PROPERTIES OF ROBOT DYNAMICS

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OUTLINE

- Introduction
- Dynamics of robot manipulator
- Dynamics: State variable representation
- Properties of dynamic equations
- Exercise: Dynamic model of 2 DOF planar robot
- Dynamic equations of robot with actuators
- Dynamic equations with joint flexibility

Introduction

- Properties of robot dynamics are utilized at the derivation and stability proof of most of the robot controllers.
- Only basic background required for study of robot control will be considered.
- Case study: 2 DOF planar robot. This robot will be used as test object for designed controllers in this course.

Dynamics of robot manipulator

Robot dynamic model (n-DOF, revolution joints):

$$\tau(t) = M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q) + \tau_D =$$

$$= M(q)\ddot{q} + N(q,\dot{q}) + \tau_D$$

$q(t) \in \Re^n$ vector of joint positions

$\tau(t)$ vector of control torque, $\Re^n$

$\tau_D(t)$ disturbance torque(unknown dynamics)

$M(q)$ inertia matrix, $\Re^{nxn}$

$V(q,\dot{q})$ Coriolis, centripetal torque vector, $\Re^n$

$G(q)$ gravity torque vector, $\Re^n$

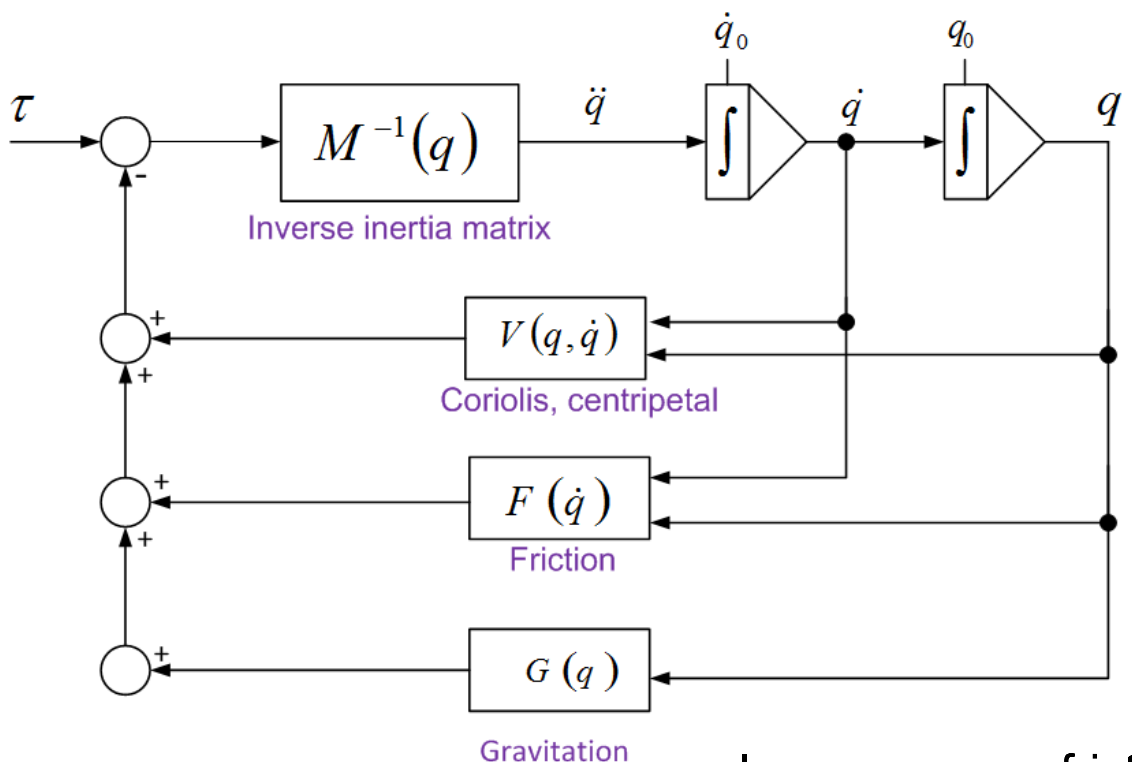
$F_v\dot{q} + F_d(\dot{q})$ friction torque vector, $\Re^n$

