Active Powered Ascent Guidance

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Nomenclature

Symbol	SI-Dimension	Quantity
a	$L^1\ T^{-2}$	Acceleration due to thrust
c_e	$L^1\ T^{-1}$	Effective exhaust velocity
e	1	Euler's number, $e \approx 2.7183$
h_{ap}	L^1	height of apoapsis above mean earth radius
h_{pe}	L^1	height of periapsis above mean earth radius
k	L^{-1}	target speed parameter
m	M^{-1}	craft current mass
m_0	M^{-1}	craft gross mass
R	L^1	Earth radius, $R \approx 6.371 \cdot 10^6$ m = 6371 km
t	T^1	time
t_b	T^1	estimated time to burn end
v_h	$L^1\ T^{-1}$	horizontal speed
v_v	$L^1\ T^{-1}$	vertical speed
θ	1	thrust vector pitch angle
$\overline{\theta}$	1	medium thrust vector pitch angle
θ_0	1	Ansatzparameter
θ_1	M^{-1}	Ansatzparameter
μ	$L^3\ T^{-2}$	Earth's standard gravitational parameter, $\mu \approx 398.600 \cdot 10^{12}\ m^3s^{-2} = 398\ 600\ km^3s^{-2}$

Powered orbital motion

The selected coordinate system is a planar horizontal-vertical system. As result, the spherical earth is converted into a flat earth, which results in pseudo forces in the equations, but simplifies a semi-analytical solution for powered flight. Because the atmosphere at the operational heights of upper stages is very thin, drag is neglected too.

$$\dot{h} = v_v$$

$$\dot{v_v} = a\ \sin\theta + \frac{v_h^2}{R+h} - \frac{\mu}{(R+h)^2}$$

$$\dot{v_h} = a\ \cos\theta - \frac{v_h\ v_v}{R+h}$$

where $-\frac{\mu}{(R+h)^2}$ is the acceleration due to gravity, $\frac{v_h^2}{R+h}$ the centrifugal pseudo acceleration and $-\frac{v_h\ v_v}{R+h}$ the pseudo acceleration due to conservation of angular momentum.

The acceleration is dependent from time. For constant thrust and fuel flow (like most upper stages):

$$a = \frac{F}{m} = \frac{F}{m_0 - \dot{m}\ t} = \frac{\dot{m}\ c_e}{m_0 - \dot{m}\ t}$$

To deliver the payload into the correct orbit, with the vis-viva equation the state vector at burn end must be

$$\begin{bmatrix} h \\ v_v \\ v_h \end{bmatrix}(t_b) = \begin{bmatrix} h_{pe} \\ 0 \\ \sqrt{\mu\ k} \end{bmatrix}$$

with

$$k = \frac{2}{R + h_{pe}} - \frac{2}{2\ R + h_{pe} + h_{ap}}$$

Solution

The control variables are time of burn end t_b and thrust vector orientation θ(t). As ansatz, θ(t) is defined as $\theta(t) = \bar{\theta} + \Delta\theta(t)$ where $\bar{\theta}$ is θ averaged and Δθ(t) is the yet unspecified time dependent deviation from $\bar{\theta}$.

First, the estimated time of burn end is calculated with the constraint $v_h(t_b) = \sqrt{\mu\, k}$. Because pseudo acceleration due to conservation of angular momentum is very small compared to thrust acceleration at ascent, it is neglected.

$$\Delta v_h = \sqrt{\mu\, k} - v_h = \int_0^{t_b} \frac{\dot{m}\, c_e}{m_0 - \dot{m}\, t} \cos(\bar{\theta})\ dt = c_e \cos(\bar{\theta})\ \ln \frac{m_0}{m_0 - \dot{m}\, t_b}$$

$$\rightarrow t_b = \frac{m_0}{\dot{m}} \left(1 - e^{-\frac{\Delta v_h}{c_e \cos(\bar{\theta})}}\right) \ [1]$$

Though $\bar{\theta}$ is unknown, $\bar{\theta}$=0 as first guess is a good approximation.

