1 | PROJECTION

The following result which is also referred to as elimination or Finsler's lemma can be used to eliminate variables from an LMI constraint, but also to artificially add variables. The latter is referred to as S(lack)-variable approach and plays an important role in reducing the conservatism of several controller design problems[1,2].

Lemma 1 (Projection Lemma)**.** Let U, V be real matrices and let Q be symmetric. Further, let $U_\perp$ and $V_\perp$ be basis matrices of $\ker(U)$ and $\ker(V)$, respectively. Then there exists some K satisfying

$$Q + U^\top K V + (\bullet)^\top \prec 0 \tag{1}$$

if and only if

$$U_\perp^\top Q U_\perp \prec 0 \quad \text{and} \quad V_\perp^\top Q V_\perp \prec 0. \tag{2}$$

2 | DISCRETE-TIME

Lemma 2 (Bounded real lemma)**.** The discrete-time system

$$\begin{pmatrix} x(t+1) \\ e(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \end{pmatrix}$$

is stable and has an energy gain $\sup_{d \in \ell_2 \setminus \{0\}} \frac{\|e\|_{\ell_2}}{\|d\|_{\ell_2}}$ smaller than γ if and only if there exists a matrix $X \succ 0$ satisfying

$$(\bullet)^\top \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} = (\bullet)^\top \left(\begin{array}{cc|c} X & & \\ & -X & \\ \hline & & P_\gamma \end{array} \right) \begin{pmatrix} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{pmatrix} \prec 0 \tag{3}$$

where $P_\gamma := \begin{pmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -\gamma I \end{pmatrix}$.

Next we apply Lemma 1 in order to introduce a slack variable G and to equivalently reformulate (3) while keeping $X \succ 0$ and $\gamma > 0$ fixed.

Lemma 3. Let $X \succ 0$ and $\gamma > 0$ be given. Then the inequality (3) holds if and only if there exists a matrix G satisfying

$$\underbrace{\left(\begin{array}{c:c} X & 0 \\ \hdashline 0 & \begin{pmatrix} -X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \end{array} \right)}_{:= Q} + \left(\underbrace{\begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}}_{:= U^\top} G \underbrace{\begin{pmatrix} -I & A & B \end{pmatrix}}_{:= V} \right) + (\bullet)^\top \prec 0. \tag{4}$$

Note that the left hand side of (4) can be alternatively expressed as

$$(\bullet)^\top \left(\begin{array}{ccc|c} 0 & G^\top & 0 & \\ G & X - G - G^\top & 0 & \\ 0 & 0 & -X & \\ \hline & & & P_\gamma \end{array} \right) \begin{pmatrix} 0 & A & B \\ I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & C & D \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} X - G - G^\top & GA & GB \\ (\bullet)^\top & -X + C^\top C \frac{1}{\gamma} & C^\top D \frac{1}{\gamma} \\ (\bullet)^\top & (\bullet)^\top & D^\top D \frac{1}{\gamma} - \gamma I \end{pmatrix} = \begin{pmatrix} X - G - G^\top & GA & GB \\ (\bullet)^\top & -X & 0 \\ (\bullet)^\top & (\bullet)^\top & -\gamma I \end{pmatrix} + (\bullet)^\top \frac{1}{\gamma} \begin{pmatrix} 0 & C & D \end{pmatrix}.$$

Proof. Observe at first that basis matrices of the kernels of U and V are given by

$$U_\perp := \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad V_\perp := \begin{pmatrix} A & B \\ I & 0 \\ 0 & I \end{pmatrix},$$

respectively. Next, note that we have the following identities

$$V_\perp^\top Q V_\perp = (\bullet)^\top \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}$$

and

$$U_\perp^\top Q U_\perp = \begin{pmatrix} -X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} = V_\perp^\top Q V_\perp - (\bullet)^\top X \begin{pmatrix} A & B \end{pmatrix}.$$

After these preparations, the proof is as follows.

"If": This follows from applying the projection lemma 1 and from observing that $V_\perp^\top Q V \prec 0$ coincides with (3).