Dynamics of robot manipulator

$$\tau(t) = M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q) + \tau_D =$$

$$= M(q)\ddot{q} + N(q,\dot{q}) + \tau_D$$

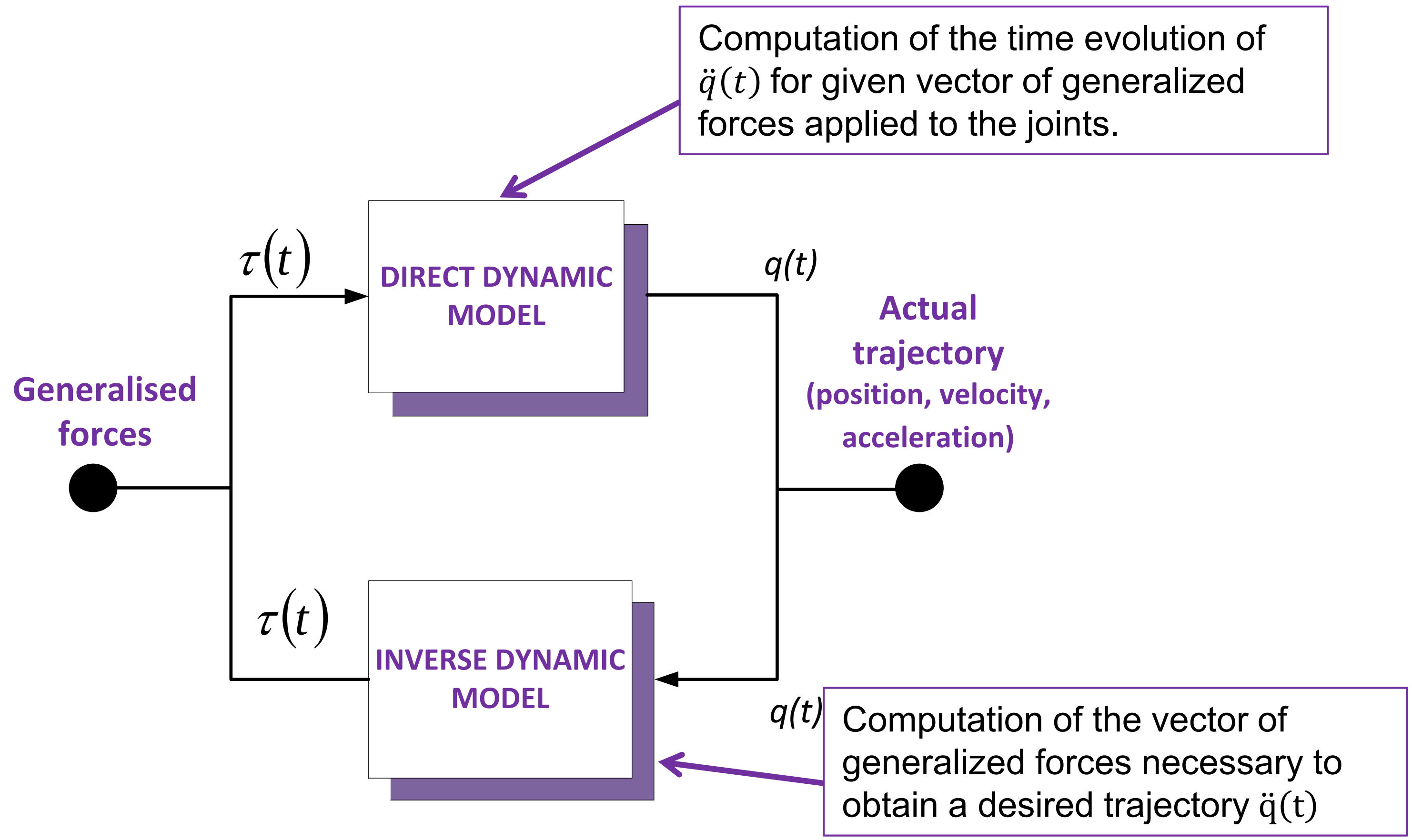
Draw the block diagram!



