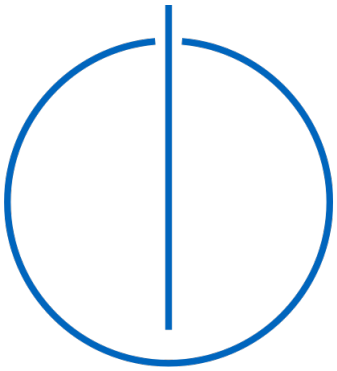
Technical University of Munich

Department of Informatics

Bachelor's Thesis in Informatics

Polynomial Time Competitive Repartitioning of Dynamic Graphs

Tobias Forner



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Polynomial Time Competitive Repartitioning of Dynamic Graphs

Kompetitive Repartitionierung Dynamischer Graphen in polynomieller Zeit

Tobias Forner

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I confirm that this bachelor's thesis is my own work and I have documented all sources and material used.

Date	Tobias Forner

Abstract

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1 Introduction

2 Problem Definition

The task is to maintain a partitioning of a dynamic graph consisting of $n = k \cdot l$ nodes that communicate with each other into k parts, each of size l while minimizing both the cost due to communication and due to node migrations defined as follows. The communication cost is zero if both nodes are located on the same server at the time the request needs to be served and it is normalized to one if they are mapped to different servers. An algorithm may perform node migrations in order to change the mapping of nodes to servers prior to serving the communication request at time t. Such a move of one vertex incurs cost $\alpha > 1$.

More formally we are given l servers $V_0, ..., V_{l-1}$, each with capacity k and an initial perfect mapping of $n = k \cdot l$ nodes to the l servers, i.e. each server is assigned exactly k nodes. An input sequence $\sigma = (u_1, v_1), (u_2, v_2), ...(u_i, v_i), ...$ describes the sequence of communication requests: the pair (u_t, v_t) represents a communication request between the nodes u_t and v_t arriving at time t. At time t the algorithm is allowed to perform node migrations at a cost of $\alpha > 1$ per move. After the migration step the algorithm pays cost 1 if u_t and v_t are mapped to different servers and does not pay any cost otherwise. Note that an algorithm may also choose to perform no migrations at all.

We are in the realm of competitive analysis and as a result we compare an online algorithm Onl to the optimal offline algorithm Opt. Onl only learns of the requests in the input sequence σ as they happen and as a result only knows about the partial sequence $(u_1, v_1), ..., (u_t, v_t)$ at time t whereas Opt has perfect knowledge of the complete sequence σ at all times.

The goal is to design an online algorithm Onl with a good competitive ratio with regard to Opt defined as follows.

An online algorithm Onl is $\rho - competitive$ if there exists a constant β such that

$$\text{Onl}(\sigma) \leq \rho \cdot \text{Opt}(\sigma) + \beta \, \forall \sigma$$

where $\text{Onl}(\sigma)$ and $\text{Opt}(\sigma)$ denote the cost of serving input sequence σ of Onl and Opt respectively.

Often we allow the online algorithm to use larger capacities per server. In this case we speak of an *augmentation* of δ in the case where the online algorithm is allowed to assign $\delta \times n/k$ nodes to each server where $\delta > 1$. This augmented online algorithm is than compared with the optimal offline algorithm Opt which is not allowed to use any augmentation.

3 Related Work

3.1 Dynamic Balanced RePartitioning

Avin et al.[1] initiated the study of the online variant of the Balanced RePartitioning (BRP) problem that is at the core of this also the topic of this thesis. We discovered flaws in their competitive analysis which makes us question their results. These flaws are discussed in greater detail in Section 4.

3.2 Learning Variant of BRP

[5] studies a special *learning variant* of the Dynamic Balanced Graph Partitioning problem specified above. In this version it is assumed that the input sequence σ eventually reveals a perfect balanced partitioning of the n nodes into l parts of size k such that the edge cut is zero. In this case the communication patterns reveal connected components of the communication graph of which each forms one of the partitions. Algorithms are tasked to *learn* this partition and to eventually collocate nodes according to the partition while minimizing communication and migration costs.

[5] present an algorithm for the case where the number of servers is $l = 2$ that achieves a competitive ratio of $O((\log n)/\epsilon)$ with augmentation ϵ, i.e. each server has capacity $(1+\epsilon)n/2$ for $\epsilon \in (0, 1)$.

For the general case of l servers of capacity $(1+\epsilon)n/2$ the authors construct an exponential-time algorithm that achieves a competitive ratio of $\emptyset((l \log n \log l)/\epsilon)$ for $\epsilon \in (0, 1/2)$ and also provide a distributed version. Additionally the authors describe a polynomial-time $O((l^2 \log n \log l)/\epsilon^2)$-competitive algorithm for the case with general l, servers of capacity $(1+\epsilon)n/l$ and $\epsilon \in (0, 1/2)$.

It is important to stress that the assumption that the requests reveal a perfect partitioning of the communication nodes is not applicable for most practical applications and thus it is important to study the general BRP problem without restricting σ.

