Online Balanced Repartitioning of Dynamic Communication Patterns in Polynomial Time

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Abstract

This paper revisits the online balanced repartitioning problem (introduced by Avin et al. at DISC 2016) which asks for a scheduler that dynamically collocates frequently communicating nodes, in order to reduce communication costs while minimizing migrations in distributed systems. More specifically, communication requests arrive online and need to be served, either remotely across different servers at cost 1, or locally within a server at cost 0; before serving a request, the online scheduler can change the mapping of nodes to servers, i.e., migrate nodes, at cost α per node move. Avin et al. presented a deterministic $O(n \log n)$-competitive algorithm, which is optimal up to a logarithmic factor; however, their algorithm has the drawback that it relies on expensive repartitioning operations which result in a super-polynomial runtime. Our main contribution is a different deterministic algorithm pCrep which achieves the same competitive ratio, but runs in polynomial time. Our algorithm monitors the connectivity of communication requests over time, rather than the density as in prior work; this enables the polynomial runtime. We analyze pCrep both analytically and empirically.

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1 Introduction

Most distributed systems critically rely on an efficient interconnecting communication network. With the increasing scale of these systems, the network traffic often grows accordingly: applications related to distributed machine learning, batch processing, or scale-out databases, spend a considerable fraction of their runtime shuffling data [15]. An interesting method to improve the efficiency in these systems is to exploit their resource allocation flexibilities: many distributed systems are highly virtualized today and support to relocate (or migrate) communication partners (e.g. virtual machines). By collocating frequently communicating nodes on the same server, slow and costly inter-server communication can be reduced. However, relocations also come at a cost, and the number of migrations should be kept low.

This paper revisits the online balanced repartitioning problem [5] which models the tradeoff between the benefits and the costs of dynamic relocations. The goal is to design an algorithm which maintains, at any time, a mapping of n communication nodes (virtual machines) to ℓ servers of fixed equal size k; in the absence of augmentation, $n = \ell k$. The communication pattern can be seen as a dynamic graph, from which communication requests arrive in an online manner; the online algorithm does not have prior knowledge of future communication requests. The goal is to strike a balance between the benefits and the costs of migrations. More specifically, the cost model is as follows: if a communication request is served remotely, i.e., between nodes mapped to different servers, it incurs a communication cost of 1; communication requests between nodes located on the same server are free of cost. Before the cost for the current request is payed, an algorithm has the option to migrate nodes at a cost of α for each node move.

The problem can be seen as a symmetric version of caching: two nodes can be "cached" together on any server.

1.1 Contributions

Our main result is a deterministic online algorithm pCrep for the dynamic balanced graph partitioning problem which achieves a competitive ratio of $O(k \log k)$ and runs in *polynomial time*, for a constant augmentation.

A $O(k \log k)$-competitive algorithm was already given by Avin et al. in [5], together with an almost tight lower bound $\Omega(k)$. However, the algorithm relies on expensive repartitioning which results in a super-polynomial runtime. Our algorithm, pCrep, is similar to the algorithm by Avin et al., but it comes with a twist: rather than considering the *density* of emerging communication patterns when deciding the repartitioning, we consider the *connectivity*. The latter allows for polynomial-time approximations, as we will show, but also requires a new analysis.

We not only evaluate our algorithm analytically but also in simulations, based on real datacenter workloads. We will make our implementation publicly available as open source, together with this paper.

1.2 Preliminaries

Let us introduce some definitions and notations that will be used throughout the paper. We define a graph $G = (V, E, w)$ where V is the set of vertices, E the set of (undirected) edges, and $w : E \to \mathbb{N}$ assigns each edge an (integer) weight. Given a graph $G = (V, E, w)$, we define an *(edge) cut* of G as a pair of two disjoint subsets X, Y of V such that $X \cup Y = V$. The value of this cut is the sum of the weight of edges between nodes from X and Y, i.e. $\sum_{e=\{u,v\} \in E : u \in X, v \in Y} w(e)$ is the value of the cut (X, Y). Note that such a cut can also be defined by the set of the edges connecting X and Y that are cut. We call a cut a *minimum (edge) cut* of G if it is one of the cuts with minimum value.

The *connectivity* of a graph G is equal to the value of a minimum edge cut of G. This definition will be used in order to define the communication components our algorithm maintains as these are subsets of V which induce subgraphs of high connectivity. We explain the concept of components in greater detail later.

Furthermore we define the term (s, t)*-cut* as a cut (X, Y) for which $s \in X$ and $t \in Y$, i.e. a (s, t)-cut separates the nodes s and t in G. Then a *minimum* (s, t)*-cut* is a (s, t)-cut of minimum value. Note that a minimum (s, t)-cut is not necessarily a minimum cut.

Finally we call an injective function $m : X \to Y$ a *mapping of* X *to* Y. We use this terminology for example when we talk about the assignment of nodes to servers.

2 Model

We consider the problem of maintaining a partitioning of a set of $n = k \cdot \ell$ nodes (e.g., processes or virtual machines) that communicate with each other, into ℓ servers (henceforth sometimes also called clusters) of size k each, while minimizing both the cost due to communication and due to node migrations. More formally we are given ℓ servers $V_0, ..., V_{\ell-1}$, each with capacity k and an initial perfect mapping of $n = k \cdot \ell$ nodes to the ℓ servers, i.e. each server is assigned exactly k nodes. An input sequence $\sigma = (u_1, v_1), (u_2, v_2), ...(u_i, v_i), ...$ describes the sequence of communication requests: the pair (u_t, v_t) represents a communication request between the nodes u_t and v_t arriving at time t. At time t the algorithm is allowed to perform node migrations at a cost of $\alpha > 1$ per move. After the migration step, the algorithm pays cost 1 if u_t and v_t are mapped to different servers and does not pay any cost otherwise. Note that an algorithm may also choose to perform no migrations at all.

We are in the realm of competitive analysis and as a result we compare an online algorithm Onl to the optimal offline algorithm Opt. Onl only learns of the requests in the input sequence σ as they happen and as a result only knows about the partial sequence $(u_1, v_1), ..., (u_t, v_t)$ at time t whereas Opt has perfect knowledge of the complete sequence σ at all times.

The goal is to design an online algorithm Onl with a good competitive ratio with regard to Opt defined as follows. An online algorithm Onl is ρ-competitive if there exists a constant β such that

$$\textsc{Onl}(\sigma) \leq \rho \cdot \textsc{Opt}(\sigma) + \beta \, \forall \sigma$$

where $\textsc{Onl}(\sigma)$ and $\textsc{Opt}(\sigma)$ denote the cost of serving input sequence σ of Onl and Opt respectively.

We consider a model with augmentation (as in prior work [5]), and allow the online algorithm to use larger capacities per server. In particular, the online algorithm is allowed to assign $(2 + \epsilon) \cdot n/\ell$ nodes to each server where $\epsilon > 0$. This augmented online algorithm is then compared with the optimal offline algorithm Opt which is not allowed to use any augmentation. Throughout this paper, we will also use $1 + \epsilon$ as the basis for the logarithm.

3 Basic Algorithm

We first describe the basic algorithm underlying our approach, before presenting the polynomial-time implementation later in this paper. In general, pCrep relies on a second-order partitioning of the communication nodes into *communication components* which represent node-induced sub-graphs of the original communication graph given by the requests from the input sequence σ. Initially each node forms a singleton component, but as the input sequence σ is revealed, new communication patterns unfold. The algorithm keeps track of these patterns by maintaining a graph in which the nodes represent the actual communication nodes and the weighted edges represent the number of communication requests between nodes that were part of different components at the time of the request; that is, for edge $e = \{u, v\}$, $w(e)$ represents the number of paid communication requests between u and v. We say that a communication request between nodes u and v is *paid* if the nodes are located on different servers at the time of the request. The algorithm merges a set S of components into a new component C if the connectivity of the component graph induced by the components in S is at least α. After each edge insertion the algorithm checks whether there exists a new component set S with $|S| > 1$ which fulfills this requirement.

If after any request and the insertion of the resulting edge the algorithm discovers a new subset S of nodes whose induced subgraph has connectivity at least α and which is of cardinality at most k, it merges the components that form this set into one new component and collocates all the nodes in the resulting set on a single server. The algorithm reserves additional space $\min\{\lfloor \epsilon \cdot |C| \rfloor, k - |C|\}$ for each component on the server it is currently located on. Note that the additional reservation may be zero for components smaller than $1/\epsilon$. This reservation guarantees that nodes are not migrated too often for the analysis to work. This also limits the total space a component can use to a maximum of k. This makes sense as a component whose size exceeds k is deleted (rather than merged). To this end pCrep keeps track of the reservations for each component.

The algorithm uses augmentation $2 + \epsilon$ in order to guarantee that the collocation of such component sets of at most k individual communication nodes is always possible without moving a node not in C. This guarantees by an averaging argument that there is always at least one cluster with capacity at least k, which a newly merged component can be moved to.