In many cases friction is given only as function of velocity!

Dynamics of robot manipulator

$$\tau(t) = M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q) + \tau_D =$$
$$= M(q)\ddot{q} + N(q,\dot{q}) + \tau_D$$



Dynamics of robot manipulator: State-variable representation

- Nonlinear system described with:

$$\frac{d^n y(t)}{dt^n} = h\left[y(t), y^{(1)}(t), .., y^{(n-1)}(t), u(t), \dots u^{(n)}(t)\right]$$

  - o $y(t)$ is output
  - o $u(t)$ is input

o State vector and output

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$..$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = h\left[y(t), y^{(1)}(t), .., y^{(n-1)}(t), u(t), \dots u^{(n)}(t)\right]$$

$$y(t) = x_1(t)$$

o Compact description:

$$\dot{x}(t) = f[x(t), U(t)]$$
$$y(t) = cx(t)$$

$$U(t) = \left[u(t), \dots u^{(n-1)}(t)\right]^T$$
$$c = [1,0,0 \dots 0]$$

Dynamics of robot manipulator: State-variable representation

$$\tau = M(q)\ddot{q} + V(q,\dot{q}) + G(q)$$

State variable description:

$$\dot{x}(t) = f[x(t), U(t)]$$
$$y(t) = cx(t)$$

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -M^{-1}(V+G) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tau$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Dynamics of robot manipulator: State-variable representation

Alternatively 'linear' state equations can be used

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u$$

$$u(t) = -M^{-1}(q)(V + G\ ) + M^{-1}(q)\tau =$$
$$= -M^{-1}(q)N(q,\dot{q}) + M^{-1}(q)\tau$$

Where $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$.

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Properties of dynamic equations

- Inertia matrix $M(q)\epsilon R^{nxn}$:
  - o Symmetric and positive definite.
  - o Upper and lower bounded $\varepsilon_1 I < M(q) < \varepsilon_2 I$.
  - o $m_1 < \|M(q)\| < m_2$, for (1,2,∞), $m_{1,2}$ pos. scalars.
  - o Inverse matrix (when it exist) is bounded: $\frac{1}{\mu_2} I < M^{-1}(q) < \frac{1}{\mu_1} I$, for revolute joints are μ_2, μ_2 const.
  - o Elements:
    - o Diagonal elements $M_{ii}(q)$ are the moment of inertia about the *k*-th joint axis, in a given configuration and considering blocked all the other joints.
    - o $M_{ij}(q)$ is the inertia coupling, accounting the effect of acceleration of joint *j* on joint *i.*

Properties of dynamic equations

- Centripetal, Coriolis term $V(q,\dot{q}) = V_m(q,\dot{q})\dot{q}$;
  - Complex term, a quadratic function of $\dot{q} \in \Re^n$.
  - $S(q,\dot{q}) \equiv \dot{M}(q) - 2V_m(q,\dot{q})$, $S(q,\dot{q})$ is skew symmetric and, accordingly, $x^T S(q,\dot{q})x = 0, \forall x$.
  - $\|V_m(q,\dot{q})\dot{q}\| \leq v_b \|\dot{q}^2\|$ in case of revolute joints is v_b constant (not function of q).
  - If the speed of robot arm is low then this term is becomes small.
  - It only one joint is moving at the time this term is zero.

Properties of dynamic equations

H. Olsson, K.J. Åström, C. Canudas de Wit, M. Gäfvert, P. Lischinsky: Friction Models and Friction Compensation

- Friction $F(\dot{q}) = F_v\dot{q} + F_d(\dot{q})$,
  - Friction term needs to be added to Lagrange equations of motion.
  - It can be assumed that friction is uncoupled, local effect.
  - $F_v = diag\{v_i\}$ coefficient matrix of viscous friction.
  - $F_d(\dot{q}) = K_d sign(\dot{q})$; $K_d = diag\{k_i\}$ dynamic friction. Discontinuity at $\dot{q} = 0$.
  - Bound on friction term: $\|F_v\dot{q} + F_d(\dot{q})\| \leq v\|\dot{q}\| + k$.