4 Problems With Analysis in Previous Work

In this section we point out problems in the analysis in two versions of [2] which share a similar approach to ours.

The problem in the analysis of [1] arises due to their usage of the concept of $F(c)$: This contains only edges incident to components from $S(c)$ that arrived after the involved components from $S(c)$ were created by Crep.In the analysis it is assumed that only those edges that are contained in $F(c)$ can contribute to the creation of component c. As we will show that is not the case and as a result it is very challenging to separate the costs of Opt that are due to the requests that actually led to the creation of c.

In order for the approach in [1] to work the different sets $F(c)$ that occur as Crep handles input σ need to form a partition of all requests such that each request can be mapped to one unique set $F(c)$. Only in this case it is possible to lower bound the cost of Crep via the requests in $\bigcup_{c \in \text{DEL}} F(c)$.

Figure 1 shows an example sequence of requests for which this approach does not work. The diagram shows horizontal lines, each representing one of the vertices. A vertical line represents a communication request between its end points. For example the sequence shown contains a communication request between nodes 1 and 2 at time $t = 5$.

In this sequence the first two requests are not contained in $F(c)$ for any component c as nodes 3 and 4 were part of other components that were eventually deleted by Crep before finally being merged at time $t = 15$.

In an older version ([2]) of the paper mentioned above we have also discovered some problems. In this version a similar component structure is maintained and merges are performed similarly to our approach as upon the deletion of a component

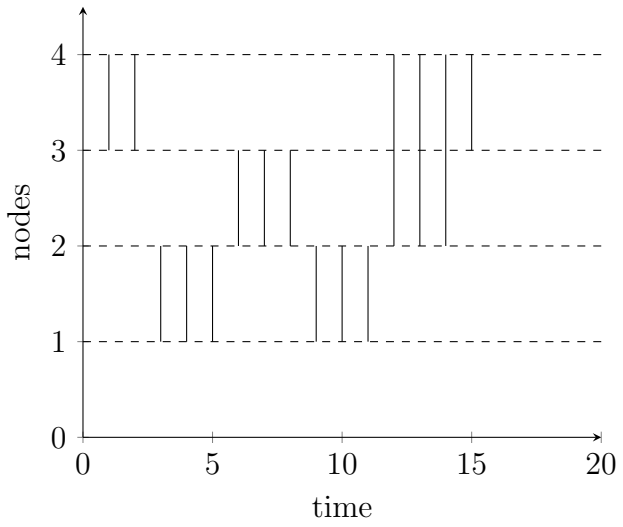

Figure 1: problem in the new version

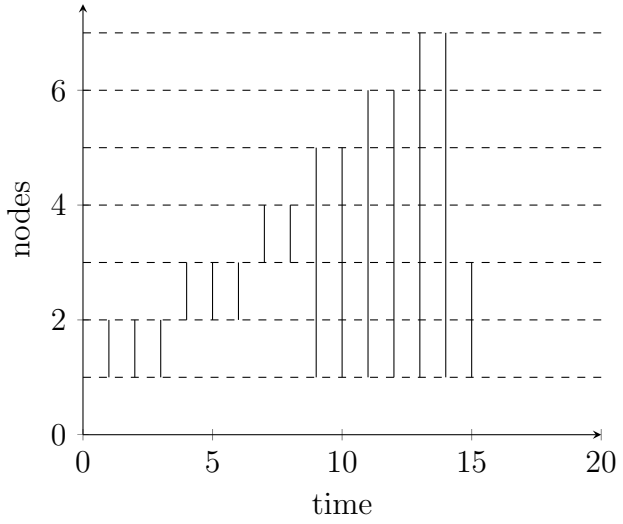

Figure 2: problem in the older version

the algorithm not only deletes inner edges but also those that leave the component, i.e. all edges $e = u, v$ are deleted where u or v are inside the component. However the additional deletions of edges that are leaving the component are not accounted for in the analysis. We show that this approach may lead to scenarios where the amount of edges that are deleted that leave the component may greatly exceed the amount of inner edges. We show in Section 6 how the analysis can be adapted while preserving the competitive ratio.

Figure 2 illustrates an example of the problems mentioned above for this older version of Crep. In the illustration it is assumed that $\alpha = 3$ and $k = 3$. In this case the input sequence leads Crep to first merge nodes 1 and 2 at time $t = 3$ and then to add node 3 to the resulting component at time $t = 6$. The next two requests do not quite lead to the merge of the node 4 with the component. Instead a series of requests follows where node 1 communicates with other nodes that are outside of its component without any merges. Note that this sequence can be extended until node 1 has communicated with every node except from the nodes $1, 2, 3, 4$. Finally the first 4 nodes are merged at time $t = 15$ at which point the resulting component is deleted as well as *all* edges in the sequence.

5 Algorithm Overview

In this section we give an overview of our algorithm Crep which builds upon the ideas of [2] and [1]. Implementation details will be discussed in greater detail in Section 7.

