ON MINIMAL NON-σ-SCATTERED LINEAR ORDERS

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ABSTRACT. The purpose of this article is to give new constructions of linear orders which are *minimal* with respect to being *non-σ-scattered*. Specifically, we will show that Jensen's principle $\diamondsuit$ implies that there is a minimal Countryman line, answering a question of Baumgartner [5]. We will also construct the first consistent examples of minimal non-σ-scattered linear orders of cardinality greater than $\aleph_1$. In fact this can be achieved at any successor cardinal κ^+, both via forcing constructions and via axiomatic principles which hold in Gödel's Constructible Universe. These linear orders of cardinality κ^+ moreover have the property that their square is the union of κ-many chains.

1. INTRODUCTION

The class $\mathfrak{M}$ of *σ-scattered linear orders* was considered by Galvin as a natural generalization of the classes of countable linear orders and well orders. On the one hand $\mathfrak{M}$ is quite rich, and on the other it is amenable to refined structural analysis. Recall that a linear order is *scattered* if it does not contain a copy of the rational line $(\mathbb{Q}, \leqslant)$ and is *σ-scattered* if it is a union of countably many scattered suborders. Both of these classes include the well orders and are closed under lexicographic sums $\sum_{i \in K} L_i$ and the converse operation $L \mapsto L^*$ which reverses the order on L; in fact Hausdorff [11] showed that the scattered orders form the least class with these closure properties.

The σ-scattered orders form the least class with these closure properties and the additional property of closure under countable unions. In [20], Laver proved Fraïssé's conjecture that the countable linear orders are *well quasi-ordered*: whenever L_i $(i < \infty)$ is a sequence of countable linear orders, there is an $i < j$ such that L_i embeds into L_j. In fact, his proof established the following celebrated result.

Theorem 1.1. *(Laver [20]) The class $\mathfrak{M}$ is well quasi-ordered by the embeddability relation.*

Empirically, $\mathfrak{M}$ is the largest class of linear orders which is immune to set-theoretic independence phenomena. It is therefore natural to study those linear orders which lie just barely outside of $\mathfrak{M}$. In general, given a class $\mathfrak{C}$ of linear orders, we will say that a linear order L is a *minimal* element of $\mathfrak{C}$ if L is in $\mathfrak{C}$ and embeds into all of its suborders which are in $\mathfrak{C}$. In this paper we will investigate those linear orders L which are minimal with respect to not being in $\mathfrak{M}$. More precisely, we will prove that it is consistent that for each infinite cardinal κ, there

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is a linear order of cardinality κ^+ which is minimal with respect to being non-σ-scattered. Previously it was not known if it was consistent to have a minimal non-σ-scattered order of cardinality greater than $\aleph_1$. Moreover, even our construction of a minimal non-σ-scattered order of cardinality $\aleph_1$ is novel and answers a question of Baumgartner [5, p. 275].

Mathematical and historical background. One of the first results on scattered linear orders is the following result of Hausdorff.

Theorem 1.2. *(Hausdorff [11], see also [27]) If κ is a regular cardinal and L is a scattered linear order of cardinality κ, then either κ or κ^* embeds into L.*

While σ-scattered linear orders were not considered until [20], Theorem 1.2 immediately generalizes to the class of σ-scattered linear orders. Since neither ω_1 nor ω_1^* embed into $\mathbb{R}$, it follows that no uncountable set of reals is σ-scattered. For brevity, we will say that a linear order is a *real type* if it is isomorphic to an uncountable suborder of the real line.

The properties of real types are already sensitive to set theory. On one hand, a classical diagonalization argument yields the following result of Dushnik and Miller.

Theorem 1.3. *(Dushnik and Miller [8]) Assume CH. For any uncountable $X \subseteq \mathbb{R}$ there is a $Y \subseteq X$ such that Y^2 does not contain the graph of any uncountable strictly monotone function other than the identity (and hence does not embed into any proper suborder).*

On the other hand, Baumgartner demonstrated that if $X, Y \subseteq \mathbb{R}$ are $\aleph_1$-dense[1] and CH holds, then there is a c.c.c. forcing which makes X and Y order isomorphic [4]. In particular, he showed that there is always a forcing extension in which every two $\aleph_1$-dense sets of reals are isomorphic. This result is now often phased axiomatically as follows.

Theorem 1.4. *(Baumgartner [4]) Assume PFA. Any two $\aleph_1$-dense subsets of $\mathbb{R}$ are isomorphic. In particular, any real type of cardinality $\aleph_1$ is minimal.*

Here the Proper Forcing Axiom (PFA) is a powerful generalization of the Baire Category Theorem. It plays an important role in the broader analysis of non-σ-scattered linear orders as we will see momentarily. More information on PFA in the context of linear orders can be found in [32]; see e.g. [1], [7], [25], [33] for an introduction to PFA and its consequences.

Another class of non-σ-scattered linear orders is provided by the *Aronszajn lines*:[2] uncountable linear orders with the property that they do not contain uncountable suborders which are either separable or scattered. Aronszajn lines were first constructed by Aronszajn and Kurepa (see [15] [34]) in the course of analyzing Souslin's Problem [31]. By Theorem 1.2, they are necessarily non-σ-scattered.

In the 1970s, R. Countryman introduced a class of linear orders now known as *Countryman lines.* These are the uncountable linear orders C such that $C \times C$ is a union of countably many chains. Such orders are necessarily Aronszajn and have the property that no uncountable linear order can embed into both C and C^*. They were first constructed by Shelah [28], with a simplified construction later

[1] A linear order is *κ-dense* if it has no first or last elements and each interval has cardinality κ.

[2] Aronszajn lines are also known as *Specker types.*

being given by Todorcevic [35]. Notice that being Countryman is clearly inherited by uncountable suborders.

Abraham and Shelah proved the analog of Theorem 1.4 for Countryman lines.

Theorem 1.5. *(Abraham and Shelah [2]) Assume PFA. Any Countryman line embeds into all of its uncountable suborders. Moreover, any two regular[3] Countryman lines are either isomorphic or reverse isomorphic.*

The next results give a complete classification of the Aronszajn lines under PFA.

Theorem 1.6. *(Moore [23]) Assume PFA. Every Aronszajn line has a Countryman suborder.*

Theorem 1.7. *(Martinez-Ranero [21]) Assume PFA. The Aronszajn lines are well quasi-ordered by embeddability.*

The next theorem gives a complete characterization of the minimal non-σ-scattered linear orders under PFA$^+$, a strengthening of PFA.

Theorem 1.8. *(Ishiu and Moore [12]) Assume* PFA$^+$*. Every minimal non-σ-scattered linear order is isomorphic to either a set of reals of cardinality $\aleph_1$ or a Countryman line. Furthermore, any non-σ-scattered linear order contains a non-σ-scattered suborder of cardinality $\aleph_1$.*

Since PFA and PFA$^+$ are rather strong assumptions, it is natural to ask what is possible in other models of set theory. While it is reasonable to think that some enumeration principle such as CH or $\diamondsuit$ might allow one to prove an analog of Theorem 1.3 for Aronszajn lines, Baumgartner showed that this is not the case (Baumgartner's construction contained an error which was later corrected by D. Soukup).

Theorem 1.9. *(Baumgartner [5], D. Soukup [30]) Assume $\diamondsuit^+$. There is a Souslin line which embeds into all of its uncountable suborders.*

While Baumgartner's construction produces a minimal Aronszajn line, it should be noted that Souslin lines are necessarily not Countryman. In [5], Baumgartner asked if $\diamondsuit^+$ could be weakened to $\diamondsuit$ in his construction and if his argument could be adapted to construct a minimal Aronszajn line which was not Souslin.

In [24], the third author proved that it is consistent that there are no minimal Aronszajn lines. This was achieved by obtaining a model of CH which also satisfied a certain combinatorial consequence of PFA. That CH held in this model also yielded the following stronger result.

Theorem 1.10. *(Moore [24]) It is consistent (with CH) that ω_1 and ω_1^* are the only minimal uncountable linear orders.*

In [29], D. Soukup adapts this argument to show that the existence of a Souslin line does not imply the existence of a minimal Aronszajn line.

The strategy in [24] was combined with the analysis of [12] to yield the following result.

Theorem 1.11. *(Lamei Ramandi and Moore [19]) If there is a supercompact cardinal, there is a forcing extension in which CH holds and there are no minimal non-σ-scattered linear orders.*

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[3] An Aronszajn line L is *regular* if L is $\aleph_1$-dense and the collection of all countable subsets of L which are closed in the order topology contains a closed and unbounded set in $[L]^\omega$.

On the other hand, Lamei Ramandi has shown that $\diamondsuit$ is consistent with the existence of a minimal non-σ-scattered linear order which is neither a real nor Aronszajn type. In fact he has produced two qualitatively different constructions.

Theorem 1.12. *(Lamei Ramandi [17]) It is consistent with $\diamondsuit$ that there is a minimal non-σ-scattered linear order L with cardinality $\aleph_1$ which is a dense suborder of a Kurepa line.*

Theorem 1.13. *(Lamei Ramandi [16]) It is consistent with $\diamondsuit$ that there is a minimal non-σ-scattered order with the property that every uncountable suborder contains a copy of ω_1.*

Main results. Up to this point though, all consistent examples of minimal non-σ-scattered linear orders are of cardinality $\aleph_1$. In order to state our main result, we need to introduce another definition. A linear order L is *κ^+-Countryman* if L has cardinality κ^+ and L^2 is a union of κ chains.

Theorem 1.14. *Assume $\mathbf{V} = \mathbf{L}$. For each infinite cardinal κ, there is a κ^+-Countryman line which is minimal with respect to being non-σ-scattered.*

The κ^+-Countryman lines are interesting in their own right. Although they have almost exclusively been studied when $\kappa = \aleph_0$ (in which case they are known simply as *Countryman lines*), their remarkable properties readily generalize to the higher cardinal case:

- κ^+-Countryman lines do not contain a copy of κ^+ or its converse.
- If L is κ^+-Countryman and $X \subseteq L$ has cardinality κ^+, then there is a family of pairwise disjoint intervals of X of cardinality κ^+. In particular X has density κ^+.
- If L is κ^+-Countryman, then no linear order of cardinality κ^+ embeds into both L and L^*.

Like many results which use $\mathbf{V} = \mathbf{L}$ as a hypothesis, the construction in Theorem 1.14 factors through a family of combinatorial principles; these will be defined in Section 7 and are (seemingly) slight strengthenings of the principles $\boxed{\diamondsuit}_\kappa$ considered in [3], [26].

