Simple Linear Regression

Lecture 07 - CPSC392

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  - The 2 variables are a dependent variable (denoted by y) and an independent variable (denoted by x)
- Linear regression is in fact a comparison of 2 models

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- Let’s plot the data

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- Our best estimate, in this case, is the mean
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“ with only one variable, and no other information, the best prediction for the next measurement is the mean of the sample”

Goodness of Fit

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- This is very similar to the concept of standard deviation

Residuals/Errors

- Distance between the best fit line to the observed values

Sum of Squared Errors (SSE)

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- Calculated by squaring all errors and summing them up

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- If the regression model is “significant”, it will take away a large chunk of the SSE
- The model should “fit” the data better and minimize the residuals once we introduce an independent variable

The SSE of the model with just test scores is 20. Let’s introduce a new independent variable, total hours of study, and see if we can create a linear regression model using this attribute

Lines

Lines

- **$y = mx + b$**

- **m** = slope (rise/run)
- **b** = y-intercept (point where $x = 0$)

Lines

- $y = mx + b$
- $y = \beta_0 + \beta_1 x + \varepsilon$

- β_1 = slope parameter
- β_0 = y-intercept parameter
- ε = error term (unexplained variation in y)

Lines

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- $y = \beta_0+\beta_1 x+\varepsilon$ (for population data)

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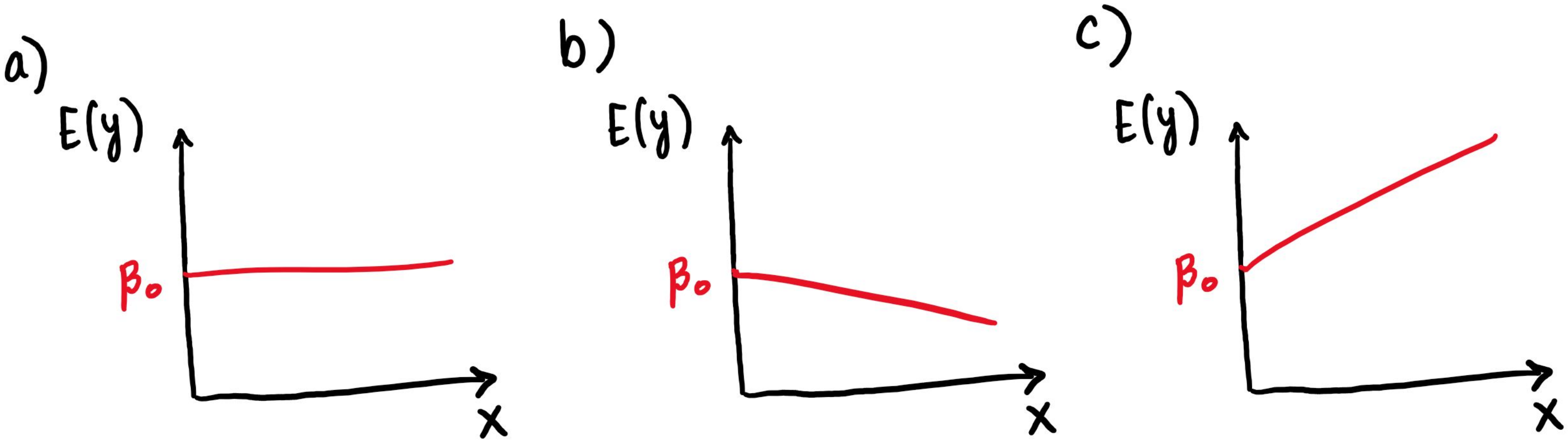
- ε = error term (unexplained variation in y)

Lines (Simple Linear Regression)

- $E(y) = \beta_0+\beta_1 x$

- β_1 = slope parameter
- β_0 = y-intercept parameter
- **E(y)** = mean or expected value of y, given some x

What are the equations for these lines?



