

Algorithms and Datastructures

Hash Map, Universal Hashing

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Structure

Associative Arrays

- Introduction

- Hash Map

Universal Hashing

- Introduction

- Probability Calculation

- Proof

- Examples

Associative Arrays

How do we build a Map?

Reminder:

- ▶ An associative array is like a normal array, only that the indices are not $0, 1, 2, \dots$, but different, e.g. telephone numbers

Problem:

- ▶ Quickly find a element with a specific key
- ▶ Naive solution: Store pairs of key and value in a normal field
- ▶ For n keys searching requires $\Theta(n)$ time
- ▶ With a **hash map** this just requires $\Theta(1)$ in the best case, ... regardless how many elements are in the map!

Associative Arrays

The Hash Map

Idea:

- ▶ Mapping the keys onto indices with a **hash function**
- ▶ Store the values at the calculated indices in a normal array

Example:

- ▶ Key set: $x = \{3904433, 312692, 5148949\}$
- ▶ Hash function: $h(x) = x \bmod 5$, in the range $[0, \dots, 4]$
- ▶ We need an array **T** with **5** elements.
A “hashtable” with 5 “buckets”
- ▶ The element with the key **x** is stored in $T[h(x)]$

Associative Arrays

The Hash Map

Storage:

- ▶ `insert(3904433, "A")`: $h(3904433) = 3 \Rightarrow T[3] = (3904433, "A")$
- ▶ `insert(312692, "B")`: $h(312692) = 2 \Rightarrow T[2] = (312692, "B")$
- ▶ `insert(5148949, "C")`: $h(5148949) = 4 \Rightarrow T[4] = (5148949, "C")$

Figure: Hashtable T



Associative Arrays

The Hash Map

Searching:

- ▶ `search(3904433): $h(3904433) = 3 \Rightarrow T[3] \rightarrow (3904433, "A")$`
- ▶ `search(123459): $h(123459) = 4 \Rightarrow T[4]$`
 \Rightarrow Value with key 123459 does not exist
- ▶ Search time for this example: $\mathcal{O}(1)$

Figure: Hashtable T

0	
1	
2	← 312692 B
3	← 3904433 A
4	← 5148949 C

Associative Arrays

Hash Collisions

Further inserting:

- ▶ `insert(876543, "D")`: $h(876543) = 3$
 $\Rightarrow T[3] = (876543, "D") \Rightarrow$ Collision
- ▶ This happens more often than expected
 - ▶ **Birthday problem:** With 23 people we have the probability of 50 % that 2 of them have birthday at the same day

Figure: Hashtable T

0	
1	
2	← 312692 B
3	← 3904433 A
4	← 5148949 C

Associative Arrays

Hash Collisions

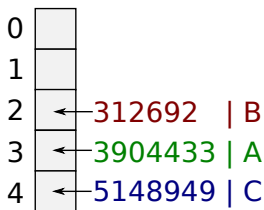
Problem:

- ▶ Two keys are equal $h(x) = h(y)$ but not the values $x \neq y$

Easiest Solution:

- ▶ Represent each bucket as list of key value pairs
- ▶ Append new values to the end of the list

Figure: Hashtable T



Associative Arrays

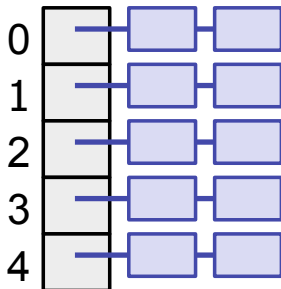
Expected Runtime

Best case:

- ▶ We have n keys which are equally distributed over m buckets
- ▶ We have $\approx \frac{n}{m}$ pairs per bucket
- ▶ The runtime for searching is nearly $\mathcal{O}(1)$ when **not** $n \gg m$

Best case

$(m = 5, n = 10)$



Associative Arrays

Expected Runtime

Worst case:

- ▶ All n keys are mapped onto the same bucket
- ▶ The runtime is $\Theta(n)$ for searching



Universal Hashing

Thought Experiment

Thought Experiment:

- ▶ A **hash function** is defined for a given **key set**
- ▶ Find a **set of keys** resulting in a degenerated **hash table**
 - ▶ *The **hash function** stays fixed*
 - ▶ *For table size of 100: Try $100 \times (99 + 1)$ different numbers*
 - ▶ *Worst case: All 100 **key sets** map to one bucket*
- ▶ **Now:** Find a solution to avoid that problem

Universal Hashing

Idea

Solution: universal hashing

- ▶ Out of a set of hash functions we randomly choose one
- ▶ The **expected result** of the hash function is an equal distribution over the buckets
- ▶ This hash function stays fixed for the lifetime of table
Optional: copy table with new hash when degenerated



Figure: Hash func. 1



Figure: Hash func. 2



Figure: Hash func. coll.

