Excercise 5, problem 3
Poisson equation 1D derivation

$$\frac{\partial^2 \Phi}{\partial x^2} = -\frac{\rho(x)}{\varepsilon_0}, \quad \rho(x) = \varepsilon_0 \pi^2 \sin(\pi x)$$

Dirichlet boundary conditions

Analytical Solution

$$\Rightarrow \frac{\partial^2 \Phi}{\partial x^2} = -\pi^2 \sin(\pi x), \quad \Phi(0) = \Phi(1) = 0, \quad \Phi(x) = \sin(\pi x)$$

OK!

$$\frac{\partial^2}{\partial x^2}(\sin(\pi x)) = \frac{\partial}{\partial x}\pi\cos(\pi x) = -\pi^2 \sin(\pi x)$$

- $$\begin{cases} \frac{\partial^2 \Phi}{\partial x^2} = -\pi^2 \sin(\pi x), \quad x \in (0,1) \\ \Phi(0) = 0 \\ \Phi(1) = 0 \end{cases}, \quad f_j = -\pi^2 \sin(\pi x_j)$$

n = number of points

h = spacing $= \frac{1}{n+1}$

$x_j = jh, \quad j = 1, 2, \dots, n$

$f_j = f(x_j)$

Φ_j = numerical solution at x_j

Φ = the true solution $= \sin(\pi x)$

- $$-\frac{\partial^2}{\partial x^2}\Phi_j = \frac{-\Phi_{j-1} + 2\Phi_j - \Phi_{j+1}}{h^2} = f_j$$

$j = 1, 2, \dots, n$ $\quad x_0 = 0, \quad x_{n+1} = 1$

$$U_h = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \quad \hat{U} = \begin{bmatrix} \Phi(x_1) \\ \Phi(x_2) \\ \vdots \\ \Phi(x_n) \end{bmatrix}, \quad K = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

$j = 1, \quad \Phi(0) = 0$
$\rightarrow -\Phi_{j-1} = 0$

$j = n, \quad \Phi_{j+1} = 0$

Matrix equation:

- $$\frac{1}{h^2} K U_h = F, \quad \Rightarrow U_h = \text{np.linalg.solve}\left(\frac{1}{h^2}K, F\right)$$