Algorithm 1 pCrep

initialize an empty graph on n nodes
turn each of the n nodes into a singleton component
for all $r = \{u, v\} \in \sigma$ **do**
  if $comp(v) \neq comp(u)$ **then**
    $w(\{u, v\}) \leftarrow w(\{u, v\}) + 1$
  end if
  if $\exists$ component set X with connectivity at least α and $|X| > 1$ and $nodes(X) \leq k$ **then**
    mergeAndRes(X)
  end if
  if $\exists$ component set Y with connectivity at least α and $nodes(Y) > k$ **then**
    delete(Y) // *to be specified later*
  end if
end for

Algorithm 2 mergeAndRes(X)

for all $C \in X$ **do**
  $cap(serv(C)) \leftarrow cap(serv(C)) + res(C)$
end for
$N \leftarrow collocate(X)$ // *moves all components from* X *to the same server as described*
// N *contains the newly created component*
if $|N| > 2/\epsilon$ **then**
  // *reserve additional space*
  $res(N) \leftarrow \min\{\lfloor \epsilon \cdot |N| \rfloor, k - |N|\}$
  $cap(serv(N)) \leftarrow cap(serv(N)) - res(N)$
end if

If the subset has cardinality greater than k, the resulting component is deleted. This general structure of our algorithm is also summarized in the form of pseudocode in Algorithm 1 and Algorithm 2. In the pseudocode description we denote the reservation of a component C by $res(C)$ and the current server it is mapped to by $serv(C)$. The free capacity of a server i is denoted by $cap(i)$.

The main differentiating factor of this approach compared to prior work [2, 5] is that we merge once a component set reaches connectivity α, while prior approaches do so once the component set reaches a certain density threshold. More specifically earlier algorithms merge a component set S once it fulfills $w(S) \geq (|S| - 1) \cdot \alpha$ where $w(S)$ denotes the cumulative weight of the edges between nodes contained in the components of S.

pCrep resets all the edges contained in the deleted component C and also resets the weights of edges *adjacent* to C, i.e. all edges $e = \{u, v\}$ are reset to zero if u or v were contained in component C at the time of its deletion. The deletion method is also described in pseudocode in Algorithm 3. Furthermore the merge and deletion process is illustrated in Figure 1.

As we will see, the idea for the competitive analysis is to relate the cost of both Opt and pCrep to the deleted components in the solution of pCrep. The fact that pCrep also resets adjacent edges means that we can uniquely identify requests with the deleted component whose deletion led to the reset of the corresponding edge weights to zero.

Algorithm 3 delete(Y) of pCrep

for all $e = \{u, v\} \in E$ **do**
  if $u \in Y$ or $v \in Y$ **then**
    $w(e) \leftarrow 0$
  end if
end for
for all $C \in Y$ **do**
  $cap(serv(C)) \leftarrow cap(serv(C)) + res(C)$
  $res(C) \leftarrow 0$
end for

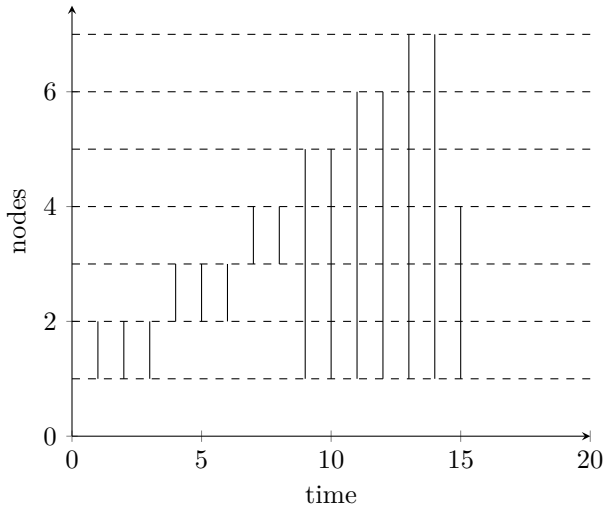

Figure 1 Illustration for the pCrep approach: The horizontal lines represent the nodes over time, vertical lines represent requests between the respective end points. We consider the case where $\alpha = 3$ and $k = 3$. The first six requests lead pCrep to merge the nodes $\{0, 1, 3\}$ into a new component C. The following eight requests connect the nodes 4, 5, 6 and 7 to C, but the respective cuts have value 2, hence no merge is performed. At time $t = 15$ finally the node 4 gets merged with C which results in a component of size 4 which gets deleted. During this deletion all edges shown in the figure are deleted, both those that led to the merges and the remaining ones.

4 Competitive Analysis

We analyze the competitive ratio of pCrep with augmentation $(2 + \epsilon)$ and show that pCrep is $O(2/\epsilon \cdot k \log k)$-competitive. We will use the following general definitions.

▶ **Definition 1.** *For any subset S of components, let $w(S)$ be the total weight of all edges between nodes of S.*

▶ **Definition 2.** *We call a set of components of size at least 2 and of connectivity α mergeable.*

▶ **Definition 3.** *An α-connected component is a maximal set of vertices that is α-connected.*

Despite the algorithmic differences, some claims from [2] can be adapted for pCrep. In Section A.1, we introduce some preliminaries which can easily be derived from prior work. In particular, we can show that there is always at most one mergeable component set after the insertion of a new edge which pCrep then merges. The earliest point in time a new

mergeable component set can emerge is after the next edge is inserted. In the following, we will now focus on the novel aspects of the competitive analysis.

4.1 Upper Bound on pCrep

We start the analysis with upper bounding the cost of pCrep by introducing several notions that we will use throughout the analysis. We define the set $\text{DEL}(\sigma)$ as the set of components that were deleted by pCrep during its execution given the input sequence σ. We define the following notions for a deleted component $C \in \text{DEL}(\sigma)$. Let $\text{EPOCH}(C)$ denote the (node, time) pairs of nodes in C starting at the time after the time $\tau(node)$ when *node* was last turned into a singleton component, i.e.

$$\text{EPOCH}(C) = \bigcup_{n \in nodes(C)} \{n\} \times \{\tau(n)+1, ..., \tau(C)\}.$$

Note that for $C \in \text{DEL}(\sigma)$, $\tau(C)$ denotes both the time of the creation as well as the time of deletion of C. We can use this definition of a component epoch $\text{EPOCH}(C)$ to uniquely assign each node to a deleted component C at each point in time t (except for nodes in components that persist until the end of sequence σ). We assign all requests to $\text{EPOCH}(C)$ whose corresponding requests are deleted because of the deletion of component C and call the set of those requests $\text{REQ}(C)$. We split the requests from $\text{REQ}(C)$ into two sets: $\text{CORE}(C)$ and $\text{HALO}(C)$. $\text{CORE}(C)$ contains all requests for which both nodes have already been assigned to C at the time of the request, i.e.

$$\text{CORE}(C) = \{r = \{u, v\} \in \sigma | (u, \text{TIME}(r)) \in \text{EPOCH}(C) \text{ and } (v, \text{TIME}(r)) \in \text{EPOCH}(C)\}.$$

These are the requests that led to the creation of component C by increasing the connectivity within the corresponding subgraph.

We define $\text{HALO}(C)$ as the set of all requests from $\text{REQ}(C)$ for which exactly one end point was associated with C at the time of the request. Note that this means that $\text{HALO}(C) = \text{REQ}(C) \backslash \text{CORE}(C)$. These definitions allow us to differentiate between the highly-connected sub-graph induced by the nodes of C which are connected by requests from $\text{CORE}(C)$ and the edges leaving C from $\text{HALO}(C)$ which are relatively less dense as pCrep has not merged any outer node with the component.

We start the analysis by bounding the communication cost of pCrep that is due to serving requests from $\text{CORE}(C)$ for $C \in \text{DEL}(\sigma)$.

▶ **Lemma 4.** *With augmentation $2+\epsilon$,* pCrep *pays at most communication cost $|C| \cdot \alpha$ for requests in $\text{CORE}(C)$ where $C \in \text{DEL}(\sigma)$.*

Proof. First note that due to Lemma 14, pCrep merges mergeable component sets as soon as they emerge. Whenever pCrep performs a merge of a mergeable component set S, Lemma 17 states that there was at most total edge weight $(|S|-1) \cdot \alpha$ between the merged components, i.e. $w(S) \leq (|S|-1) \cdot \alpha$. Each such merge decreases the number of components that need to be merged in order to form component C by $|S|-1$. Hence pCrep has payed at most $|C| \cdot \alpha$ communication cost for requests in $\text{CORE}(C)$. ◀

We define $\text{FIN-WEIGHTS}(\sigma)$ as the total amount of edge weight between the components $\text{FIN-COMPS}(\sigma)$ which are present after the execution of pCrep given input sequence σ. Together with the fact that pCrep pays for all requests in $\text{HALO}(C)$ for deleted components C we use these definitions as well as the previous lemma to bound the total communication cost of pCrep in the following lemma.

▶ **Lemma 5.** *The cost of serving communication requests that* pCrep *has to pay, denoted by* $\text{pCrep}^{req}(\sigma)$ *given input sequence* σ *is bounded by*

$$\text{pCrep}^{req}(\sigma) \leq \sum_{C \in DEL(\sigma)} (|C| \cdot \alpha + |HALO(C)|) + \sum_{C \in FIN\text{-}COMPS(\sigma)} |C| \cdot \alpha + FIN\text{-}WEIGHTS(\sigma).$$

Proof. The number of communication requests that led to the creation of a component C is bounded by $|C| \cdot \alpha$ due to Lemma 17. If component C was deleted by pCrep then also the edge weights corresponding to requests from $\text{halo}(C)$ were reset to zero. All other edge weights were not changed. The remaining communication requests that have not been accounted for so far have either led to the creation of component $C \in \text{fin-comps}(\sigma)$ and are hence also bounded by $|C| \cdot \alpha$ or have not led pCrep to any merge and are hence contained in $\text{fin-weights}(\sigma)$. This concludes the proof. ◀

We continue our analysis by bounding the migration cost of pCrep.