Universal Hashing

Definition

Definition:

- ▶ We call \mathbb{U} the set (universum) of possible keys
- ▶ The size m of the hash table T
- ▶ Set of hash functions $\mathbb{H} = \{h_1, h_2, \dots, h_n\}$ with $h_i : \mathbb{U} \rightarrow \{0, \dots, m-1\}$
- ▶ Idea: runtime should be $O(1 + \frac{|\mathbb{S}|}{m})$, where $\frac{|\mathbb{S}|}{m}$ is the table load



Figure: Hash function h_1

Universal Hashing

Definition

- ▶ We choose two random keys $x, y \in \mathbb{U} \mid x \neq y$
- ▶ An average of 3 out of 15 functions produce collisions



Figure: Set of hash functions \mathbb{H}

Universal Hashing

Definition

Definition: \mathbb{H} is c -universal if $\forall x, y \in \mathbb{U} \mid x \neq y :$

Number of hash functions that create collisions

$$\frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}, \quad c \in \mathbb{R}$$

Number of hash functions

- With other words, given a arbitrary but fixed pair x, y .
If $h \in \mathbb{H}$ is chosen randomly then

$$\text{Prob}(h(x) = h(y)) \leq c \cdot \frac{1}{m}$$

Note: If the hash function assigns each key x and y randomly to buckets then:

$$\text{Prob}(\text{Collision}) = \frac{1}{m} \Leftrightarrow c = 1$$

Universal Hashing

Definition

- ▶ \mathcal{U} : Key universe
- ▶ \mathcal{S} : Used Keys
- ▶ $\mathcal{S}_i \subseteq \mathcal{S}$: Keys mapping to Bucket i (“synonyms”)
- ▶ Ideal would be $|\mathcal{S}_i| = \frac{|\mathcal{S}|}{m}$



Figure: Hash function $h \in \mathbb{H}$

Universal Hashing

Definition

- ▶ Let \mathbb{H} be a c -universal class of hash functions
- ▶ Let \mathbb{S} be a set of keys and $h \in \mathbb{H}$ selected randomly
- ▶ Let \mathbb{S}_i be the key x for which $h(x) = i$
- ▶ The expected average number of elements to search through per bucket is

$$\mathbb{E}[|\mathbb{S}_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m}$$

- ▶ Particular: If $(m = \Omega(|\mathbb{S}|))$ then $\mathbb{E}[|\mathbb{S}_i|] = \mathcal{O}(n)$

Universal Hashing

Probability Calculation

- ▶ We just discuss the discrete case
- ▶ Probability space Ω with elementary (simple) events
- ▶ Events e have probabilities ...

$$\sum_{e \in \Omega} P(e) = 1$$

- ▶ The probability for a subset of events $E \subseteq \Omega$ is

$$P(E) = \sum_{e \in E} P(e)$$

Table: Throwing a dice

e	$P(e)$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

Universal Hashing

Probability Calculation

Example:

- ▶ Rolling a dice twice ($\Omega = \{1, \dots, 6\}^2$)
- ▶ Each event $e \in \Omega$ has the probability
 $P(e) = 1/36$
- ▶ $E =$ if both results are even, then
 $P(E) =$

Table: Throwing a dice twice

e	$P(e)$
(1, 1)	1/36
(1, 2)	1/36
(1, 3)	1/36
...	...
(6, 5)	1/36
(6, 6)	1/36

Universal Hashing

Probability Calculation

Example:

- ▶ Random variable
 - ▶ Assigns a number to the result of an experiment
 - ▶ For example: X = Sum of results for rolling twice
 - ▶ $X = 12$ and $X \geq 7$ are regarded as events
 - ▶ Example 1: $P(X = 2) =$
 - ▶ Example 2: $P(X = 4) =$