Problem of robot control is often problem of friction (identification, compensation).

Properties of dynamic equations

Linearity in parameters

$$\tau(t) = M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q) =$$

$$= M(q)\ddot{q} + N(q,\dot{q}) = \boldsymbol{W(q,\dot{q},\ddot{q})\varphi}$$

with:

- φ parameter vector of geometrical/mechanical parameters of the links (i.e. masses, inertia, friction, ...).
- $W(q,\dot{q},\ddot{q})$ is an matrix of (known) robot dynamics. It can be computed for any robot.
- Note! τ_D is not included in equation.
- This property will be used at the derivation of controllers, especially an adaptive controller.

Properties of dynamic equations

Transformation to Cartesian space:

$$y = h(q), \dot{y} = J\dot{q}, \ddot{y} = J\ddot{q} + \dot{J}\dot{q}$$

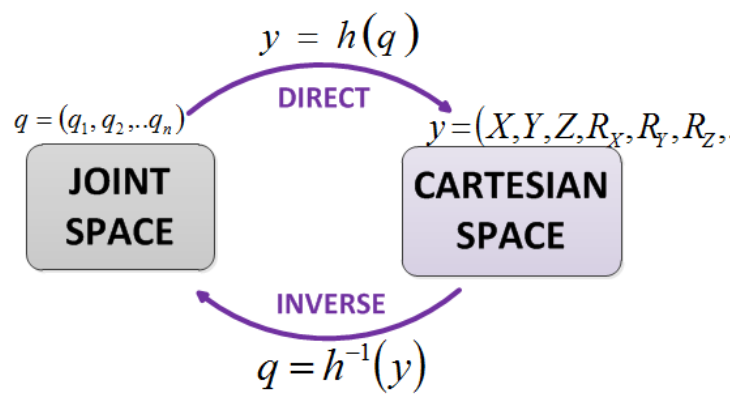
$$\dot{y} = [v^T, \omega^T]^T, \ \tau = J^T F$$

Joint space dynamics

$$\tau(t) = M(q)\ddot{q} + N(q,\dot{q}) + \tau_d$$

Cartesian dynamics

$$\tau(t) = \bar{M}(q)\ddot{y} + \bar{N}(q,\dot{q}) + f_d$$



$$\bar{M} = J^{-T} M J^{-1}$$
$$\bar{N} = J^{-T}(N - MJ^{-1}\dot{J}\dot{q}) = J^{-T}(N - MJ^{-1}\dot{J}J^{-1}\dot{y})$$
$$f_d = J^{-T}\tau_d$$

Note, most of the elements here are vectors and matrices

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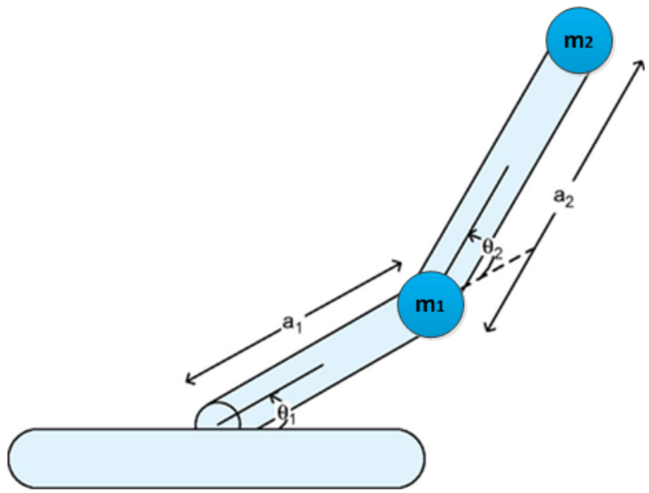
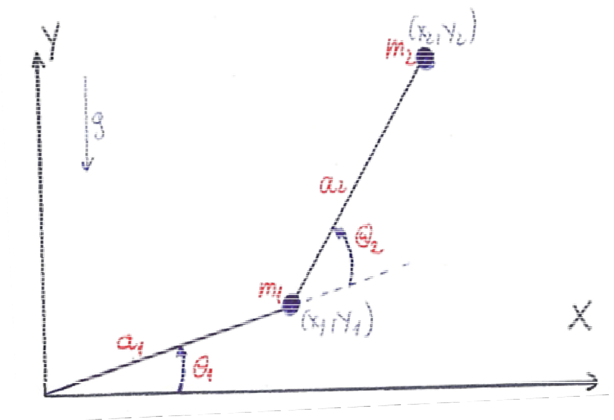
2 DOF planar robot