5.1 Overview

The algorithm maintains a second-order partitioning of the nodes into *communication components* which are sets of nodes that communicated frequently. As more requests from the input sequence σ are revealed to the algorithm the components grow in size until the algorithm discovers a component that is too large and hence decides to delete it.

More formally, initially each node forms a singleton component, but as the algorithm the input sequence σ is revealed to Crep new communication patterns unfold which the algorithm keeps track of by maintaining a graph in which the nodes represent the actual communication nodes and the weighted edges represent the number of communication requests between nodes that were part of different components at the time of the request, i.e. for edge $e = \{u, v\}$ $w(e)$ represents the number of paid communication requests between u and v. We say that a communication request between nodes u and v is *paid* if the nodes are located o different servers at the time of the request.

Crep merges a set S of components into a new component C if the connectivity of the component graph induced by the components in S is at least α. After each edge insertion the algorithm checks whether there exists a new component set S with $|S| > 1$ which fulfills this requirement.

If after any request and the insertion of the resulting edge the algorithm discovers a new subset S of nodes whose induced subgraph has connectivity at least α and which is of cardinality at most k it merges the components that form this set into one new component and collocates all the nodes in the resulting set on a single server. If the resulting component has size at least $2/\epsilon$ the algorithm reserves additional space $\min\{\epsilon \cdot |C|, k\}$.

If the subset has cardinality greater than k the resulting component is split into singleton components and all edges $e = \{u, v\}$ are reset to zero if u or v were contained in component C at the time of its deletion.

The collocation of such component sets of at most k individual communication nodes is always possible without moving a node in not in C due to the allowed augmentation of $2 + \epsilon$. This guarantees by an averaging argument that there is almost at least one cluster with capacity at least k. We have also included a pseudocode description of the algorithm in Algorithm 1.

6 Analysis

We analyse the competitive ratio of Crep with augmentation $(2 + \epsilon)$ and show in Theorem 15 that Crep is $O(k \log k)$-competitive.

Algorithm 1 DynamicDecomp

```
Initialize an empty graph on n nodes
turn each of the n nodes into a singleton component
for all r = {u, v} ∈ σ do
  if comp(v) ≠ comp(u) then
    w({u, v}) ← w({u, v}) + 1
  end if
  if ∃ component set X with connectivity at least α and |X| > 1 and nodes(X) ≤
  k then
    merge(X)
  end if
  if ∃ component set Y with connectivity at least α and nodes(Y) > k then
    delete(Y)
  end if
end for
```

6.1 Algorithm Definitions

We begin our analysis by introducing two general definitions that we will use throughout the analysis.

Definition 1. Define for any subset S of components $w(S)$ as the total weight of all edges between nodes of S.

Note that such an edge can only have positive weight if its endpoints are in different components.

Definition 2. Let a set of components of size at least 2 and of connectivity α be a *mergeable* component set.

6.2 Structural Properties

Note: These properties are changed to use the connectivity based approach which generally simplifies them, but guarantees slightly less minimum edge weight within mergeable component sets.

Definition 3. An *α-connected component* is a maximal set of vertices that is α-connected.

Lemma 4. At any time t after Crep performed its merge and delete actions all subsets S of components with $|S| > 1$ have connectivity less than α, i.e. there exist no mergeable component sets after Crep performed its merges.

Proof. We proof the lemma by an induction on steps. The lemma holds trivially at time 0.

Now assume that at some time $t > 0$ the lemma does not hold, i.e. there is a subset S of components with connectivity at least α and $|S| > 1$. We may assume that t is the earliest time for which S has connectivity α.

Then the incrementation of the weight of edge e at time t raised the connectivity of S, but S was not merged into a new α-connected component C. if no new component was created at time t then we arrive at a contradiction as Crep always merges if there exists a mergeable component set.

Now assume that a component C was created at time t. This means that C must also contain the endpoints of e. But then the conjunction of C and S forms an even larger subset of components with connectivity at least α which is a contradiction to the maximality of C and S.

Lemma 5. Fix any time t and consider weights right after they were updated by CREP but before any merge or delete actions. Then all subsets S of components with $|S| > 1$ have connectivity at most α and a mergeable component set S has connectivity exactly α.

Proof. This lemma follows directly from lemma 2 as connectivities can only increase by at most 1 at each time t.

Lemma 6. The weight between the components of a component subset S of connectivity α is at least $|S|/2 \cdot alpha$.

Proof. Consider the sum of the weighted degrees of all components:

$$\sum_{c \in S} deg_S(c) = 2 \sum_{e \in S} w(e)$$

The equality follows as the left sum counts each edge twice, once for each endpoint. Now consider the fact that each component must have degree at least α with respect to the edges in S as S has connectivity α and hence the lemma follows.

Lemma 7. The weight between the components of a component subset S of connectivity α is at most $(|S| - 1) \cdot \alpha$.

Proof. We iteratively partition S into subsets via minimum cuts with regard to edge weight, i.e. we consider a minimum edge cut of S which partitions S into the subsets S_1 and S_2 and iteratively partition the resulting sets until all sets contain only one component each. As this required at most $|S| - 1$ cuts of value at most α the lemma follows.

