Chapter 6 - Integration

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Identities to learn

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x = e^x + C$$

$$\int \frac{1}{x} = ln|x| + C$$

$$\int cos(x) = sinx(x) + C$$

$$\int sin(x) = -cos(x) + C$$

$$\int sec^2(x) = tan(x) + C$$

$$\int cosec(x)cot(x) = -cosec(x) + C$$

$$\int cosec^2(x) = -cot(x) + C$$

$$\int sec(x)tan(x) = sec(x) + C$$

$$\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$$

$$\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} ln|ax+b| + C$$

$$\int cos(ax+b)dx = \frac{1}{a}sin(ax+b)+C$$

$$\int sin(ax+b)dx = -\frac{1}{a}cos(ax+b)+C$$

$$\int sec^2(ax+b)dx = \frac{1}{a}tan(ax+b)+C$$

$$\int cosec(ax+b)cot(ax+b)dx = -\frac{1}{a}cosec(ax+b)+C$$

$$\int cosec(ax+b)^2 dx = -\frac{1}{a}cot(ax+b)+C$$

$$\int sec(ax+b)tan(ax+b)dx = \frac{1}{a}sec(ax+b)+C$$

$$\int tan(x)dx = ln|sec(x)| + C$$

$$\int sec(x)dx = ln|sec(x)+tan(x)|+C$$

$$\int cot(x)dx = ln|sin(x)| + C$$

$$\int cosec(x)dx = -ln|cosec(x)+cot(x)|+C$$

Simple intergration

You can already do simple integration from earlier core modules. Here's a reminder.

Example - simple integration

Find the integral of $\int 2cosx + \frac{3}{x} - \sqrt{x}$.

Integrate each term seperately:

$$\int 2cosx = 2sinx$$

$$\int \frac{3}{x} = 3ln|x|$$

$$\begin{aligned}\int \sqrt{x} &= \int x^{\frac{1}{2}} \\ &= \frac{2}{3}x^{\frac{3}{2}}\end{aligned}$$

Then simply add the terms together:

$$\int 2cosx + \frac{3}{x} - \sqrt{x}dx = \underline{\mathbf{2sinx + 3ln|x| - \frac{2}{3}x^{\frac{3}{2}} + C}}$$

The reverse chain rule

You can integrate functinos such as $f(ax+b)$ using the reverse of the chain rule.

The general rule is as follows:

$$\int f'(ax+b)dx = \frac{1}{a}f(ax+b) + C$$

Example - reverse chain rule

$$\int cos(2x+3)dx = \underline{\mathbf{\frac{1}{2}sin(2x+3) + C}}$$

Using trigonometric identities in integration

You can use trigonometric identities in order to rearrange equations into a form that is easier to integrate.

Example - trigonometric identities in integration

Find $\int tan^2 x dx$

Use $sec^2 x = 1 + tan^2 x$:

$$\begin{aligned} \int tan^2 x dx &= \int (sec^2 x - 1) dx \\ &= \int sec^2 x dx - \int 1 dx \\ &= \underline{\mathbf{tanx - x + C}} \end{aligned}$$

Intergration with partial fractions

You can use partial fractions to intergrate expressions. You will usualy end up with a load of natural logs that you must then combine using the laws of logs.

Example - integrating with partial fractions

Find the integral of $\int \frac{x-5}{(x+1)(x-2)} dx$

First, we need to convert the experssion into a partial fraction:

$$\frac{x-5}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$
$$x - 5 = A(x-2) + B(x+1)$$

Let $x = -1$:

$$-6 = A(-3) \therefore A = 2$$

Let $x = 2$:

$$-3 = B(3) \therefore B = -1$$

So

$$\int \frac{x-5}{(x+1)(x-2)}dx = \int \left(\frac{2}{x+1} - \frac{1}{x-2}\right) dx$$
$$= 2ln|x+1| - ln|x-2| + C$$
$$= \underline{\mathbf{ln}\left|\frac{\mathbf{(x+1)^2}}{\mathbf{x-2}}\right| + \mathbf{C}}$$

Standard patterns

Some expressions fit into general patterns that can be integrated. For these, you must try fitting them into a pattern, and then differentiate them afterwards to check that they fit. Adjust any constants.