▶ **Lemma 6.** *With augmentation* $2 + \epsilon$, pCrep *pays at most migration costs of*

$$\text{pCrep}^{mig}(\sigma) \leq \sum_{C \in DEL(\sigma) \cup FIN\text{-}COMPS(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha.$$

Proof. First note that pCrep only performs migrations when it merges components. We fix a component $C \in \text{del}(\sigma) \cup \text{fin-comps}(\sigma)$ and bound the number of times each node of C is moved as pCrep processes the requests that led to the creation of C.

As pCrep only reserves additional space $\lfloor \epsilon \cdot |B| \rfloor$ for each component B and only moves component B when a merge results in a component of size more than $(1+\epsilon) \cdot |B|$ each node of C is moved at most $(2/\epsilon + 1) + \log k$ times. Summing over all nodes in C that were actually moved by pCrep bounds the number of migrations by $|C| \cdot ((2/\epsilon + 1) \log k)$ as components get deleted without migrations once they contain more than k nodes. This leads to the desired bound on the migration costs as each node migration incurs cost α to pCrep. ◀

We combine our results from Lemma 5 and Lemma 6 in the following lemma in order to obtain the final upper bound on the cost of pCrep.

▶ **Lemma 7.** *With augmentation* $2 + \epsilon$, pCrep *pays at most total cost*

$$2 \cdot \sum_{C \in COMPS(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha + \sum_{C \in DEL(\sigma)} |HALO(C)| + FIN\text{-}WEIGHTS(\sigma).$$

where $COMPS(\sigma) = DEL(\sigma) \cup FIN\text{-}COMPS(\sigma)$.

Proof. We use the results from Lemma 5 and Lemma 6 to obtain the lemma:

$$\begin{aligned}
\text{pCrep}(\sigma) &\leq \text{pCrep}^{req} + \text{pCrep}^{mig} \\
&\leq \sum_{C \in \text{DEL}(\sigma)} (|C| \cdot \alpha + |\text{halo}(C)|) + \sum_{C \in \text{FIN-COMPS}(\sigma)} |C| \cdot \alpha + \text{fin-weights}(\sigma) \\
&\quad + \sum_{C \in \text{COMPS}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha \\
&\leq 2 \cdot \sum_{C \in \text{COMPS}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha + \sum_{C \in \text{DEL}(\sigma)} |\text{halo}(C)| \\
&\quad + \text{fin-weights}(\sigma).
\end{aligned}$$

◀

4.2 Lower Bound on Opt

We next bound the cost on Opt by assigning cost to Opt based on the size of the components C that pCrep deletes and the associated adjacent edges $\text{halo}(C)$ which pCrep resets to zero during the deletion of C. In order to achieve this we introduce some additional notions. First we define the term *offline interval* of a node v to be the time between two migrations of v in the solution of Opt. More specifically an offline interval of node v either starts at time zero (if it is the first offline interval of v) or after a migration of v and ends with the next migration of node v that Opt performs.

Furthermore we say that an offline interval is contained in the epoch $\text{epoch}(C)$ of a component $C \in \text{del}(\sigma)$ if it ends before the time $\tau(C)$. Note that $\tau(C)$ is both the time of the creation of C in the solution of pCrep and the time of its deletion as $C \in \text{del}(\sigma)$. We assign a request r involving the node v to an offline interval of v if it is both the first offline interval of one of the end points of r that ends and if the offline interval ends before the deletion of the edge representing r due to a component deletion. The requests from $\mathcal{H} = \bigcup_{C \in \text{DEL}(\sigma)} \text{halo}(C)$ that are not assigned to any offline interval are then those which are deleted due to the deletion of a component that took place before the corresponding offline interval ended. Let P denote the set of edges from $\bigcup_{C \in \text{DEL}(\sigma)} \text{halo}(C)$ that both pCrep and Opt pay for and let I denote the set of requests we have assigned to offline intervals.

These definitions are illustrated in Figure 2. Note that we only show some requests explicitly for the sake of readability. The grey horizontal lines represent the nodes at each time t. The red outline surrounds the (node,time) pairs of $\text{epoch}(C)$. Blue dots mark migrations of the corresponding node performed by Opt while red dots mark deletions of the component the node was assigned to at that time. The dashed vertical lines in black mark requests that are assigned to another component because it is deleted before component C. The dashed green line is a request from $\text{halo}(C)$ assigned to the offline interval of node 5 between the two blue dots. The regular green lines are assigned to an offline interval which is not contained in $\text{epoch}(C)$. We define this concept more formally at a later point in the analysis. The lines in magenta are sample requests from $\text{core}(C)$.

We start by bounding the total edge weight (the total number of requests) we assign to any one offline interval when limiting ourselves to requests from $\mathcal{H}$ which pCrep pays for but Opt does not. We denote the set of these requests by N, i.e. $N = \mathcal{H} \backslash P$. Note that $\mathcal{H}$ only contains requests which pCrep payed for due to the definition of $\text{halo}(C)$.

▶ **Lemma 8.** *We assign at most $k \cdot \alpha$ requests from N to any one offline interval.*

Proof. We fix an arbitrary offline interval of node v. Observe that none of the nodes involved in the assigned requests are moved by Opt during the offline interval, hence all the requests in question involve only nodes that Opt has placed on the same server as v during the offline interval.

The number of such nodes is hence limited by the server capacity k. As we only examine requests from $\mathcal{H}$ we know that none of these requests have led pCrep to perform any merges, hence there were at most α requests between v and any one of the other nodes on its server during the offline interval. This bounds the number of requests assigned to the offline interval by $k \cdot \alpha$. ◀

Let $R(C)$ denote the set of requests from $\text{halo}(C)$ that were not assigned to any offline interval for a deleted component $C \in \text{del}(\sigma)$. We say that a migration of node v at time

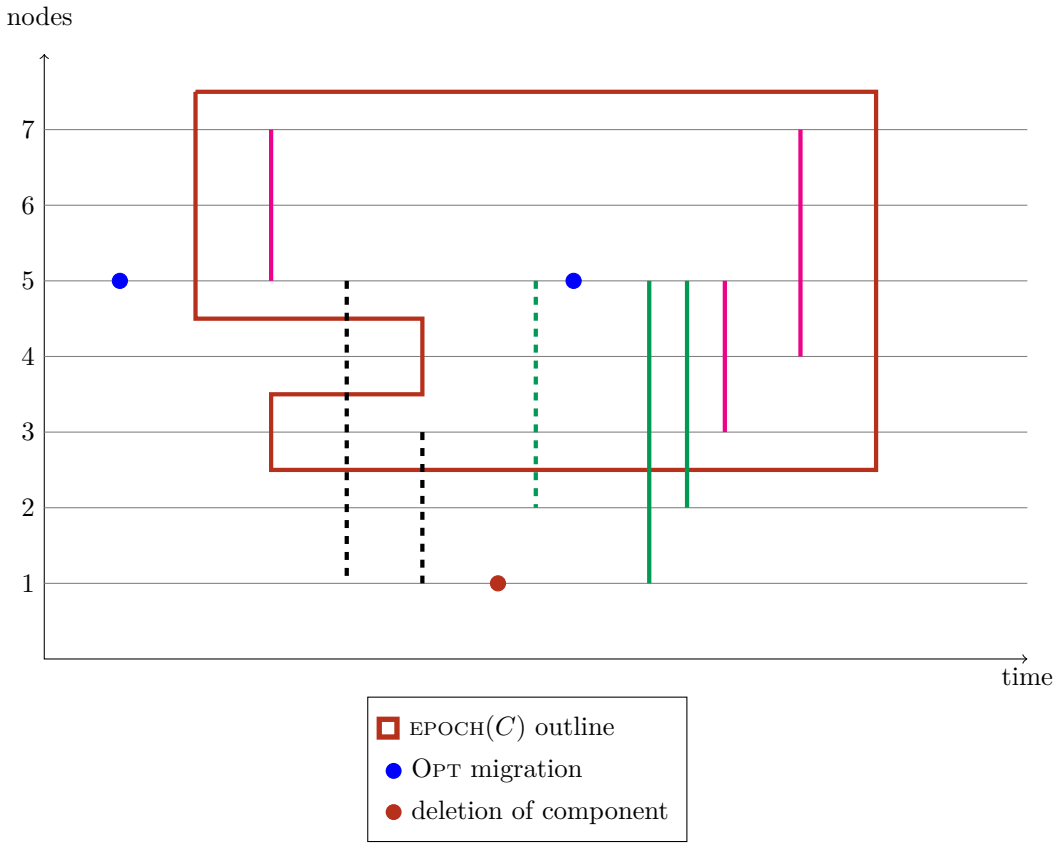

Figure 2 Illustration of definitions used in the analysis. Vertical lines represent requests between the corresponding nodes, dashed black requests are assigned to another component. The dashed green line is assigned to the offline interval of node 5; the regular green lines are assigned to an offline interval which is not contained in EPOCH(C).

t in the solution of OPT is *contained* in EPOCH(C) if $(v,t) \in$ EPOCH(C). Let OPT-MIG(C) denote the cost of OPT due to migrations of nodes from component C that are contained in EPOCH(C) and let OPT-REQ(C) denote the cost of OPT due to serving requests from CORE(C). We show the following lower bound on the cost of OPT for migrations from OPT-MIG(C) and requests from OPT-REQ(C) for all deleted components C.