Second, $\bar{\theta}$ is determined with the constraint $v_v(t_b) = 0$. Because R >> h, R+h is assumed as nearly constant with $h = h_{pe}$ to make the calculations as precise as possible when $t_b \rightarrow 0$.

$$0 = v_v(t_b) = v_{v0} + \int_0^{t_b} a\ \sin\bar{\theta} + \frac{v_h^2}{R+h} - \frac{\mu}{(R+h)^2}\ dt$$

$$= v_{v0} + \int_0^{t_b} \frac{\dot{m}\, c_e}{m_0 - \dot{m}\, t} \sin\bar{\theta} + \frac{v_h^2}{R+h_{pe}} - \frac{\mu}{\left(R+h_{pe}\right)^2}\ dt$$

$$= v_{v0} + c_e \sin\bar{\theta}\ \ln\frac{m_0}{m_0 - \dot{m}\, t_b} + \int_0^{t_b} \frac{v_h^2(t)}{R+h_{pe}}\ dt - \frac{t_b\, \mu}{\left(R+h_{pe}\right)^2}$$

$$= v_{v0} + c_e \sin\bar{\theta}\ \frac{\Delta v_h}{c_e\ \cos\bar{\theta}} + \int_0^{t_b} \frac{v_h^2(t)}{R+h_{pe}}\ dt - \frac{t_b\, \mu}{\left(R+h_{pe}\right)^2}$$

$$= v_{v0} + \Delta v_h\ \tan\bar{\theta} + \int_0^{t_b} \frac{v_h^2(t)}{R+h_{pe}}\ dt - \frac{t_b\, \mu}{\left(R+h_{pe}\right)^2} = 0$$

$$\rightarrow \Delta v_h\ \tan\bar{\theta} = \frac{t_b\, \mu}{\left(R+h_{pe}\right)^2} - v_{v0} - \int_0^{t_b} \frac{v_h^2(t)}{R+h_{pe}}\ dt\ \ [2]$$

with

$$v_h(t) = v_{h0} + c_e \cos\bar{\theta}\ \ln\left(\frac{m_0}{m_0 - \dot{m}\, t}\right)$$

where $\int_0^{t_b} \frac{v_h^2(t)}{R+h}\, dt$ should be solved numerical due to its very complex analytical solution.

Third, Δθ(t) has to be determined with the constraint $h(t_b) = h_{pe}$. Δθ is assumed small enough, that its affect on v_h and t_b is negligibly, but large enough to affect vertical acceleration (which is necessary to have an impact on $h(t_b)$). So that Δθ has no affect on [2], which is needed to guarantee the constraint $v_v(t_b) = 0$, and because the other terms in [2] are not directly dependent from θ, it must apply for the vertical speed at t_b due to thrust:

$$\int_0^{t_b} a\,\sin\bar{\theta}\;dt = \int_0^{t_b} a\,\sin\left(\bar{\theta} + \Delta\theta(t)\right)\,dt$$

$$\int_0^{t_b} \frac{\dot{m}\,c_e}{m_0 - \dot{m}\,t}\sin\bar{\theta}\;dt = \int_0^{t_b} \frac{\dot{m}\,c_e}{m_0 - \dot{m}\,t}\sin\left(\bar{\theta} + \Delta\theta(t)\right)\,dt$$

Taylor expansion of sin(θ): $\sin\left(\bar{\theta} + \Delta\theta(t)\right) \approx \sin\bar{\theta} + \left(\frac{\partial}{\partial\theta}\sin\theta\right)(\bar{\theta})\,\Delta\theta(t) = \sin\bar{\theta} + \cos\bar{\theta}\;\Delta\theta(t)$

$$\int_0^{t_b} \frac{\sin\bar{\theta}}{m_0 - \dot{m}\,t}\,dt = \int_0^{t_b} \frac{\sin\bar{\theta} + \cos\bar{\theta}\;\Delta\theta(t)}{m_0 - \dot{m}\,t}\,dt = \int_0^{t_b} \frac{\sin\bar{\theta}}{m_0 - \dot{m}\,t}\,dt + \int_0^{t_b} \frac{\cos\bar{\theta}\;\Delta\theta(t)}{m_0 - \dot{m}\,t}\,dt$$

$$\rightarrow \int_0^{t_b} \frac{\Delta\theta(t)}{m_0 - \dot{m}\,t}\,dt = 0\ [3]$$

To deliver the payload into the correct orbit, it must apply for the height at burn time:

$$h(t_b) = h_{pe} = h_0 + \int_0^{t_b} v_{v0} + \int_0^{\tilde{t}} a\,\sin\theta + \frac{v_h^2(t)}{R+h} - \frac{\mu}{(R+h)^2}\,dt\,d\tilde{t}$$

$$h_{pe} = h_0 + v_{v0}\,t_b + \int_0^{t_b}\int_0^{\tilde{t}} \frac{\dot{m}\,c_e}{m_0 - \dot{m}\,t}\left(\sin\bar{\theta} + \cos\bar{\theta}\;\Delta\theta(t)\right) + \frac{v_h^2(t)}{R+h_{pe}} - \frac{\mu}{\left(R+h_{pe}\right)^2}\,dt\,d\tilde{t}$$