"Only if": By the above computations, we infer $V_\perp^\top Q V_\perp \prec 0$ from (3). Moreover, due to $X \succ 0$, we can conclude that

$$U_\perp^\top Q U_\perp = V_\perp^\top Q V_\perp - (\bullet)^\top X \begin{pmatrix} A & B \end{pmatrix} \prec 0$$

holds as well. We can then again apply the projection lemma 1 to infer the claim. □

Next, note that by applying the Schur complement, (4) is equivalent to

$$\begin{pmatrix} X-G-G^\top & GA & GB & 0 \\ (\bullet)^\top & -X & 0 & C^\top \\ (\bullet)^\top & (\bullet)^\top & -\gamma I & D^\top \\ 0 & C & D & -\gamma I \end{pmatrix} = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & -X & 0 & C^\top \\ 0 & 0 & -\gamma I & D^\top \\ 0 & C & D & -\gamma I \end{pmatrix} + \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} G \begin{pmatrix} -I & A & B & 0 \end{pmatrix} + (\bullet)^\top = (\bullet)^\top \left(\begin{array}{cccc:c} X & 0 & 0 & 0 & -I \\ 0 & -X & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hdashline -I & A & B & 0 & 0 \end{array}\right) \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hdashline G^\top & 0 & 0 & 0 \end{array}\right) \prec 0. \tag{5}$$

We can now apply the Lemma 1 one more in order to introduce another slack variable F.

Lemma 4. Let $X \succ 0$, $\gamma > 0$ and G be given. Then the inequality (5) holds if and only if there exists a matrix $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ satisfying

$$\underbrace{\left(\begin{array}{cccc:c} X & 0 & 0 & 0 & -I \\ 0 & -X & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hdashline -I & A & B & 0 & 0 \end{array}\right)}_{:=Q} + \underbrace{\begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}}_{=:U^\top} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \underbrace{\begin{pmatrix} G^\top & 0 & 0 & 0 & -I \end{pmatrix}}_{=:V} + (\bullet)^\top = \left(\begin{array}{cccc:c} X & 0 & 0 & 0 & -I \\ 0 & -X & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hdashline -I & A & B & 0 & 0 \end{array}\right) + \begin{pmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ F_2 \end{pmatrix} \begin{pmatrix} G^\top & 0 & 0 & 0 & -I \end{pmatrix} + (\bullet)^\top \prec 0 \tag{6}$$

Proof. Observe at first that basis matrices of the kernels of U and V are given by

$$\begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hdashline G^\top & 0 & 0 & 0 \end{array}\right),$$

respectively. Next, one notes that $V_\perp^\top Q V_\perp \prec 0$ coincides with (5) and that

$$U_\perp^\top Q U_\perp = \begin{pmatrix} -X & 0 & C^\top \\ 0 & -\gamma I & D^\top \\ C & D & -\gamma I \end{pmatrix}.$$

"If": Follows from applying projection 1 in order to eliminate F which yields $V_\perp^\top Q V_\perp \prec 0$ as desired.

"Only if": If (5) (or equivalently $V_\perp^\top Q V_\perp \prec 0$) holds, we infer from before that the initial bounded real inequality 3 holds as well. Note that, by $\gamma > 0$ and applying Schur, the remaining to-be-shown inequality $U_\perp^\top Q U_\perp \prec 0$ holds if and only if

$$0 \succ \begin{pmatrix} -X & 0 \\ 0 & -\gamma I \end{pmatrix} + (\bullet)^\top \frac{1}{\gamma} \begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} -X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} = \underbrace{(\bullet)^\top \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}}_{\prec 0} - \underbrace{(\bullet)^\top X \begin{pmatrix} A & B \end{pmatrix}}_{\succcurlyeq 0}.$$

As indicated, the latter is always true under the given assumptions. □

Finally, we arrive at an inequality with an interesting structure that forms the basis of[3] and the follow-up papers. It has the benefit that the (potentially closed-loop) system matrices and the Lyapunov certificate enter in an affine fashion. Nonconvexity (even for analysis!) appears due to the product of the slack variables. However, it is not too difficult to develop an iterative algorithm based on the following inequality.

Lemma 5. Suppose that $X \succ 0$ and $\gamma > 0$ hold. Then the inequality (3) holds if and only if there exist $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ and $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ satisfying

$$\left(\begin{array}{cccc|c} X & 0 & 0 & 0 & -I \\ 0 & -X & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hline -I & A & B & 0 & 0 \end{array}\right) + \begin{pmatrix} F_1 \\ 0 \\ 0 \\ 0 \\ F_2 \end{pmatrix}\begin{pmatrix} G_1 \\ 0 \\ 0 \\ 0 \\ G_2 \end{pmatrix}^\top + (\bullet)^\top \prec 0.$$

Proof. "Only if": This is clear from the previous results and from defining $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} := \begin{pmatrix} G \\ -I \end{pmatrix}$.