There are two general patterns to remember when integrating:

1. In order to integrate expressions of the form $\int \mathbf{k}\frac{\mathbf{f'(x)}}{\mathbf{f(x)}}\mathbf{dx}$ try $\mathbf{ln}|\mathbf{f(x)}|$.
2. In order to integrate expressions of the form $\int \mathbf{kf'(x)[f(x)]^n dx}$ try $\mathbf{[f(x)]^{n+1}}$.

Example - integrating using general pattern #1

Find the integral of $\int \frac{cosec^2x}{(2+cotx)^3}dx$.

Let $I = \int \frac{cosec^2x}{(2+cotx)^3}dx$.

Let $y = (2+cotx)^{-2}$.

N.b. Here, we are picking the integral of what would be $f(x)$ in the general pattern. We then differentiate it to check that it fits.

$$\frac{dy}{dx} = -2(2+cotx)^{-3} \times (-cosec^2x)$$
$$= 2(2+cotx)^{-3}cosec^2x$$
$$= 2\frac{cosec^2x}{(2+cotx)^3}$$

The expression must be multiplied by a factor of $\frac{1}{2}$ in order to obtain the correct answer.

$$I = \underline{\frac{\mathbf{1}}{\mathbf{2}}\mathbf{(2+cotx)^{-2}} + \mathbf{C}}$$

Example - integrating using general pattern #2

Find the integral of $\int 5tan(x)sec^4(x)dx$**.**

Let $I = \int 5tan(x)sec^4(x)dx$.

Let $y = sec^4(x)$.

$$\begin{aligned}\frac{dy}{dx} &= 4sec^3(x) \times sec(x)tan(x)\\ &= 4sec^4(x)tan(x)\end{aligned}$$

Since the value of $\frac{dy}{dx}$ is $\frac{4}{5}$ times the required answer, we must multiply it by $\frac{5}{4}$.

$$I = \underline{\mathbf{\frac{5}{4}sec^4(x) + C}}$$

Integration by substitution

It is possible to simplify an integral by changing an expression for a variable (similar to the chain and product rule in differentiation).

The trick is to choose a substitution that is able to simplify a large part of the equation.

Example - integration by substitution #1

Use the substitution $u = 2x + 5$ **to find** $\int x\sqrt{(2x+5)}dx$

Let $I = \int x\sqrt{(2x+5)}dx$

Let $u = 2x + 5$

Differentiate u to get $\frac{du}{dx}$

$$\frac{du}{dx} = 2$$

Rearrange to get an expression for dx

$$dx = \frac{du}{2}$$

Find x in terms of u

$$x = \frac{u-5}{2}$$

Replace the x terms in the equation with u and integrate the expression:

$$\begin{aligned} I &= \int \left(\frac{u-5}{2}\right) u^{\frac{1}{2}} \frac{du}{2} \\ &= \int \frac{1}{4}(u-5)u^{\frac{1}{2}} du \\ &= \int \frac{1}{4}(u^{\frac{3}{2}} - 5u^{\frac{1}{2}}) du \\ &= \frac{1}{4}\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{5u^{\frac{3}{2}}}{4 \times \frac{3}{2}} + C \\ &= \frac{u^{\frac{5}{2}}}{10} - \frac{5u^{\frac{3}{2}}}{6} + C \end{aligned}$$

Now we can sub in $u = 2x + 5$ to get the final answer:

$$I = \underline{\mathbf{\frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + C}}$$

Example - integration by substitution #2

Use integration by substitution to find $\int 6xe^{x^2} dx$

Let $I = \int 6xe^{x^2} dx$

Let $u = x^2$

Differentiate u:

$$\frac{du}{dx} = 2x$$

Replace x with u:

$$\begin{aligned} I &= \int 3e^u du \\ &= 3e^u + C \end{aligned}$$

Then sub in $u = x^2$

$$\underline{\mathbf{I = 3e^{x^2} + C}}$$

Integration by subsitution and implicit differentiaton

Sometimes you must differentiate the term that you are substituting with implicit differentiation. This is usually the case when u is raised to a power other than 1.