Derive Lagrange equations of motion for two link robot. It is assumed that the link masses are concentrated at the end of the links in *m1* and *m2*. Derive also direct kinematics and calculate Jacobian matrix.

Joint variables:

$$q = [\theta_1, \theta_2]^T$$

Generalized force vector

$$\tau = [\tau_1, \tau_2]^T$$

Dynamics of robot manipulator

Euler-Lagrange equations for conservative system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

- q is a vector of generalized coordinates q_i
- τ is a vector of generalized forces τ_i
- *Lagrangian* is difference between kinetic and potential energy: $L = K - P$

For a robot:

- q is a vector of joint variables (angles θ_i and offsets d_i)
- τ is a vector that has as components torques (for angles) and forces (for offsets).

2 DOF planar robot

Result

$$\tau = M(q)\ddot{q} + N(q,\dot{q}) + \tau_D$$

Inertia term

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}(m_1+m_2){a_1}^2 + m_2{a_2}^2 + 2m_2a_1a_2c(\theta_2) & m_2{a_2}^2 + m_2a_1a_2c(\theta_2)\\ m_2{a_2}^2 + m_2a_1a_2c(\theta_2) & m_2{a_2}^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}$$

Centrifugal/Coriolis term

$$+\begin{bmatrix}-m_2a_1a_2\left(2\dot{\theta}_1\dot{\theta}_2 + {\dot{\theta}_2}^2\right)s(\theta_2)\\ m_2a_1a_2{\dot{\theta}_1}^2 s(\theta_2)\end{bmatrix}$$

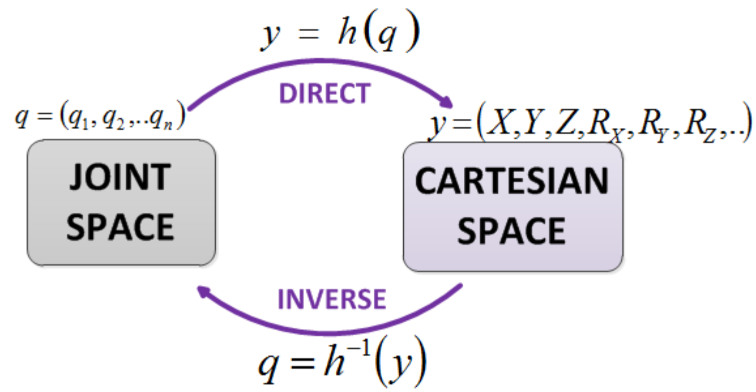
Gravitation term

$$+\begin{bmatrix}(m_1+m_2)ga_1c(\theta_1) + m_2ga_2c(\theta_1+\theta_2)\\ m_2ga_2c(\theta_1+\theta_2)\end{bmatrix}$$

2 DOF planar robot

Direct geometric model

$$X = f1(\theta_1, \theta_2)$$
$$\mathrm{Y} = f2(\theta_1, \theta_2)$$



Direct kinematic model

$$\dot{X} = g1\left(\dot{\theta}_1, \dot{\theta}_2\right)$$
$$\dot{Y} = g2\left(\dot{\theta}_1, \dot{\theta}_2\right)$$
$$\ddot{X} = h1\left(\ddot{\theta}_1, \ddot{\theta}_2\right)$$
$$\ddot{Y} = h2\left(\ddot{\theta}_1, \ddot{\theta}_2\right)$$