"If": From the $(5,5)$ if the given LMI, we infer $F_2 G_2 + (\bullet)^\top \prec 0$ which implies that both F_2 and G_2 are nonsingular because both these matrices are square. Then one easily finds that (6) is true for $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and G replaced by $-\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} G_2^\top$ and $-G_2^{-1} G_1$. This yields the claim. □

3 | CONTINUOUS-TIME

The continuous-time case is almost as the discrete-time case with the difference that an additional scalar parameter is involved when applying projection.

Lemma 6. The continuous-time system

$$\begin{pmatrix} \dot{x}(t) \\ e(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x(t) \\ d(t) \end{pmatrix}$$

is stable and has an energy gain $\sup_{d \in L_2 \setminus \{0\}} \frac{\|e\|_{L_2}}{\|d\|_{L_2}}$ smaller than γ if and only if there exists a matrix $X \succ 0$ satisfying

$$(\bullet)^\top \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} = (\bullet)^\top \left(\begin{array}{cc|c} 0 & X & \\ X & 0 & \\ \hline & & P_\gamma \end{array}\right)\left(\begin{array}{cc} A & B \\ I & 0 \\ \hline C & D \\ 0 & I \end{array}\right) \prec 0 \tag{7}$$

where $P_\gamma := \begin{pmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -\gamma I \end{pmatrix}$.

Next, we apply Lemma 1 in order to introduce a slack variable G and to equivalently reformulate (7) while keeping $X \succ 0$ and $\gamma > 0$ fixed.

Lemma 7. Let $Y \succ 0$ and $\gamma > 0$ be given. Then the inequality (7) holds if and only if there exists a matrix G and a scalar $\rho > 0$ satisfying

$$\underbrace{\left(\begin{array}{c|c} 0 & \begin{pmatrix} X & 0 \end{pmatrix} \\ \hline (\bullet)^\top & (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} \end{array}\right)}_{:= Q} + \left(\underbrace{\begin{pmatrix} \rho I \\ I \\ 0 \end{pmatrix}}_{:= U^\top} G \underbrace{\begin{pmatrix} -I & A & B \end{pmatrix}}_{:= V}\right) + (\bullet)^\top \prec 0 \tag{8}$$

Note that the left hand side of (8) can be alternatively expressed as

$$(\bullet)^\top \left(\begin{array}{ccc|c} 0 & \rho G^\top & G^\top & \\ \rho G & -\rho(G + G^\top) & X - G^\top & \\ G & X - G & 0 & \\ \hline & & & P_\gamma \end{array}\right)\left(\begin{array}{ccc} 0 & A & B \\ I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & C & D \\ 0 & 0 & I \end{array}\right) = \begin{pmatrix} -\rho G - \rho G^\top & \rho G A + X - G^\top & \rho G B \\ (\bullet)^\top & GA + (\bullet)^\top + C^\top C \frac{1}{\gamma} & GB + C^\top D \frac{1}{\gamma} \\ (\bullet)^\top & (\bullet)^\top & D^\top D \frac{1}{\gamma} - \gamma I \end{pmatrix}$$

$$= \begin{pmatrix} -\rho G - \rho G^\top & \rho G A + X - G^\top & \rho G B \\ (\bullet)^\top & GA + (\bullet)^\top & GB \\ (\bullet)^\top & (\bullet)^\top & -\gamma I \end{pmatrix} + (\bullet)^\top \frac{1}{\gamma} \begin{pmatrix} 0 & C & D \end{pmatrix}.$$

Proof. Observe at first that basis matrices of the kernels of U and V are given by

$$U_\perp := \begin{pmatrix} -\frac{1}{\rho}I & 0 \\ I & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad V_\perp := \begin{pmatrix} A & B \\ I & 0 \\ 0 & I \end{pmatrix},$$

respectively. Next, note that we have the following identities

$$V_\perp^\top Q V_\perp = (\bullet)^\top \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}$$

and

$$U_\perp^\top Q U_\perp = \begin{pmatrix} -\frac{2}{\rho}X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}$$

After these preparations, the proof is as follows.

"If": This follows from applying the projection lemma 1 and from observing that $V_\perp^\top Q V \prec 0$ coincides with (7).

