodes **x** and **y** have been swapped

# Balanced Trees

## AVL-Tree - Rebalancing

### AVL-Tree:

- ▶ If a height difference of  $\pm 2$  occurs on an **insert** or **remove** operation the tree is rebalanced
- ▶ Many different cases of rebalancing
- ▶ **Example:** **insert** of 1, 2, 3, ...



Figure: Inserting 1, ..., 10 into an AVL-tree [Gna]

# Balanced Trees

## AVL-Tree - Summary

### Summary:

- ▶ Historical the first search tree providing guaranteed **insert**, **remove** and **lookup** in  $O(\log n)$
- ▶ However not amortized update costs of  $O(1)$
- ▶ Additional memory costs: We have to save a height difference for every node
- ▶ Better (and easier) to implement are **(a,b)**-trees

# (a,b)-Trees

## Introduction

### (a,b)-Tree:

- ▶ Also known as **b-tree** (b for “balanced”)
- ▶ Used in databases and file systems

### Idea:

- ▶ Save a varying number of elements per node
- ▶ So we have space for elements on an **insert** and balance operation

# (a,b)-Trees

## Introduction

### (a,b)-Tree:

- ▶ All leaves have the same depth
- ▶ Each inner node has  $\geq a$  and  $\leq b$  nodes  
(Only the root node may have less nodes)



- ▶ Each node with  $n$  children is called “node of degree  $n$ ” and holds  $n - 1$  sorted elements
- ▶ Subtrees are located “between” the elements
- ▶ We require:  $a \geq 2$  and  $b \geq 2a - 1$

# (a,b)-Trees

## Introduction

### (2,4)-Tree:



Figure: Example of an (2,4)-tree

- ▶ (2,4)-tree with depth of 3
- ▶ Each node has between 2 and 4 children (1 to 3 elements)



# (a,b)-Trees

## Introduction

**Not an (2,4)-Tree:**



Figure: **Not** an (2,4)-tree

- ▶ Invalid sorting
- ▶ Degree of node too large / too small
- ▶ Leaves on different levels

# (a,b)-Trees

## Implementation - Lookup

### Searching an element: (lookup)

- ▶ The same algorithm as in [BinarySearchTree](#)
- ▶ Searching from the root downwards
- ▶ The keys at each node set the path



Figure: (3,5)-Tree [Gna]

# (a,b)-Trees

## Implementation - Insert

### Inserting an element: (`insert`)

- ▶ Search the position to insert the key into
- ▶ This position will always be an leaf
- ▶ Insert the element into the tree
- ▶ **Attention:** As a result node can overflow by one element (Degree  $b + 1$ )
- ▶ Then we **split** the node

# (a,b)-Trees

## Implementation - Insert

Inserting an element: (`insert`)



Figure: Splitting a node

- ▶ If the degree is higher than  $b + 1$  we split the node
- ▶ This results in a node with  $\text{ceil}(\frac{b-1}{2})$  elements, a node with  $\text{floor}(\frac{b-1}{2})$  elements and one element for the parent node
- ▶ That's why we have the limit  $b \geq 2a - 1$

# (a,b)-Trees

## Implementation - Insert

### Inserting an element: (`insert`)

- ▶ If the degree is higher than  $b + 1$  we split the node
- ▶ Now the parent node can be of a higher degree than  $b + 1$
- ▶ We `split` the parent nodes the same way
- ▶ If we split the root node we create a new parent root node  
(The tree is now one level deeper)

# (a,b)-Trees

## Implementation - Remove

### Removing an element: (remove)

- ▶ Search the element in  $O(\log n)$  time
- ▶ **Case 1:** The element is contained by a leaf
  - ▶ Remove element
- ▶ **Case 2:** The element is contained by an inner node
  - ▶ Search the **successor** in the right subtree
  - ▶ The **successor** is always contained by a leaf
  - ▶ Replace the element with its **successor** and delete the **successor** from the leaf
- ▶ **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree

# (a,b)-Trees

## Implementation - Remove

### Removing an element: (remove)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree
- **Case a:** If the left or right neighbour node has a degree **greater than a** we **borrow** one element from this node



Figure: Borrow an element

# (a,b)-Trees

## Implementation - Remove

### Removing an element: (remove)

- **Attention:** The leaf might be too small (degree of  $a - 1$ )  
⇒ We **rebalance** the tree
- **Case b:** We **merge** the node with its right or left neighbour