Example - integration by substitution using implicit differentiation

Use the substitution $u^2 = 2x + 5$ **to find** $\int x\sqrt{2x+5}$

Let $I = \int x\sqrt{2x+5}$

Let $u^2 = 2x + 5$

We can use implicit differentiation on u^2 to find an expression for du.

$$2\frac{du}{dx} = 2$$

We can rearrange $u^2 = 2x + 5$ to get x in terms of u:

$$x = \frac{u^2 - 5}{2}$$

We can also get u in terms of x:

$$u = \sqrt{2x+5}$$

So

$$\begin{aligned} I &= \int \left(\frac{u^2-5}{2}\right) u \times u du \\ &= \int \left(\frac{u^4}{2} - \frac{5u^2}{2}\right) du \\ &= \frac{u^5}{10} - \frac{5u^3}{6} + C \end{aligned}$$

Replacing u with x:

$$I = \underline{\mathbf{\frac{(2x+5)^{\frac{5}{2}}}{10} - \frac{5(2x+5)^{\frac{3}{2}}}{6} + C}}$$

Example - integrating by substitution with limits

Find the integral of $\int_0^2 x(x+1)^3 dx$

Let $I = \int_0^2 x(x+1)^3 dx$

Let $u = x + 1$

Differentiate u in order to find an expression for dx:

$$\frac{du}{dx} = 1$$
$$\therefore dx = du$$

We can also find x in terms of u:

$$x = u - 1$$

Because we have subbed in u for x, we must change the limits of the integral to ensure we get the correct answer. To do this, find the value of u where x is each of the limits:

x	u
2	3
0	1

So:

$$\begin{aligned} I &= \int_0^2 x(x+1)^3 dx \\ &= \int_1^3 (u-1)u^3 du \\ &= \int_1^3 (u^4 - u^3) du \\ &= \left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^3 \\ &= \left(\frac{3^5}{5} - \frac{3^4}{4}\right) - \left(\frac{1^5}{5} - \frac{1^4}{4}\right) \\ &= 48.4 - 20 \\ &= \underline{\mathbf{28.4}} \end{aligned}$$

Integration by parts

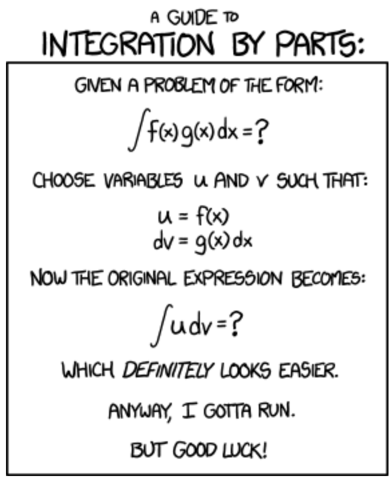
The formulea for integration by parts is as follows:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

The derivation of the formulea is in the textbook. It's not needed for the exam.

In general, let u equal any x^n term, unless there is a $ln(x)$ term, in which case, let that be u instead.

Even xkcd think it's useful:



Example - integration by parts

Find $\int xcos(x)dx$

Let $I = \int xcos(x)dx$

Let $u = x \therefore \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = cos(x) \therefore v = sin(x)$

Now we can use the integration by parts formulea:

$$I = xsin(x) - \int sin(x) \times 1dx$$
$$= \underline{\mathbf{xsin(x) + cos(x) + C}}$$

Example - integration by parts using $u = ln(x)$

Find $\int x^2 ln(x) dx$

Let $I = \int x^2 ln(x) dx$

Let $u = ln(x) \therefore \frac{du}{dx} = \frac{1}{x}$

Let $\frac{dv}{dx} = x^2 \therefore v = \frac{x^3}{3}$

Now lets apply the formulea:

$$\begin{aligned} I &= \frac{x^3}{3} ln(x) - \int \frac{x^3}{3} \times \frac{1}{x} dx \\ &= \frac{x^3}{3} ln(x) - \int \frac{x^2}{3} dx \\ &= \underline{\mathbf{\frac{x^3}{3} ln(x) - \frac{x^3}{9} + C}} \end{aligned}$$

Sometimes, it is nececarry to integrate use integration by parts again in order to differentiate the expression obtained by integration by parts the first time.