Table: Throwing a dice twice

e	$P(e)$	X
(1, 1)	$1/36$	2
(1, 2)	$1/36$	3
(1, 3)	$1/36$	4
...
(6, 5)	$1/36$	11
(6, 6)	$1/36$	12

Universal Hashing

Probability Calculation

Expected value is defined as $\mathbb{E}(X) = \sum (k \cdot P(X = k))$

- Intuitive: The weighted average of possible values of X , where the weights are the probabilities of the values

Table: Throwing a dice once

X	$P(X)$
1	$1/6$
2	$1/6$
3	$1/6$
4	$1/6$
5	$1/6$
6	$1/6$

Table: Throwing a dice twice

X	$P(X)$
2	$1/36$
3	$2/36$
4	$3/36$
...	...
11	$2/36$
12	$1/36$

- Example rolling once: $\mathbb{E}(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = 3.5$
- Example rolling twice: $\mathbb{E}(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \dots + 12 \cdot \frac{1}{36} = 7$

Universal Hashing

Probability Calculation

Sum of expected values: For arbitrary discrete random variables X_1, \dots, X_n we can write:

$$\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_n)$$

Example: Throwing two dice

- ▶ X_1 : Expected result of dice 1: $\mathbb{E}(X_1) = 3.5$
- ▶ X_2 : Expected result of dice 2: $\mathbb{E}(X_2) = 3.5$
- ▶ $X = X_1 + X_2$: Expected total number:

$$\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

Universal Hashing

Probability Calculation

Corollary:

The probability of the event E is $p = P(E)$. Let X be the occurrences of the event E and n be the number of executions of the experiment. Then $\mathbb{E}(X) = n \cdot P(E) = n \cdot p$

Example (Rolling the dice 60 times:)

$$\mathbb{E}(\text{occurrences of } 6) = \frac{1}{6} \cdot 60 = 10$$

Universal Hashing

Probability Calculation

Proof Corollary:

Indicator variable: X_i

$$X_i = \begin{cases} 1, & \text{if event occurs} \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow X = \sum_{i=1}^n X_i$$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbb{E}(X_i) \stackrel{\text{def. } \mathbb{E}\text{-value}}{=} \sum_{i=1}^n p = n \cdot p$$

□

Def. \mathbb{E} -value: $\mathbb{E}(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = P(X_i = 1)$

Universal Hashing

Proof

Given:

- ▶ We pick two random keys $x, y \in \mathbb{S} \mid x \neq y$ and a random hash function $h \in \mathbb{H}$
- ▶ We know the probability of a collision:

$$P(h(x) = h(y)) \leq c \cdot \frac{1}{m}$$

To proof:

$$\mathbb{E}[|S_i|] \leq 1 + c \cdot \frac{|\mathbb{S}|}{m} \quad \forall i$$

Universal Hashing

Proof

We know:

$$\mathbb{S}_i = \{x \in \mathbb{S} : h(x) = i\}$$

If $\mathbb{S}_i = \emptyset \Rightarrow |\mathbb{S}_i| = 0$ otherwise, let $x \in \mathbb{S}_i$ be any key

We define an indicator variable:

$$I_y = \begin{cases} 1, & \text{if } h(y) = i \\ 0, & \text{else} \end{cases} \quad y \in \mathbb{S} \setminus \{x\}$$

$$\Rightarrow |\mathbb{S}_i| = 1 + \sum_{y \in \mathbb{S} \setminus x} I_y$$

$$\Rightarrow \mathbb{E}(|\mathbb{S}_i|) = \mathbb{E}\left(1 + \sum_{y \in \mathbb{S} \setminus x} I_y\right) = 1 + \sum_{y \in \mathbb{S} \setminus x} \mathbb{E}(I_y)$$

