List 4

Introductory to analytic combinatorics course at Wroclaw University of Science and Technology 2020/2021

Deadline:

21.12.2020

Exercise 1 (2 points)

Let structure α of size 3 be labeled with permutation $2-1-3$ and structure β of size 2 be labeled with permutation $2-1$. Describe $\alpha * \beta$.

$\alpha * \beta$ is labeled with integers $\{1,2,3,4,5\}$

$$|\alpha * \beta| = \binom{5}{2} = 10$$

$$\begin{gathered}\alpha * \beta = [(② - ① - ③, ⑤ - ④), (② - ① - ④, ⑤ - ③), (③ - ① - ④, ⑤ - ②),\\ (③ - ② - ④, ⑤ - ①), (② - ① - ⑤, ④ - ③), (③ - ① - ⑤, ④ - ②),\\ (③ - ② - ⑤, ④ - ①), (④ - ① - ⑤, ③ - ②), (④ - ② - ⑤, ③ - ①),\\ (④ - ③ - ⑤, ② - ①)]\end{gathered}$$

Exercise 2 (3 points)

Describe EGF for a class of permutations, that have at most 5 cycles.

Before we get the final EGF function we need to start by preparing its parts, so:

$$\mathcal{P} = SET(CYC(\mathcal{Z}))$$
$$Cyc(\mathcal{A}) = log\frac{1}{1-\mathcal{A}}$$
$$Set(\mathcal{A}) = exp(\mathcal{A})$$

Number of premutations with r cycles is

$$\mathcal{P}^{(r)} = SET_r(CYC(\mathcal{Z}))$$

So if we want to get permutations, that have at most 5 cycles. We need to consider

$$\bigcup_{i=1}^{5} \mathcal{P}^{(i)}$$

Now we can get to writing down EGF function.

$$\left(\bigcup_{i=1}^{5} \mathcal{P}^{(i)}\right)(z) = \sum_{i=1}^{5} \left(\mathcal{P}^{(i)}(z)\right) =$$

$$\sum_{i=1}^{5} SET_i(CYC(\mathcal{Z}))(z) = \sum_{i=1}^{5} SET_i\left(log\frac{1}{1-z}\right)(z) =$$

$$\sum_{i=1}^{5} \frac{1}{i!}\left(log\frac{1}{1-z}\right)^i(z)$$

THe exact number of such permutations is a sum of Stirling numbers of the first kind.

$$\sum_{i=1}^{5} \begin{bmatrix} n \\ i \end{bmatrix}$$

Exercise 3 (3 points)

Describe EGF for k–surjection of set with n elemnts onto set with r elements, that is, functions such that each element form [1...r] set has at least k-element inverse image.

Firstly lets consider all surections with r images.

$$\mathcal{S}_n^{(r)} = SEQ_r(SET_{\geq 1}(\mathcal{Z}))$$

Now lets ensure that we count only those that k-element inverse image condition.

$$\mathcal{S}_n^{(r)} = SEQ_r(SET_{\geq k}(\mathcal{Z}))$$

Let's get inside this function firstly by expanding SET function. We need to remember to remove 'by hand' sets of size smaller than k.

$$\mathcal{S}_n^{(r)}(z) = SEQ_r\left(exp(z) - \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)\right)$$

And now expand SEQ function

$$\mathcal{S}_n^{(r)}(z) = \left(e^z - \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)\right)^r$$

Let's simplify it a bit more

$$[z^n]\sum_{i=0}^{r}\binom{r}{i}e^{zi}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6}\right)^{n-i}$$