Example - integration by parts (twice)

Find $\int x^2 e^x dx$

Let $I = \int x^2 e^x dx$

We can assign the following variables:

$$u = x^2 \therefore \frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = e^x \therefore v = e^x$$

We can apply the integration by parts formula ($\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$):

$$I = x^2 e^x - \int 2xe^x dx$$

In order to integrate $\int 2xe^x dx$, we must apply the integration by parts formulea again, using another set of variables:

$$u = 2x \therefore \frac{du}{dx} = 2$$

$$\frac{dv}{dx} = e^x \therefore v = e^x$$

Applying the formula for integration by parts, we get:

$$\int 2xe^x dx = 2xe^x - \int 2e^x dx$$

Therefore:

$$\begin{aligned} I &= x^2e^x - \left[2xe^x - \int 2e^x dx\right] \\ &= x^2e^x - 2xe^x + \int 2e^x dx \\ &= \underline{\mathbf{x^2e^x - 2xe^x + 2e^x + C}} \end{aligned}$$

Example - integration by parts with limits

Find $\int_1^2 ln(x)dx$ leaving your answer in terms of natural logarithms

First we must choose our u and v variables:

$$u = ln(x) \therefore \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = 1 \therefore v = x$$

We can treat dx as $1 \times dx$ so $\frac{dv}{dx} = 1$

Now we can apply the integration by parts formulea:

$$\begin{aligned} I &= (xln(x))_1^2 - \int_1^2 x\frac{1}{x}dx \\ &= (xln(x))_1^2 - \int_1^2 1dx \\ &= (xln(x))_1^2 - [x]_1^2 \\ &= 2ln(2) - ln(1) - 2 + 1 \\ &= \underline{\mathbf{2ln(2) - 1}} \end{aligned}$$

Integration, the trapezium rule and percentage error

You may be asked to find the exact integral of an expression, and compare it with an approximation of the integral found using the trapezium rule.

The trapezium rule is as follows:

$$\int_a^b y \approx \frac{1}{2}h[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

Where $h = \frac{b-a}{n}$

Example - integration with the trapezium rule

For the integral $I = \int_0^{\frac{\pi}{3}} sec(x)dx$

1. Find the exact value of I
2. Use the trapezium rule with four strips to find an approximation for the value of I
3. Find the percentage error for this approximation

1.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{3}} sec(x)dx \\ &= [ln|sec(x) + tan(x)|]_0^{\frac{\pi}{3}} \\ &= (ln|2 + \sqrt{3}|) - (ln|1 + 0|) \\ &= (ln|2 + \sqrt{3}|) \end{aligned}$$

2.

x	y
0	1
$\frac{\pi}{12}$	1.035
$\frac{\pi}{6}$	1.155
$\frac{\pi}{4}$	1.414
$\frac{\pi}{3}$	2

Therefore:

$$\begin{aligned} I &\approx \frac{1}{2}\frac{\pi}{3\times 4}[1+2(1.035+1.155+1.414)+2] \\ &\approx \frac{\pi}{24}[10.208] \\ &\approx 1.34 \end{aligned}$$

3. Percentage error:

$$\frac{1.34 - ln|2+\sqrt{3}|}{ln|2+\sqrt{3}|} \times 100 = 1.5\%$$

The percentage error formulea is: $\frac{two-one}{one} \times 100$ where *one* and *two* are the values to be compared.

Integration to find volumes

In order to find the volume of the shape produced if a curve was rotated 360° we must use the formula:

$$Volume = \pi \int_a^b y^2 dx$$

Example - finding the volume of a curve rotated through 360°

Find the volume of the solid formed when the region bounded by the curve $y = sin(2x)$ between the points $x = 0$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis.

$$\begin{aligned} Volume &= \pi \int_0^{\frac{\pi}{2}} sin(2x)^2 dx \\ &= \pi \int_0^{\frac{\pi}{2}} sin^2(2x) dx \\ &= \pi \int_0^{\frac{\pi}{2}} (1 - cos(4x) dx \\ &= \pi \left[\frac{1}{2}x - \frac{1}{8}sin(4x)\right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\left(\frac{\pi}{4} - 0\right) - (0)\right] \\ &= \underline{\boldsymbol{\frac{\pi^2}{4}}} \end{aligned}$$

Integration of parametric equations

Integrating parametric equations is similar to integrating by substitution. Use the following:

$$Area = \int y dx = \int y \frac{dx}{dt} dt$$

Then you can integrate in terms of t.

