Aufgabe 10.1:

a)
$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ 1 & 3 & 1 & 3 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 4 & 8 & 8 \\ 8 & 20 & 16 \\ 8 & 16 & 20 \end{pmatrix} \qquad A^T b = \begin{pmatrix} 10 \\ 22 \\ 22 \end{pmatrix}$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 4 & 2 & 2 \end{pmatrix}$$

$$Ly = A^T b$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 4 & 2 & 2 \end{pmatrix} y = \begin{pmatrix} 10 \\ 22 \\ 22 \end{pmatrix}$$

$$2y_1 = 10 \Rightarrow y_1 = 5$$
$$\Rightarrow y_2 = 1 \qquad y = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$
$$\Rightarrow y_3 = 0$$

$$L^T x = y$$

$$\begin{pmatrix} 2 & 4 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} x = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

$$2x_3 = 0 \Rightarrow x_3 = 0$$
$$2x_2 + 2x_3 = 1 \Rightarrow 2x_2 = 1 \Rightarrow x_2 = \frac{1}{2}$$
$$2x_1 + 4x_2 + 4x_3 = 5 \Rightarrow 2x_1 = 3 \Rightarrow x_1 = \frac{3}{2}$$
$$x = \begin{pmatrix} 3/2 \\ 1/2 \\ 0 \end{pmatrix}$$

b)
$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_1 = a_1 + \text{sign}(a_1[1]) \|a_1\|_2 \, e_1$$
$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \sqrt{4} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_1 = \frac{1}{\|v_1\|_2} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{12}} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$H_1 = I - 2 v_1 v_1^T = I - 2 \frac{1}{12} \begin{pmatrix} 9 & 3 & 3 & 3 \\ 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 3/2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1/2 & 1/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

$$H_1 A = \begin{pmatrix} -1/2 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -1/2 & 5/6 & -1/6 & -1/6 \\ -1/2 & -1/6 & 5/6 & -1/6 \\ -1/2 & -1/6 & -1/6 & 5/6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -4 & -4 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 2 & 2 \end{pmatrix} \qquad a_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$v_2 = a_2 + \operatorname{sign}(a_2[1]) \|a_2\|_2 \, e_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \sqrt{9} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

$$v_2 = \frac{1}{\|v_2\|_2} \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{24}} \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

$$H_2 = I - 2 v_2 v_2^T = I - 2 \frac{1}{24} \begin{pmatrix} 16 & 8 & 8 \\ 8 & 4 & 4 \\ 8 & 4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 4/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$$

$$H_2 A' = \begin{pmatrix} -3 & -2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$Q = H_1 H_2 \qquad R = \begin{pmatrix} -2 & -4 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R x = Q^T b$$

$$Q^T b = H_2 H_1^T b = \begin{pmatrix} -5{,}5 \\ -2 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -2 & -4 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5{,}5 \\ -2 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} 6{,}75 \\ -2 \\ 0 \end{pmatrix}$$

Aufgabe 10.2:

a) $$r = \frac{p}{1 - e\cos(\phi)}$$

$$r(1 - e\cos(\phi)) = p$$

$$r - re\cos(\phi) = p$$

$$r = p + re\cos(\phi)$$

$r_1 = 10 \qquad \cos(\phi_1) \approx 0{,}63$

$r_2 = 5 \qquad \cos(\phi_2) \approx 0{,}39$

$r_3 = 2{,}5 \qquad \cos(\phi_3) \approx 0{,}12$

$r_4 = 1{,}3 \qquad \cos(\phi_4) \approx -0{,}31$

$r_5 = 1 \qquad \cos(\phi_5) \approx -0{,}59$

$10 = p + 10\, e\; 0{,}63$

$5 = p + 5\, e\; 0{,}39$

$2{,}5 = p + 2{,}5\, e\; 0{,}12$

$1{,}3 = p + 1{,}3\, e\,(-0{,}31)$

$1 = p + 1\, e\,(-0{,}59)$

$$\begin{pmatrix} 1 & 10 \cdot 0{,}63 \\ 1 & 5 \cdot 0{,}39 \\ 1 & 2{,}5 \cdot 0{,}12 \\ 1 & 1{,}3 \cdot (-0{,}31) \\ 1 & 1 \cdot (-0{,}59) \end{pmatrix} \begin{pmatrix} p \\ e \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 2{,}5 \\ 1{,}3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6{,}3 \\ 1 & 1{,}95 \\ 1 & 0{,}3 \\ 1 & -0{,}403 \\ 1 & -0{,}59 \end{pmatrix} \begin{pmatrix} p \\ e \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 2{,}5 \\ 1{,}3 \\ 1 \end{pmatrix}$$

b) $$A^T A x = A^T b$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6{,}3 & 1{,}95 & 0{,}3 & -0{,}403 & -0{,}59 \end{pmatrix} \begin{pmatrix} 1 & 6{,}3 \\ 1 & 1{,}95 \\ 1 & 0{,}3 \\ 1 & -0{,}403 \\ 1 & -0{,}59 \end{pmatrix} = \begin{pmatrix} 5 & 7{,}557 \\ 7{,}557 & 45{,}8299 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6{,}3 & 1{,}95 & 0{,}3 & -0{,}403 & -0{,}59 \end{pmatrix} \begin{pmatrix} 10 \\ 5 \\ 2{,}5 \\ 1{,}3 \\ 1 \end{pmatrix} = \begin{pmatrix} 19{,}8 \\ 119{,}8759 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 7{,}557 \\ 7{,}557 & 45{,}8299 \end{pmatrix} \begin{pmatrix} p \\ e \end{pmatrix} = \begin{pmatrix} 19{,}8 \\ 119{,}8759 \end{pmatrix}$$

$$\begin{pmatrix} p \\ e \end{pmatrix} \approx \begin{pmatrix} 0{,}266 \\ 2{,}317 \end{pmatrix}$$

$\Rightarrow e \approx 2{,}317$. Da $e > 1$ ist die Bahn des Kometen eine Hyperbel.

Aufgabe 10.3:

a) Sei $A \in \mathbb{R}^{n \times n}$ eine nichtsinguläre quadratische Matrix. Dann existieren orthonormale Matrizen $U, V \in \mathbb{R}^{n \times n}$ und eine Diagonalmatrix $\Sigma = \operatorname{diag}(\sigma_1, \sigma_n) \in \mathbb{R}^{n \times n}$ mit $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$, sodass gilt:

$$A = U \Sigma V^T$$

Sei $x \in \mathbb{R}^n$ ein bel. Vektor

$$x = \sum_{k=1}^{n} c_k v_k$$

$, v_1, v_2, \ldots, v_n$ ortho. Spaltenvek. von V

$, c_k$ Koeff. der Linearkombination

$$Ax = A\left(\sum_{k=1}^{n} c_k v_k\right)$$
$$= \sum_{k=1}^{n} c_k A v_k$$

$Av_k = U \Sigma V^T v_k$ — Da V orthonormal gilt $V^T v_k = e_k$

$Av_k = U \Sigma e_k$ — $\Sigma e_k = \sigma_k e_k$

$Av_k = U \sigma_k e_k$ — $U e_k = u_k$, k-te Spalte von U

$Av_k = \sigma_k u_k$

$$\Rightarrow Ax = \sum_{k=1}^{n} c_k \sigma_k u_k$$

b) $\|A\|_2 = \sigma_1$

$\|A^{-1}\|_2 = \dfrac{1}{\sigma_n}$

$$\operatorname{cond}_2(A) = \|A\|_2 \, \|A^{-1}\|_2 = \sigma_1 \frac{1}{\sigma_n} = \frac{\sigma_1}{\sigma_n}$$