Universal Hashing

Proof

Auxiliary calculation:

$$\begin{aligned}\mathbb{E}[I_y] &= P(I_y = 1) \\ &= P(h(y) = i) \\ &= P(h(y) = h(x)) \\ &\leq c \cdot \frac{1}{m}\end{aligned}$$

$$\begin{aligned}\textbf{Hence: } \mathbb{E}[|S_i|] &= 1 + \sum_{y \in S \setminus x} \mathbb{E}[I_y] \leq 1 + \sum_{y \in S \setminus x} c \cdot \frac{1}{m} \\ &= 1 + (|S| - 1) \cdot c \cdot \frac{1}{m} \\ &\leq 1 + |S| \cdot c \cdot \frac{1}{m} \\ &= 1 + c \cdot \frac{|S|}{m} \quad \square\end{aligned}$$

Universal Hashing

Examples

Negative example:

- ▶ The set of all h for which $h_a(x) = (a \cdot x) \bmod m$, for a $a \in \mathbb{U}$
- ▶ Is not c -universal.
- ▶ If universal:

$$\forall x, y \quad x \neq y: \frac{|\{h \in \mathbb{H} : h(x) = h(y)\}|}{|\mathbb{H}|} \leq c \cdot \frac{1}{m}$$

- ▶ Which x, y lead to a relative collision count bigger than $\frac{c}{m}$?

Universal Hashing

Examples

Positive example:

- ▶ Let p be a big prime number, $p > m$ and $p \geq |\mathbb{U}|$
- ▶ Let \mathbb{H} be the set of all h for which:

$$h_{a,b}(x) = ((a \cdot x + b) \bmod p) \bmod m,$$

where $1 \leq a < p$, $0 \leq b < p$

- ▶ This is ≈ 1 -universal, see Exercise 4.11 in Mehlhorn/Sanders
- ▶ E.g.: $U = \{0, \dots, 99\}$, $p = 101$, $a = 47$, $b = 5$
- ▶ Then $h(x) = ((47 \cdot x + 5) \bmod 101) \bmod m$
- ▶ Easy to implement but hard to proof
- ▶ Exercise: show empirically that it is 2-universal

Universal Hashing

Examples

Positive example:

- ▶ The set of hash functions is c -universal:

$$h_a(x) = a \bullet x \mod m, \quad a \in \mathbb{U}$$

- ▶ We define:

$$a = \sum_{0, \dots, k-1} a_i \cdot m^i, \quad k = \text{ceil}(\log_m |\mathbb{U}|)$$

$$x = \sum_{0, \dots, k-1} x_i \cdot m^i$$

- ▶ **Intuitive:** Scalar product with base m

$$a \bullet x = \sum_{0, \dots, k-1} a_i \cdot x_i$$

Universal Hashing

Examples

Example ($\mathbb{U} = \{0, \dots, 999\}$, $m = 10$, $a = 348$)

With $a = 348$: $a_2 = 3$, $a_1 = 4$, $a_0 = 8$

$$\begin{aligned}h_{348}(x) &= (a_2 \cdot x_2 + a_1 \cdot x_1 + a_0 \cdot x_0) \mod m \\&= (3x_2 + 4x_1 + 8x_0) \mod 10\end{aligned}$$

With $x = 127$: $x_2 = 1$, $x_1 = 2$, $x_0 = 7$

$$\begin{aligned}h_{348}(127) &= (3 \cdot x_2 + 4 \cdot x_1 + 8 \cdot x_0) \mod 10 \\&= (3 \cdot 1 + 4 \cdot 2 + 8 \cdot 7) \mod 10 \\&= 7\end{aligned}$$

Further Literature

► General for this Lecture

[CRL01] Thomas H. Cormen, Ronald L. Rivest, and Charles E. Leiserson.

Introduction to Algorithms.

MIT Press, Cambridge, Mass, 2001.

[MS08] Kurt Mehlhorn and Peter Sanders.

Algorithms and data structures, 2008.

<https://people.mpi-inf.mpg.de/~mehlhorn/ftp/Mehlhorn-Sanders-Toolbox.pdf>.

Further Literature

► Hash Map - Theory

[Wik] [Hash table](#)

https://en.wikipedia.org/wiki/Hash_table

► Hash Map - Implementations / API

[Cpp] [C++ - hash_map](#)

http://www.sgi.com/tech/stl/hash_map.html

[Jav] [Java - HashMap](#)

<https://docs.oracle.com/javase/7/docs/api/java/util/HashMap.html>

[Pyt] [Python - Dictionaries \(Hash table\)](#)

https://en.wikipedia.org/wiki/Hash_table