Figure: Merge two nodes



# (a,b)-Trees

## Implementation - Remove

### Removing an element: (remove)

- ▶ Now the parent node can be of degree  $a - 1$
- ▶ We merge parent nodes the same way
- ▶ If the root has only a single child
  - ▶ Remove the root
  - ▶ Define sole child as new root
  - ▶ The tree shrinks by one level

# (a,b)-Trees

## Runtime Complexity

### Runtime complexity of lookup, insert and remove:

- ▶ All operations in  $O(d)$  with  $d$  being the depth of the tree
- ▶ Each node (except the root) has more than  $a$  children  
 $\Rightarrow n \geq a^{d-1}$  and  $d \leq 1 + \log_a n = O(\log_a n)$

### In detail:

- ▶ lookup always takes  $\Theta(d)$
- ▶ insert and remove often require only  $O(1)$  time
- ▶ **Worst case:** split or merge all nodes on path up to the root
- ▶ Therefore instead of  $b \geq 2a - 1$  we need  $b \geq 2a$

# (a,b)-Trees

Runtime Complexity - Counter-example for (2,3)-Tree

## Counter example (2,3)-Tree:

- Before executing `delete(11)`

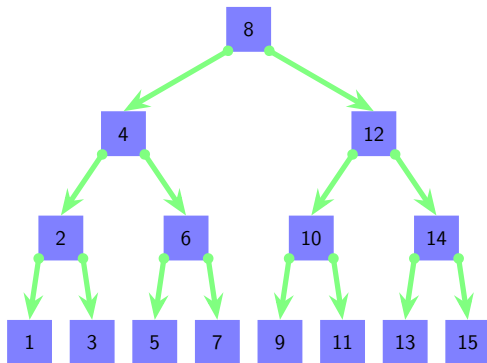


Figure: Normal (2,3)-Tree

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 1

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executing `delete(11)`



Figure: (2,3)-Tree - Delete step 2

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executing `delete(11)`

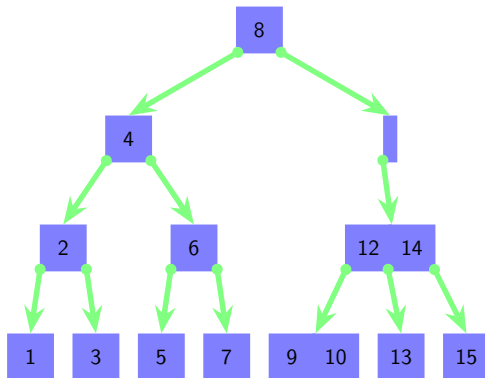


Figure: (2,3)-Tree - Delete step 3

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executed `delete(11)`

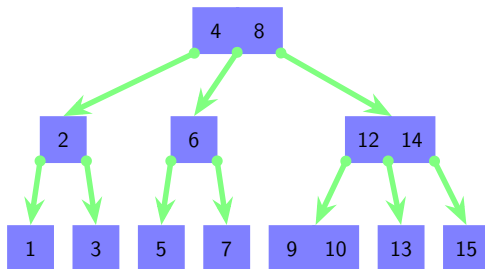


Figure: (2,3)-Tree - Delete step 4

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executing `insert(11)`



Figure: (2,3)-Tree - Insert step 1



# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executing `insert(11)`

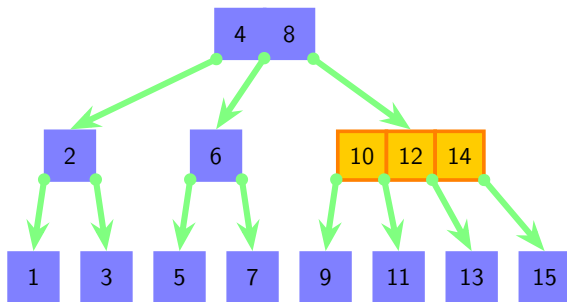


Figure: (2,3)-Tree - Insert step 2

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executing `insert(11)`

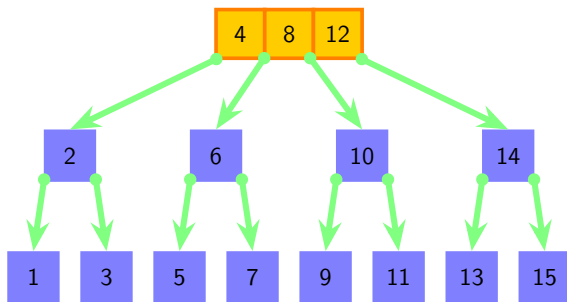


Figure: (2,3)-Tree - Insert step 3

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ Executed `insert(11)`

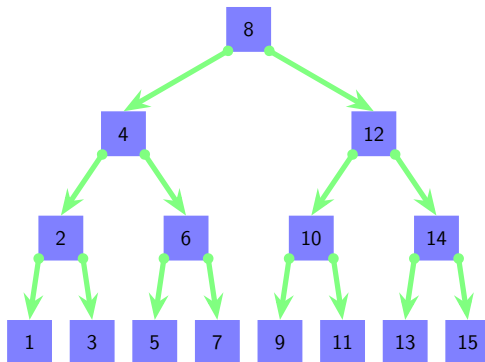


Figure: (2,3)-Tree - Insert step 4

# (a,b)-Trees

Runtime Complexity - Counter example for (2,3)-Tree

## Counter example (2,3)-Tree:

- ▶ We are exactly where we started
- ▶ If  $b = 2a - 1$  then we can create a sequence of **insert** and **remove** operations where each operation costs  $O(\log n)$
- ▶ We need  $b \geq 2a$  instead of  $b \geq 2a - 1$

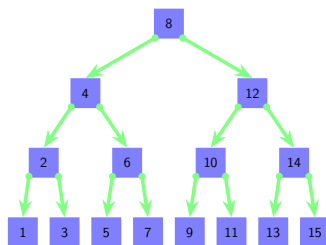


Figure: (2,3)-Tree

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

### (2,4)-Tree:

- ▶ If all nodes have 2 children we have to merge the nodes up to the root on a remove operation