Our argument for $\kappa = \aleph_0$ is somewhat simpler and of independent interest as it answers Baumgartner's question mentioned above.

Theorem 1.15. *Assume $\diamondsuit$. There is a Countryman line which embeds into all of its uncountable suborders.*

Organization. Section 2 will contain a review of the basic analysis of trees and linear orders which we will need. In Section 3, we will show that $\diamondsuit$ is sufficient to construct a minimal Countryman line. Section 4 contains the basic analysis of κ^+-Countryman lines for arbitrary infinite cardinals κ. A framework for constructing κ^+-Countryman lines which are minimal with respect to being non-σ-scattered is introduced in Section 5. This framework is then put to use in Sections 6 and 7 where we present forcing and axiomatic constructions of such linear orders. Finally, Section 8 contains some concluding remarks.

2. Preliminaries

We will begin with a brief review of some notation, terminology, and concepts from set theory which we will need. None of the material in this section is new or due to the authors. Further information on trees and linear orders can be found in [27] and [34]. Both [13] and [14] are standard references for set theory ([13] is encyclopaedic whereas [14] is more detail oriented).

All counting starts at 0. As is standard, we will use ω to denote the set of finite ordinals, which we take to coincide with the nonnegative integers. A *sequence* is a function whose domain is an ordinal. The domain of a sequence s is typically referred to as its *length* and denoted $|s|$. If s and t are sequences of ordinals, we define $s \leqslant_{\text{lex}} t$ if either s is an initial part of t or else there is a $\xi < \min(|s|, |t|)$ with $s(\xi) \neq t(\xi)$ and, for the least such ξ, $s(\xi) < t(\xi)$. We will generally identify a function with its graph. In particular if f and g are functions, $f \subseteq g$ exactly when f is a restriction of g (including the possibility $f = g$).

We note that any linear ordering is isomorphic to a set of sequences of ordinals ordered by $\leqslant_{\text{lex}}$. If one closes this set of sequences under initial segments, the structure of this set equipped with the extension partial order captures important aspects of the linear order. For this reason, it is fruitful to abstract this concept. A *tree* is a partially ordered set $(T, \leqslant_T)$ in which the set $\{s \in T : s <_T t\}$ of predecessors of t is well-ordered by $<_T$ for any $t \in T$. The order-type of this set is called the *height of* t. The collection of all elements of T of a given height δ will be denoted by T_δ, referred to as the δ^{th} *level of* T. The *height* of the tree T is the least δ such that T contains no elements of height δ. Notation such as $T_{\leqslant \delta}$ should be given the obvious interpretation. If κ is an infinite cardinal, a tree is a κ-*tree* if the height of T is κ and all levels of T have cardinality less than κ.

We say that T is *Hausdorff* if whenever $s, t \in T$ have limit height and are distinct, they have distinct sets of predecessors. If T is a set of sequences which is downwards closed with respect to $\leqslant$, then $(T, \leqslant)$ is a Hausdorff tree and moreover T_α consists of the sequences in T of length α; we say that T is a *tree of sequences*. Conversely, any Hausdorff tree is isomorphic to a tree of sequences.

In this paper we will work with trees of sequences which moreover have the property that sequences of limit length δ are extended by a unique element of the tree of length $\delta + 1$. For this reason, we will typically work with trees consisting of sequences of successor length and which are closed under taking initial segments of successor length.

An *antichain* in a tree T is a collection of pairwise incomparable elements. It is worth noting that in a tree if s and t are incomparable, they have no common upper bound (i.e. they are *incompatible*). If T is a tree, S is a *subtree*[4] of T if $S \subseteq T$, S is downward closed in T, and S has the same height as T. A subtree of T which is a chain is a *branch* of T.

A κ-*Aronszajn tree* is a κ-tree with no branches. A linear ordering L is a κ-*Aronszajn line* if it does not contain a copy of κ or κ^* and whenever $X \subseteq L$ has cardinality κ, its density is κ. It is a standard fact that the lexicographic ordering on a κ-Aronszajn tree is a κ-Aronszajn line and any κ-Aronszajn line is isomorphic to the lexicographic ordering of some subset of a κ-Aronszajn tree of sequences. If $\kappa = \aleph_1$, then we just write "Aronszajn" instead of "κ-Aronszajn." A linear order C is κ-Countryman if C has cardinality κ and C^2 is a union of fewer than κ chains (in the coordinatewise order); we will write "Countryman" to mean "$\aleph_1$-Countryman." The basic analysis of κ-Countryman lines can be found in Section 4.

[4]This meaning of "subtree" and "branch" are not completely standard.

Finally, we recall a useful characterization of σ-scattered linear orders which follows easily from Galvin's analysis of $\mathfrak{M}$ (see [20]). If γ is an infinite cardinal, consider the collection $\mathbb{Q}_\gamma \subseteq \mathbb{Q}^\gamma$ consisting of all x which change their values finitely often: there exist $0 = \xi_0 < \ldots < \xi_n = \gamma$ such that if $i < n$, x is constantly q_i on $[\xi_i, \xi_{i+1})$. We equip $\mathbb{Q}_\gamma$ with the lexicographic order. Since $|\mathbb{Q}_\gamma| = |\gamma|$, neither γ^+ nor its converse embed into $\mathbb{Q}_\gamma$. It is also easily checked by induction that any interval in $\mathbb{Q}_\gamma$ contains copies of δ and δ^* for any $\delta < \gamma^+$. Thus by Theorem 3.3 of [20], $\mathbb{Q}_\gamma$ is σ-scattered and any σ-scattered linear order of cardinality $|\gamma|$ embeds into $\mathbb{Q}_\gamma$ (if L is σ-scattered and has cardinality at most $|\gamma|$, then $L \times \mathbb{Q}_\gamma$ and $\mathbb{Q}_\gamma$ satisfy (i)-(iii) of [20, 3.3] for $\alpha = \beta = \gamma^+$ and hence are biembeddable). Rephrasing this, we have the following.

Proposition 2.1. If γ is an infinite ordinal, then a linear order of cardinality at most $|\gamma|$ is σ-scattered if and only if it embeds into $\mathbb{Q}_\gamma$. In particular $\mathbb{Q}_\gamma$ is biembeddable with $\mathbb{Q}_{|\gamma|}$.

3. Baumgartner's Question

In this section, we prove Theorem 1.15, thus answering Baumgartner's questions by showing that from $\diamondsuit$, one may construct a minimal Countryman line. As already noted, such a linear order is Aronszajn but not Souslin.

It will be useful to define some notation and terminology before proceeding.

Definition 3.1. $\mathbb{S}$ is the set of all $s \in {}^{<\omega_1}\omega$ of successor length which are finite-to-one.

We will view $\mathbb{S}$ as being equipped with the order of extension, making it a tree. Define $f : \mathbb{S} \to \omega \times \omega$ by

$$f(s) := (s(\xi), |\{\eta < \xi : s(\eta) = s(\xi)\}|)$$

where $|s| = \xi + 1$. Observe that if $f(s) = f(s')$, then s and s' are incomparable. Thus $\mathbb{S}$ is special.

The next definitions abstract the aspects of the tree of sequences associated to the function ϱ_2 of [35].

Definition 3.2. A *ϱ_2-modifier* is a continuous integer-valued sequence of successor length. If $s, t \in \mathbb{S}$, we say that s is a *ϱ_2-modification* of t if $|s| = |t|$ and $s - t$ is a ϱ_2-modifier. If $X \subseteq \mathbb{S}$, we will say that X is *closed under ϱ_2-modifications* if whenever $s \in X$ and t is a ϱ_2-modification of s, $t \in X$. We say X is *ϱ_2-full* if it is uncountable and closed under initial segments of successor length and ϱ_2-modifications.

We will sometimes drop the prefix "ϱ_2-" from ϱ_2-modifier for brevity. Notice that if m is a ϱ_2-modifier of length $\alpha + 1$ for α limit, then m is uniquely determined by its restriction to α. In fact any continuous sequence s taking values in $\mathbb{Z}$ that is eventually constant and having limit length can be uniquely extended to a ϱ_2-modifier of length $|s| + 1$. It will sometimes be useful to regard such sequences s as ϱ_2-modifiers by identifying them with the minimum modifier with extends them.

Proposition 2.1 yields the following proposition.

Proposition 3.3. For any successor ordinal α, the set of modifiers s of length α is σ-scattered when ordered by $\leqslant_{\text{lex}}$.

Definition 3.4. A subset $X \subseteq \mathbb{S}$ is *ϱ_2-coherent* if whenever $s, t \in X$ with $|s| \leqslant |t|$, $t \restriction |s|$ is a ϱ_2-modification of s. For ease of reading we will write "full" instead of "ϱ_2-full" in the context of "ϱ_2-coherent."

Notice that there are only countably many ϱ_2-modifications of an element of $\mathbb{S}$ and therefore any ϱ_2-coherent full subset of $\mathbb{S}$ is a subtree which has countable levels and hence is an Aronszajn tree. The following theorem is essentially due to Todorcevic (see [35, 3.4]); see the proof of the more general Proposition 4.3 below.

Theorem 3.5. *If $C \subseteq \mathbb{S}$ is uncountable and ϱ_2-coherent, then $(C, \leqslant_{\text{lex}})$ is Countryman.*

We will prove Theorem 1.15 by showing that $\diamondsuit$ implies the existence of a full ϱ_2-coherent tree $T \subseteq \mathbb{S}$ with the property that for any uncountable antichain X of T and any uncountable subset Y of T, there is an embedding of $(X, \leqslant_{\text{lex}})$ into $(Y, \leqslant_{\text{lex}})$.

Lemma 3.6. Suppose T is a ϱ_2-coherent subtree of $\mathbb{S}$ such that for any uncountable subtree S of T there is an embedding $\phi : T \to S$ that preserves both the lexicographic order $\leqslant_{\text{lex}}$ and incompatibility with respect to T's tree order. Then given any uncountable antichain X of T and uncountable subset Y of T, there is an embedding of $(X, \leqslant_{\text{lex}})$ into $(Y, \leqslant_{\text{lex}})$.

Proof. Observe that by replacing Y with an uncountable subset if necessary, we may assume Y is an antichain in T. Let S be the downward closure of Y in T and let $\phi : T \to S$ be the hypothesized embedding. We define a function $f : X \to Y$ by letting $f(x)$ be some element of Y that extends $\phi(x)$; this is possible by our choice of S. Given $x <_{\text{lex}} y$ in X, we know that $\phi(x)$ and $\phi(y)$ must be incompatible in S, and $\phi(x) <_{\text{lex}} \phi(y)$. But this implies $f(x) <_{\text{lex}} f(y)$ as well, and we are done. $\square$

How does this previous lemma help our project? It tells us that it will be sufficient to build a ϱ_2-coherent T that admits suitable embeddings into any of its subtrees. Our strategy is to build T so that every subtree will contain a tree of a canonical form that will render the existence of the required ϕ obvious. This provides the motivation for the next set of definitions, which capture a crucial ingredient in our proof.