▶ **Lemma 9.**

$$\sum_{C \in DEL(\sigma)} \mathit{OPT\text{-}MIG}(C) + \mathit{OPT\text{-}REQ}(C) \geq |C|/k \cdot \alpha + |R(C)|/k$$

Proof. For the following part of the proof we fix an arbitrary component $C \in$ DEL(σ). Note that the nodes involved in requests from $R(C)$ were not moved by OPT during the processing of requests from $R(C)$ until the time of deletion of C as otherwise they would be assigned to an offline interval.

The number of nodes contained in C or connected to C via edges representing requests from R is at least $|C| + |R(C)|/\alpha$ since requests from $R(C)$ have not led PCREP to perform any migrations. As OPT does not pay for any requests from $R(C)$ it follows that OPT must have placed the nodes from C in $(|C| + |R(C)|/(\alpha k)$ different servers.

We first examine the case in which Opt does not move any nodes from C during $\text{epoch}(C)$. In this case Opt must partition a graph containing the nodes from C which are connected via edges representing the requests from $\text{core}(C)$. As stated earlier Opt placed those nodes in $(|C| + |R(C)|/(\alpha k)$ different servers at time $\tau(C)$. As pCrep merged component C this graph is α-connected and hence Lemma 16 gives that Opt has to cut at least edges of total weight $(|C| + |R(C)|/(\alpha k) \cdot \alpha = |C|/k \cdot \alpha + |R(C)|/k$.

For the more general case in which Opt may perform node migrations during $\text{epoch}(C)$ we adapt the graph construction from above as follows: we add a vertex representing each (node, time) pair from $\text{epoch}(C)$. We connect each (node, time) pair p with edges of weight α to the pairs of the same node that represent the time step directly before and directly after p (if they exist in the graph). These edges represent the fact that Opt may choose to migrate a node between any two time steps in $\text{epoch}(C)$. Additionally we add an edge of weight one for each request $r = \{u, v\}$ from $\text{core}(C)$ by connecting the nodes in the graph that represent the pairs (u, t) and (v, t), respectively. Opt once again has to partition this graph into $\frac{|C|+|R(C)|/\alpha}{k}$ parts. Note that we only added edges of weight α to the graph and hence this graph is also α-connected. We conclude that once again Opt has to cut edges of weight at least $\frac{|C|+|R(C)|/\alpha}{k} \cdot \alpha = |C|/k \cdot \alpha + |R(C)|/k$.

In both cases only edges representing either requests from $\text{opt-req}(C)$ or migrations from $\text{opt-mig}(C)$ were cut. As the sets $\text{core}(C)$, $R(C)$, $\text{core}(D)$ and $R(D)$ are disjoint for two different components $C, D \in \text{del}(\sigma)$ per their definition we conclude that

$$\sum_{C \in \text{DEL}(\sigma)} \text{opt-mig}(C) + \text{opt-req}(C) \geq |C|/k \cdot \alpha + |R(C)|/k.$$

◀

In the following lemma we combine the results of the previous lemmas in order to bound the cost of Opt given input sequence σ, denoted by $\text{Opt}(\sigma)$.

▶ **Lemma 10.** *The cost of the solution of Opt given input sequence σ is bounded by*

$$Opt(\sigma) \geq 1/2 \cdot \sum_{C \in DEL(\sigma)} |C|/k \cdot \alpha + |halo(C)|/k.$$

Proof. We combine the results from Lemma 8 and Lemma 9. Note that the cost from Lemma 9 may contain migration costs. In this case the corresponding migrations represent the end of an offline interval. We denote the number of offline intervals by o. This gives us that

$$2\text{Opt}(\sigma) \geq \sum_{C \in \text{DEL}(\sigma)} \text{opt-mig}(C) + \text{opt-req}(C) + o \cdot \alpha + |P|$$

as we account for each migration at most twice.

Consider that due to Lemma 8 we have the inequality $o \geq |N|/k$. We repeat that $\mathcal{H} = \bigcup_{C \in \text{DEL}(\sigma)} \text{halo}(C)$. Note that N is the subset of requests of $\mathcal{H}$ for which Opt does not pay while P is the subset of $\mathcal{H}$ Opt pays for. It follows that the disjoint union of N and P is $\mathcal{H}$. Hence we obtain

$$
\begin{aligned}
2\textsc{Opt}(\sigma) &\geq \sum_{C \in \textsc{del}(\sigma)} \textsc{opt-mig}(C) + \textsc{opt-req}(C) + o \cdot \alpha + |P| \\
&\geq \sum_{C \in \textsc{del}(\sigma)} |C|/k \cdot \alpha + |R(C)|/k + (|N| + |P|)/k \\
&\geq \cdot \sum_{C \in \textsc{del}(\sigma)} |C|/k \cdot \alpha + |\textsc{halo}(C)|/k.
\end{aligned}
$$

◀

4.3 Competitive Ratio

We can now combine the results of Lemma 7 and Lemma 10 to obtain the following theorem which gives us the desired competitive ratio.

▶ **Theorem 11.** *With augmentation* $(2+\epsilon)$ *the competitive ratio of* pCrep *is in* $O(2/\epsilon \cdot k \log k)$.

Proof. We arbitrarily fix an input sequence σ and use our previous results to bound the competitive ratio of pCrep. We define $\textsc{comps}(\sigma) := \textsc{del}(\sigma) \cup \textsc{fin-comps}(\sigma)$ in order to improve readability. Let P denote the set of edges from $\bigcup_{C \in \textsc{del}(\sigma)} \textsc{halo}(C)$ that both pCrep and Opt pay for.

$$
\begin{aligned}
&\frac{\textsc{pCrep}(\sigma) - \textsc{fin-weights}(\sigma)}{\textsc{Opt}(\sigma)} \\
&\leq \frac{2 \cdot \sum_{C \in \textsc{comps}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha + \sum_{C \in \textsc{del}(\sigma)} |\textsc{halo}(C)|}{1/2 \cdot \sum_{C \in \textsc{del}(\sigma)} |C|/k \cdot \alpha + |\textsc{halo}(C)|/k + |P|} \\
&\leq k \log k \frac{2 \cdot \sum_{C \in \textsc{del}(\sigma)} |C| \cdot (2/\epsilon + 1) \cdot \alpha + \sum_{C \in \textsc{del}(\sigma)} |\textsc{halo}(C)|}{1/2 \sum_{(C \in \textsc{del}(\sigma)} |C| \cdot \alpha/2 + |\textsc{halo}(C)|)} + \beta \\
&= O(2/\epsilon \cdot k \log k) + \beta
\end{aligned}
$$

where

$$
\beta = \sum_{C \in \textsc{fin-comps}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha
$$

Let $\beta' = \beta + \textsc{fin-weights}(\sigma)$. Then it follows that

$$
\frac{\textsc{pCrep}(\sigma)}{\textsc{Opt}(\sigma)} \leq O(2/\epsilon \cdot k \log k) + \beta'.
$$

To obtain the bound on β' we observe that the components in $\textsc{fin-comps}(\sigma)$ each are of size at most k since they were not deleted by pCrep. This allows us to derive the bound $\sum_{C \in \textsc{fin-comps}(\sigma)} |C| \cdot ((2/\epsilon + 1) + \log k) \leq \ell \cdot k \cdot ((2/\epsilon + 1) + \log k)$. Since at the end of the execution of pCrep there can be at most $k \cdot \ell$ components, Lemma 17 allows us to bound $\textsc{fin-weights}(\sigma)$ by $k \cdot \ell \cdot \alpha$. Hence we conclude that $\beta' \leq \ell \cdot k \cdot ((2/\epsilon + 1) + \log k) \cdot \alpha + k \cdot \ell \cdot \alpha \in O(2/\epsilon \cdot k \log k)$. ◀

5 Polynomial-Time Implementation

So far we have only shown that the algorithm pCrep described in Section 3 has a competitive ratio $O(2/\epsilon \cdot k \log k)$. We now show that it can also be implemented in polynomial time.

In order to limit the section of the graph G maintained by pCrep that needs to be updated upon a new request between nodes of different components, we maintain a decomposition tree defined as follows: the root represents the whole graph and is assigned the connectivity of the entire graph. Given a node v in the tree that represents a subgraph G' of G, we decompose G' into subgraphs whose connectivity is strictly larger than that of G' and add children to v for each such subgraph. We do not decompose sub-graphs of connectivity at least α any further as we only need to identify whether a new subgraph of connectivity at least α was created by the insertion of the most recent request. Additionally we keep track of the connectivity of each such subgraph. In the decomposition tree we have labelled each node with the corresponding subset of vertices and the connectivity of the graph induced by these vertices.

If a new request is revealed to pCrep then we only need to update the smallest subtree of the decomposition tree which still contains both end points of the request. This is correct because we can view each decomposition of a subgraph G' into smaller graphs of a higher connectivity as a set of cuts that separates the nodes of G'. Inserting a new edge within a subgraph G' may only increase the value of the cuts which result in the decomposition of G', but do not affect cuts separating G' itself from other subgraphs. If a new request led to the creation of a new component this means that two old components that were at least α-connected were merged and hence the number of leaves in the decomposition tree decreased. If this is the case then the algorithm checks whether the new component contains more than k nodes. In this case the component is deleted and split into singleton components, each containing one node from the deleted component.