Jacobian matrix

$$J = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}} = \begin{bmatrix} \dfrac{\partial x}{\partial \theta_1} & \dfrac{\partial x}{\partial \theta_2} \\ \dfrac{\partial y}{\partial \theta_1} & \dfrac{\partial y}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -a_1 s(\theta_1) - a_2 s(\theta_1 + \theta_1) & -a_2 s(\theta_1 + \theta_1) \\ a_1 c(\theta_1) + a_2 c(\theta_1 + \theta_1) & a_2 c(\theta_1 + \theta_1) \end{bmatrix}$$

2 DOF planar robot

General properties of robot dynamics implemented for 2DOF robot

Centripetal, Coriolis term $\boldsymbol{V(q,\dot{q}) = V_m(q,\dot{q})\dot{q}}$ can be calculated as:

$$\begin{bmatrix} -m_2 a_1 a_2 \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2{}^2\right) s(\theta_2) \\ m_2 a_1 a_2 \dot{\theta}_1{}^2 s(\theta_2) \end{bmatrix}$$

Centrifugal/Coriolis term

$$V(q,\dot{q}) = V_m \dot{q}$$

This is kronecker product

$$V_m = \frac{1}{2}\left(M + U^T - U\right) \; ; \quad U = \left(I \otimes \dot{q}^T\right)\frac{\partial M}{\partial q}$$

2 DOF planar robot

$$V_m = \begin{bmatrix} -\dot{\theta}_2 m_2 a_1 a_2 s\theta_2 \, , & -(\dot{\theta}_1 + \dot{\theta}_2) m_2 a_1 a_2 s\theta_2 \\ \dot{\theta}_1 m_2 a_1 a_2 s\theta_2 \, , & 0 \end{bmatrix}$$

2 DOF planar robot

General properties of robot dynamics implemented for 2DOF robot

$$\tau(t) = W(q, \dot{q}, \ddot{q})\varphi$$

$friction1 = v_1 \dot{\Theta}_1 + k_1 sgn(\dot{\Theta}_1)$

For the case of unknown payload masses and friction parameters.

$$\varphi = [m_1 \quad m_2 \quad k_1 \quad v_1 \quad k_2 \quad v_2]$$

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} & 0 & 0 \\ 0 & w_{22} & 0 & 0 & w_{25} & w_{26} \end{bmatrix}$$

$$w_{11} = a_1^2 \ddot{\Theta}_1 + g a_1 c\Theta_1$$

$$w_{12} = [a_1^2 + a_2^2 + 2a_1 a_2 c\Theta_2]\ddot{\Theta}_1 + [a_2^2 + a_1 a_2 c\Theta_2]\ddot{\Theta}_2 - a_1 a_2 (2\dot{\Theta}_1 \dot{\Theta}_2 + \dot{\Theta}_2^2) s\Theta_2 + g a_1 c\Theta_1 + g a_2 c(\Theta_1 + \Theta_2)$$

$$w_{13} = sgn(\dot{\Theta}_1)$$

$$w_{14} = \dot{\Theta}_1$$

$$w_{22} = [a_2^2 + a_1 a_2 c\Theta_2]\ddot{\Theta}_1 + a_2^2 \ddot{\Theta}_2 + a_1 a_2 \dot{\Theta}_1^2 s\Theta_2 + g a_2 c(\Theta_1 + \Theta_2)$$