Definition 3.7. Suppose $n < \omega$ and s and t are in $\mathbb{S}$. We say that t *is an n-extension of s*, written $s \subseteq_n t$, if $s \subseteq t$ and whenever $|s| \leqslant \xi < |t|$, $t(\xi) \geqslant n$.

Definition 3.8. Suppose that $T \subseteq \mathbb{S}$ is ϱ_2-coherent and full.

(1) The *cone of T determined by s*, denoted $T[s]$, is defined as usual by
$$T[s] := \{t \in T : t \subseteq s \text{ or } s \subseteq t\}.$$
(2) The *frozen cone of T determined by s and n*, denoted $T[s, n]$, is defined by
$$T[s, n] := \{t \in T : t \subseteq s \text{ or } s \subseteq_n t\}.$$
(3) Given an ordinal δ, we let $T_\delta[s]$ denote the elements of $T[s]$ of height δ, and similarly for $T_\delta[s, n]$.

It is clear that any cone of T is also a frozen cone, as $T[s]$ is just $T[s, 0]$. Since T is ϱ_2-full, frozen cones of T are also subtrees of T.

Lemma 3.9. If $T \subseteq \mathbb{S}$ is ϱ_2-coherent and full, then for any $s \in T$ and $n < \omega$ there is an embedding ϕ of T into $T[s, n]$ that preserves the lexicographic order $\leqslant_{\mathrm{lex}}$ and incompatibility with respect to the tree ordering.

Before we begin the proof of the lemma, it will be useful to introduce two operations on $\mathbb{S}$.

Definition 3.10. If $s, t, u \in \mathbb{S}$, $|s| < |t| = |u|$, and

$$u(\xi) := \begin{cases} s(\xi) & \text{if } \xi < |s|, \text{ and} \\ t(\xi) & \text{if } |s| \leqslant \xi < |t| \end{cases}$$

then we say that u is obtained by *writing s over t.*

Definition 3.11. If $t \in T$, $\beta < |t|$, and $n < \omega$, then the sequence v defined by

$$v(\xi) := \begin{cases} t(\xi) & \text{if } \xi < \beta, \text{ and} \\ t(\xi) + n & \text{if } \beta \leqslant \xi < |t| \end{cases}$$

is *the result of translating t by n beyond β.*

Proof of Lemma 3.9. Observe that if $T \subseteq \mathbb{S}$ is ϱ_2-coherent and full, then it is closed under these two operations. We prove the lemma in two stages. First, we prove that for any $s \in T$ and $n < \omega$ there is such an embedding from the (ordinary) cone $T[s]$ into the frozen cone $T[s, n]$. Doing this is straightforward: given $t \in T[s]$ extending s, we translate t by n beyond $|s|$. This function has the required properties, and by our assumption on T the range is contained in $T[s, n]$.

Next, we show for any $s \in T$ that T can be embedded into the (ordinary) cone $T[s]$ preserving the lexicographic order and incompatibility. To do this, define $\delta := |s| + \omega + 1$. Every open interval of $(T_\delta[s], <_{\mathrm{lex}})$ contains copies of ω and ω^*. Thus by Propositions 2.1 and 3.3, there is a $\leqslant_{\mathrm{lex}}$-preserving embedding

$$\phi_0 : T_{\leqslant \delta} \to T_\delta[s].$$

Notice that ϕ_0 trivially preserves incompatibility since $T_\delta[s]$ is an antichain. We extend ϕ_0 to a function $\phi : T \to T[s]$ by letting $\phi(t)$ be the result of writing $\phi_0(t \upharpoonright \delta)$ over t for t of height greater than δ. Again, our assumptions imply that the range of ϕ is contained in $T[s]$, and the function preserves both $\leqslant_{\mathrm{lex}}$ and incompatiblity. $\square$

Now we come to the point: if we can build a tree as in Lemma 3.9 with the property that any uncountable subtree contains a frozen cone, then we will have what we need to establish Theorem 1.15. We have therefore reduced our task to establishing the following proposition.

Proposition 3.12. Assume $\diamondsuit$. There is a $T \subseteq \mathbb{S}$ which is ϱ_2-coherent, full, and has the property that every subtree of T contains a frozen cone.

We will pause to introduce some notation and terminology which, while a little gratuitous now, anticipates the greater complexities of the higher cardinal constructions in later sections. Let $\equiv$ denote the equivalence relation on $\mathbb{S}$ defined by $s \equiv t$ if t is a ϱ_2-modification of s. Define $\mathbb{P} := \{[s] : s \in \mathbb{S}\}$ to be the collection of all $\equiv$-equivalence classes of functions in $\mathbb{S}$, and order $\mathbb{P}$ in the natural way: given q and p in $\mathbb{P}$, we define $q \leqslant_{\mathbb{P}} p$ to mean that some element of q extends some element of p in $\mathbb{S}$. We extend the notion of "height" to elements of $\mathbb{P}$ in the obvious way: the height of p is the height of any of its elements.

Our construction will depend on the interplay between the partially ordered sets $(\mathbb{S}, \supseteq)$ and $(\mathbb{P}, \leqslant_{\mathbb{P}})$, and we explore that relation a little with the following observations. We start by recording some easy facts about the interaction between the equivalence relation $\equiv$ and the operations on sequences from Lemma 3.9.

Lemma 3.13. The following are true:

(1) Let $s_0, t_0, s_1, t_1 \in \mathbb{S}$ with $s_0 \equiv s_1$, $t_0 \equiv t_1$ and $|s_0| = |s_1| < |t_0| = |t_1|$. Let r_i be the result of writing s_i over t_i. Then $r_0 \equiv r_1$.

(2) Let $t \in \mathbb{S}$, let $\beta < |t|$ and let r be the result of translating t by n beyond $\beta + 1$. Then $r \equiv t$.

(3) For each $s \in \mathbb{S}$ and countable $\beta \geqslant |s|$, there is a $t \in \mathbb{S}$ such that $s \subseteq t$ and $|t| = \beta + 1$.

Proof. Routine. □

Lemma 3.14. The following are true:

(1) If $\langle s_n : n < \omega \rangle$ is a sequence in $\mathbb{S}$ with $s_n \subseteq_n s_{n+1}$ then $(\bigcup_{n<\omega} s_n)^\frown\langle i \rangle \in \mathbb{S}$ for all $i < \omega$.

(2) If $s, t \in \mathbb{S}$ with $s \equiv t \upharpoonright \alpha$, then for any $n < \omega$ there is an $s \subseteq_n r$ such that $r \equiv t$.

(3) Any decreasing sequence $\langle p_n : n < \omega \rangle$ in $\mathbb{P}$ has a lower bound.

Proof. For (1), let $s = (\bigcup_n s_n)^\frown\langle i \rangle$. Clearly $|s|$ is a countable successor ordinal, so we need only verify that s is finite-to-one. For $k < n < \omega$,

$$s^{-1}(\{k\}) \subseteq s_n^{-1}(\{k\}) \cup \{|s| - 1\}$$

by the choice of the sequence $\langle s_n : n < \omega \rangle$, and this set is finite because $s_n \in \mathbb{S}$. To see (2), let r' be the result of writing s over t, and let r be the result of translating r' by n beyond α. By definition $s \subseteq_n r$, and by Lemma 3.13 $r \equiv t$. For (3), observe that by (2), we may recursively choose $s_n \in \mathbb{S}$ such that $s_n \subseteq_n s_{n+1}$ and $p_n = [s_n]$. By (1), $[(\bigcup_{n<\omega} s_n)^\frown\langle 0 \rangle]$ is a lower bound for $\langle p_n : n < \omega \rangle$. □

Using Lemma 3.14 it is straightforward to build $\leqslant_{\mathbb{P}}$-decreasing sequences $\langle p_\alpha : \alpha < \omega_1 \rangle$ in $\mathbb{P}$ such that p_α is the $\equiv$-class of some $t_\alpha : \alpha + 1 \to \omega$. Given such a sequence, define

$$T := \{t \in \mathbb{S} : t \in p_\alpha \text{ for some } \alpha < \omega_1\} = \bigcup_{\alpha<\omega_1} p_\alpha.$$

Clearly T is ϱ_2-coherent and full and hence an Aronszajn tree by remarks made after Definitions 3.1 and 3.4.

Our general strategy to prove Theorem 1.15 now comes into focus. What we need to do is to use $\diamondsuit$ to build a sequence $\langle t_\alpha : \alpha < \omega_1 \rangle$ of elements of $\mathbb{S}$ such that:

- $t_\alpha : \alpha + 1 \to \omega$,
- the sequence $\langle [t_\alpha] : \alpha < \omega_1 \rangle$ is $\leqslant_{\mathbb{P}}$-decreasing in $\mathbb{P}$, and
- the associated tree has the property that any uncountable subtree contains a frozen cone.

If we can do this, then Theorem 1.15 follows.

Proof of Theorem 1.15. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence, which we will assume is tailored to guess initial segments of ω_1-trees from $\mathbb{S}$—i.e. if $S \subseteq \mathbb{S}$ is an ω_1-tree, then there are stationarily many limit ordinals $\delta < \omega_1$ with $A_\delta = S_{<\delta}$, the initial segment of S of all levels prior to level δ (see discussion in the proof of Theorem 7.2 below).

Part of the construction is trivial: we let $t_0 = \langle 0 \rangle$, and if we are given t_α then we set $t_{\alpha+1} := t_\alpha {}^\frown \langle 0 \rangle$. Thus, the interesting case occurs when $\delta < \omega_1$ is a limit ordinal and we have constructed $\langle t_\alpha : \alpha < \delta \rangle$. Under these circumstances, we will know what T looks like below level δ, and our choice of $t_\delta : \delta + 1 \to \omega$ will determine which branches through this initial segment of T will have continuations at level δ.

The $\diamondsuit$-sequence presents us with a countable subtree A_δ of $\mathbb{S}$, and we ask if A_δ is a subtree of $T_{<\delta}$ that does not contain a frozen cone of $T_{<\delta}$. If the answer to this question is "no," then we need not worry about A_δ and let $t_\delta : \delta + 1 \to \omega$ be any element of $\mathbb{S}$ such that $[t_\delta]$ is a lower bound of $\langle [t_\alpha] : \alpha < \delta \rangle$ in $\mathbb{P}$. If the answer is "yes," then we will need to choose $t_\delta : \delta + 1 \to \omega$ in $\mathbb{S}$ so that for each $s \equiv t_\delta$, there is $\alpha < \delta$ such that $s \restriction \alpha + 1$ is not in A_δ.

