In this note we present a characterization of the Continuum Hypothesis which we learned from the book *Problems and Theorems in Classical Set Theory* by Komjath and Totik.

Theorem 1. *The Continuum Hypothesis fails if and only if for every partition of $\mathbb{R}$ into countably many pieces, we can find distinct x, y, u, and v in one piece of the partition such that $x+y=u+v$. (We say that x, y, u, and v are a* non-trivial monochromatic solution to $x+y=u+v$.*)*

Proof. For one direction, assume that the Continuum Hypothesis holds and fix an increasing and continuous sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ such that

- each A_α is a countable subgroup of $(\mathbb{R}, +)$, and
- $\mathbb{R} = \bigcup_{\alpha<\omega_1} A_\alpha$.

By recursion, we build a sequence $\langle c_\alpha : \alpha < \omega_1\rangle$ such that

- $c_\alpha : A_\alpha \to \omega$,
- $\beta < \alpha \Longrightarrow c_\beta = c_\alpha \restriction A_\beta$, and
- A_α does not contain a non-trivial monochromatic (with respect to c_α) solution to $x+y=u+v$.

We start by letting $c_0 : A_0 \to \omega$ be one-to-one, and clearly if δ is a limit ordinal then we may take $c_\delta = \bigcup_{\alpha<\delta} c_\alpha$. Thus, assume $\alpha = \beta+1$ and we have defined suitable $c_\beta : A_\beta \to \omega$. We define c_α so that it extends c_β and is one-to-one on $A_\alpha \setminus A_\beta$.

Now suppose we are given four distinct x, y, u, and v in A_α with $x+y=u+v$. If three of these are in A_β, then because A_β is a subgroup of $(\mathbb{R},+)$ the fourth is in A_β as well and we are done by properties of c_β. Thus, at least two of these are in $A_\alpha \setminus A_\beta$, and we defined c_α so that they are sent to different values. In any case, c_α takes on at least two values on the set $\{x,y,u,v\}$. To finish, we define $c = \bigcup_{\alpha<\omega_1} c_\alpha$ and so CH implies the existence of a partition of $\mathbb{R}$ into countably many pieces for which there is no non-trivial monochromatic solution to $x+y=u+v$.

For the other direction, assume CH fails and $c : \mathbb{R} \to \omega$. Fix a sequence $\langle r_\alpha : \alpha < \omega_2\rangle$ of distinct reals that are linearly independent over $\mathbb{Q}$, and define $d : [\omega_2]^2 \to \omega$ by

$$d(\alpha,\beta) = c(r_\alpha + r_\beta). \tag{0.1}$$

Claim. There are distinct α, α', β, and β' such that d is constant on $\{\alpha,\alpha'\} \times \{\beta,\beta'\}$.

Proof. Suppose not, and choose disjoint subsets A and B of ω_2 such that $|A| = \omega_1$ and $|B| = \omega_2$. Given a pair $\{\alpha,\alpha'\}$ of ordinals from A and an $n<\omega$, there is at most one $\beta \in B$ for which

$$d(\alpha,\beta) = d(\alpha',\beta) = n, \tag{0.2}$$

because of our assumption. Since $|[A]^2| = \aleph_1 < \aleph_2 = |B|$, we can find a $\beta \in B$ such that $d(\alpha,\beta) \neq d(\alpha',\beta)$ whenever $\alpha \neq \alpha'$ are in A. But this means the function d is one-to-one on $A \times \{\beta\}$, which is impossible as A is uncountable. □

So fix α, α', β, and β' as in the claim, so d is constant on $\{\alpha,\alpha'\} \times \{\beta,\beta'\}$. and define $x = r_\alpha + r_\beta$, $y = r_{\alpha'} + r_{\beta'}$, $u = r_\alpha + r_{\beta'}$, and $v = r_{\alpha'} + r_\beta$. These four are all distinct because our collection of reals is linearly independent, and clearly

$$x+y = r_\alpha + r_{\alpha'} + r_\beta + r_{\beta'} = u+v. \tag{0.3}$$

Finally, if d is constant on $\{\alpha,\alpha'\} \times \{\beta,\beta'\}$ with value n then we have by definition

$$c(x) = c(y) = c(u) = c(v) = n. \tag{0.4}$$

Thus, if $|\mathbb{R}| \geq \aleph_2$ then for every $c : \mathbb{R} \to \omega$ we can find a non-trivial monochromatic solution to $x+y=u+v$. □