"Only if": By assumption and the above computations, we have $V_\perp^\top Q V_\perp \prec 0$. Due to $X \succ 0$ and $(\bullet)^\top P_\gamma \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \prec 0$, we can then find some small $\rho > 0$ such that $U_\perp^\top Q U_\perp \prec 0$ holds as well. We can then again apply the projection lemma 1 to infer the claim. □

Next, note that by applying the Schur complement, (8) is equivalent to

$$\begin{pmatrix} 0 & X & 0 & 0 \\ X & 0 & 0 & C^\top \\ 0 & 0 & -\gamma I & D^\top \\ 0 & C & D & -\gamma I \end{pmatrix} + \begin{pmatrix} \rho I \\ I \\ 0 \\ 0 \end{pmatrix} G \begin{pmatrix} -I & A & B & 0 \end{pmatrix} + (\bullet)^\top = (\bullet)^\top \left(\begin{array}{cccc|c} 0 & X & 0 & 0 & -I \\ X & 0 & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hline -I & A & B & 0 & 0 \end{array}\right) \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline \rho G^\top & G^\top & 0 & 0 \end{array}\right) \prec 0. \tag{9}$$

Lemma 8. Let $X \succ 0$, $\gamma > 0$, $\rho > 0$ and G be given. Then the inequality (9) holds if and only if there exists some $\left(\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right)$ satisfying

$$\underbrace{\left(\begin{array}{cccc|c} 0 & X & 0 & 0 & -I \\ X & 0 & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hline -I & A & B & 0 & 0 \end{array}\right)}_{:=Q} + \underbrace{\begin{pmatrix} \sigma I & 0 \\ I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}}_{=:U^\top} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \underbrace{\begin{pmatrix} \rho G^\top & G^\top & 0 & 0 & -I \end{pmatrix}}_{=:V} + (\bullet)^\top = \left(\begin{array}{cccc|c} 0 & X & 0 & 0 & -I \\ X & 0 & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hline -I & A & B & 0 & 0 \end{array}\right) + \begin{pmatrix} \sigma F_1 \\ F_1 \\ 0 \\ 0 \\ F_2 \end{pmatrix} \begin{pmatrix} \rho G^\top & G^\top & 0 & 0 & -I \end{pmatrix} + (\bullet)^\top \prec 0. \tag{10}$$

Proof. Observe at first that basis matrices of the kernels of U and V are given by

$$\begin{pmatrix} -\frac{1}{\sigma}I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ \hline \rho G^\top & G^\top & 0 & 0 \end{array}\right),$$

respectively. Next one notes that $V_\perp^\top Q V_\perp \prec 0$ coincides with (9) and that

$$U_\perp^\top Q U_\perp = \begin{pmatrix} -\frac{2}{\sigma}X & 0 & C^\top \\ 0 & -\gamma I & D^\top \\ C & D & -\gamma I \end{pmatrix}.$$

"If": Follows from applying projection 1 in order to eliminate $\left(\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right)$ which yields $V_\perp^\top Q V_\perp \prec 0$ as desired.

"Only if": If (9) (or equivalently $V_\perp^\top Q V_\perp \prec 0$) holds, then we infer from before that the initial bounded real inequality (7) holds as well. Next, note that, by $\gamma > 0$ and applying Schur, the remaining to-be-shown inequality $U_\perp^\top Q U_\perp \prec 0$ holds if and

only if

$$0 \succ \begin{pmatrix} -\frac{2}{\sigma}X & 0 \\ 0 & 0 \end{pmatrix} + (\bullet)^\top P_\gamma \begin{pmatrix} C & D \\ 0 & I \end{pmatrix}.$$

By $(\bullet)^\top P_\gamma \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \prec 0$, the latter is true for σ sufficiently small. □

With the same arguments as in the discrete-time case, one ends up with the following inequality that can be used to construct an iterative algorithm.

Lemma 9. Suppose that $X \succ 0$ and $\gamma > 0$ hold. Then the inequality (7) holds if and only if there exist $\left(\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right)$, $\rho > 0$ and $\sigma > 0$ satisfying

$$\left(\begin{array}{cccc:c} 0 & X & 0 & 0 & -I \\ X & 0 & 0 & C^\top & A^\top \\ 0 & 0 & -\gamma I & D^\top & B^\top \\ 0 & C & D & -\gamma I & 0 \\ \hdashline -I & A & B & 0 & 0 \end{array}\right) + \begin{pmatrix} \sigma F_1 \\ F_1 \\ 0 \\ 0 \\ F_2 \end{pmatrix} \begin{pmatrix} \rho G_1 \\ G_1 \\ 0 \\ 0 \\ G_2 \end{pmatrix}^\top + (\bullet)^\top \prec 0. \tag{11}$$

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