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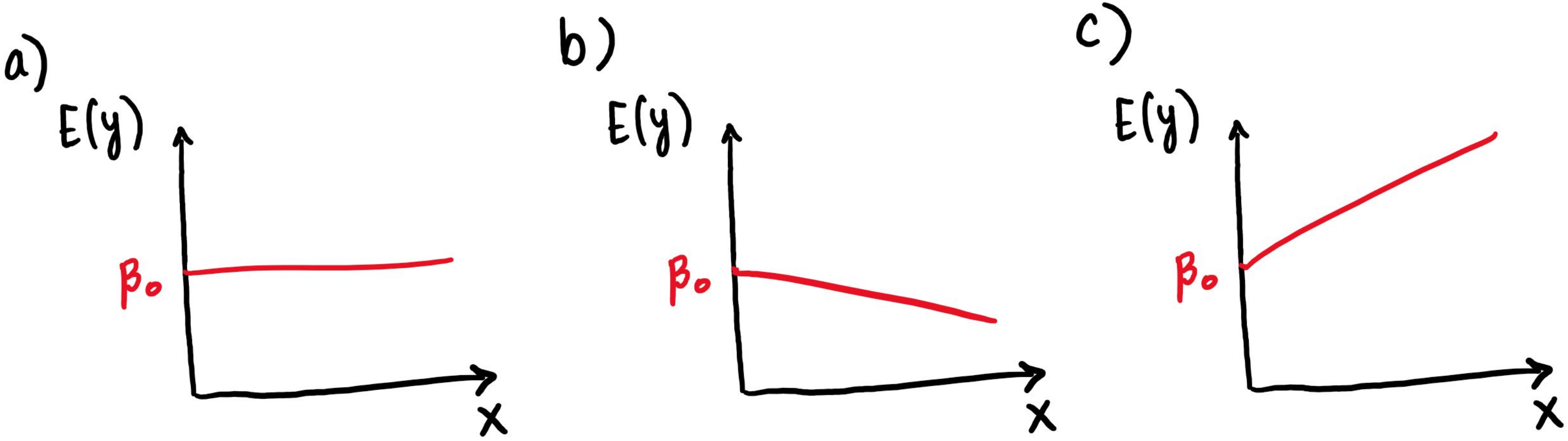

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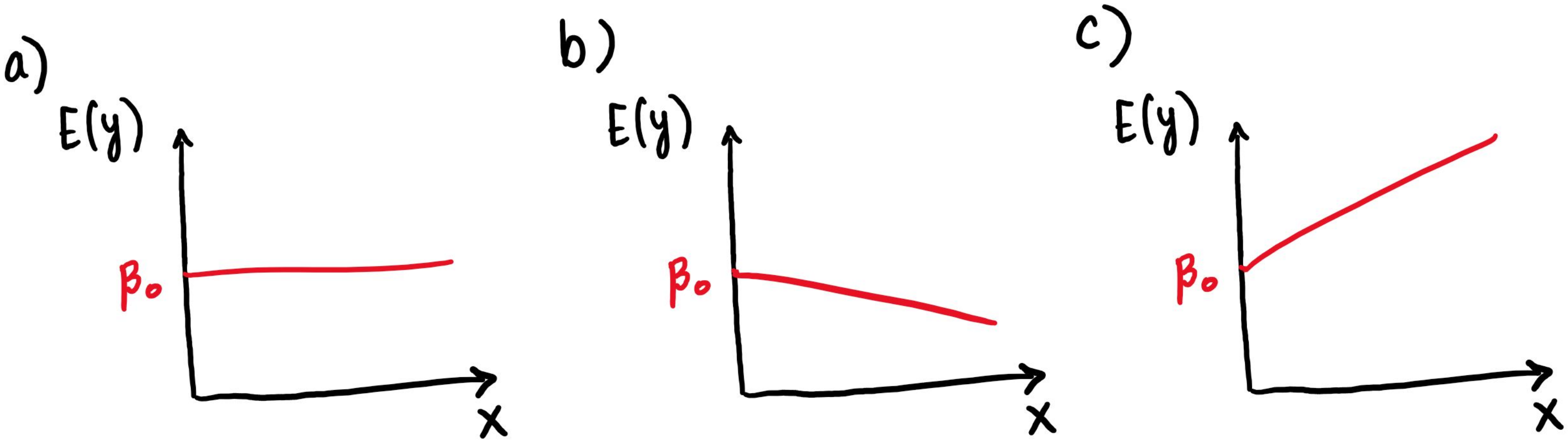