We do this in countably many steps. First, we let $\langle \delta_n : n < \omega \rangle$ be an increasing sequence cofinal in δ. In our construction, we will be choosing ordinals α_n and corresponding $s_n \in p_{\alpha_n}$ such that:

- $\delta_n \leqslant \alpha_n < \delta$, and
- $s_n \subseteq_n s_{n+1}$.

This guarantees that

$$t_\delta := \left(\bigcup_{n<\omega} s_n \right) {}^\frown \langle 0 \rangle$$

will be in $\mathbb{S}$ and of the right length.

We start with $\alpha_0 = \delta_0$, and let $s_0 = t_{\alpha_0}$. Once we have constructed s_n and α_n, we assume that some bookkeeping process hands us a ϱ_2-modifier $m_n : \delta + 1 \to \mathbb{Z}$. The function m_n should be thought of as coding a member of the equivalence class of the t_δ we are building. Thus, we look at the function $s_n + m_n$ defined on $|s_n|$ given by

$$(s_n + m_n)(\xi) := s_n(\xi) + m_n(\xi)$$

and ask if this is a member of A_δ. If the answer is "no" (this includes the case in which $s_n + m_n$ has negative values), then we choose α_{n+1} to be greater than δ_{n+1}, and let s_{n+1} be some n-extension of s_n in the $\equiv$-equivalence class $[t_{\alpha_{n+1}}]$. If the answer is "yes," then we finally need to use our assumption that A_δ, when considered as a subtree of $T_{<\delta}$, does not contain a frozen cone of $T_{<\delta}$.

Define

$$M := \max\{|m_n(\xi)| : \xi < \delta\} \qquad \text{and} \qquad N := M + n + 1.$$

Our assumption says that $s_n + m_n$ will have an N-extension r in $T_{<\delta}$ that is not in A_δ. Extending r will not change this situation, so we may assume that $|r| = \alpha_{n+1}+1$ where

$$\alpha_{n+1} \geqslant \delta_{n+1}.$$

Now the idea is that we should define

$$s_{n+1} := r - m_n.$$

Notice that if $\xi < \alpha_n$, then

$$s_{n+1}(\xi) = r(\xi) - m_n(\xi) = s_n(\xi) + m_n(\xi) - m_n(\xi) = s_n(\xi),$$

and so s_{n+1} extends s_n. If $\alpha_n \leqslant \xi < \alpha_{n+1}$, then $r(\xi) \geqslant N$ and hence

$$s_{n+1}(\xi) = r(\xi) - m_n(\xi) \geqslant n + 1.$$

Thus s_{n+1} is in fact an n-extension of s_n. Finally, $s_{n+1} \equiv t_{\alpha_{n+1}}$ because s_{n+1} is equivalent to r and $r \in T_{<\delta}$. The key point is that if t is any extension of s_{n+1} in T_δ, then applying the modification m_n to t results in some s such that $s \restriction \alpha_n + 1$ is not in A_δ.

Since we made sure to arrange $s_n \subseteq_n s_{n+1}$, we know

$$t_\delta := \Big(\bigcup_{n<\omega} s_n\Big)^\frown\langle 0\rangle,$$

is in $\mathbb{S}$ and of height δ. Thus, $[t_\delta]$ will be a lower bound for $\langle [t_\alpha] : \alpha < \delta\rangle$ in $\mathbb{P}$, and the ϱ_2-modifications of t_δ will be the δ^{th} level of T.

The construction described above will produce a decreasing sequence $\langle [t_\alpha] : \alpha < \omega_1\rangle$ in $\mathbb{P}$. It remains to show that every uncountable subtree of T contains a frozen cone. Suppose $S \subseteq T$ is downward closed and does not contain a frozen cone. By the choice of our $\diamondsuit$-sequence, there must be a $\delta < \omega_1$ such that $A_\delta = S_{<\delta}$ and $(T_{<\delta}, <_T, S_{<\delta}) \prec (T, <_T, S)$. In particular δ is a limit ordinal, $A_\delta \subseteq T_{<\delta}$, and $A_\delta = S_{<\delta}$ contains no frozen cone of $T_{<\delta}$.

It suffices to show that $S \subseteq T_{<\delta}$. This follows from our construction, though: if t is any element of level δ of T, then during our construction of t_δ there was a stage where the function $t - t_\delta$ appeared as m_n. Since S does not contain a frozen cone, s_{n+1} was chosen so that $s_{n+1} + m_n$ is in $T_{<\delta} \backslash S_{<\delta}$. Because t extends $s_{n+1} + m_n$, t is also not in S. Thus, the height of S is at most δ and so S is countable. We conclude that any subtree of T contains a frozen cone, as required. $\square$

4. Countryman lines at higher cardinals

In the remainder of the paper, our aim is to adapt the construction in the previous section to higher cardinals. While this is of interest in its own right, our main motivation is to produce examples of minimal non-σ-scattered linear orders of cardinality $\kappa^+ > \aleph_1$. In fact these orders will be κ^+-Countryman lines.

We will begin recording some some basic facts about κ^+-*Countryman lines*, when κ is an infinite cardinal. Here a linear order C is κ^+-Countryman if its cardinality is κ^+ and C^2 is the union of κ chains with respect to the coordinatewise partial order on C^2.

Lemma 4.1. Suppose that L is a linear order of cardinality κ^+ and that whenever $Z \subseteq L \times L$ is a chain, there are at most κ elements $x \in L$ such that

$$Z_x := \{y \in L : (x, y) \in Z\}$$

has cardinality κ^+. Then L is not κ^+-Countyman.

Proof. Suppose that $\mathscr{X}$ is a collection of chains in $L \times L$ with $|\mathscr{X}| = \kappa$. Since κ^+ is not a union of κ sets of cardinality κ, our assumption implies there is an $x \in L$ such that for every $Z \in \mathscr{X}$, Z_x has cardinality at most κ. Again using the regularity of κ^+, there is a $y \in L$ such that $y \notin Z_x$ for every $Z \in \mathscr{X}$. But now $(x, y) \in L \times L$ is not covered by $\mathscr{X}$. Since $\mathscr{X}$ was arbitrary, L is not κ^+-Countryman. $\square$

Proposition 4.2. Suppose that C is κ^+-Countryman. The following are true:

(1) C^* is κ^+-Countryman and any suborder of C of cardinality κ^+ is κ^+-Countryman.
(2) C is not a well order.
(3) C is has no dense suborder of cardinality κ.

(4) C is κ^+-Aronszajn.
(5) If L is a linear order which embeds into C and C^*, $|L| \leqslant \kappa$.

Proof. Item (1) is trivial and (4) is an immediate consequence of (1)–(3). To see (2), observe that by (1), it suffices to show that κ^+ is not Countryman. Notice that if $Z \subseteq \kappa^+ \times \kappa^+$ is a chain and some section Z_α has cardinality κ^+, then it is cofinal in κ^+ and hence $Z_{\alpha'}$ is empty whenever $\alpha < \alpha'$. In particular, there is at most one α such that Z_α has cardinality κ^+. By Lemma 4.1 κ^+ is not Countyman.

To see (3), suppose that C has cardinality κ^+ and yet has a dense subset D of cardinality κ. If $Z \subseteq C \times C$ is a chain, let X be the set of all $x \in C$ such that the section Z_x contains at least three elements $a_x < d_x < b_x$; since D is dense we may choose $d_x \in D$. Since $x < x'$ implies $Z_x < Z'_x$, it also implies $d_x \neq d_{x'}$. Thus $|X| \leqslant |D| \leqslant \kappa$. Again, by Lemma 4.1, C is not κ-Countryman.

Finally, to see (5), notice that if L is any linear order and $f : L \to C$ and $g : L \to C^*$ are order preserving, then $\{(f(x), g(x)) : x \in L\}$ meets any chain in C^2 in at most one point. In particular, if C is κ^+-Countryman, $|L| \leqslant \kappa$. $\square$

Notice that the definitions of ϱ_2-*modification* and ϱ_2-*coherent* which we made previously makes sense in the generality of ${}^{<\kappa^+}\omega$. A subset X of ${}^{<\kappa^+}\omega$ is ϱ_2-*full with respect to* κ^+ if it has cardinality κ^+ and is closed under initial segments of successor length and ϱ_2-modifications. If κ^+ is clear from the context, we will sometimes abuse notation and write "ϱ_2-full" (or just "full") to mean "ϱ_2-full with respect to κ^+." The next proposition provides a useful criterion for demonstrating that a linear order is κ^+-Countryman. The proof is a routine modification of arguments of Todorcevic [35] and is included for completeness. Recall that a tree of height κ^+ is *special* if it is a union of κ antichains.

Proposition 4.3. Suppose that $T \subseteq {}^{<\kappa^+}\omega$ is ϱ_2-coherent and has cardinality κ^+. If T is special, then $(T, \leqslant_{\text{lex}})$ is κ-Countryman.

Proof. It suffices to cover $\{(s,t) \in T^2 : |s| \leqslant |t|\}$ by κ many chains. Given $(s,t) \in T^2$ with $|s| \leqslant |t|$, let $n = n(s,t)$ and $\xi_i = \xi_i(s,t)$ for $i \leqslant n$ be such that:

- $\xi_0 = 0 < \xi_1 < \ldots < \xi_n = |s|$,
- $t(\xi_i) - s(\xi_i) \neq t(\xi_{i+1}) - s(\xi_{i+1})$, and
- if $\xi_i < \eta < \xi_{i+1}$, then $t(\eta) - s(\eta) = t(\xi_i) - s(\xi_i)$.

Let $f : T \to \kappa$ be such that $f^{-1}(\alpha)$ is an antichain for each $\alpha < \kappa$. Define $\sigma(s,t)$ and $\phi(s,t)$ to be the sequences of length $n(s,t)$ given by

$$\sigma(s,t)(i) := t(\xi_i) - s(\xi_i) \qquad\qquad \phi(s,t)(i) := f(s \restriction \xi_{i+1})$$

whenever $i < n(s,t)$.

Since the sets of possible values of σ and ϕ have cardinality κ, it suffices to show that if $\sigma(s,t) = \sigma(s',t')$ and $\phi(s,t) = \phi(s',t')$, then either:

- $s \leqslant_{\text{lex}} s'$ and $t \leqslant_{\text{lex}} t'$ or
- $s' \leqslant_{\text{lex}} s$ and $t' \leqslant_{\text{lex}} t$.