6.3 Upper Bound On CREP

We define the set DEL(σ) as the set of components that were deleted by CREP during its execution given the input sequence σ.

We define the following notions for a component $C \in$ DEL(σ), i.e. the subgraph induced by the nodes of C has connectivity at least α and C consists of more than k nodes:

Let EPOCH(C) denote the (node, time) pairs of nodes in C starting at the time after the time $\tau(node)$ when *node* was last turned into a singleton component, i.e. EPOCH(C) $= \bigcup_{n \in nodes(C)} \{n\} \times \{\tau(n) + 1, ..., \tau(C)\}$. Note that for $C \in$ DEL(σ), $\tau(C)$ denotes both the time of the creation as well as the time of deletion of C. We can use this definition of a component epoch EPOCH(C) to uniquely assign each node to a deleted component C at each point in time t (except for nodes in components that persist until the end of sequence σ).

We assign all requests to EPOCH(C) whose corresponding requests are deleted because of the deletion of component C and call the set of those requests REQ(C). We split the requests from REQ(C) into two sets: CORE(C) contains all requests for which both nodes have already been assigned to C at the time of the request, i.e.

$$\text{CORE}(C) = \{r = \{u, v\} \in \sigma | (u, \text{TIME}(r)) \in \text{EPOCH}(C) \text{ and } (v, \text{TIME}(r)) \in \text{EPOCH}(C)\}.$$

These are the requests that led to the creation of component C. $\textsc{halo}(C)$ contains all requests from $\textsc{req}(C)$ for which exactly one end point was associated with C at the time of the request. Note that this means that $\textsc{halo}(C) = \textsc{req}(C) \backslash \textsc{core}(C)$.

We start the analysis by bounding the communication cost of Crep that is due to serving requests from $\textsc{core}(C)$ for $C \in \textsc{del}(\sigma)$.

Lemma 8. With augmentation $2+\epsilon$, Crep pays at most communication cost $|C| \cdot \alpha$ for requests in $\textsc{core}(C)$ where $C \in \textsc{del}(\sigma)$.

Proof. The lemma follows directly from Lemma 7 due to the fact that component sets of connectivity α get merged immediately by Crep as shown in Lemma 5. □

We define $\textsc{final-weights}(\sigma)$ as the total amount of edge weight between the components $\textsc{final-comps}(\sigma)$ which are present after the execution of Crep given input sequence σ.

Together with the fact that Crep pays for all requests in $\textsc{halo}(C)$ for deleted components C we use these definitions as well as the previous lemma to bound the total communication cost of Crep in the following lemma.

Lemma 9. The cost of serving communication requests that Crep has to pay, denoted by $\textsc{Crep}^{req}(\sigma)$ given input sequence σ is bounded by $\textsc{Crep}^{req}(\sigma) \leq \sum_{C \in \textsc{del}(\sigma)} (|C| \cdot \alpha + |\textsc{halo}(C)|) + \sum_{C \in \textsc{final-comps}(\sigma)} |C| \cdot \alpha + \textsc{final-weights}(\sigma)$.

Proof. The number of communication requests that led to the creation of a component C is bounded by $|C| \cdot \alpha$ due to Lemma 7. If component C was deleted by Crep then also the requests from $\textsc{halo}(C)$ were deleted. All other edge weights were not changed. The remaining communication requests that have not been accounted for so far have either led to the creation of component $C \in \textsc{final-comps}(\sigma)$ and are hence also bounded by $|C| \cdot \alpha$ or have not let Crep to any merge and are hence contained in $\textsc{final-comps}(\sigma)$. This concludes the proof. □

We continue our analysis by bounding the migration cost of Crep in the following lemma.

Lemma 10. With augmentation $2+\epsilon$, Crep pays at most migration costs of $\sum_{C \in \textsc{del}(\sigma) \cup \textsc{final-comps}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha$.

Proof. First note that Crep only performs migrations when it merges components. We fix a component $C \in \textsc{del}(\sigma) \cup \textsc{final-comps}(\sigma)$ and bound the number of times each node of C is moved as Crep processes the requests that led to the creation of C.

As Crep only reserves additional space $\lfloor \epsilon \cdot |B| \rfloor$ for components of size at least $2/\epsilon$ for each component B and only moves component B when a merge results in a component of size more than $(1+\epsilon) * |B|$ each node of C is moved at most $(2/\epsilon+1)+\log k$ times. Summing over all nodes in C that were actually moved by Crep bounds the number of migrations by $|C| \cdot ((2/\epsilon + 1) \log k)$ as components get deleted without migrations once they contain more than k nodes. This leads to the desired bound on the migration costs as each node migration incurs cost α to Crep. □

Finally we summarize our results in the following lemma.

Lemma 11. With augmentation $2+\epsilon$, Crep pays at most total cost $2 \cdot \sum_{C \in \textsc{del}(\sigma) \cup \textsc{final-comps}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha + \sum_{C \in \textsc{del}(\sigma)} |\textsc{halo}(C)| + \textsc{final-weights}(\sigma)$.