$$\rightarrow \int_0^{t_b}\int_0^{\tilde{t}} \frac{\dot{m}\,c_e\,\cos\bar{\theta}\;\Delta\theta(t)}{m_0 - \dot{m}\,t}\,dt\,d\tilde{t} = c_e\cos\bar{\theta}\int_0^{t_b}\int_0^{\tilde{t}} \frac{\dot{m}\,\Delta\theta(t)}{m_0 - \dot{m}\,t}\,dt\,d\tilde{t}$$
$$= h_{pe} - h_0 - t_b\left(v_{v0} + c_e\sin\bar{\theta}\right) + \frac{t_b^2\,\mu}{2\left(R+h_{pe}\right)^2} - \Delta v_h\tan\bar{\theta}\left(t_b - \frac{m_0}{\dot{m}}\right)$$
$$- \int_0^{t_b}\int_0^{\tilde{t}} \frac{v_h^2(t)}{R+h_{pe}}\,dt\,d\tilde{t}\ [4]$$

with

$$v_h(t) = v_{h0} + c_e\cos\bar{\theta}\,\ln\left(\frac{m_0}{m_0 - \dot{m}\,t}\right)$$

where $\int_0^{t_b}\int_0^{\tilde{t}} \frac{v_h^2(t)}{R+h_{pe}}\,dt\,d\tilde{t}$ should be solved numerical due to its very complex analytical solution.

An ansatz linear in mass was chosen. Because mass decreases linear with time, this ansatz is also linear in time. In addition, an ansatz linear in mass leads to relatively simple integrals in [3] and [4]. This ansatz, called “mass-law”, is:

$$\Delta\theta(t) = \theta_0 + \theta_1\, m(t) = \theta_0 + \theta_1\, (m_0 - \dot{m}\, t)$$

$$\rightarrow \dot{\theta} = \frac{\partial}{\partial t}\left(\bar{\theta} + \Delta\theta(t)\right) = \frac{\partial}{\partial t}\left(\theta_0 + \theta_1\, (m_0 - \dot{m}\, t)\right) = -\theta_1\, \dot{m}$$

$$[3]\colon \int_0^{t_b} \frac{\theta_0 + \theta_1\, (m_0 - \dot{m}\, t)}{m_0 - \dot{m}\, t}\, dt = 0$$

$$\int_0^{t_b} \frac{\theta_0}{m_0 - \dot{m}\, t} + \theta_1\, dt = 0$$

$$\frac{\theta_0}{\dot{m}} \ln \frac{m_0}{m_0 - \dot{m}\, t_b} + \theta_1\, t_b = \frac{\theta_0}{\dot{m}} \frac{\Delta v_h}{c_e\, \cos\bar{\theta}} + \theta_1\, t_b = 0$$

$$\rightarrow \theta_1 = -\theta_0\, \frac{\Delta v_h}{\dot{m}\, c_e\, t_b\, \cos\bar{\theta}}\ [5]$$

$$[4]\colon \int_0^{t_b} \int_0^{\tilde{t}} \frac{\dot{m}\, c_e\, \cos\bar{\theta}\, \Delta\theta(t)}{m_0 - \dot{m}\, t}\, dt\, d\tilde{t} = \cdots$$

$$= c_e\, \cos\bar{\theta} \int_0^{t_b} \int_0^{\tilde{t}} \frac{\dot{m}\left(\theta_0 + \theta_1\, (m_0 - \dot{m}\, t)\right)}{m_0 - \dot{m}\, t}\, dt\, d\tilde{t} = c_e\, \cos\bar{\theta} \int_0^{t_b} \int_0^{\tilde{t}} \theta_0\, \frac{\dot{m}}{m_0 - \dot{m}\, t} + \theta_1\, \dot{m}\, dt\, d\tilde{t}$$

$$= c_e\, \cos\bar{\theta} \int_0^{t_b} \theta_0\, \ln \frac{m_0}{m_0 - \dot{m}\, \tilde{t}} - \theta_0\, \frac{\Delta v_h\, \tilde{t}}{c_e\, t_b\, \cos\bar{\theta}}\, d\tilde{t}$$

$$= \theta_0\, c_e \cos\bar{\theta} \left(\left(t_b - \frac{m_0}{\dot{m}}\right) \ln \frac{m_0}{m_0 - \dot{m}\, t_b} + t_b - \frac{\Delta v_h\, t_b}{2\, c_e \cos\bar{\theta}}\right)$$