Upon such a component deletion the edges inside of and adjacent to the component are deleted, i.e. their weight is reset to zero. This means that the decomposition tree needs to be recomputed in order to reflect this change. If however the resulting component C contains at most k nodes the algorithm tries to collocate the nodes of the component while minimizing migration costs, i.e. looking for a cluster which contains as many nodes of the newly merged component as possible but which also has enough free capacity for the remaining nodes to be moved there and for additional reservation $\min\{\lfloor \epsilon \cdot |C| \rfloor, k - |C|\}$.

The detailed algorithms appear in Section A.2.

5.1 Subgraph Decomposition

We next describe our algorithm for the decomposition of a given subgraph represented by a node in the decomposition tree.

Given a node v of the decomposition tree, first a *partition graph* is constructed which is a graph consisting of the nodes in the subgraph represented by v and the edges which are between the nodes of the subgraph. This partition graph also supports merges and cuts of its nodes. More specifically the partition graph is initialized as a graph $P = (V, E)$ with $V = nodes(v)$ and $E = \{e = \{u, w\} \in E' | u \in nodes(v) \text{ and } w \in nodes(v)\}$ where E' represents the set of edges in the graph maintained by pCrep. Additionally we maintain a mapping M which assigns each node from V a set of the nodes in the subgraph represented by the decomposition tree node v. Initially M assigns each node in V the subset containing only the node itself.

We now run a *maximum adjacency search* algorithm, sometimes also called *maximum*

cardinality search algorithm [17], in order to obtain an arbitrary minimum (s,t)-cut of the graph. The maximum adjacency search algorithm is defined as follows: We start with an empty list L to which we add an arbitrary node of P. We then continually add the most tightly connected node from V to L, i.e. the node which is connected to the nodes in L via edges of the most total weight. Stoer and Wagner [17] have shown that the edges between the last two nodes s and t added to L form a minimum (s,t)-cut. We use the value of this cut in order to decide whether to merge the nodes s and t or whether to separate them. If the cut has value less than c we separate the nodes, otherwise we perform a merge. Here the separation of the nodes s and t means that we remove all edges in the cut from the edges E of the partition graph P. In the case of a merge we combine the nodes s and t and merge the outgoing edges, i.e. we replace the set of nodes V of P by the set $V' = V \backslash \{s,t\} \cup \{v'\}$. The edges E of P are modified by removing all edges adjacent to s and t and adding an edge $e' = \{v', u\}$ of weight $w(\{s,u\}) + w(\{t,u\})$ where $w(e)$ denotes the weight of edge e if it exists and is equal to zero otherwise. Furthermore we adjust the mapping M by setting $M(v') = M(s) \cup M(t)$.

We continually run this algorithm until P contains no edges, i.e. until $E = \emptyset$. The sets of nodes mapped to each ot the nodes of P by M now represent candidate subgraphs for the decomposition. Note though that we have only cut and merged according to minimum (s,t)-cuts and not according to minimum cuts. This means that the specific sequence in which we have performed the cuts may influence the result, e.g. if we merged based on a minimum (s,t)-cut which is not a minimum cut. This can be remedied by repeating the procedure on the resulting subgraphs until it returns a subgraph of only one node, i.e. until no separation step is performed during the decomposition. This is due to the fact that this procedure always cuts a subgraph of connectivity less than c at least once, as Chang et al. have shown (see Cutability Property in [8]). In order to speed up this computation we use the heap data structure proposed and analyzed by Chang et al. [8].

5.2 Running Time

The main bottleneck of the algorithm lies in the decomposition updates; it is easy to see that the other parts of the algorithm can be implemented in polynomial time.

▶ **Lemma 12.** *The subroutine decompose which decomposes a subgraph can be implemented in polynomial time* $O(\alpha|V|^2|E|)$.

Proof. The worst case is given when the whole tree has to be recomputed. We first discuss the time complexity of decomposing a single tree node v. Let the corresponding subgraph be denoted by $G_v = (V_v, E_v)$. Since each iteration of the subroutine *decompose* performs at least one cut as long as the connectivity of the given graph is smaller than the current threshold c we conclude that after at most $|V_v|$ iterations of decompose a correct decomposition is found.

Each step of *decompose* can be performed in $O(|V_v| \cdot |E_v|)$ as the maximum adjacency search algorithm finds an arbitrary minimum (s,t)-cut in time $O(|E_v|)$ as shown in theorem 4.1 in [8] and as there are at most $|V_v|$ minimum (s,t)-cuts computed for each invocation of *decompose*.

Hence the complexity of decomposing the subgraph represented by a tree node v is in $O(|V_v|^2|E_v|)$. Let C_v denote the time needed for the decomposition of the subgraph represented by decomposition tree node v.

We now sum this complexity over the nodes for each connectivity level of the decomposition tree. To this end let $level(i)$ denote all nodes in the decomposition tree which are of connectivity exactly i.

$$\sum_{i=0}^{\alpha} \sum_{v \in \text{level}(i)} C_v \leq \sum_{i=0}^{\alpha} O(|V|^2|E|) \in O(\alpha|V|^2|E|).$$

We conclude our analysis of the time complexity by observing the polynomial-time complexity of $O(\alpha|V|^2|E|)$. ◀

Since the remaining subroutines can be implemented in polynomial time:

▶ **Theorem 13.** *The algorithm* pCrep *can be implemented in polynomial time.*

6 Related Work

The closest work to ours is by Avin et al. [2] who initiated the study of the dynamic balanced graph partitioning problem. The authors present a $O(k \log k)$-competitive algorithm with augmentation $2 + \epsilon$ for any $\epsilon > 1/k$; this algorithm however has a super-polynomial runtime, which we improve upon in this paper. In their paper, Avin et al. also show a lower bound of $k - 1$ for the competitive ratio of any online algorithm on two clusters via a reduction to online paging. Restricted variants of the balanced repartitioning problem have also been studied. Here one assumes certain restrictions of the input sequence σ and then studies online algorithms for these cases. Avin et al. [3] assume that an adversary provides requests according to a fixed distribution of which the optimal algorithm Opt has knowledge while an online algorithm that is compared with Opt has not. Further the authors restrict the communication pattern to form a ring-like pattern, i.e. for the case of n nodes $0, ..., n-1$ only requests r of the form $r = \{i \mod n, (i+1) \mod n\}$ are allowed. For this case they present a competitive online algorithm which achieves a competitive ratio of $O(\log n)$ with high probability. Henzinger et al. [12] study a special *learning variant* of the problem where it is assumed that the input sequence σ eventually reveals a perfect balanced partitioning of the n nodes into ℓ parts of size k such that the edge cut is zero. In this case the communication patterns reveal connected components of the communication graph of which each forms one of the partitions. Algorithms are tasked to *learn* this partition and to eventually collocate nodes according to the partition while minimizing communication and migration costs. The authors of [12] present an algorithm for the case where the number of servers is $\ell = 2$ that achieves a competitive ratio of $O((\log n)/\epsilon)$ with augmentation ϵ, i.e. each server has capacity $(1 + \epsilon)n/2$ for $\epsilon \in (0, 1)$. For the general case of ℓ servers of capacity $(1 + \epsilon)n/\ell$ the authors construct an exponential-time algorithm that achieves a competitive ratio of $O((\ell \log n \log \ell)/\epsilon)$ for $\epsilon \in (0, 1/2)$ and also provide a distributed version. Additionally the authors describe a polynomial-time $O((\ell^2 \log n \log \ell)/\epsilon^2)$-competitive algorithm for the case with general ℓ, servers of capacity $(1 + \epsilon)n/\ell$ and $\epsilon \in (0, 1/2)$.

The dynamic balanced graph partitioning problem can be seen as a generalization (or symmetric version) of online paging. In the online paging problem [9, 10] one is given a scenario with a fast cache of k pages and $n - k$ pages in slow memory. Pages are requested in an online manner, i.e. without prior knowledge of future requests. If a requested page is in the cache at the time of the request it can be served without cost. If it is in slow memory however, then a *page fault* occurs and the requested page needs to be moved into the cache. If the cache is full then a page from the cache needs to be evicted, i.e. moved to the slow memory in order to make space for the requested one. The goal is to design algorithms which minimize the number of such page faults. However, the standard version of online paging has no equivalent to the option of serving a request remotely as is possible in the dynamic

balanced graph partitioning problem. The variant *with bypassing* allows an algorithm to access pages in slow memory without moving them into the cache, thus providing such an equivalent. It is worth stressing however that in our problem requests involve two nodes while in online paging the nodes themselves are requested.

The static balanced graph partitioning problem is the static offline variant of the problem of this paper. In this version an algorithm may not perform any migrations, but has perfect knowledge of the request sequence σ and then needs to provide a perfectly balanced partitioning of the $n = k \cdot \ell$ nodes into ℓ sets of equal size k that minimizes cost, i.e. the weight of edges between the servers. This scenario can be modelled as a graph partitioning problem where the weight of an edge corresponds to the number of requests between its end points in the input sequence σ. An algorithm then has to provide a partition of the nodes into sets of exactly k nodes each while minimizing the total edge weights between partitions, i.e. an algorithm needs to minimize the edge cut of the graph. This problem is NP-complete and for the case where $\ell \geq 3$, Andreev and Räcke [1] have shown that there is no polynomial time approximation algorithm which guarantees a finite approximation factor unless P=NP.

There are several algorithms and frameworks for graph partitioning problems. Usually these frameworks employ heuristics in order to achieve their results. The most successful such heuristic is *Multilevel Graph Partitioning* [7]. This method consists of three phases. Initially the graph is repeatedly coarsened into a hierarchy of smaller graphs in such a way that cuts in the coarse graphs also correspond to cuts in the finer graphs. On the coarsest level a (potentially expensive) algorithm is used in order to compute an initial partition. This partitioning is then transferred to the finer graphs. In this process one usually uses other local heuristics in order to improve the partition quality even further with every step. METIS [13, 14] and Jostle [18, 19] are examples of libraries that utilize this multilevel approach. We choose METIS as a reference for our empirical evaluation.