Proof. The lemma follows by summing the results from Lemma 9 and Lemma 10.

6.4 Lower Bound on Opt

We split the analysis of the cost Opt has to pay into two parts. First we examine the cost incurred to Opt due to requests that led Crep to create (and delete) a component $C \in \text{DEL}(\sigma)$, i.e. those from $\text{CORE}(C)$. Then we provide a lower bound on the cost incurred to Opt due to requests from $\text{HALO}(C)$ for $C \in \text{DEL}(\sigma)$.

Lemma 12. Opt pays at least $\lceil \frac{|C|}{k} \rceil \cdot \alpha/2$ due to requests from $\text{CORE}(C)$ for any $C \in \text{DEL}(\sigma)$.

Proof. We fix $C \in \text{DEL}(\sigma)$ arbitrarily and observe that any solution that does not move any vertices of C during the epoch of C incurs cost at least $\lceil \frac{|C|}{k} \rceil \cdot \alpha/2$ as the nodes of C need to be distributed on at least $\lceil \frac{|C|}{k} \rceil$ servers. Lemma 6 gives a suitable lower bound on the weight between these servers and hence also on the number of communication requests an algorithm which chooses this static configuration must pay for.

We show that it is not possible for Opt to improve upon this static solution via migrations. Observe that a migration involving only one node that is mapped to a server from the static configuration may only increase the communication costs for requests in $\text{CORE}(C)$ as it increases the number of partitions in the result of Lemma 6. A migration between two servers used in the static configuration may only decrease the weight between the two servers involved and incurs cost 2α to Opt. Note that the proof of Lemma 6 only requires the weight between one component set S and all the other component sets that together form a mergeable set to be at least α. If Opt now performs a swap of two nodes we can attribute cost 2α to Opt for the connection between these two servers which may only increase the cost we can attribute to Opt. Hence the lemma follows. □

Lemma 13. Opt pays at least $\sum_{C \in \text{DEL}(\sigma)} |\text{HALO}(C)|/k$ for serving requests from $\bigcup_{C \in \text{DEL}(\sigma)} \text{HALO}(C) =: H$.

Proof. First note that we may ignore requests from H that both Crep and Opt pay for as including these may only give Crep an advantage in our analysis.

In the following we only consider those requests from H that Crep pays for while Opt does not. We relate the number of those requests that Opt does not pay to the number of migrations Opt performs by assigning each request from H to the next migration performed by Opt that moves a node involved in the request.

We now bound the number of requests from H Opt may be able to serve without cost by assigning nodes to servers and performing migrations optimally. Consider all requests from H that happen between nodes that Opt has placed on the same server between Opt migrations that involve nodes from that server. As the requests from H have not led Crep to perform a migration there can be at most $(k-1) \cdot \alpha$ such requests before Opt changes the nodes mapped to the server by performing a migration. As this migration incurs cost 2α to Opt we conclude that Opt may only improve its cost by a factor of k over the cost paid by Crep for serving requests from H. □

We obtain the final lower bound on the cost of Opt by combining the results of the previous two lemmas. Note that migrations may allow Opt to both reduce the cost associated with $\text{CORE}(C)$ and those from $\text{HALO}(B)$ for $B, C \in \text{DEL}(\sigma)$. Hence we only account half of the cost of Opt and conclude our analysis on the lower bound on Opt with the following lemma.

Lemma 14. $\textsc{Opt}(\sigma) \geq 1/2 \sum_{C \in \text{DEL}(\sigma)} (\lceil \frac{|C|}{k} \rceil \cdot \alpha/2 + |\textsc{halo}(C)|/k)$.

Proof. The proof follows from the two previous lemmas and the thoughts on these.

6.5 Competitive Ratio

In this section we combine the results of Lemma 11 and Lemma 14 to obtain the following theorem.

Theorem 15. With augmentation $(2 + \epsilon)$ the competitive ratio of Crep is in $O(k \log k)$.

Proof. We arbitrarily fix an input sequence σ and use our previous results to bound the competitive ratio of Crep. We define $\textsc{comps}(\sigma) := \textsc{del}(\sigma) \cup \textsc{final-comps}(\sigma)$ in order to improve readability.

$$\begin{aligned}
\frac{\textsc{Crep}(\sigma)}{\textsc{Opt}(\sigma)} &\leq \frac{2 \cdot \sum_{C \in \text{COMPS}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha + \sum_{C \in \text{DEL}(\sigma)} |\textsc{halo}(C)| + \textsc{final-weights}(\sigma)}{1/2 \sum_{C \in \text{DEL}(\sigma)} (\lceil \frac{|C|}{k} \rceil \cdot \alpha/2 + |\textsc{halo}(C)|/k)} \\
&\leq k \log k \frac{2 \cdot \sum_{C \in \text{DEL}(\sigma)} |C| \cdot (2/\epsilon + 1) \cdot \alpha + \sum_{C \in \text{DEL}(\sigma)} |\textsc{halo}(C)|}{1/2 \sum_{(C \in \text{DEL}(\sigma)} |C| \cdot \alpha/2 + |\textsc{halo}(C)|)} + \beta \\
&= O(k \log k) + \beta
\end{aligned}$$

where $\beta = \sum_{C \in \text{FINAL-COMPS}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha + \textsc{final-weights}(\sigma)$.