Notice that this is vacuously true if either $s = s'$ or $t = t'$. For ease of reading, we will write ξ_i for $\xi_i(s,t)$ and ξ'_i for $\xi_i(s',t')$. Let $i \leqslant n$ be maximal such that $\xi_i = \xi'_i$. If $i = n$ and $s = s'$, then the desired conclusion follows. Otherwise set $\zeta = |s|$ if $i = n$ and $\zeta = \min(\xi_{i+1}, \xi'_{i+1})$ if $i < n$.

Claim 4.4. $s \restriction \zeta \neq s' \restriction \zeta$.

Proof. If $i = n$ then $\xi_n = \xi'_n = |s| = |s'| = \zeta$, and we are done since $s \neq s'$. Thus we may assume that $i < n$. Since $s \restriction \xi_{i+1} \neq s' \restriction \xi'_{i+1}$ and $f(s \restriction \xi_{i+1}) = f(s' \restriction \xi'_{i+1})$, it follows that $s \restriction \xi_{i+1}$ is incompatible with $s' \restriction \xi'_{i+1}$ and therefore that $s \restriction \zeta \neq s' \restriction \zeta$. $\square$

By exchanging the roles of s and s' if necessary assume that $s <_{\text{lex}} s'$. Observe that since $\sigma(s,t) = \sigma(s',t')$,

$$t(\eta) - s(\eta) = t'(\eta) - s'(\eta)$$

and hence

$$t(\eta) - t'(\eta) = s(\eta) - s'(\eta) \tag{4.1}$$

whenever $\eta < \zeta$. Let δ be minimal such that $s(\delta) \neq s'(\delta)$. Since $\delta < \zeta$, (4.1) implies $t \restriction \delta = t' \restriction \delta$ and $t(\delta) < t'(\delta)$. Thus $t <_{\text{lex}} t'$, as desired. $\square$

Proposition 4.5. Suppose that $C \subseteq {}^{<\kappa^+}\omega$ is ϱ_2-coherent and full. If $X \subseteq C$ has cardinality at most κ, then $(X, \leqslant_{\text{lex}})$ is σ-scattered.

Proof. By adding 1 to all of the values of elements of X if necessary, we may assume that no element of X takes the value 0. Let $t \in C$ be such that $|t|$ is an upper bound for the lengths of elements of X, and let Y be the set of all ϱ_2-modifications of t. Define $f : X \to Y$ by

$$f(s)(\xi) := \begin{cases} s(\xi) & \text{if } \xi < |s| \\ 0 & \text{if } \xi = |s| \\ t(\xi) & \text{if } \xi > |s| \end{cases}$$

and observe that f preserves $\leqslant_{\text{lex}}$ (since we've arranged s only takes positive values, 0 effectively serves as a terminating symbol for the sequence and we've defined $\leqslant_{\text{lex}}$ so that the terminating symbol is less than all other symbols). Since $y \mapsto y - t$ also preserves $\leqslant_{\text{lex}}$ and maps Y into the set of ϱ_2-modifiers of length $|t|$, we are done by Proposition 3.3. $\square$

5. Higher ϱ_2-coherence and the tree $\mathbb{S}_\kappa$

In order to apply Proposition 4.3, it will be helpful to construct an analog $\mathbb{S}_\kappa \subseteq {}^{<\kappa^+}\omega$ of $\mathbb{S}$ for higher cardinals κ such that any $T \subseteq \mathbb{S}_\kappa$ which is ϱ_2-coherent and full is special. Toward this end, let us assume that κ is a (possibly singular) infinite cardinal. If there is a $\square_\kappa$-sequence, then the tree $T(\varrho_2)$ defined using minimal walks down the $\square_\kappa$-sequence has many nice coherence properties. Our plan is to capture some of this structure in an abstract way.

Definition 5.1. Define $\mathbb{S}_\kappa$ to consist of all functions $t \in {}^{<\kappa^+}\omega$ which satisfy the following conditions:

(1) $|t| = \delta + 1$ for some $\delta < \kappa^+$ (which we denote as $\text{top}(t)$),
(2) for each $n < \omega$ the set $C^t_n := \{\alpha < |t| : t(\alpha) \leqslant n\}$ is closed,
(3) if $\alpha < |t|$ is a limit ordinal, then $t(\alpha)$ is the least n such that C^t_n is unbounded in α (noting that $C^t_{-1} = \varnothing$), and
(4) if I is a maximal open interval of $|t|$ that is disjoint to C^t_{n-1} then

$$\text{otp}(C^t_n \cap I) < \kappa \cdot \omega.$$

If $t \in \mathbb{S}_\kappa$, then we let $\text{last}(t)$ (the last value of t) be given by

$$\text{last}(t) := t(\text{top}(t)).$$

Setting $n = 0$ and $I = |t|$ in (4), $\operatorname{otp}(C_0^t) < \kappa \cdot \omega$. An easy induction (break up $|t|$ into blocks demarcated by elements of the closed set C_n^t) now shows that $\operatorname{otp}(C_n^t) < (\kappa \cdot \omega)^{n+1}$ for all $n < \omega$.

Observe that if $s \neq t$ are in $\mathbb{S}_\kappa$, $\operatorname{last}(s) = \operatorname{last}(t) =: n$, and $s \subseteq t$, then C_n^s is a proper initial segment of t and hence $\operatorname{otp}(C_n^s) < \operatorname{otp}(C_n^t)$. In particular,

$$s \mapsto (\operatorname{last}(s), \operatorname{otp}(C^s_{\operatorname{last}(s)}))$$

is a specializing function for $\mathbb{S}_\kappa$. (The use of the specific ordinal $\kappa\cdot\omega$ in the definition is not critical: $\kappa\cdot\omega$ is large enough to guarantee that $\mathbb{S}_\kappa$ will be closed under certain types of increasing unions, but small enough to ensure that our specializing function takes values in a set of cardinality κ.)

The definition of $\subseteq_n$ given in Section 3 generalizes without change to $\mathbb{S}_\kappa$, as does the definition of *frozen cone*. The following proposition summarizes what we have shown so far; the proof of the later statement is obtained from the arguments in Section 3 *mutatis mutandis*.

Proposition 5.2. If $T \subseteq \mathbb{S}_\kappa$ is ϱ_2-coherent and full, then $(T, \leqslant_{\mathrm{lex}})$ is a κ^+-Countryman line and any suborder of cardinality at most κ is σ-scattered. Moreover, if every subtree of T contains a frozen cone, then $(C, \leqslant_{\mathrm{lex}})$ is a minimal non-σ-scattered linear order, whenever $C \subseteq T$ is an antichain of cardinality κ^+.

Unlike in Section 3, it need not be the case that in a given model of set theory that there is a subset T of $\mathbb{S}_\kappa$ which is ϱ_2-coherent and full when $\kappa > \aleph_0$—after all such a T is a κ^+-Aronszajn tree and hence witnesses the failure of the *tree property* at κ^+ (see [22]). On the other hand, if ϱ_2 is defined from a $\square_\kappa$-sequence as in [35], then the collection of all ϱ_2-modifications of

$$\{\varrho_2(\cdot, \beta)\restriction\alpha + 1 : \alpha < \beta < \kappa^+\}$$

is a subset of $\mathbb{S}_\kappa$ which is ϱ_2-coherent and full [35].

We will now establish some basic properties of $\mathbb{S}_\kappa$ and define some terminology and notation.

Lemma 5.3. Suppose $t \in \mathbb{S}_\kappa$ and $\delta = \operatorname{top}(t)$.

1. The sequence $\langle C_n^t : n < \omega\rangle$ is an increasing sequence of closed sets with union $\delta + 1 = |t|$.
2. If $\alpha < |t|$ is a limit ordinal, then $t(\alpha) = n$ implies that t is constant with value n on a closed unbounded subset of α.
3. For each $n < \omega$ the set $\operatorname{nacc}(C_n^t)$ of non-accumulation points of C_n^t consists of successor ordinals.
4. The function t is determined by its values on successor ordinals.

Proof. Item (1) is immediate from the definitions. For (2), assume that $t(\alpha) = n$. Both $C_{n-1}^t \cap \alpha$ and $C_n^t \cap \alpha$ are closed in α, but the former is bounded below α while the latter is not. Thus $C_n^t \cap \alpha \setminus \sup(C_{n-1}^t)$ is closed and unbounded in α. But since this is contained in the set of $\beta < \alpha$ for which $t(\beta) = n$, we are done. Statements (3) and (4) now follow immediately. $\square$

The collection $\mathbb{S}_\kappa$ is closed under some natural operations. For example, it clear that this set is closed under restrictions to successor ordinals. Also if $t \in \mathbb{S}_\kappa$, then so is $t^\frown\langle n\rangle$ for every $n < \omega$. Most important for us, though, is that $\mathbb{S}_\kappa$ is essentially closed under certain types of increasing unions. The next definition will help us analyze the situation.

Definition 5.4. Given a limit ordinal $\delta < \kappa^+$, a function $t : \delta \to \omega$ is an $\mathbb{S}_\kappa$-*limit* if $t \restriction \alpha + 1$ is in $\mathbb{S}_\kappa$ for every $\alpha < \delta$.

The point is that any strictly $\subseteq$-increasing union of elements of $\mathbb{S}_\kappa$ is an $\mathbb{S}_\kappa$-limit. If t is an $\mathbb{S}_\kappa$-limit with domain some limit ordinal δ, then t will possess many of the characteristics of an element of $\mathbb{S}_\kappa$ automatically. For example, the definition of C^t_n makes sense for each n, and these sets will each be closed in δ because all of their proper initial segments are closed. We also note that if t does have an extension $s \in \mathbb{S}_\kappa$ with $\operatorname{top}(s) = \delta$, then in fact this extension is *unique*, because the value $s(\delta)$ must be the least n for which C^t_{n+1} is unbounded in δ. We will encounter this idea many times, so it will be convenient to give this particular n a name.

Definition 5.5. Suppose $t : \delta \to \omega$ for some limit ordinal $\delta < \kappa^+$. The limit infimum of t, denoted $\liminf(t)$ is defined to be the least $n < \omega$ with pre-image unbounded in δ if such an n exists, and is said to be ∞ otherwise.

Notice that if δ has uncountable cofinality, then any $t : \delta \to \omega$ will have a finite limit infimum by a simple counting argument. Thus, the situation $\liminf(t) = \infty$ is possible only if $\operatorname{cf}(\delta) = \omega$.

