DP for Hamming weight Small Space Source

Prithviraj Ghosh

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1 Report

We have a Small Space Source which is the uniform distribution over the width $n+1$ where any node s representing the hamming weight in layer i is connected with node s and node $s+1$ in layer $i+1$ each representing not accepting and accepting w_{i+1} respectively.

We define $\tau(i, a, s)$ as the minimum capacity with the accepting probability over all possible binary strings is at least a and present state in the small space source is s at layer i for weights $w_1, w_2, \ldots, w_i$.

Claim

$$\tau(i, a, s) = \min_{\substack{\alpha, \beta \in [0,1] \\ \alpha p_1 + \beta p_2 = a}} \max \begin{cases} \tau(i-1, \alpha, s) \\ \tau(i-1, \beta, s-1) + w_i \end{cases}$$

where p_1 & p_2 are the probabilities associated with the edge from s and $s-1$ in layer $i-1$ of the small space source respectively.

Proof

Fix any $\alpha \in [0, 1]$ and automatically a β is fixed.

Call $B = \max\{\tau(i-1, \alpha, s),\ \tau(i-1, \beta, s-1) + w_i\}$. There is at least α probability with weights $w_1, w_2, \ldots, w_{i-1}$ with present state s and capacity B. Also there is at least β probability with weights $w_1, w_2, \ldots, w_{i-1}$ with present state $s-1$ and capacity $B - w_i$. Since $\alpha p_1 + \beta p_2 = a$ there is at least a probability with weights $w_1, w_2, \ldots, w_i$ with present state s and capacity B as all solutions are disjoint. So,

$$\tau(i, a, s) \leq \min_{\substack{\alpha, \beta \in [0,1] \\ \alpha p_1 + \beta p_2 = a}} \max \begin{cases} \tau(i-1, \alpha, s) \\ \tau(i-1, \beta, s-1) + w_i \end{cases}$$

Calling $\tau(i,a,s) = C$. Let α be such that αp_1 be the fraction of a which does not include w_i and β accordingly. So, $\tau(i-1,\alpha,s) \leq C$ and $\tau(i-1,\beta,s-1) \leq C-w_i$. Hence $\max\{\tau(i-1,\alpha,s),\ \tau(i-1,\beta,s-1)+w_i\} \leq C$

$$\tau(i,a,s) \geq \min_{\substack{\alpha,\beta\in[0,1] \\ \alpha p_1+\beta p_2=a}} \max \left\{ \begin{array}{l} \tau(i-1,\alpha,s) \\ \tau(i-1,\beta,s-1)+w_i \end{array} \right.$$

Giving the equality.

Solution

The solution probability $a^* = max\{a : \tau(n,a,s) \leq C \forall s \in [n+1]\}$