# Separatrices and their sensitivities

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# Dynamical system and separatrix

- System

$$\dot{x}(t) = f(x(t), p) \tag{1}$$

$x : \mathbb{R} \to \mathbb{R}^2$, $p \in \mathbb{R}$, $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$

- Steady states

$$0 = f(x_s, p) \tag{2}$$

- Jacobian in saddle point

$$A_s = \frac{\partial f}{\partial x}(x_s, p) \tag{3}$$

- Eigenvalues and eigenvectors

$$A_s v_s = \lambda_s v_s \tag{4}$$

- Let $x_s$ be a saddle point (one eigenvalue with negative real part and one with positive real part). Then, the separatrix [1] is computed by solving the final value problem

$$x(t_f) = x_s, \tag{5a}$$
$$\dot{x}(t) = f(x(t), p) \tag{5b}$$

## Computation of separatrix and sensitivities

- We approximate the solution to the final value problem (5) by solving the initial value problem

$$x(\tau_0) = x_s + \epsilon v_s, \tag{6a}$$
$$\dot{x}(\tau) = -f(x(\tau), p) \tag{6b}$$

  corresponding to the parametrization of time $t = -\tau$. Here, $v_s$ is the eigenvector corresponding to the stable eigenvalue of the saddle point, and $\epsilon$ is a small number, e.g., $10^{-3}$ or $10^{-6}$.
- Sensitivities, $S = \frac{\partial x}{\partial p}$

$$S(\tau_0) = \frac{\partial x_s}{\partial p} + \epsilon \frac{\partial v_s}{\partial p}, \tag{7a}$$
$$\dot{S}(\tau) = -\left( \frac{\partial f}{\partial x}(x(\tau), p) S(t) + \frac{\partial f}{\partial p}(x(\tau), p) \right) \tag{7b}$$

- Sensitivity of steady state

$$\frac{\partial f}{\partial x}(x_s, p) \frac{\partial x_s}{\partial p} + \frac{\partial f}{\partial p}(x_s, p) = 0 \tag{8}$$

  or

$$\frac{\partial x_s}{\partial p} = -\left( \frac{\partial f}{\partial x}(x_s, p) \right)^{-1} \frac{\partial f}{\partial p}(x_s, p) \tag{9}$$

## Computation of separatrix and sensitivities

- Derivatives of eigenvector [2, 3]. We compute the Moore-Penrose inverse $(\,\cdot\,)^+$ using Matlab's `pinv`.

$$\frac{\partial v_s}{\partial p} = (\lambda_s I - A_s)^+ \frac{\partial A_s}{\partial p} v_s \tag{10}$$

- At any given point on the separatrix, it is only the component of the sensitivity that is orthogonal to the separatrix that indicates a change in its shape
- In contrast, the component of the sensitivity that is parallel to the seperatrix only indicates a change in its parametrization, i.e., on its dependency on the underlying parameter, which is $\tau$
- Orthogonal part of sensitivities (the tangent of the separatrix is $-f(x(\tau), p)$)

$$\bar{S}(\tau) = S(\tau) - \left( \frac{S^T(\tau) f(x(\tau), p)}{\|f(x(\tau), p)\|^2} \right) f(x(\tau), p) \tag{11}$$

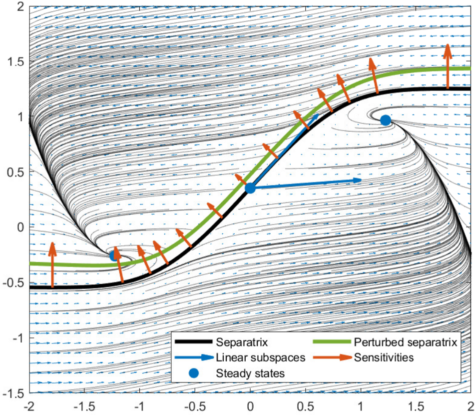
# FitzHugh-Nagumo model

- System

$$\dot{v}(t) = v(t) - \frac{v^3(t)}{3} - w + e, \quad e = 0.35, \tag{12a}$$

$$\dot{w}(t) = \frac{1}{\tau}v(t) + \frac{a}{\tau} - \frac{b}{\tau}w(t), \quad a = 0.7, \quad b = 2.0, \quad \tau = 12.5 \tag{12b}$$

- Numerical results (the parameter is $e$, it is increased by 30%, and the sensitivities have been normalized)



# Bibliography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