$$= \theta_0\, c_e \cos\bar{\theta} \left(\left(t_b - \frac{m_0}{\dot{m}}\right) \frac{\Delta v_h}{c_e \cos\bar{\theta}} + t_b - \frac{\Delta v_h\, t_b}{2\, c_e \cos\bar{\theta}}\right) = \theta_0\, \Delta v_h \left(\frac{t_b}{2} + \frac{c_e\, t_b\, \cos\bar{\theta}}{\Delta v_h} - \frac{m_0}{\dot{m}}\right)$$

$$\rightarrow \theta_0 \left(\frac{t_b}{2}\, \Delta v_h + c_e\, t_b\, \cos\bar{\theta} - \frac{m_0}{\dot{m}}\, \Delta v_h\right)$$

$$= h_{pe} - h_0 - t_b\, (v_{v0} + c_e \sin\bar{\theta}) + \frac{t_b^2\, \mu}{2\left(R + h_{pe}\right)^2} - \Delta v_h \tan\bar{\theta}\, \left(t_b - \frac{m_0}{\dot{m}}\right)$$

$$- \int_0^{t_b} \int_0^{\tilde{t}} \frac{v_h^2(t)}{R + h_{pe}}\, dt\, d\tilde{t}\ [6]$$

Therefore, the pitch law for ascent is:

$$\theta(t) = \bar{\theta} + \theta_0 + \theta_1 m(t) = \bar{\theta} + \theta_0 + \theta_1\, (m_0 - \dot{m}\, t)$$

$$\rightarrow \theta = \bar{\theta} + \theta_0 + \theta_1 m$$

Implementation

When t_b is calculated with [1] (and the first assumption $\bar{\theta}$=0), $\bar{\theta}$ can be calculated with [2]. With $\bar{\theta}$, t_b can calculated more precisely, so a more precise $\bar{\theta}$ can be calculated. Experience has shown, that three iterations of [1] and [2] are entirely sufficient to produce stable and consistent results.

The program consists of an inner loop and an outer loop. The inner loop calculates θ and ∂θ/∂t with a frequency of 1Hz to compensate the errors due to approximations and numeric calculation, the outer loop runs continuously and controls the pitch angle, which is delivered from the inner loop. The outer loop also contains terminal guidance.

To avoid oscillations due to approximations and numerical errors when the craft is near burn end, the program switches to terminal guidance when the inner loop determines t_b < 10s. As part of the outer loop, terminal guidance runs continuously and holds pitch angle at $\bar{\theta}$ until $v_h \geq \sqrt{\mu\, k}$. Then, terminal guidance cuts off the engine and terminates the guidance program.

$\int_0^{t_b} \frac{v_h^2(t)}{R+h}\, dt$ and $\int_0^{t_b} \int_0^{\tilde{t}} \frac{v_h^2(t)}{R+h_{pe}}\, dt\, d\tilde{t}$ are calculated with 3-point Gauss–Legendre quadrature:

$$\int_{-1}^{+1} f(x)\, dx \approx w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3)$$

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2} \left(w_1\, f\left(\frac{b-a}{2} x_1 + \frac{a+b}{2}\right) + w_2\, f\left(\frac{b-a}{2} x_2 + \frac{a+b}{2}\right) + w_3\, f\left(\frac{b-a}{2} x_3 + \frac{a+b}{2}\right) \right)$$

$$\int_0^a f(x)\, dx \approx \frac{a}{2} \left(w_1\, f\left(\frac{a}{2}\, (x_1 + 1)\right) + w_2\, f\left(\frac{a}{2}\, (x_2 + 1)\right) + w_3\, f\left(\frac{a}{2}\, (x_3 + 1)\right) \right)$$

$$\int_0^a \int_0^{\tilde{x}} f(x)\, dx\, d\tilde{x} \approx \frac{a}{2} \left(2 \int_0^{\frac{a}{2}(x_1+1)} f(x)\, dx + \frac{13}{9} \int_{\frac{a}{2}(x_1+1)}^{\frac{a}{2}(x_2+1)} f(x)\, dx + \frac{5}{9} \int_{\frac{a}{2}(x_2+1)}^{\frac{a}{2}(x_3+1)} f(x)\, dx \right)$$

i	x_i	w_i
1	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
2	0	$\frac{8}{9}$
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

Variable quantities required: $m, \dot{m}, c_e, v_h, v_v, h$

Orbit elements required: h_{pe}, h_{ap}

Constants required: R, μ

Functions required: sqrt, sin, cos, tan, atan, exp, ln