$$w_{25} = sgn(\dot{\Theta}_2)$$

$$w_{26} = \dot{\Theta}_2$$

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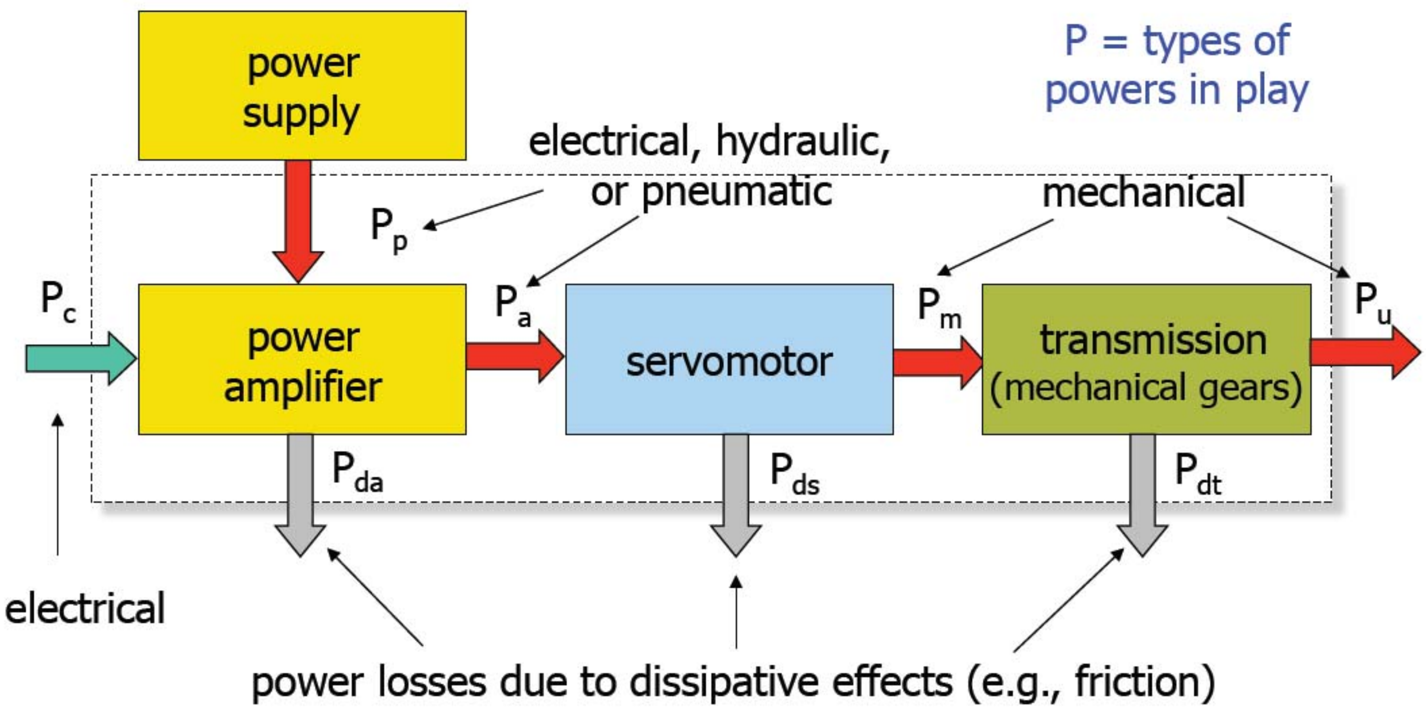
Dynamic equations of robot with actuators

Robot 'drivetrain'



Dynamic equations of robot with actuators

- Robot dyn. equation: $\tau(t) = M(q)\ddot{q} + V(q,\dot{q}) + F(\dot{q}) + G(q)$
- Robot is in most cases driven by electric drives. Dynamics of electric drives is given by *n* decoupled equations:

$$M_M\ddot{q}_M + F_M(\dot{q}_M) + \tau_{Md} = \tau_M$$

  - o $q_M = [q_{M1,}\dots, q_{Mn}]^T \epsilon R^n$ drives positions angles (measured)
  - o $\tau_M \epsilon R^n$drives' torque (control input)
  - o $F_M(\dot{q}_M)\ \epsilon R^n$ drives' friction
  - o $M_M\ \epsilon R^n$ drives moment of inertia
  - o τ_{Md} disturbance, let us suppose it goes to zero (control of servomotor is solved)
- Gear model $q_{Mi} = n_i q_i,\ q_M = Nq,$

$\tau_i = n_i\tau_{Mi},\ \tau = N\tau_M,$

$\mathrm{N} = diag\{n_i\}$, reductor $n_i > 1$

Robot dynamics with reductors:

$$\left(M_M + \frac{1}{N^2}M\right)\ddot{q} + \frac{1}{N}V\left(\frac{q_M}{N},\frac{\dot{q}_M}{N}\right) + \left(F_M(\dot{q}_M) + \frac{1}{N}F\left(\frac{q_M}{N}\right)\right) + \frac{1}{N}G\left(\frac{q_M}{N}\right) = \tau_M$$