More generally, clustering has been studied within a variety of different contexts from data mining to image segmentation [6, 20, 16], and is the process of generating subsets of elements with high similarity [11]. However, we consider an online problem, i.e. algorithms need to react dynamically to changes in the graph and need to maintain their data structures and adapt accordingly whereas clustering considers complete data sets which are static.

7 Future Work

Our work leaves upon several interesting directions for future research. On the theoretical front, it would be interesting to explore how to close the gap between upper and lower bound on the competitive ratio, and to study randomized algorithms. On the practical front, we believe that our algorithm can be further engineered and optimized to achieve a lower runtime in practice, as well as an improved empirical competitive ratio under real (non worst-case) workloads. These kinds of adjustments may be achieved for example by changing certain algorithm parameters such as the connectivity threshold. There may also be potential in adapting and improving our decomposition tree data structure in order to improve running times.

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A Deferred Technical Details

A.1 Preliminaries: Adaptions from [2]

▶ **Lemma 14.** *At any time t after* pCrep *performed its merge and delete actions, all subsets S of components with $|S| > 1$ have connectivity less than α, i.e. there exist no mergeable component sets after* pCrep *performed its merges and deletions.*

Proof. We prove the lemma by induction. The lemma holds trivially at time 0. Now assume that at some time $t > 0$ the lemma does not hold, i.e., there is a subset S of components with connectivity at least α and $|S| > 1$. We may assume that t is the earliest time for which S has connectivity α. The incrementation of the weight of some edge e at time t raised the connectivity of S, but S was not merged into a new α-connected component C. If no new component was created at time t we arrive at a contradiction as pCrep always merges if there exists a mergeable component set.

Now assume that a component C was created at time t. This means that C must also contain the endpoints of e. But then the conjunction of C and S forms an even larger subset of components with connectivity at least α which is a contradiction to the maximality of C and S. ◀

The following lemma is adapted for our connectivity-based approach from Corollary 4.2 in [2].

▶ **Lemma 15.** *Fix any time t and consider weights right after they were updated by* pCrep *but before any merge or delete actions. Then all subsets S of components with $|S| > 1$ have connectivity at most α and a mergeable component set S has connectivity exactly α.*

Proof. This lemma follows directly from Lemma 14 as connectivities can only increase by at most 1 at each time t and Lemma 14 guarantees that mergeable component sets are merged by pCrep directly after they emerge before a new request is revealed. ◀

The following two lemmas combined give us a result similar to Lemma 4.3 in [2]: bounds on the edge weight that is cut when partitioning a mergeable component set, i.e. a set of components of connectivity at least α.

We start by establishing a lower bound on this edge weight in the following lemma.

▶ **Lemma 16.** *Given a mergeable set of components S and a partition of S into $g > 1$ parts $S_1, ..., S_g$. Then the weight between the parts of the partition is at least $g/2 \cdot \alpha$.*

Proof. We construct a graph G with the different parts $S_i, i \in \{1, ..., g\}$ of the partition as nodes. Note that this graph is α-connected. We insert an edge for each edge between the parts S_i. Now consider the sum of the weighted degrees of all such nodes S_i in the constructed graph:

$$\sum_{i \in \{1,...,g\}} deg_G(S_i) = 2 \sum_{e \in G} w(e)$$

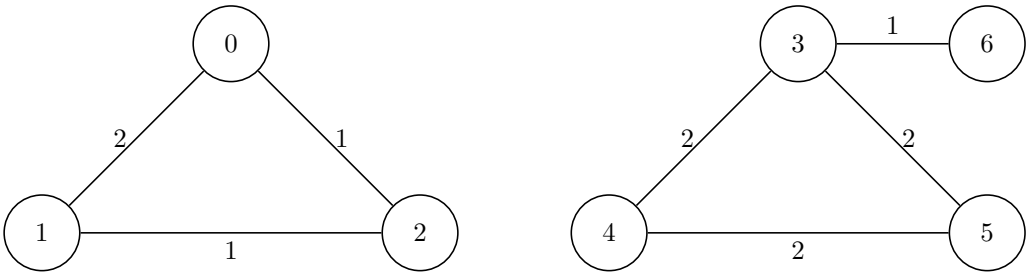


Figure 3 Example graph illustrating the decomposition described in Section 5.

The equality follows as the left sum counts each edge twice, once for each endpoint. Now consider the fact that each node S_i must have degree at least α with respect to the edges in G because G is α-connected. Hence

$$2\sum_{e\in G} w(e) = \sum_{i\in\{1,\dots,g\}} deg_G(S_i) \leq \sum_{i\in\{1,\dots,g\}} \alpha = g\cdot\alpha$$

which gives us that $\sum_{e\in G} w(e) \geq g/2\cdot\alpha$. ◀

In the following lemma we establish the upper bound on the cut edge weight when partitioning a mergeable set of components S into $g \geq 2$ parts.

▶ **Lemma 17.** *Given a mergeable set of components S and a partitioning of S into $g \geq 2$ parts $S_1, \dots, S_g$. The weight between the parts S_i is at most $(g-1)\cdot\alpha$ during the execution of* pCrep.

Proof. Similarly to before we construct a graph $G = (V, E)$ with the different parts $S_i, i \in \{1, \dots, g\}$ of the partition as nodes and we insert an edge for each edge between the parts S_i. Note again that this graph is α-connected. We iteratively partition G into subsets via minimum cuts with regard to edge weight, i.e. we consider a minimum edge cut of G which partitions the nodes of G into the subsets V_1 and V_2. We continue to iteratively partition the resulting sets until all sets contain only one node of G each. As this required at most $|V| - 1$ cuts of value at most α and $|V| = g$ by definition of G the lemma follows. ◀

Algorithm 4 insertAndUpdate(a,b)

```
if comp[a] == comp[b] then
  return
end if
addEdge(a, b)
updateDecomposition(a, b)
del ← updateMapping(alphaConnectedComponents)
delComponents(del)
```

A.2 Detailed Algorithms

We now present our algorithms in more details. Algorithm 4 is the main function that is called upon each new request. It checks whether the new request is between different

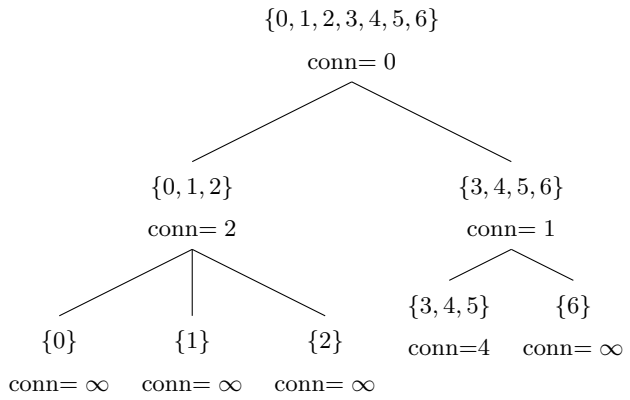


Figure 4 Decomposition tree for the graph from Figure 3 for $\alpha = 4$

α-connected components. If this is not the case it determines that this request cannot change the decomposition and returns. Otherwise the weight of the corresponding edge is increased and other routines are called that update the decomposition based on this new edge. Algorithm 5 first determines the smallest sub-graph in the decomposition tree that contains both end points of the request and decomposes this sub-graph. For this decomposition step we use the algorithm proposed by Chang et al. [8]. We explain this algorithm in greater detail in Section 5.1.

Afterward, the routine *updateMapping* is called which compares the number of components to the number of components before the arrival of the request. Only if the number of components has decreased, it checks for the new component; otherwise there was no component merge. If there was a merge then the routine examines the size of the newly created (merged) component and decides whether to delete or to collocate based on the logic described in the previous section.

If a deletion has to be performed, then this step is done in the routine *delComponents* (Algorithm 6) which resets all edge weights both of edges between nodes of the component as well as all adjacent edges and finally starts the decomposition of the whole graph in order to arrive at a new decomposition that follows the definition from the previous section. In the case of a collocation, the nodes are moved to a cluster that has enough space while updating the reservations and cluster capacities accordingly.

B Empirical Evaluation

In order to complement our analytical results and shed light on the performance of our algorithm in practice, we implemented pCrep and conducted experiments under real-world traffic traces from datacenters and high-performance computing clusters. We also discuss an algorithm engineering approach to improve the practical performance of pCrep, and compare the algorithm to different reference algorithms and heuristics.