For an $\mathbb{S}_\kappa$-limit t, the question of whether t can be extended to an element of $\mathbb{S}_\kappa$ hinges on the existence of a finite limit infimum whose pre-image is not too large. The following lemma makes this precise.

Lemma 5.6. Suppose t is an $\mathbb{S}_\kappa$-limit with domain some limit ordinal $\delta < \kappa^+$. Then the following two statements are equivalent:

- t has an extension $s \in \mathbb{S}_\kappa$ with $\operatorname{top}(s) = \delta$.
- $\liminf(t)$ is some finite $n < \omega$, and the pre-image of n under t has a tail of order-type less than $\kappa \cdot \omega$.

In particular, if t is an $\mathbb{S}_\kappa$-limit and $|t| = \delta$ has uncountable cofinality, then t has an extension $s \in \mathbb{S}_\kappa$ with $\operatorname{top}(s) = \delta$.

Proof. For the forward implication, suppose $s \in \mathbb{S}_\kappa$ is an extension of t with $\operatorname{top}(s) = \delta$. Since $s \in \mathbb{S}_\kappa$, $\liminf(t) = s(\delta)$ is finite. If $\alpha < \delta$ is such that $t \geqslant s(\delta)$ on the interval $(\alpha, \delta]$, then

$$\operatorname{otp}(\{\eta \in (\alpha, \delta) : t(\eta) = s(\delta)\}) < \kappa \cdot \omega$$

because s satisfies requirement (4) in Definition 5.1.

For the reverse implication assume t satisfies $\liminf(t) = n$. We want to show that the function $s := t^\frown\langle n\rangle$ is in $\mathbb{S}_\kappa$. Since t is an $\mathbb{S}_\kappa$-limit and $s(\delta) = \liminf(t) = n$, requirements (1)–(3) of Definition 5.1 are easily satisfied.

For requirement (4), let m be given and $I \subseteq \delta + 1$ be an open interval on which $s > m$. If $m \geqslant n$, then since $s^{-1}(n)$ is cofinal in δ, it must be that $\beta := \sup(I) < \delta$. Since $t \restriction \beta + 1$ is in $\mathbb{S}_\kappa$, it follows that

$$\operatorname{otp}(C^s_{m+1} \cap I) = \operatorname{otp}(C^{t\restriction\beta+1}_{m+1} \cap I) < \kappa \cdot \omega.$$

If $m < n - 1$, then $s^{-1}(m+1) = t^{-1}(m+1)$ is bounded by some $\beta < \delta$ and we are again done by virtue of $t \restriction \beta + 1$ being in $\mathbb{S}_\kappa$. Finally, if $m = n - 1$, then by our hypothesis we may write $I \cap \delta = I_0 \cup I_1$, where I_0 is an initial segment of $I \cap \delta$ which is bounded in δ, and I_1 is a tail of $I \cap \delta$ such that $\operatorname{otp}(\{\eta \in I_1 : t(\eta) = n\}) < \kappa \cdot \omega$. Since t is an $\mathbb{S}_\kappa$-limit, we have $\operatorname{otp}(\{\eta \in I_0 : t(\eta) = n\}) < \kappa \cdot \omega$. Since $\kappa \cdot \omega$ is closed under ordinal addition, it follows that $\operatorname{otp}(\{\eta \in I : s(\eta) = n\}) < \kappa \cdot \omega$ as required. $\square$

This simplifies the project of building $\subseteq$-increasing sequences in $\mathbb{S}_\kappa$ immensely: we just need to worry about what happens at limit stages of countable cofinality. In particular, we need to guarantee that the limit infimum is finite and that the order-type of its pre-image does not grow to ordertype $\kappa \cdot \omega$. This turns out to be relatively easy to arrange provided we are careful at successor stages. The following definition formulates a straightforward way of doing this.

Definition 5.7. Suppose $s, t \in \mathbb{S}_\kappa$. We say that t is a *capped* extension of s if:

- $s \subset t$ (so t properly extends s),
- $\mathrm{last}(t) = 0$ (so t terminates with the value 0), and
- $t(\xi) > 0$ for all $|s| \leqslant \xi < |t| - 1$ (so $\mathrm{top}(t)$ is the only place beyond s where t returns the value 0).

The motivation for this definition is as follows. Suppose that $\langle s_n : n < \omega\rangle$ is a sequence of elements of $\mathbb{S}_\kappa$ and that s_{n+1} is a capped extension of s_n for all $n < \omega$. The definition guarantees that the union t of the chain will satisfy $\liminf(t) = 0$, and

$$\mathrm{otp}(t^{-1}(\{0\})) = \mathrm{otp}(s_0^{-1}(\{0\})) + \omega < \kappa \cdot \omega.$$

Thus, the sequence $\langle s_n : n < \omega\rangle$ can be continued in a canonical way: we can define

$$s_\omega := t^\frown\langle 0\rangle.$$

The function s_ω so defined is in fact a *least* upper bound for the sequence in $\mathbb{S}_\kappa$, as any such extension must take on the value 0 at $\mathrm{top}(s)$. We now extend this notion to longer sequences in the obvious way.

Definition 5.8. A $\subset$-increasing sequence $\bar{s} = \langle s_\beta : \beta < \alpha\rangle$ of elements of $\mathbb{S}_\kappa$ is *capped* if:

- $s_{\beta+1}$ is a capped extension of s_β for all $\beta < \alpha$.
- for $\gamma < \alpha$ a limit ordinal, we have
$$s_\gamma = \Big(\bigcup_{\beta<\gamma} s_\beta\Big)^\frown\langle 0\rangle.$$
(So for limit γ, s_γ is the canonical extension of the sequence $\langle s_\beta : \beta < \gamma\rangle$ in $\mathbb{S}_\kappa$.)

We now have all the pieces we need to easily get our sufficient condition for building $\subseteq$-increasing sequences in $\mathbb{S}_\kappa$ that are guaranteed to have upper bounds.

Lemma 5.9. A capped sequence in $\mathbb{S}_\kappa$ of length at most κ has a least upper bound in $(\mathbb{S}_\kappa, \subseteq)$.

Proof. Let $\langle s_\alpha : \alpha < \gamma\rangle$ be a capped sequence in $\mathbb{S}_\kappa$ for $\gamma \leqslant \kappa$ and let $t = \bigcup_{\alpha<\gamma} s_\alpha$. Let $\delta_\alpha = |s_\alpha|$ and observe that $\{\delta_\alpha : \alpha < \gamma\}$ is a closed unbounded set in $|t|$ and moreover is a tail of $t^{-1}(0)$. In particular, $\liminf t = 0$ and a tail of $t^{-1}(0)$ has ordertype $\gamma \leqslant \kappa < \kappa \cdot \omega$. By Lemma 5.6, $t^\frown\langle 0\rangle$ is in $\mathbb{S}_\kappa$. $\square$

Next we turn to modifications of elements of $\mathbb{S}_\kappa$.

Definition 5.10. If s and t are two sequences taking values in $\mathbb{Z}$, we define $s+t$ to be the sequence of length $\min(|s|, |t|)$ obtained by adding s and t coordinatewise on the restricted domain. If t is a sequence taking values in $\mathbb{Z}$, then $-t$ is the sequence of length $|t|$ obtained by multiplying t coordinatewise by -1. As is standard, $s - t$ abbreviates $s + (-t)$.

Typically we will be interested in computing $s + m$ when s is in $\mathbb{S}_\kappa$ and m is a ϱ_2-modifier of length at least $|s|$; as before we will say that $s+m$ is a ϱ_2-modification of s. For the sake of simplicity, in the context of $\mathbb{S}_\kappa$, all modifiers will have length $\kappa^+ + 1$ and will be identified with their restriction to κ^+ as per our remark in Section 3.

It will be helpful to define some notation and terminology associated to a given modifier.

Definition 5.11. Suppose $m : \kappa^+ \to \mathbb{Z}$ is a modifier.

(1) The *height of* m, denoted $\mathrm{ht}(m)$, is the least $\zeta < \kappa^+$ for which h is constant on $[\zeta, \kappa^+)$.
(2) The *norm of* m, denoted $\|m\|$, is the the maximum value of the form $|m(\xi)|$. Equivalently $\|m\|$ is the least N such that every value of m is in $[-N, N]$.
(3) We define the *change set of* m, denoted $\Delta(m)$, to consist of the ordinals $\xi_0 = 0 < \xi_1 < \cdots \xi_n = \mathrm{ht}(m)$ such that m is constant on $[\xi_i, \xi_{i+1})$ for each $i < n$ and $m(\xi_{i+1}) \neq m(\xi_i)$; the ordinals ξ_i for $0 < i \leqslant n$ are the *change points* of m.

We say that a modifier t is *legal* for $s \in \mathbb{S}_\kappa$ if the values of $s + t$ are nonnegative. The motivation for this definition is the following lemma.

Lemma 5.12. If $s \in \mathbb{S}_\kappa$ and m is a ϱ_2-modifier which is legal for s, then $s+m \in \mathbb{S}_\kappa$.

Proof. Clearly $|s + m| = |s|$ is a successor ordinal and $s + m$ takes values in ω, so requirement (1) of Definition (5.1) is satisfied. To verify requirements (2) and (3), let $\alpha < |s|$ with α limit, and let $s(\alpha) = k$ and $m(\alpha) = l$. Since m is continuous $m(\beta) = l$ for all large $\beta < \alpha$, and since $s \in \mathbb{S}_\kappa$, $s(\beta) \geqslant k$ for all large $\beta < \alpha$ and $s(\beta) = k$ for cofinally many $\beta < \alpha$. It follows that $(s + m)(\alpha) = k + l$, $(s + m)(\beta) \geqslant k + l$ for all large $\beta < \alpha$, and $(s + m)(\beta) = k + l$ for cofinally many $\beta < \alpha$.

As for requirement (4), let I be an interval such that $(s+m)(\alpha) \geqslant n$ for all $\alpha \in I$, and break up I into finitely many disjoint subintervals I_i for $i < k$ such that m is constant on I_i with value l_i. For $\alpha \in I_i$ we have that $s(\alpha) = (s+m)(\alpha)-l_i \geqslant n-l_i$, so that

$$\mathrm{otp}(\{\alpha \in I_i : s(\alpha) = n - l_i\}) < \kappa \cdot \omega,$$

which implies that $\mathrm{otp}(\{\alpha \in I_i : (s + m)(\alpha) = n\}) < \kappa \cdot \omega$. Since the ordinal $\kappa \cdot \omega$ is closed under finite sums, $\mathrm{otp}(\{\alpha \in I : s(\alpha) = n\}) < \kappa \cdot \omega$ and we have verified requirement (4). $\square$

Notice that it follows immediately from Lemma 5.12 that if $s \in \mathbb{S}_\kappa$ and m is legal for s, then $-m$ is legal for $s + m$, in which case $(s + m) - m = s$.

