Homework 3 Part 1

Question 1

(1) $E[(y - \hat{y})^2] = E[(f(x) + \varepsilon - \hat{f}(x))^2]$

(2) $\qquad = E[(f(x) + \varepsilon - \hat{f}(x) + E[\hat{f}(x)] - E[\hat{f}(x)])^2]$

(3) $\qquad = E[((f(x) - E[\hat{f}(x)]) + (E[\hat{f}(x)] - \hat{f}(x)) + \varepsilon)^2]$

(4) $\qquad = E[(f(x) - E[\hat{f}(x)])^2] + E[(E[\hat{f}(x)] - \hat{f}(x))^2]$

$\qquad + E[\varepsilon^2] + 2E[\varepsilon(f(x) - E[\hat{f}(x)])]$

$\qquad + 2E[\varepsilon(E[\hat{f}(x)] - \hat{f}(x))] + 2E[(f(x) - E[\hat{f}(x)])$

$\qquad \cdot (E[\hat{f}(x)] - \hat{f}(x))]$

(5) $\qquad = (f(x) - E[\hat{f}(x)])^2 + Var[\hat{f}(x)] + Var[\varepsilon]$

$\qquad + 2E[\varepsilon]E[f(x) - E[\hat{f}(x)]] + 2E[\varepsilon]E[E[\hat{f}(x)] - \hat{f}(x)]$

$\qquad + 2E[(f(x) - E[\hat{f}(x)])(E[\hat{f}(x)] - \hat{f}(x))]$

(6) $\qquad = Bias[\hat{f}(x)]^2 + Var[\hat{f}(x)] + \sigma^2$ ✓

Notes:

(1) $y = f(x) + \varepsilon$ and $\hat{y} = \hat{f}(x)$

(2) Add and subtract $E[\hat{f}(x)]$

(3) Group variables

(4) Expand square (Note: $E[a+b] = E[a] + E[b]$)

(5) $E[(f(x) - E[\hat{f}(x)])^2] = (f(x) - E[\hat{f}(x)])^2$
($f(x)$ is deterministic and $E[\hat{f}(x)]$ is a constant)

$E[(E[\hat{f}(x)] - \hat{f}(x))^2] = E[(\hat{f}(x) - E[\hat{f}(x)])^2]$
$= Var[\hat{f}(x)]$

$E[\varepsilon^2] = Var[\varepsilon]$ since $E[\varepsilon] = 0$

(6) $Var[\varepsilon] = \sigma^2$

$2E[\varepsilon] E[f(x) - E[\hat{f}(x)]] = 2E[\varepsilon] E[E[\hat{f}(x)] - \hat{f}(x)] = 0$
since $E[\varepsilon] = 0$

$2E[[f(x) - E[\hat{f}(x)]](E[\hat{f}(x)] - \hat{f}(x))]$
$= 2(E[f(x) E[\hat{f}(x)] - f(x)\hat{f}(x) - E[\hat{f}(x)]^2 + \hat{f}(x)E[\hat{f}(x)]])$
$= 2(f(x)E[\hat{f}(x)] - f(x)E[\hat{f}(x)] - E[\hat{f}(x)]^2 + E[\hat{f}(x)]^2)$
$= 2(0)$
$= 0$