To obtain the bound on β we observe that the components in $\textsc{final-comps}(\sigma)$ each are of size at most k since they were not deleted by Crep. This allows us to derive to bound $\sum_{C \in \text{FINAL-COMPS}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \leq l \cdot k \cdot ((2/\epsilon + 1) + \log k)$. Since at the end of the execution of Crep there can be at most $k \cdot l$ components, Lemma 7 allows us to bound $\textsc{final-weights}(\sigma)$ by $k \cdot l \cdot \alpha$. Hence we conclude that $\beta \leq l \cdot k \cdot ((2/\epsilon + 1) + logk) \cdot \alpha + k \cdot l \cdot \alpha \in O(k \log k)$. □

7 Implementation Details

7.1 Algorithm Pseudocode

Algorithm 2 insertAndUpdate(a,b)

```
if comp[a] == comp[b] then
    return
end if
addEdge(a, b)
updateDecomposition(a, b)
del← updateMapping(alphaConnectedComponents)
delComponents(del)
```

Algorithm 3 updateDecomposition(a,b)

```
q← findSmallestSubgraph(a, b)
while q not empty do
  current← q.popFront()
  if res.connectivity==alpha then
    continue
  end if
  res← decompose(current, current.connectivity+1)//decomposition based on s-t-cuts
  current.connectivity ← value of smallest encountered cut
  if current.connectivity≥alpha then
    continue
  end if
  childrenQueue ← res
  //make sure that only subgraphs with higher connectivity are added as children

  while childrenQueue not empty do
    c←childrenQueue.pop()
    cRes←decompose(c, current.connectivity+1)
    c.connectivity ← value of smallest encountered cut
    if decompose returned only one graph then
      current.children.add(cRes)
      if cRes has connectivity smaller than alpha then
        q.push(cRes)
      end if
    else
      childrenQueue.add(cRes)
    end if
  end while
end while
```

Algorithm 4 delComponents(del)

```
delInterEdges(del)
root.connectivity=0
root.children={}
updateDecomposition(0,1)
```

7.2 Algorithm Explanations

- Algorithm 2 calls the other routines as needed
- Algorithm 3 starts at the smallest subgraph containing the nodes a and b in the decomposition tree and computes a new decomposition of the subgraph. Specifically it uses the decomposition approach from [4] to decompose one subgraph and then also computes the subgraphs with the next higher connectivity and recurses until the connectivity has reached alpha.
- updateMapping checks whether the alphaConnectedComponents were changed. If yes then it either collocates them if the resulting component is small enough or it adds the component to its return value. Then all the returned components are deleted, i.e. the edges connecting its nodes are deleted and the decomposition is recomputed
- this deletion is performed by Algorithm 4

8 Evaluation

For this evaluation we compare our implementation described in Section 7 to a static algorithm available via Metis (METIS_PartGraphRecursive) and an adaptive/dynamic algorithm (ParMETIS_V3_AdaptiveRepart) implemented in the ParMetis framework. Both frameworks are known to produce very good results and to be very fast.
As input data we use several HPC traces, the nature of the data is described in more detail by Avin, Ghobadi, Griner and Schmid in [3].

All data sets contain 1024 different communication nodes and are limited to the first 300 000 requests. The value of α is set to be 6 and the algorithm was tasked to partition the nodes into 32 clusters of size 32 each. The dynamic algorithms were allowed to use augmentation with a factor of 2.1, i.e. for the dynamic algorithms the maximum cluster capacities were $\lfloor 32 \cdot 2.1 \rfloor = 67$.
To our knowledge it is not possible to specify hard limits for the capacities used by the static algorithm implemented in Metis and as a result there are some occurences where the algorithm exceeds the capacities that are allowed by a small amount.

We will first discuss the overall results of our experiments, i.e. we will describe the quality of the results (see Figure 3) as well as the running time needed for each examined algorithm (see Figure 4).
The static algorithm is shown to give the best results in the shortest time, but the low running was also to be expected as it is only called once as opposed to the 300 000 times the other algorithms need to decide whether they want to change their partitioning. The static algorithm also has knowledge of all requests and as a result is able to produce the best results.
For data set A our decomposition algorithm beats the adaptive ParMetis algorithm by a significant amount of about one third of the cost of the latter while ParMetis is significantly faster. For data sets B and C ParMetis is shown to produce slightly better results within drastically less computation time. It is worth mentioning that ParMetis uses the ... graph description format that does not allow for easy adaptation on the fly and needed to be recomputed after every request during our tests.

