Data.ML.100, Homework, Week 1

MSE function ($\varepsilon_{MSE}(a,b)$). Optimal poin derivative at 0

$$\nabla \varepsilon_{MSE} = \bar{0} \Rightarrow \begin{pmatrix} \frac{\partial \varepsilon_{MSE}}{\partial a} \\ \frac{\partial \varepsilon_{MSE}}{\partial b} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

Chain rule: $f \to z^2$, $g \to y_i - ax_i - b$, $f(g(x))$, $\sum_{i=1}^{N}$

$$\frac{\partial \varepsilon_{MSE}}{\partial a} = \frac{\partial \frac{1}{N}\sum (y_i - \hat{y}_i)^2}{\partial a} \Rightarrow \frac{\partial \frac{1}{N}\sum (y_i - ax_i - b)^2}{\partial a} = 0$$

- $\Rightarrow \frac{1}{N} \cdot 2 \cdot \sum (y_i - ax_i - b) \cdot (-x_i) = 0 \Leftrightarrow a = \frac{\sum y_i x_i - b\sum x_i}{\sum x_i^2}$ 1^0
- Subst. 1^0 to $\frac{\partial \varepsilon_{MSE}}{\partial b} \Rightarrow$ solve $b \Rightarrow$ back subst b to $1^0 \Rightarrow$ solve a

$$\frac{\partial \varepsilon_{MSE}}{\partial b} \Rightarrow \frac{\partial \frac{1}{N}\sum (y_i - ax_i - b)^2}{\partial b} \overset{\text{subst } 1^0}{=} 0 \Leftrightarrow \frac{\partial \frac{1}{N}\sum \left(y_i - \frac{\sum y_i x_i - b\sum x_i}{\sum x_i^2} x_i - b\right)^2}{\partial b} = 0$$

$$\Rightarrow \frac{1}{N} \cdot \left(-\frac{\sum x_i}{\sum x_i^2} - 1\right) \cdot 2\sum \left(y_i - \frac{\sum y_i x_i - b\sum x_i}{\sum x_i^2} x_i - b\right) = 0 \quad | : \frac{1}{N} \cdot \left(-\frac{\sum x_i}{\sum x_i^2} - 1\right) \cdot 2$$

$$\Rightarrow \sum \left(y_i - \frac{\sum y_i x_i - b\sum x_i}{\sum x_i^2} x_i - b\right) = 0 \Leftrightarrow \sum y_i - \frac{\sum y_i x_i \sum x_i}{\sum x_i^2} + \frac{b\sum x_i \sum x_i}{\sum x_i^2} - \sum b = 0$$

$$\Leftrightarrow \sum b - \frac{b\sum x_i \sum x_i}{\sum x_i^2} = \sum y_i - \frac{\sum y_i x_i}{\sum x_i^2} \sum x_i, \quad \sum b = b\sum 1 = bN, \quad \sum_{i=1}^{N} 1 \cdot \sum x_i^2$$

$$\Leftrightarrow b\left(N\sum x_i^2 - (\sum x_i)^2\right) = \sum y_i \sum x_i^2 - \sum y_i x_i \sum x_i \quad | : \sum x_i^2 \left(N - (\sum x_i)^2\right)$$

- $b = \frac{\sum y_i \sum x_i^2 - \sum y_i x_i \sum x_i}{N\sum x_i^2 - (\sum x_i)^2}$, subst b back to 1^0 to solve a

$$a = \frac{\sum y_i x_i}{\sum x_i^2} - \frac{\sum y_i \sum x_i^2 - \sum y_i x_i \sum x_i}{N\sum x_i^2 - (\sum x_i)^2} \cdot \frac{\sum x_i}{\sum x_i^2}$$

$$a = \frac{\sum y_i x_i}{\sum x_i^2} - \frac{\sum y_i \sum x_i^2 \sum x_i - \sum y_i x_i \sum x_i \sum x_i}{\sum x_i^2 \left(N\sum x_i^2 - (\sum x_i)^2\right)}$$

$$a = \frac{\sum y_i x_i}{\sum x_i^2} - \frac{\sum y_i \sum x_i^2 \sum x_i - \sum y_i x_i (\sum x_i)^2}{\sum x_i^2 \left(N\sum x_i^2 - (\sum x_i)^2\right)}$$