Proposition 5.13. Suppose $s : \delta + 1 \to \omega$ is in $\mathbb{S}_\kappa$ and let $\alpha < \delta$.

(1) The sequence s has a modification t that extends $s \restriction \alpha$ and satisfies $t(\delta) = 0$.
(2) If δ is a successor ordinal, then s has a modification that is a capped extension of $s \restriction \alpha$.

Proof. Part (1) is immediate except for the case where δ is a limit ordinal for which $n := s(\delta) > 0$. Since $s \in \mathbb{S}_\kappa$, n is the least element of ω whose pre-image is unbounded in δ. Increasing α if necessary, we may assume that $s(\xi) \geqslant n$ for $\xi \geqslant \alpha$.

Now we can define a function $m : \kappa^+ \to \mathbb{Z}$ by

$$m(\xi) = \begin{cases} 0 & \text{if } \xi \leqslant \alpha, \text{ and} \\ -n & \text{if } \alpha < \xi < \kappa^+. \end{cases}$$

The function m is a modifier, and by the choice of α we know that it is legal for s. The function $t := s + m$ has all the required properties.

Now suppose $\delta = \gamma + 1$. Part (2) is easy if $\alpha = \gamma$, so let us assume $\alpha < \gamma$ and define a modifier $m : \kappa^+ \to \omega$ by

$$m(\xi) = \begin{cases} 0 & \text{if } \xi \leqslant \alpha, \\ 1 & \text{if } \alpha < \xi \leqslant \gamma, \text{ and} \\ -s(\delta) & \text{if } \xi = \delta. \end{cases}$$

Now m is legal for s, and $s + m$ is a capped extension of $s \restriction \alpha$ that is equivalent to s. □

6. Forcing an Example

Recall that in Section 3 we derived a poset $\mathbb{P}$ from the set $\mathbb{S}$, investigated the properties of $\mathbb{P}$ as a forcing poset, and then used $\diamondsuit$ and $\mathbb{P}$ to construct a subtree of $\mathbb{S}$ which permitted us to answer Baumgartner's question. The argument of Section 3 easily shows that forcing with $\mathbb{P}$ adds a suitable tree, and indeed we may view the $\diamondsuit$ construction as building an ω_1-sequence of elements which generates a sufficiently generic filter.

By analogy with the definition of $\mathbb{P}$ from $\mathbb{S}$, we let $[s]$ be the set of legal modifications of s for $s \in \mathbb{S}_\kappa$, let $\mathbb{P}_\kappa = \{[s] : s \in \mathbb{S}_\kappa\}$, and order $\mathbb{P}_\kappa$ by ruling that $[t] \leqslant [s]$ if and only if $|s| \leqslant |t|$ and $[s] = [t \restriction |s|]$. In this section we investigate $\mathbb{P}_\kappa$ as a notion of forcing, and show that $\mathbb{P}_\kappa$ is a $(\kappa + 1)$-*strategically closed* notion of forcing that adjoins a tree of the sort we desire.

Definition 6.1. Let $\mathbb{P}$ be a notion of forcing and let α be an ordinal. The game $G_\alpha(\mathbb{P})$ involves two players, Odd and Even, who take turns playing conditions from $\mathbb{P}$ for α many moves. Odd chooses their move at odd stages, and Even chooses their move at even stages (including all limit stages). Even is required to play $1_{\mathbb{P}}$ (the maximal element of $\mathbb{P}$) at move zero. If p_β is the condition played at move β, the player who played p_β loses immediately unless $p_\beta \leqslant p_\gamma$ for all $\gamma < \beta$. If neither player loses at any stage $\beta < \alpha$, then Even wins the game.

Definition 6.2. Let $\mathbb{P}$ be a notion of forcing and γ be an ordinal. The notion of forcing $\mathbb{P}$ is γ-*strategically closed* if and only if Even has a winning strategy in $G_\gamma(\mathbb{P})$.

We come now to one of our main points.

Theorem 6.3. *$\mathbb{P}_\kappa$ is $(\kappa + 1)$-strategically closed.*

Proof. The strategy for Even in the game is simple, and involves building a capped sequence $\langle t_\beta : \beta < \kappa \rangle$, where t_β is an element of $p_{2\beta}$. In the end, Even's victory will be assured by applying Lemma 5.9.

Whenever Odd chooses their move $p_{2\beta+1}$, Even will choose a $t \in p_{2\beta+1}$ that is a 1-extension of t_β, and define $t_{\beta+1} = t^\frown\langle 0 \rangle$ and $p_{2(\beta+1)} = [t_{\beta+1}]$. At a limit stage $\delta \leqslant \kappa$, the capped sequence $\langle t_\beta : \beta < \delta \rangle$ will have a least upper bound t_δ in $\mathbb{S}_\kappa$, and Even will then play the condition $p_\delta = [t_\delta]$. □

Note that this game is very easy for Even to win: if they are building a capped sequence $\langle t_\xi : \xi < \kappa\rangle$ in the background, then all that is required at successor stages is that their response to $p_{2\beta+1}$ must contain a capped extension of t_β. If this is done, then Even will always be able to play at limit stages. This flexibility will be an important ingredient for us, as part of our proof relies on the fact that Even has many winning moves available at successor stages.

The fact that $\mathbb{P}_\kappa$ is $(\kappa+1)$-strategically closed tells us that it adds no κ-sequences of ordinals, and therefore preserves all cardinals up to and including κ^+. If we assume $2^\kappa = \kappa^+$ as well, then all cardinals and cofinalities will be preserved.

The forcing also adds a κ^+-tree. Given a generic filter $G \subseteq \mathbb{P}_\kappa$, let us step into the extension $\mathbf{V}[G]$. An easy density argument shows us that G will consist of a decreasing sequence $\langle p_\delta : \delta < \kappa^+\rangle$ of elements of $\mathbb{P}_\kappa$, which we enumerate so that p_δ consists of sequences of length $\delta + 1$.

If we now define

$$T(G) := \bigcup_{\delta<\kappa^+} p_\delta$$

then it is straightforward to see that $T(G)$ forms a tree under extension. Moreover, by construction $T(G)$ is ϱ_2-coherent and full.

We will need to work with certain elementary submodels of cardinality κ. In the case when κ is regular and $\kappa^{<\kappa} = \kappa$ we could use such models which are closed under sequences of length $< \kappa$, but if κ is singular this is impossible because in this case $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, and in any case we do not want to make cardinal arithmetic assumptions. We will make a standard move and use a certain type of "internally approachable" model.

If χ be a sufficiently large regular cardinal, we will mildly abuse notation by writing "$N \prec H(\chi)$" as a shorthand for "$N \prec (H(\chi), \in, \prec_\chi)$" where $\prec_\chi$ is some fixed wellordering of $H(\chi)$. We claim that any parameter $x \in H(\chi)$, we can find an elementary submodel $M \prec H(\chi)$ satisfying the following:

- $x \in M$;
- M is of cardinality κ with $\kappa + 1 \subseteq M$;
- $M \cap \kappa^+$ is some ordinal $\delta < \kappa^+$;
- for every $X \in M$ with $|X| \geqslant \kappa$, there is an enumeration $\vec{x} = \langle x_i : i < \kappa\rangle$ of $X \cap M$ such that $\vec{x}\restriction j \in M$ for all $j < \kappa$.

Let $\mathrm{cf}(\kappa) = \mu$. We construct M as the union of a μ-chain $(M_i)_{i<\mu}$ where:

- $x, \kappa \in M_0$;
- for all $i < \mu$, $M_i \prec H(\chi)$ and $|M_i| < \kappa$;
- for all i and j with $i < j < \mu$, $M_i \subseteq M_j$ and $M_i \in M_j$;
- for all $j < \mu$, $\langle M_i : i \leqslant j\rangle \in M_{j+1}$;
- for all $\gamma < \kappa$ there is $i < \mu$ such that $\gamma \subseteq M_i$.

This is all possible if we choose χ sufficiently large.

We verify that if we set $M := \bigcup_{i<\mu} M_i$ then M is as required. By construction $\kappa \subseteq M$, and so $M \cap \kappa^+ \in \kappa^+$. Now suppose $X \in M$ with $|X| \geqslant \kappa$. To build $\vec{x}$, we assume without loss of generality that $X \in M_0$. We start by choosing $\langle x_i : i < \gamma_0\rangle$ to be the $<_\chi$-least enumeration of $X \cap M_0$, noting that $\gamma_0 < \kappa$ because $|M_0| < \kappa$ and $\langle x_i : i < \gamma_0\rangle \in M_1$ because $X, M_0 \in M_1$. We will now proceed inductively for μ steps, choosing $\langle x_i : i < \gamma_j\rangle$ enumerating $X \cap M_j$ with $\gamma_j < \kappa$ and $\langle x_i : i < \gamma_j\rangle \in M_{j+1}$. Given $\langle x_i : i < \gamma_j\rangle$, we choose $\langle y_i : i < \delta_j\rangle$ to be the $<_\chi$-least enumeration of $X \cap (M_{j+1}\backslash M_j)$, and then set $\gamma_{j+1} = \gamma_j + \delta_j$ and $x_{\gamma_j+i} = y_i$ for

$i < \delta$. Since $X, M_j, M_{j+1} \in M_{j+2}$ it follows that $\langle y_i : i < \delta_j \rangle \in M_{j+2}$, and so $\langle x_i : i < \gamma_{j+1} \rangle \in M_{j+2}$. When j is limit let $\gamma_j = \sup_{j_0 < j} \gamma_{j_0}$, then $\gamma_j < \kappa$ because $j < \mu = \mathrm{cf}(\kappa)$, and $\langle x_i : i < \gamma_j \rangle \in M_{j+1}$ because it can be defined from $\langle M_i : i < j \rangle$ and we have $\langle M_i : i \leqslant j \rangle \in M_{j+1}$.

