List 1

Introductory to analytic combinatorics course at Wroclaw University of Science and Technology 2020/2021

Deadline:

28.10.2020

Exercise 1 (1 points)

Sequence $(a_1, a_2, \ldots)$ corresponds to the generating function $A(z)$. Calculate sequences corresponding to:

1. $A'(z) + A(z)$**,**

Let consider how $A(z)$ looks:

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

Knowing that let's see what is $A'(z)$

$$A'(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)'$$

$$(a_n z^n)' = n a_n z^{n-1}$$

$$A'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

So finaly

$$A'(z) + A(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (a_n + (n+1) a_{n+1}) z^n$$

2. $2A(z)$,

$$2A(z) = 2\left(\sum_{n=0}^{\infty} a_n z^n\right)$$
$$2A(z) = \sum_{n=0}^{\infty} 2a_n z^n$$

3. $A^2(z)$

$$A^2(z) = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) =$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_n a_m z^n$$

Exercise 2 (3 points)

Let $\mathcal{A} = (\{\epsilon, 1, 2, *\}, |\cdot|)$ and $\mathcal{B} = (\{a\}, \{a \to 1\})$ be combinatorial class. Where $|\epsilon| = 0, |1| = 1, |2| = 2, |*| = 5$.

Describe the following classes (if they exist) and their generating functions:

(a) $\mathcal{A} + \mathcal{B}$,

Sum of classes.

$$\mathcal{A} + \mathcal{B} = (\{\epsilon, 1, a, 2, *\}, |\cdot|)$$
$$A + B(z) = A(z) + B(z)$$
$$A + B(z) = 1 + z + z^2 + z^5 + z = 1 + 2z + z^2 + z^5$$

(b) $\mathcal{A} \times \mathcal{B}$,

Cartesian product of classes

$$\mathcal{A} \times \mathcal{B} = (\{(\epsilon, a), (1, a), (2, a), (*, a)\}, |\cdot|)$$
$$\mathcal{A} \times \mathcal{B}(z) = A(z) \cdot B(z) = z + z^2 + z^3 + z^6$$

(c) $Seq(\mathcal{A})$,

Does not exist because $\mathcal{A}_0 = 0$

(d) $Seq(\mathcal{B})$,

$Seq(\mathcal{B})$ is basicly $\mathbb{N}$ - all natural numbers

$$Seq(\mathcal{B})(z) = \frac{1}{1 - \mathcal{B}(z)} = \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n$$

(e) $Seq(\mathcal{A} + \mathcal{B})$**,**

Does not exist because $\mathcal{A}_0 = 0$

(f) $Seq(\mathcal{A}) + Seq(\mathcal{B})$**,**

Does not exist because $\mathcal{A}_0 = 0$

(g) $MSet(\mathcal{B})$**,**

Multiset of $\mathcal{B}$

$$MSet(\mathcal{B})(z) = \prod_{n>=1} (1 - z^n)^{B_n}$$

Remembering that $\mathcal{B}(z) = z$, $B_1 = 1$ and for all others $B_n = 0$

$$MSet(\mathcal{B})(z) = (1 - z)^{-1} = \frac{1}{1 - z}$$

And that equals to $\mathbb{N}(z) = \sum_{n=0}^{\infty} z^n$

(h) $PSet(\mathcal{A}) + PSet(\mathcal{B})$**,**

Does not exist because $\mathcal{A}_0 = 0$

(i) $Cyc(\mathcal{A})$

Exercise 3 (1 points)

When defining $Seq(\mathcal{A})$ **we assumed that** $[z^0]A(z) = 0$**. Why?**

When defining a combinatorial class we specified 2 conditions: (i) the size of an element is a non-negative integer; (ii) the number of elements of any given size is finite.

And when $[z^0]A(z) = 0$, then we can generate an infinite number of elements of any size by appending elements of size 0.

Exercise 4 (1 points)

Calculate $[z^{30}]\frac{1}{(1-z)^7}$ **(without computer, using formulas presented in lecture).**

Based on formula nr 6

$$\frac{1}{(1 - y)^{k+1}} = \sum_{a \geq 0} \binom{k + a}{k} y^a,$$

we get:

$$\frac{1}{(1-z)^7} = \sum_{n\geq 0} \binom{6+n}{6} z^n.$$

From that we know that

$$[z^{30}]\frac{1}{(1-z)^7} = \binom{6+30}{6} =$$

$$\frac{36!}{30!\cdot 6!} = \frac{30\cdot 31\cdot 32\cdot 33\cdot 34\cdot 35\cdot 36}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} = 58433760$$

Exercise 5 (1 points)

Study and get understanding of following formulas:

1) $(x+y)^n = \sum_k \binom{n}{k} x^k y^{(n-k)}$

2) $\sum_{i=0}^{\infty} q^i = \frac{1}{1-x}$ dla $(|q| < 1)$

3) $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$

4) $\binom{n+m}{k} = \sum_j \binom{n}{j}\binom{m}{k-j}$

5) $\frac{y^k}{(1-y)^{k+1}} = \sum_{n\geq 0}\binom{n}{k} y^k$

6) $\frac{1}{(1-y)^{k+1}} = \sum_{a\geq 0}\binom{k+a}{k} y^a$

Exercise 6 (3 points)

Let $\mathcal{N}$ be a combinatoric class of natural numbers with size function $|a| = a$. Let $\mathcal{N}_{r,k}$ be a combinatoric class of natural numbers that which give a remainder of dividing r by k. Prove that

$$\mathcal{N} \simeq \mathcal{N}_{0,k} + \mathcal{N}_{1,k} + \ldots + \mathcal{N}_{r-1,k}.$$

Let's start by taking a second look at $\mathcal{N}_{r,k}$

$$\mathcal{N}_{r,k} = (\{x : (\forall i \in \mathbb{N})(\exists x_i = ik + r)\}, |\cdot|)$$

$$\mathcal{N}_{r,k} = (\{r, k+r, 2k+r, 3k+r, \ldots\}, |\cdot|)$$

Also, analyze what we need to prove:

$$\mathcal{N} \simeq \mathcal{N}_{0,k} + \mathcal{N}_{1,k} + \ldots + \mathcal{N}_{r-1,k} = \sum_{i=0}^{k-1} \mathcal{N}_{i,k}$$

Now

$$\sum_{i=0}^{k-1} \mathcal{N}_{i,k}(z) = \sum_{i=0}^{k-1} z^{(i)} + z^{(i+k)} + z^{(i+2k)} + z^{(i+3k)} + \ldots = \sum_{i=0}^{k-1}\sum_{n\geq 0} z^{(i+nk)} =$$

$$\sum_{n\geq 0}\sum_{i=0}^{k-1} z^{(i+nk)} = \sum_{n\geq 0} z^{(nk)} + z^{(1+nk)} + z^{(2+nk)}+\ldots+z^{(k-1+nk)} = \sum_{n\geq 0} z^{n} = \mathcal{N}(z).$$

For that to be a valid proof we need to denote that:

- Two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ are said to be isomorphic, which is written $\mathcal{A} \cong \mathcal{B}$, iff their counting sequences are identical.