B.1 Reference Algorithms

We compare our algorithm based on maintaining a second-order partition of the nodes into components we discussed in Section 3, i.e. pCrep where on a component deletion all edges inside of and adjacent to the component are deleted, with reference algorithms. We have

Algorithm 5 updateDecomposition(a,b)

```
q ← findSmallestSubgraph(a, b)
while q not empty do
  current← q.popFront()
  if res.connectivity==α then
    continue
  end if
  res ← decompose(current, current.connectivity+1) // decomposition based on (s, t)-cuts

  current.connectivity ← value of smallest encountered cut
  if current.connectivity≥ α then
    continue
  end if
  childrenQueue ← res
  //make sure that only subgraphs with higher connectivity are added as children
  while childrenQueue not empty do
    c ←childrenQueue.pop()
    cRes ←decompose(c, current.connectivity+1)
    c.connectivity ← value of smallest encountered cut
    if decompose returned only one graph then
      current.children.add(cRes)
      if cRes has connectivity smaller than α then
        q.push(cRes)
      end if
    else
      childrenQueue.add(cRes)
    end if
  end while
end while
```

Algorithm 6 delComponents(del)

```
delAllEdges(del)
root.connectivity=0
root.children={}
updateDecomposition(0,1)
```

trace	nodes	requests	ℓ	k	α	ϵ
NB	512	250 000	32	16	3	0.1
Mocfe	1024	250 000	32	32	3	0.1
FB	2048	10 000	64	32	3	0.1

Figure 5 Traffic traces used in the evaluation

shown the competitive ratio of pCrep in Theorem 11 and its polynomial running time in the previous section.

We consider the following baselines in our evaluation. First, we consider an alternative implementation of pCrep, called dCrep, that does not use the decomposition tree structure mentioned in Section 5. Rather, this implementation applies the MinCut-based decomposition algorithm presented in Section 5.1 to the whole graph and thus computes the new components in one decomposition step.

Furthermore, recall that pCrep always maintains a second-order partition of the nodes into components, and when a component is deleted, all edges inside of and adjacent to the component are reset. A natural alternative, is to reset internal edges only on this occasion; we will refer to this algorithmic variant as Crep-Core.

Another natural reference algorithm is a static graph partitioning; to this end, we use the METIS_PartGraphRecursive algorithm implemented in the METIS framework [13, 14], and will refer to it simply as Static. For Static we first record the communication graph resulting from the requests, and give it as input to Static. We then compute the cost of Static due to migrations from the random initial mapping of nodes to servers; in addition, we charge the remote requests, i.e. the total weight of edges between nodes of the communication graph which Static mapped to different servers. The algorithms start with randomly initialized mappings of nodes to servers.

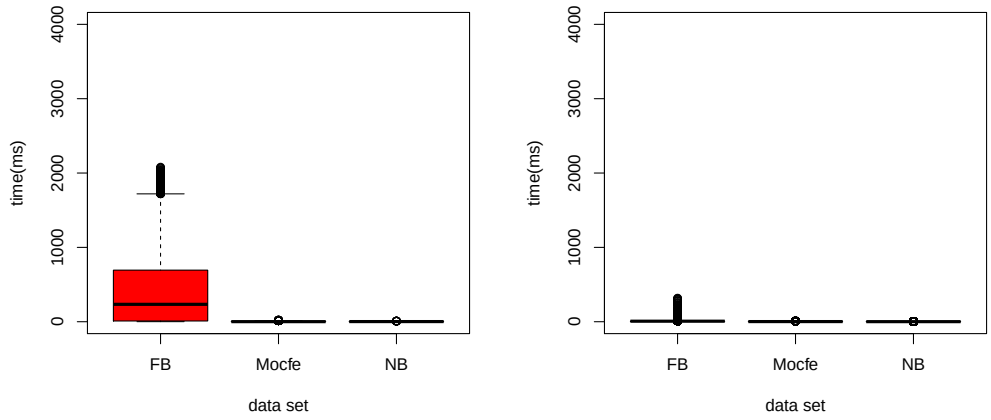
B.2 Traffic Traces and Methodology

We consider different real-world traffic traces (workloads), two hpc traces, NB and Mocfe, as well as a trace from a Facebook data center which we call FB. The traces and their characteristics are described in more detail in [4]. The traces are publicly available.

With Mocfe we study a scenario where the number of servers is $\ell = 32$, each of size $k = 32$, and we use the first 250k requests in a setting with 1024 nodes. For FB we restrict the trace to 2048 nodes. For this restriction we iteratively chose the vertex pairs that communicate the most during the first 20 million requests until we reached the number of 2048 nodes; we then added all requests between two of these nodes to our data set. For this scenario, we used a configuration with $\ell = 64$ servers is $\ell = 32$ of size $k = 32$. The runs are then performed on the first 10k of these requests. Similarly we restricted NB to 512 nodes and used the first 250k requests for this setting; this data set is then evaluated in a scenario where the number of servers is $\ell = 32$, each of size $k = 32$. For all configurations we use $\alpha = 3$ and $\epsilon = 0.1$. These configurations are summarized in Figure 5. The influence of α on the running time is investigated further in Appendix B.3.

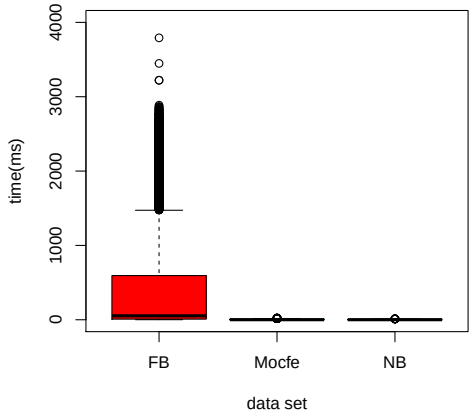
B.3 Runtime Evaluation

We evaluate the running time of pCrep by considering different traces and alternative algorithms.



(a) Runtime results for pCrep

(b) Runtime results for dCrep

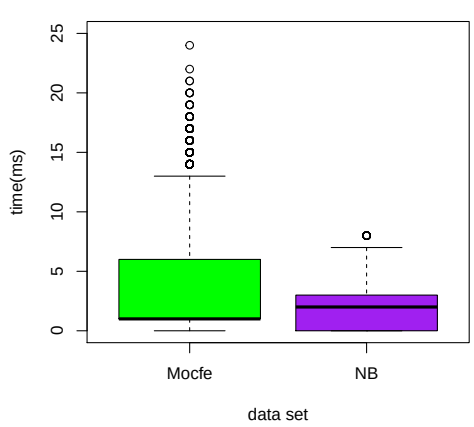


(c) Runtime results for Crep-Core

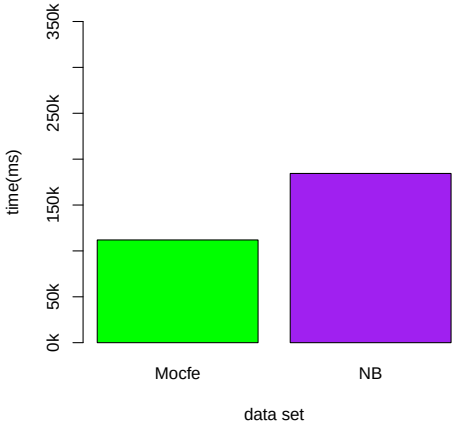
Figure 6 Comparison of the running times of pCrep (top left), dCrep (top right) and Crep-Core (bottom) for all 3 data sets

Figure 6 shows the results of all data sets for pCrep, dCrep and Crep-Core. Here we filtered the running times to only include updates which have led the respective algorithm to update, i.e. those updates where the communicating nodes belonged to different components at the time of the request. We can see that for all three algorithms FB produces by far the longest runtime and that dCrep performs significantly better than pCrep and Crep-Core for FB. We first compare the runtimes of the three algorithms for the hpc data sets before we discuss the runtimes for FB in greater detail.

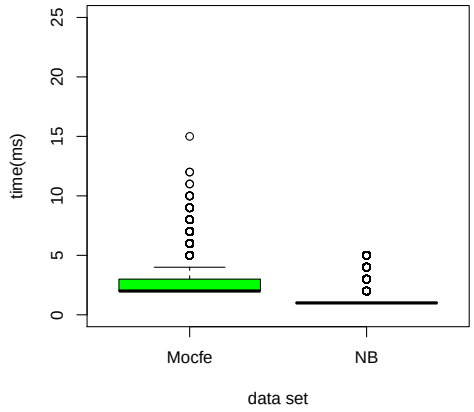
Figure 7 shows plots of run times of pCrep, dCrep and Crep-Core for the HPC traces. On the left of the figure you can see the distribution of the update times over the