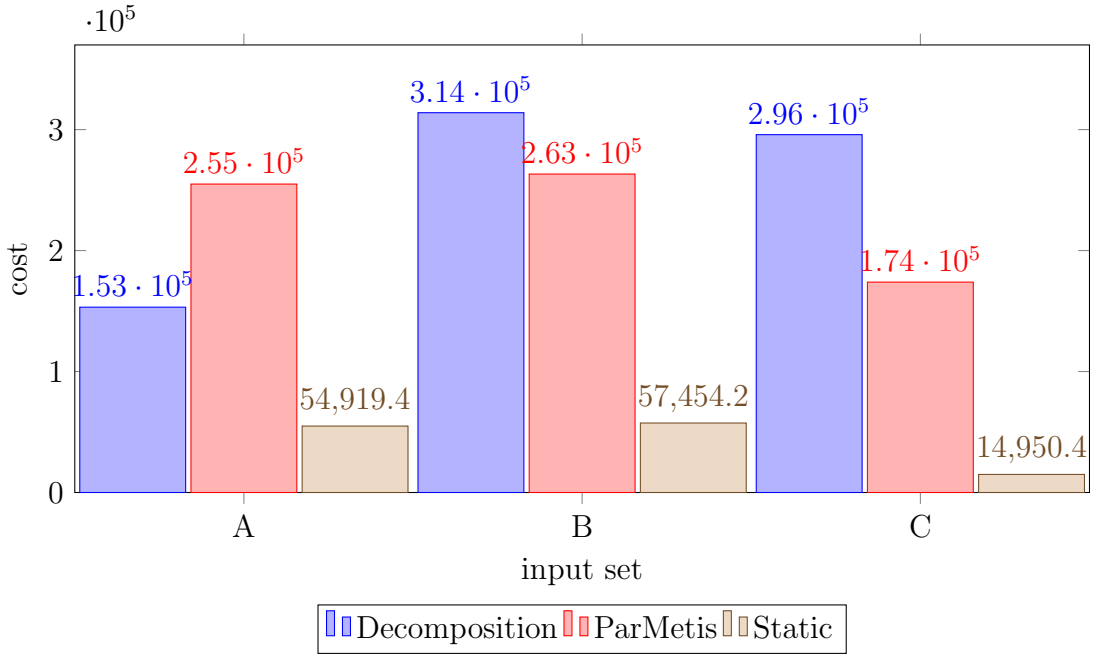

Figure 3: comparison of total cost

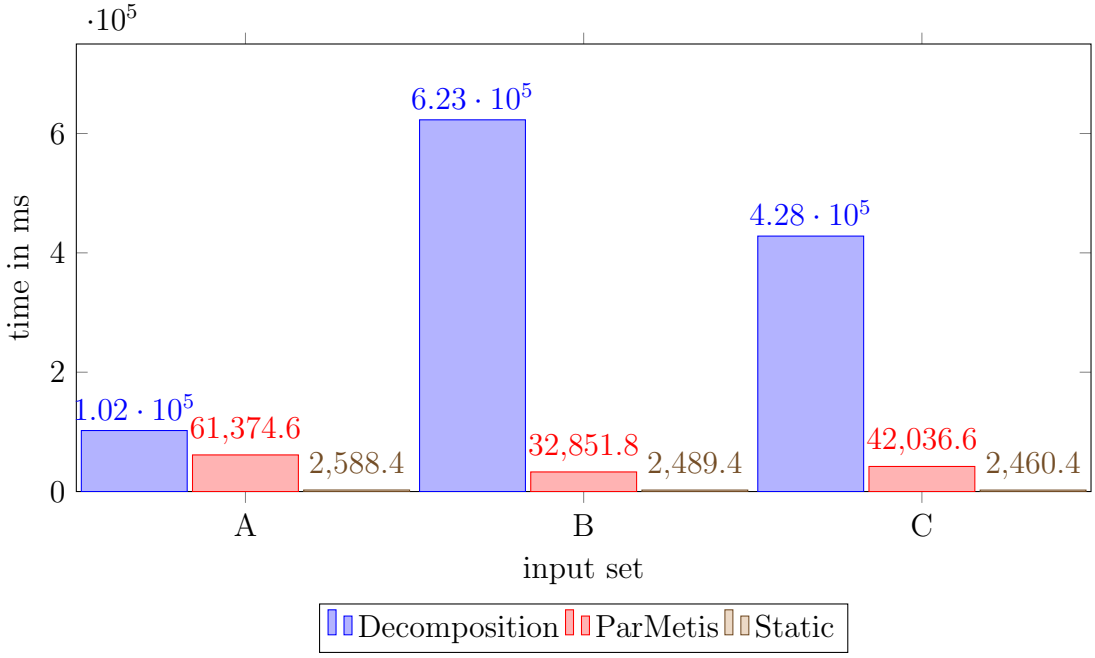

Figure 4: comparison of run time

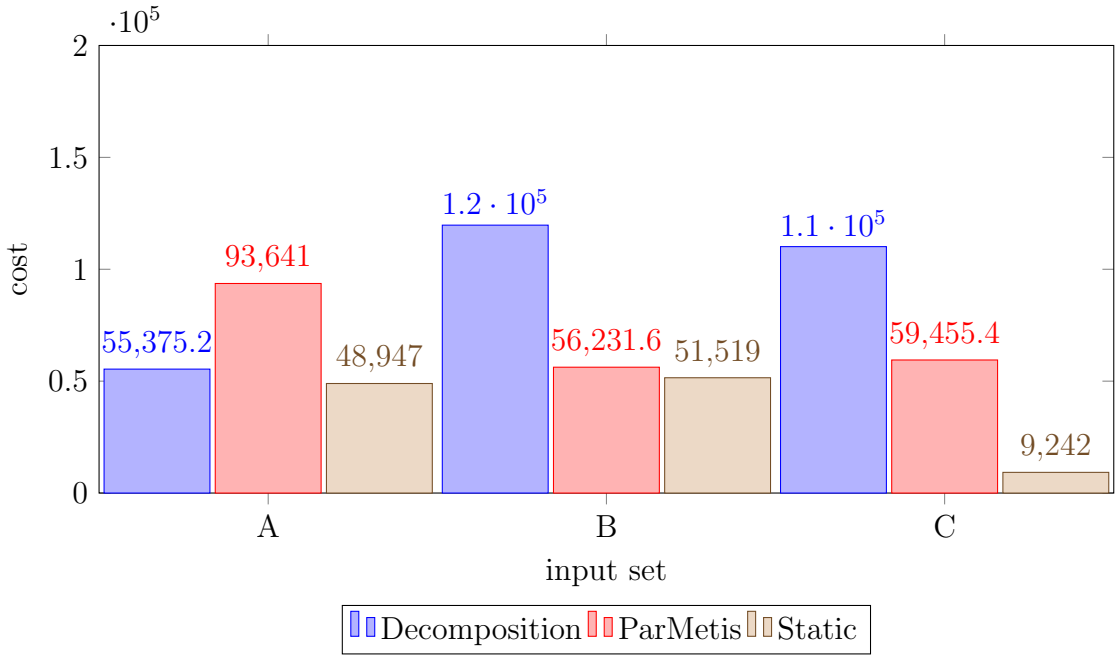

Figure 5: comparison of communication cost

However, we chose not to include this in the running time calculations.

In the next section we discuss the general distribution of the total costs of each algorithm to communication (see Figure 5) and migration costs (Figure 6).
Both ParMetis as well as our decomposition algorithm produce significantly more migration cost than communication cost