- ▶ If all nodes have 4 children we have to split the nodes up to the root on a insert operation
- ▶ If all nodes have 3 children it takes some time to reach one of the previous two states

⇒ **Nodes of degree 3 are stable**

Neither an insert nor a remove operation trigger rebalancing operations

# (a,b)-Trees

Runtime Complexity - (2,4)-Tree

## (2,4)-Tree:

- ▶ **Idea:**
  - ▶ After an expensive operation the tree is in a stable state
  - ▶ It takes some time until the next expensive operation occurs
- ▶ Like with dynamic arrays:
  - ▶ **Reallocation** is expensive but it takes some time until the next expensive operation occurs
  - ▶ If we **overallocate** clever we have an amortized runtime of  $O(1)$

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

### Terminology:

- ▶ We analyze a sequence of  $n$  operations
- ▶ Let  $\Phi_i$  be the potential of the tree after the  $i$ -th operation
- ▶  $\Phi_i$  = the number of stable nodes with degree 3
- ▶ Empty tree has 0 nodes:  $\Phi = 0$

# (a,b)-Trees

Runtime Complexity - (2,4)-Tree

## Example:

- Nodes of degree 3 are highlighted



Figure: Tree with potential  $\phi = 4$



# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

### Terminology:

- ▶ Let  $c_i$  be the costs = runtime of the  $i$ -th operation
- ▶ We will show:
  - ▶ Each operation can at most destroy one stable node
  - ▶ For each cost incurring step the operation creates an additional stable node
- ▶ The costs for operation  $i$  are coupled to the difference of the potential levels

$$c_i \leq A \cdot (\underbrace{\Phi_i - \Phi_{i-1}}_{\text{degree 3}}) + B, \quad A > 0 \text{ and } B > A$$

Number of gained stable nodes (degree 3)  $\geq -1$

- ▶ Each operation has an amortized cost of  $O(1)$  summing up to  $O(n)$  in total

# (a,b)-Trees

Runtime Complexity - (2,4)-Tree

**Case 1:**  $i$ -th operation is an **insert** operation on a full node



Figure: Splitting a node on **insert**

- ▶ Each splitted node creates a node of **degree 3**
- ▶ The parent node receives an element from the splitted node
- ▶ If the parent node is also full we have to split it too

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

**Case 1:**  $i$ -th operation is an **insert** operation on a full node

- ▶ Let  $m$  be the number of nodes split
- ▶ The potential rises by  $m$
- ▶ If the “stop-node” is of **degree 3** then the potential goes down by one

$$\begin{aligned}\Phi_i &\geq \Phi_{i-1} + m - 1 \\ \Rightarrow m &\leq \Phi_i - \Phi_{i-1} + 1\end{aligned}$$

Costs:  $c_i \leq A \cdot m + B$

$$\begin{aligned}\Rightarrow c_i &\leq A \cdot (\Phi_i - \Phi_{i-1} + 1) + B \\ c_i &\leq A \cdot (\Phi_i - \Phi_{i-1}) + \underbrace{A + B}_{B'}\end{aligned}$$