A common mistake is to forget to change the limits, don't forget!

Example - integrating parametric equations

A curve is described by:

$$x = t(1+t)$$
$$y = \frac{1}{1+t}$$

Where t is a parameter greater or equal to 0, and the curve is bounded by the lines $x = 0$ and $x = 2$.

Find the exact area bounded by the curve and the exact volume formed when the curve is rotated through 2π radians about the x-axis.

We know that the area $= \int_0^2 y dx = \int_a^b y \frac{dx}{dt} dt$

In order to find a and b we must find the values of t when $x = 0$ and $x = 2$:

$$x = t(1+t)$$
$$x = 0 \therefore t = 0 || t = -1$$
$$x = 2 \therefore t = 1 || t = -2$$

However, t cannot be less than zero, so when $x = 0$, $t = 0$ and when $x = 2$, $t = 1$.

We also need to find $\frac{dx}{dt}$ before we can use the formula:

$$x = t(t+1) \therefore \frac{dx}{dt} = 1 + 2t$$

Now we can use $\int_a^b y \frac{dx}{dt} dt$:

$$\begin{aligned}Area &= \int y\frac{dx}{dt}dt\\ &= \int_0^1 \frac{1}{1+t}(1+2t)dt\\ &= \int_0^1 \frac{1+2t}{t+1}\end{aligned}$$

Now we can use algebraic division to simplify $\frac{1+2t}{t+1}$:

$$\begin{array}{r|l} & \quad\quad 2\\ \hline t+1) & \quad 2t+1\\ & -2t-2\\ \hline & \quad\quad -1\end{array}$$

$$\begin{aligned}Area &= \int y\frac{dx}{dt}dt\\ &= \int_0^1 \frac{1}{1+t}(1+2t)dt\\ &= \int_0^1 \frac{1+2t}{t+1}\\ &= \int_0^1 \left(2-\frac{1}{1+t}\right)dt\\ &= [2t - ln|1+t|]_0^1\\ &= (2-ln(2)) - (0-ln(1))\\ Area &= \underline{\mathbf{2-ln(2)}}\end{aligned}$$

In order to find the volume of the solid, we must use the formula to find a volume.

$$\begin{aligned}Volume = \pi\int y^2dx &= \pi\int y^2\frac{dx}{dt}dt\\ &= \pi\int_0^1 \frac{1}{(1+t)^2}(1+2t)dt\\ &= \pi\int_0^1 \frac{1+2t}{(1+t)^2}dt\end{aligned}$$

Now we can use partial fractions to simplify $\frac{1+2t}{(1+t)^2}$:

$$\begin{aligned}
\frac{1+2t}{(1+t)^2} &\equiv \frac{A}{(1+t)^2} + \frac{B}{1+t} \\
1+2t &= A + B(1+t) \\
\therefore B &= 2 \\
\therefore A &= -1 \\
\therefore \frac{1+2t}{(1+t)^2} &\equiv \frac{2}{1+t} - \frac{1}{(1+t)^2}
\end{aligned}$$

We can sub that back in:

$$\begin{aligned}
Volume = \pi \int y^2 dx &= \pi \int y^2 \frac{dx}{dt} dt \\
&= \pi \int_0^1 \frac{1}{(1+t)^2}(1+2t)dt \\
&= \pi \int_0^1 \frac{1+2t}{(1+t)^2} dt \\
&= \pi \int_0^1 \frac{2}{1+t} - \frac{1}{(1+t)^2} \\
&= \pi \left[2ln|1+t| + \frac{1}{1+t} \right]_0^1 \\
&= \pi \left[\left(2ln(2) + \frac{1}{2} \right) - (0+1) \right] \\
&= \underline{\pi \left(\mathbf{2ln(2)} - \frac{\mathbf{1}}{\mathbf{2}} \right)}
\end{aligned}$$