while the static algorithm predominantly pays for communication. This shows that the dynamic algorithms tend to migrate too much while the static implementation is restricted to only migrate once to a static configuration it finds suitable and as a result has to pay more for communication. This also shows that there is potential to refine the dynamic implementations in such a way that they produce more balanced, and hopefully also less, cost overall.

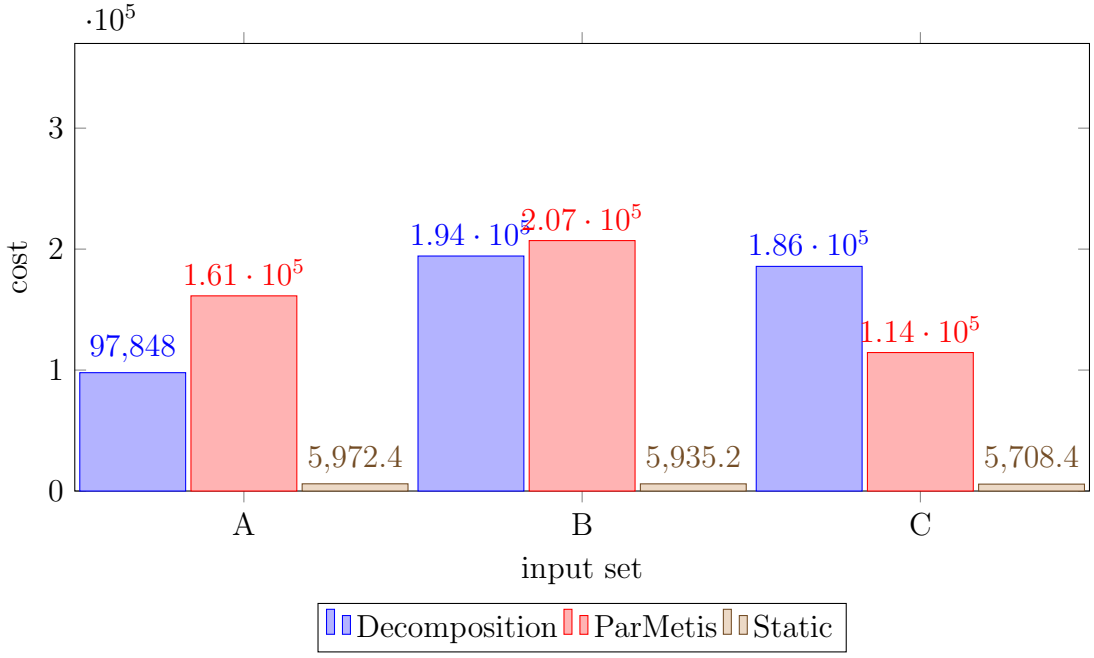

Figure 6: comparison of migration cost

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