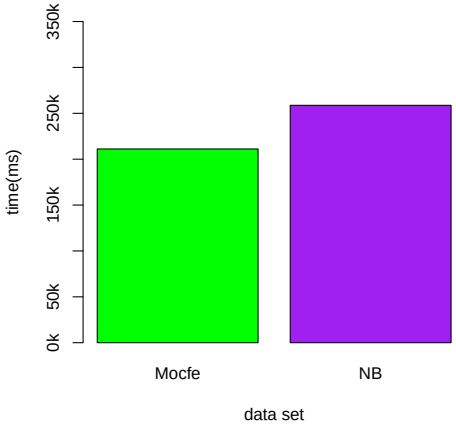
(a) Runtime results for pCrep



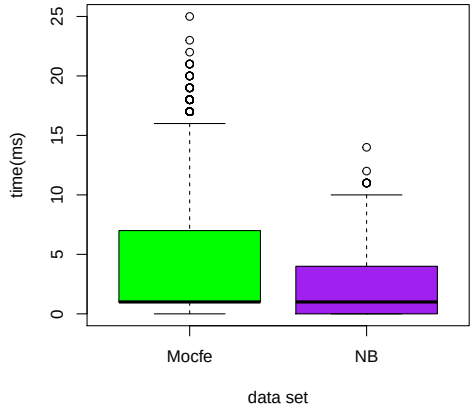
(b) Total time for pCrep



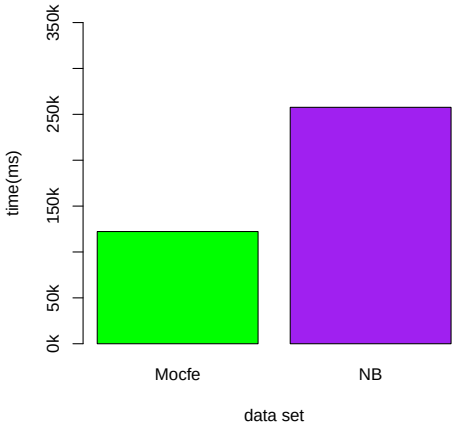
(c) Runtime results for dCrep



(d) Total time for dCrep

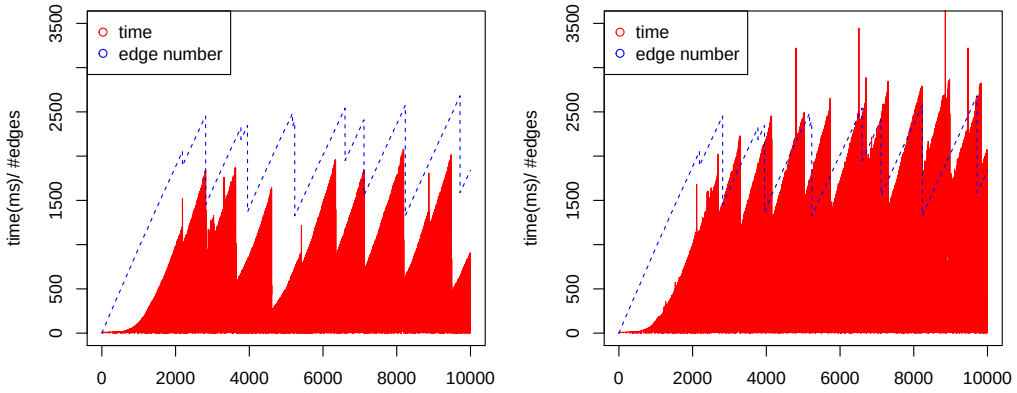


(e) Runtime results for Crep-Core



(f) Total time for Crep-Core

Figure 7 Comparison of the running times of pCrep (top), dCrep (middle) and Crep-Core(bottom) for both hpc traces



(a) Relation of size to time for FB for pCrep **(b)** Relation of size to time for FB for Crep-Core

Figure 8 Relation of size to time for FB for pCrep (left) and Crep-Core (right)

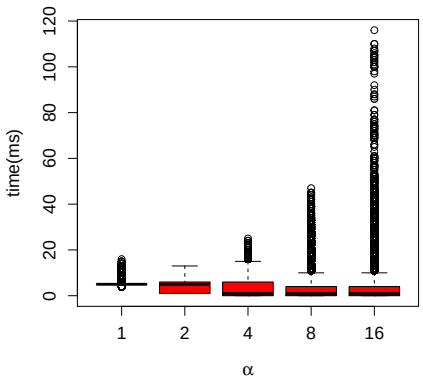
requests while the total times are shown on the right. The figure shows that pCrep is the fastest for both hpc traces while Crep-Core is faster than dCrep. This indicates that the decomposition tree data structure which both pCrep and Crep-Core use may be a significant advantage for the HPC traces.

Figure 8 illustrates the relation of the time needed for handling a request, and the number of edges in the data structure for pCrep and for Crep-Core for FB. We can see a high correlation, i.e. a large number of edges in the graph maintained by the respective algorithm seems to lead to longer update times. Similarly the update times are shorter after the number of edges decreases, i.e. after a component deletion. While we cannot prove this, we suspect that this may be due to the fact that the data used for the run shows little structure, and thus leads a large number of (costly) recomputation steps for our data structure. This is also supported by the fact that dCrep is significantly faster for FB as one can see in Figure 6.

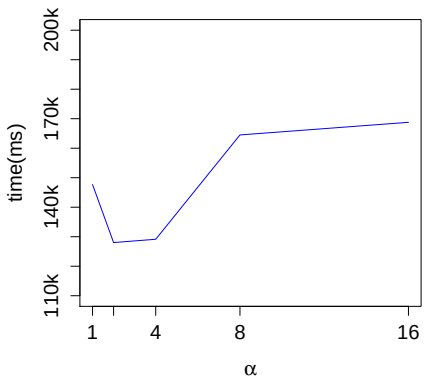
It is also interesting to study the influence of α on the running time of pCrep and Crep-Core. Figure 9 shows a plot of the running time of pCrep and Crep-Core for $\alpha \in \{1, 2, 4, 8, 16\}$. In order to improve the readability, only requests which led to an update of the data structure are included. The results show that both algorithms are slower for $\alpha = 1$ than for $\alpha \in \{2, 4\}$. This may be due to the fact that lower values of α lead the algorithms to update the tree structure more frequently. Especially for $\alpha = 1$ every inserted edge leads to a merge, as edges are only inserted between different components and α is both the cost for a node migration as well as the connectivity threshold for merging for pCrep and Crep-Core. The remaining results mirror the expectation that the increase in the potential height of the decomposition tree also increases the required time as the update times may increase. This effect increases the runtime of Crep-Core more than that pCrep leading to significantly better runtime for pCrep for $\alpha = 16$.

B.4 Cost Evaluation

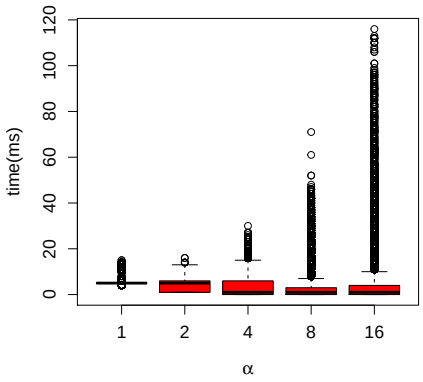
In order to shed light on the cost, in terms of communication and migration, we compare our algorithms pCrep and Crep-Core with Static.



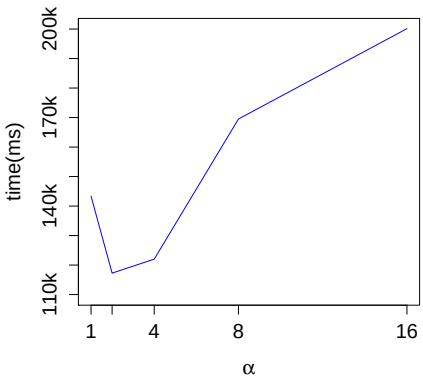
(a) Runtime of pCrep for Mocfe for different values of α



(b) Total runtime of pCrep for Mocfe for different values of α



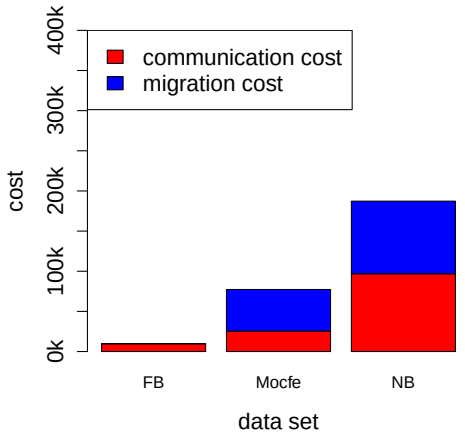
(c) Runtime of Crep-Core for Mocfe for different values of α



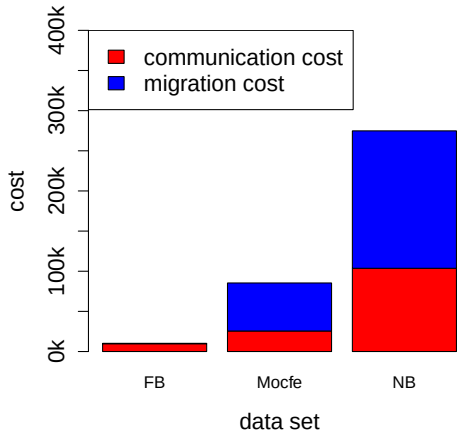
(d) Total runtime of Crep-Core for Mocfe for different values of α

Figure 9 Runtime of pCrep (top) and Crep-Core (bottom) for Mocfe for different values of α

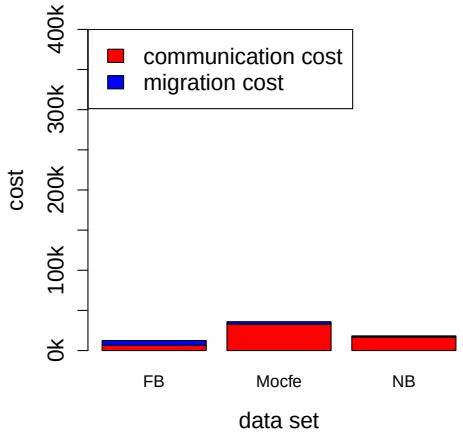
Figure 10 shows the costs of pCrep, Crep-Core and Static for FB, Mocfe and NB. Note that for FB the costs of pCrep and Crep-Core are actually lower than the cost of Static. For the HPC traces Static is able to achieve significantly lower cost than both pCrep and Crep-Core. However, keep in mind that Static is essentially an offline algorithm, which knows requests ahead of time; furthermore, we also note that Static may not produce a perfectly balanced partitioning.



(a) Cost of pCrep



(b) Cost of pCrep



(c) Cost of Static

Figure 10 Comparison of the costs of pCrep (left), Crep-Core (right) and Static (bottom) for all 3 data sets