Dynamic equations of robot with actuators

Robot dynamics with reductors:

$$\left(M_M + \frac{1}{N^2}M\right)\ddot{q} + \frac{1}{N}V\left(\frac{q_M}{N}, \frac{\dot{q}_M}{N}\right) + \left(F_M(\dot{q}_M) + \frac{1}{N}F\left(\frac{q_M}{N}\right)\right) + \frac{1}{N}G\left(\frac{q_M}{N}\right) = \tau_M$$

Conclusions:

- With high reduction (N>>) decoupled, linearised dynamic equations of second order are obtained.
- The problem of robot control actually becomes the problem of controlling actuators' dynamics.
- The reduction ration in industrial robots can be quite substantial (100-500) which means that controllers can be (mostly) quite simple.

Dynamic equations with joint flexibility

- In industrial robots the use of motion transmissions based on belts and long shafts cause flexibility between actuators (input) and driven links (output).
- However in some robots the flexibility is intentional.
  - For applications where the robot is in contact with environment require flexibility of the robot.
  - In robots for interaction with the human. Compliance in the transmissions is introduced on purpose for safety.
- Flexibility needs to be considered at the robot control design (and it causes some troubles there).

Dynamic equations with joint flexibility

Flexibility is modelled as concentrated at the joints as a stiff spring. $\tau_f(t)$ can be expressed as:

$$\tau_f(t) = B_s(\dot{q}_M - \dot{q}) + K_s(q_M - q)$$

Motor position

Joint position

$B_s = diag\{b_{si}\}$ damping constant of gear train

$K_s = diag\{k_{si}\}$ spring constant of gear train

Robot dynamics $\tau(t) = M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F_v\dot{q} + F_d(\dot{q}) + G(q)$ with modeled joints' flexibility becomes:

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + F(\dot{q}) + G(q) + B_s(\dot{q} - \dot{q}_M) + K_s(q - q_M) = 0$$

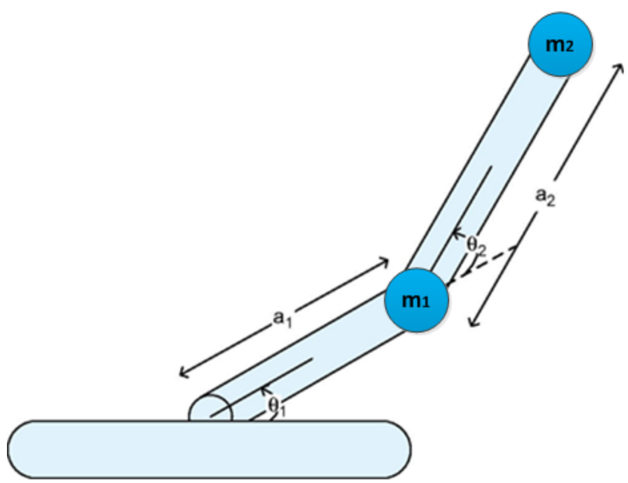
$$J_M\ddot{q}_M + B\dot{q}_M + F_M + B_s(\dot{q}_M - \dot{q}) + K_s(q_M - q) = \tau_M$$

Final notes

- Robot dynamics as a basis for model based robot controllers.
- Properties of the dynamic equations are used in derivation/stability proofs at the controllers' design.
- Important state-variable formulation.

REPORT REQUIRED

2 DOF planar robot



Build a simulation model (dynamics of the robot) in MATLAB/Simulink. Verify the model by verifying (for input torque is zero):

a) Stability of stable and unstable equilibrium position.
b) Response to initial conditions of your choice.

Additional: Properties of skew symmetric matrix

- J is skew symetric if $J_{ij} = -J_{ji}$.

- For skew symetric matrix J of dimension *nxn* following is valid:
  - o $J^T = -J$
  - o $\det(J) = \det(J^T) = \det(-J) = (-1)^n \det(J)$
    =$-x^T J x = x^T J^T x$
  - o $x^T J x = (x^T J x)^T = 0$