Solving differential equations using integration

You can use the *seperation of variables* in order to solve differential equations. The general formula for this is as follows:

When you're given a differential equation of the form

$$\frac{dy}{dx} = f(x)g(y)$$

You can write it as:

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

Example - solving differential equations with integration

Find the solution to the differential equation $(1+x^2)\frac{dy}{dx} = xtan(y)$

First, we need to write the equation in the form $\frac{dy}{dx} = f(x)g(x)$

$$\frac{dy}{dx} = \frac{x}{1+x^2}tan(y)$$

Now we can apply the formula $\int \frac{1}{g(y)}dy = \int f(x)dx$

$$\int cot(y)dy = \int \frac{x}{1+x^2}dx$$

Which gives

When integrating $\frac{x}{1+x^2}$ use standard pattern #1

$$\begin{aligned} ln|sin(y)| &= \frac{1}{2}ln|1+x^2| + C \\ ln|sin(y)| &= ln|\sqrt{1+x^2}| + ln|k| \\ ln|sin(y)| &= ln|k\sqrt{1+x^2}| \\ sin(y) &= \underline{\mathbf{k\sqrt{1+x^2}}} \end{aligned}$$

Example - solving differential equations with integration using limits

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{-3(y-2)}{(2x+1)(x+2)}$

Given that $x = 1$ **and** $y = 4$**.**

$$\int \frac{1}{y-2}dy = \int \frac{-3}{(2x+1)(x+2)}dx$$

Now we can split the right side of the equation into partial fractions so we can integrate it.

$$\begin{aligned} \frac{-3}{(2x+1)(x+2)} &\equiv \frac{A}{2x+1} + \frac{B}{x+2} \\ 3 &\equiv A(x+2) + B(2x+1) \\ \therefore A &= -2 \\ \therefore B &= 1 \end{aligned}$$

Therefore:

$$\begin{aligned}
\int \frac{1}{y-2}dy &= \int \left(\frac{1}{x+2} - \frac{2}{2x+1}\right) dx \\
ln|y-2| &= ln|x+2| - \frac{2}{2}ln|2x+1| + ln|k| \\
ln|y-2| &= ln|\frac{k(x+2)}{2x+1}| \\
y-2 &= k\frac{(x+2)}{2x+1}
\end{aligned}$$

Let $x = 1$ and $y = 4$:

$$\begin{aligned}
2 &= k(\frac{3}{3}) \\
k &= 2
\end{aligned}$$

We can sub $k = 2$ back in to get an equation in the form $y = f(x)$:

$$\begin{aligned}
y-2 &= 2\frac{x+2}{2x+1} \\
y &= 2 + 2\frac{x+2}{2x+1} \\
y &= 2 + \frac{2x+4}{2x+1} \\
y &= 2 + \frac{2x+1}{2x+1} + \frac{3}{2x+1} \\
y &= \underline{\mathbf{3} + \frac{\mathbf{3}}{\mathbf{2x+1}}}
\end{aligned}$$

Interpreting differential equations in context

Example - interpreting questions in context

The rate of a population P of microorganisms at time t is given by $\frac{dP}{dt} = kP$, where k is a positive constant. Given that when $t = 0$ the population is 8, and when $t = 1$ it was 56, find the size of the population when $t = 2$.

$$\frac{dP}{dt} = kP$$
$$\int \frac{1}{P} dP = \int k dt$$
$$ln|P| = kt + C$$

When $t = 0$, $P = 8 \therefore ln|8| = C$

So

$$ln|P| = kt + ln|8|$$
$$ln|\frac{P}{8}| = kt$$

When $t = 1$, $P = 56$

So

$$ln|\frac{56}{8}| = 1(k)$$
$$\therefore k = ln|7|$$

When $t = 2$:

$$ln|\frac{P}{8}| = 2ln|7|$$
$$ln|\frac{P}{8}| = ln|49|$$
$$\frac{P}{8} = 49$$
$$P = 49 \times 8$$
$$P = \underline{\mathbf{392}}$$