# (a,b)-Trees

Runtime Complexity - (2,4)-Tree

**Case 2:**  $i$ -th operation is an **remove** operation

► **Case 2.1:** Inner node

- Searching the successor in a tree is  $O(d) = O(\log n)$
- Normally the tree is coupled with a doubly linked list  
⇒ We can find the successor in  $O(1)$



Figure: Tree with doubly linked list

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

**Case 2:**  $i$ -th operation is an **remove** operation

- ▶ **Case 2.1:** Borrow a node
  - ▶ Creates no additional operations
  - ▶ Case 2.1.1: Potential rises by one



Figure: Case 2.1.1: Borrow an element

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

**Case 2:**  $i$ -th operation is an **remove** operation

► **Case 2.1:** Borrow a node

- Creates no additional operations
- Case 2.1.2: Potential is lowered by one



Figure: Case 2.1.2: Borrow an element

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

**Case 2:**  $i$ -th operation is an **remove** operation

► **Case 2.2:** Merging two nodes

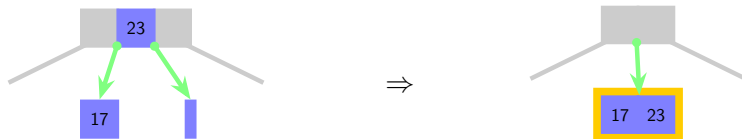


Figure: Merging two nodes

- Potential rises by one
- Parent node has one element less after the operation
- This operation propagates upwards until a node of degree  $> 2$  or a node of degree 2, which can borrow from a neighbour

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree

**Case 2:**  $i$ -th operation is an **remove** operation

► **Case 2.2:** Merging two node



Figure: Merging two nodes

- The potential rises by  $m$
- If the “stop-node” is of **degree 2** then the potential eventually goes down by one
- Same costs as **insert**



# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree - Lemma

### Lemma:

- ▶ We know:

$$c_i \leq A \cdot (\Phi_i - \Phi_{i-1}) + B, \quad A > 0 \text{ and } B > A$$

- ▶ With that we can conclude:

$$\sum_{i=0}^n c_i \in O(n)$$

# (a,b)-Trees

## Runtime Complexity - (2,4)-Tree - Lemma - Proof

**Proof:**

$$\begin{aligned}\sum_{i=0}^n c_i &\leq \underbrace{A \cdot (\phi_1 - \phi_0) + B}_{\leq c_1} + \underbrace{A \cdot (\phi_2 - \phi_1) + B}_{\leq c_2} + \cdots + \underbrace{A \cdot (\phi_n - \phi_{n-1})}_{\leq c_n} \\ &= A \cdot (\phi_n - \phi_0) + B \cdot n && | \text{ telescope sum} \\ &= A \cdot \phi_n + B \cdot n && | \text{ we start with an empty tree} \\ &< A \cdot n + B \cdot n \in O(n) && | \text{ number of degree 3 nodes} \\ &&& | \text{ number of nodes}\end{aligned}$$

# Red-Black-Trees

## Introduction

### Red-Black Tree:

- ▶ Binary tree with red and black nodes
- ▶ Number of black nodes on path to leaves is equal
- ▶ Can be interpreted as (2,4)-tree (also named 2-3-4-tree)
- ▶ Each (2,4)-tree-node is a small red-black-tree with a black root node

# Red-Black-Trees

## Introduction



Figure: Example of an red-black-tree [Gna]

## ► General

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

*Introduction to Algorithms.*

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

## ► Gnarley Trees

[Gna] Gnarley Trees

<https://people.ksp.sk/~kuko/gnarley-trees/>

► **AVL-Tree**

[Wik] [AVL tree](#)

[https://en.wikipedia.org/wiki/AVL\\_tree](https://en.wikipedia.org/wiki/AVL_tree)

► **(a,b)-Tree**

[Wika] [2-3-4 tree](#)

[https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934\\_tree](https://en.wikipedia.org/wiki/2%E2%80%933%E2%80%934_tree)

[Wikb] [\(a,b\)-tree](#)

[https://en.wikipedia.org/wiki/\(a,b\)-tree](https://en.wikipedia.org/wiki/(a,b)-tree)

## ► Red-Black-Tree

[Wik] [Red-black tree](https://en.wikipedia.org/wiki/Red%E2%80%93black_tree)

`https://en.wikipedia.org/wiki/Red%E2%80%93black\_tree`