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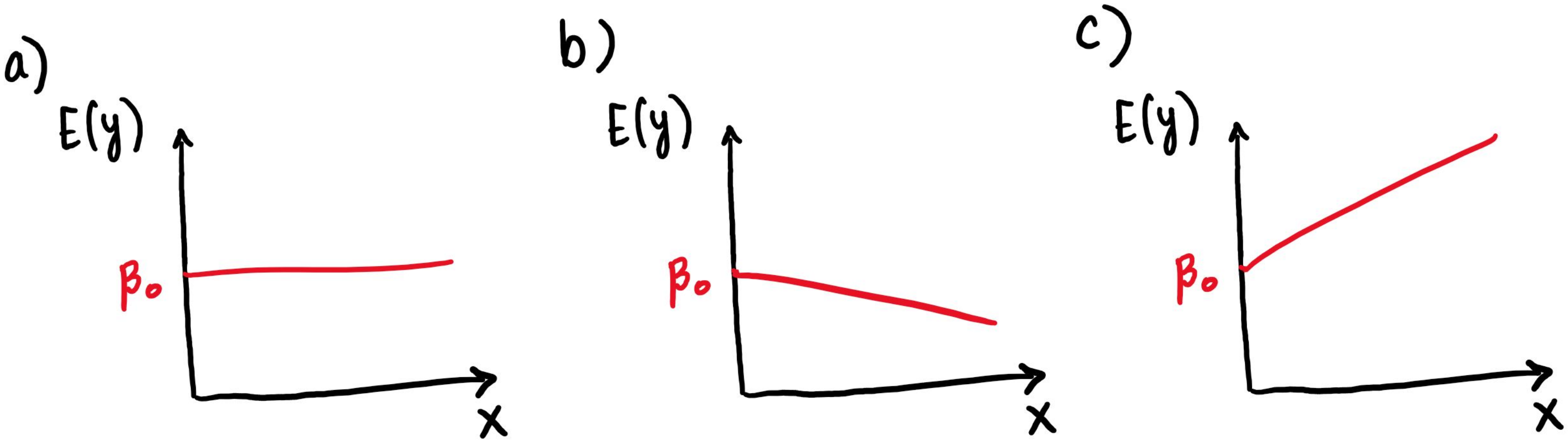

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Linear Regression for a Sample

- $E(y) = \beta_0+\beta_1 x$
- $\hat{y} = b_0+b_1 x$

- **$\hat{y}$ (y-hat)** = estimator of $E(y)$

Data (with hours of study)

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- Does the plot of hours of study vs test scores show some relationship?
- If yes, then we can fit a linear regression line to predict future scores
- If no, then the linear regression model might be useless

$$\hat{y} = b_0 + b_1 x$$

$$\hat{y}_i = b_0 + b_1 x_i$$

$$b_1 = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2} \quad , \quad b_0 = \hat{y}_i - b_1 x_i$$

$\bar{x}$ = mean of independent variable
x_i = value of independent variable
$\bar{y}$ = mean of dependent variable
y_i = value of dependent variable.

Best-fit Line

$$\hat{y}_i = 3.2 + 0.95x_i$$

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- For every 1 hour increase in study time, you expect to get an increase in score by 0.95 points
- If you don't study, x = 0, then you will end up with a score of 3.2 (practical?)

Least Square Criterion

$$\min \Sigma (y_i - \hat{y}_i)^2$$

y_i = observed value of test score

$\hat{y}_i$= predicted value of test score

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- Goal is to minimize the sum of the squared differences between the actual value of dependent variable and the estimated (predicted) value
- We can find this sum and compare with the SSE of Model 1 to see how much linear regression minimizes the distance

SSR & SST

- SSE = sum of squared errors
- SST = sum of squared total
    - Equals to SSE when no independent variable is used in model
- SSR = sum of squared regression
    - SSR = SST - SSE

Coefficient of Determination (r^2)

$r^2 = SSR / SST$

- Proportion of the variance in the dependent variable that is predictable from the independent variable

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- **GOOD FIT!**

Mean Square Error (MSE)

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- MSE is an estimator of the variance of error, ε
- MSE = SSE / (n-degrees of freedom)

- SSE = sum of squared errors
- n = number of observations (data points)
- Degrees of freedom = how many parameters are being used in the linear model

Mean Square Error (MSE)

- MSE is an estimator of the variance of error, ε
- MSE = SSE / (n-degrees of freedom)
- MSE = SSS / (n-2) (because of two parameters being used: b_0,b_1)

Standard Error

- Similar to standard deviation, measure of actual spread of data from the best-fit line

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- What does it mean?

Standard Error

- Similar to standard deviation, measure of actual spread of data from the best-fit line
- Standard Error = sqrt (MSE)
- For our model, Standard Error = 0.42
- So the average distance of the observed test scores from the fitted line is 0.42 points.

Multiple Linear Regression

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- We can’t just always be using one variable to predict the behaviour of another variable
- For our example, test scores could be dependent upon the hours you study, your average grade in the class, and if you had breakfast that morning

Multiple Linear Regression

- Extension of Simple Linear Regression which models a one-to-one relationship

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- Multiple Linear Regression models a many-to-one linear relationship

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- Extension of Simple Linear Regression which models a one-to-one relationship
- Multiple Linear Regression models a many-to-one linear relationship
    - You can have multiple independent variables and a single dependent variable (x_1, x_2, etc. and y)

Things to Consider (Multiple LR)

- Having more independent variable will always increase the r^2 value

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- Having more independent variable will always increase the r^2 value
  - Because you are explaining more and more variation in the dependent variable using multiple independent variables
- But that does not mean that your regression is better or will predict with more accuracy!

Overfitting

- Occurs when the model is too complex

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- Occurs when the model is too complex
    - By that we mean that the model has too many independent variables being used
- Here, instead of explaining the relationship between the variables, the model starts to predict the random error in the data

Multicollinearity

- If you use too many independent variables, there is a possibility that some of those variables depend on each other too

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- If you use too many independent variables, there is a possibility that some of those variables depend on each other too
    - Think in terms of derived attributes (age and age group, BMI and height etc.)

Gradient Descent

$$\hat{y}_i = b_0 + b_1 x_i$$

$$y = h(x, \theta)$$ – hypothesis function.

$$\hat{y} = h(x, \theta) + \varepsilon$$

$$\hat{y} = \theta_0 + \theta_1 x$$

$$\boxed{\hat{y} = \theta^T \bar{x}}$$

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \Rightarrow \theta^T = [\theta_0, \theta_1]$$

$$\bar{x} = [1 \ x]$$

Gradient Descent

Cost Function

$$J(x,\theta,y) = \frac{1}{2m}\sum_{i=1}^{m}(\hat{y}_i - y_i)^2$$

where m = no. of data points.

$J(x,\theta,y)$ tells me how much I am penalized when I don't predict well.

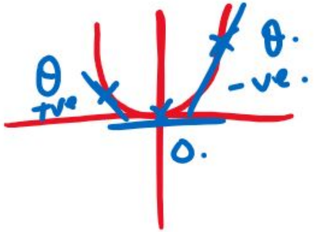
$$\frac{1}{2m}\sum_{i=1}^{m}(\underbrace{\hat{y}_i - y_i}_{})^2$$

$(\text{big difference.})^2 \Rightarrow$ big cost.

Gradient Descent

Gradient Descent.

- Algorithm that computes gradient of the cost
- Use it to change the θ^T.



$$\theta^+ = \theta^- - \alpha \nabla_\theta J$$

θ^+ = new value of θ

θ^- = old value of θ

α = step value, < 1.

Gradient Descent

Gradient = Derivative

$$J(x,\theta,y) = \frac{1}{2m}\sum_{i=1}^{m}(\hat{y}_i - y_i)^2$$

$$\frac{dJ}{d\theta} = \frac{1}{\cancel{2}m}\cancel{2}\left(\sum_{i=1}^{m}(\hat{y}_i - y_i)^2\right)\frac{d}{d\theta}\left(\theta^T\bar{x}_i - y_i\right)$$

derivative of outside

derivative of inside

$$\nabla J = \frac{1}{m}\sum_{i=1}^{m}(\hat{y}_i - y_i)^2\,\bar{x}_i$$

∇J can be plugged into $\theta^+ = \theta^- - \alpha\nabla_\theta J$, to get new θ^+ value.

Gradient Descent (Steps)

1. Take the derivative of the cost function for each parameter
2. Pick random values for the parameters
3. Plug the values into the derivatives
4. Calculate the step size
5. Calculate the new parameters
6. Repeat 3. Until step size = 0

Summary

- Linear Regression has 4 key assumptions:
    - Linear relationship (matrix correlation plot)
    - Multivariate normality (histogram)
    - No or little multicollinearity
    - Homoscedasticity (residuals are equal across the best fit line)

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- Linear Regression has 4 key assumptions:
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    - Multivariate normality (histogram)
    - No or little multicollinearity
    - Homoscedasticity (residuals are equal across the best fit line)
- Preprocess data first
    - Remove or impute outliers and missing values
    - Transform, standardize data if needed

Summary

- Simple Linear Regression can be done using the OLS and the Gradient Descent method
    - Each method tries to optimize the slope and intercept parameters
    - OLS tries to minimize the SSE
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Summary

- Simple Linear Regression can be done using the OLS and the Gradient Descent method
    - Each method tries to optimize the slope and intercept parameters
    - OLS tries to minimize the SSE
    - GD minimizes the cost function
- Linear Regression is applied using continuous data, to predict continuous data
- You can compare R-squared values, split the data into training and testing sets, and conduct k-fold validation to assess the model performance