Observation 6.4. *If we require that the set $X \cap M$ be enumerated with repetitions, then we replace X by $\kappa \times X$ and let $\langle (\alpha_i, x_i) : i < \kappa \rangle$ be an enumeration of $(\kappa \times X) \cap M$ with all its proper initial segments in M. Then $\langle x_i : i < \kappa \rangle$ enumerates X with repetitions and all its proper initial segments lie in M.*

Let M be a submodel of this type, and note that since $\kappa \in M$ any set which is definable from the parameter κ is also in M: in particular the set $\mathbb{S}_\kappa$, the forcing poset $\mathbb{P}_\kappa$, the winning strategy for the game $G_{\kappa+1}(\mathbb{P}_\kappa)$, the set of ϱ_2-modifiers of length κ^+, and the set of all dense subsets of $\mathbb{P}_\kappa$ are all elements of M. Given any $p \in M \cap \mathbb{P}_\kappa$, we can use our game $G_{\kappa+1}(\mathbb{P}_\kappa)$ to build an $(M, \mathbb{P}_\kappa)$-generic subset G of $M \cap \mathbb{P}_\kappa$ together with a lower bound for G, that is to say a totally $(M, \mathbb{P}_\kappa)$-generic condition. To this we fix an enumeration $\vec{D}$ of the dense subsets of $\mathbb{P}_\kappa$ which lie in M in order type κ, such that that every proper initial segment of $\vec{D}$ is in M. We then build a run of the game $G_{\kappa+1}(\mathbb{P}_\kappa)$ where Even uses the winning strategy, and player Odd plays by choosing $p_{2\beta+1}$ as the $<_\chi$-least extension of $p_{2\beta}$ that lies in D_β. The key point is that for every $\gamma < \kappa$, the sequence of moves up to γ is defined from the strategy and an initial segment of $\vec{D}$, hence it is in M: in particular $p_\gamma \in M$ for all $\gamma < \kappa$. It is now clear that the final move p_κ is totally $(M, \mathbb{P}_\kappa)$-generic. In particular, p_κ induces an $(M, \mathbb{P}_\kappa)$-generic filter which determines our generic tree up to level $\delta = M \cap \kappa^+$, and the same will occur if $\dot{S}$ is a name in M for a subtree of $\dot{T}$. We leverage this to establish that the generic tree $T(G)$ added by $\mathbb{P}_\kappa$ has the frozen cone property.

Theorem 6.5. *The generic tree $T(G)$ adjoined by $\mathbb{P}_\kappa$ has the frozen cone property. Thus, there is a minimal non-σ-scattered linear order of cardinality κ^+ in the generic extension.*

Proof. Let $\dot{T}$ be a $\mathbb{P}_\kappa$-name for the generic tree $T(G)$, and suppose

$$p \Vdash \text{“}\dot{S} \text{ is a subtree of } \dot{T} \text{ that does not contain a frozen cone”.} \tag{6.1}$$

We will find $\delta < \kappa^+$ and $q \leqslant p$ such that

$$q \Vdash \text{“}\dot{S} \subseteq \dot{T}_{<\delta}.\text{”} \tag{6.2}$$

Let χ be some sufficiently large regular cardinal, and let M be an elementary submodel of $H(\chi)$ as discussed above, containing all parameters of interest to us. We let δ be $M \cap \kappa^+$. As in the preceding discussion let $\vec{D} = \langle D_i : i < \kappa \rangle$ be an enumeration of the dense open subsets of $\mathbb{P}_\kappa$ that lie in M with every proper initial segment of $\vec{D}$ in M, and let $\vec{m} = \langle m_i : i < \kappa \rangle$ be an enumeration with repetitions of the modifiers that lie in M with every proper initial segment of $\vec{m}$ in M. We play the game $G_{\kappa+1}(\mathbb{P}_\kappa)$ to produce the required q. The initial moves are as expected: Even must open with $[\varnothing]$, and we let Odd respond with p.

Suppose now that we are playing the game, and it is Odd's turn to play. In this situation, we have collaboratively built $\langle p_\gamma : \gamma \leqslant 2\beta \rangle$, while Even has been building their auxiliary sequence $\langle t_\gamma : \gamma \leqslant \beta \rangle$ on the side. Our construction will be guided by $\vec{D}$ and $\vec{m}$, so that as in our prior construction of a totally generic condition we have that $\langle p_\gamma : \gamma \leqslant 2\beta \rangle$ and $\langle t_\gamma : \gamma \leqslant \beta \rangle$ are both in M. The sequence $\langle t_\gamma : \gamma \leqslant \beta \rangle$ will be topped, in particular $\mathrm{last}(t_\beta) = 0$.

We now consider the modifier m_β, noting that since $m_\beta \in M$ we have $h(m_\beta) < \delta$. We ask first if m_β is legal for t_β with $h(m_\beta) < \mathrm{top}(t_\beta)$. If the answer is "no," then Odd doesn't need to take any special action, and chooses $p_{2\beta+1} := p_{2\beta}$. In this case Even responds by choosing $t_{\beta+1} := t_\beta$.

If the answer is "yes," then we will ask Odd to do some additional work. Note that $t_\beta + m_\beta \in M$ because $m_\beta, t_\beta \in M$. Also observe that the eventual constant value of m_β is non-negative because m_β is legal for t_β, $h(m_\beta) < \mathrm{top}(t_\beta)$ and $\mathrm{last}(t_\beta) = 0$. In particular m_β is automatically legal for any extension of t_β. We choose $q \leqslant p_{2\beta}$ to be the $<_\chi$-minimal extension of $p_{2\beta}$ deciding "$t_\beta + m_\beta \in \dot{S}$." Since q is definable from parameters in M, it is in M. If q forces "$t_\beta + m_\beta \notin \dot{S}$" we let $p_{2\beta+1} = q$. In this case Even choose $t_{\beta+1}$ as the $<_\chi$-least capped extension of t_β with $[t_{\beta+1}] \leqslant q$.

If q forces "$t_\beta + m_\beta \in \dot{S}$" we take q and follow the procedure described above to extend it to a totally $(M, \mathbb{P}_\kappa)$-generic condition, generating an $(M, \mathbb{P}_\kappa)$-generic filter G_β on $M \cap \mathbb{P}_\kappa$. Of course G_β itself is not in M, but we see shortly that this is not a problem. Using G_β we can interpret names for the initial segments of $\dot{S}$ and $\dot{T}$ that are in M and thus decide the identities of $T_{<\delta}$ and $S_{<\delta}$: these objects will depend on G_β, but for any $\alpha < \delta$ there will be a condition in G_β forcing that the information is valid through level α. Since $q \in G_\beta$ we have that $t_\beta + m_\beta \in S_{<\delta}$.

Since $p \in G_\beta$, our assumption (6.1) implies that for any $s \in T_{<\delta}$ and $n < \omega$ there is an n-extension t of s in $T_{<\delta}$ that is not in $S_{<\delta}$. This is the key ingredient of our argument. Let N be the norm of our modifier m_β. Since $t_\beta + m_\beta \in S_{<\delta}$, $t_\beta + m_\beta$ has an N-extension s in $T_{<\delta}$ such that $s \notin S_{<\delta}$. This situation is forced to be true for this particular s by some condition in G_β which extends q.

We have shown that there exist an N-extension s of $t_\beta + m_\beta$ and an extension r of q such that r forces "$s \notin \dot{S}$." Let (s', r') be the $<_\chi$-least pair with these properties, where as usual this pair is in M, and let $p_{2\beta+1} = r'$. We note that there is no reason to believe that $p_{2\beta+1} \in G_\beta$ or that $s' = s$. Note also that we can just look at s' and tell that it is an N-extension of $t_\beta + m_\beta$ without reference to the forcing at all, so the point is that $p_{2\beta+1}$ contains enough information to determine that s' is in T but not in S. This has some consequences, because the only way $p_{2\beta+1}$ can force s' to be in T is if $p_{2\beta+1}$ extends the equivalence class of s' in $\mathbb{P}_\kappa$.

By the closure properties of $p_{2\beta+1}$, $p_{2\beta+1}$ contains an N-extension s'' of s'. By the definition of T, $p_{2\beta+1}$ forces that $s'' \in T$ and since S is forced to be downwards closed, $p_{2\beta+1}$ forces that $s'' \notin S$. In summary, $p_{2\beta+1}$ contains an N-extension s'' of $t_\beta + m_\beta$ that is forced by $p_{2\beta+1}$ to lie outside of S.

Now define $t := s'' - m_\beta$. Since N is the norm of m_β and s'' is an N-extension of $t_\beta + m_\beta$, we know $-m_\beta$ is legal for s'' and t will be a 1-extension of t_β. Now Even defines

$$t_{\beta+1} := t^\frown\langle 0 \rangle$$

and $p_{2(\beta+1)} := [t_{\beta+1}]$, and play continues. As we observed above, m_β is legal for $t_{\beta+1}$.

We summarise the results of this round of the construction, keeping in mind that there were various cases. We claim that in all cases where m_β is legal for t_β with $h(m_\beta) < \mathrm{top}(t_\beta)$, $t_{\beta+1}$ is a capped extension of t_β and

$$p_{2(\beta+1)} \Vdash \text{"}t_{\beta+1} + m_\beta \notin \dot{S}.\text{"}$$

If we are in the case where q forces "$t_\beta + m_\beta \notin \dot{S}$," then we set $p_{2\beta+1} := q$ and the claim is immediate because $t_{\beta+1} + m_\beta$ extends $t_\beta + m_\beta$. If we are in the case where q forces "$t_\beta + m_\beta \in \dot{S}$," then we arranged that $t_{\beta+1} + m_\beta$ extends s'' and that $p_{2\beta+1}$ forces "$s'' \notin S$."

Because we were careful to make all choices at the successor stages using the wellordering $<_\chi$, $\langle p_\gamma : \gamma \leqslant 2\beta\rangle$ and $\langle t_\gamma : \gamma \leqslant \beta\rangle$ are both in M for all $\gamma < \kappa$. If Even follows this strategy, then they will end up winning the game by Lemma 5.9, because the sequence $\langle t_\beta : \beta < \kappa\rangle$ is a capped sequence. Let q be the corresponding final move p_κ for Even, and now we claim

$$q \Vdash \text{“}\dot{S} \subseteq \dot{T}_{<\delta}.\text{”}$$

To see this, let us define

$$t := \Big(\bigcup_{\beta<\kappa} t_\beta\Big)^\frown\langle 0\rangle.$$

Observe that $t \in \mathbb{S}_\kappa$ is a bound of the capped sequence $\langle t_\beta : \beta < \kappa\rangle$ that Even built during our run of the game. We know $q = [t]$, so it suffices to show for any ϱ_2-modifier m that is legal for t that

$$q \Vdash \text{“}t + m \notin \dot{S}.\text{”}$$

It suffices to check this for modifiers m that are in M, as $t + m$ is completely determined by $m \restriction \delta$ and m must be constant on a tail of δ. Since we enumerated the modifications in M with repetitions, during our play of the game we came to a stage $2\beta + 1$ for which $m_\beta = m$ and $\mathrm{ht}(m_\beta) < \mathrm{top}(t_\beta)$. Since m is legal for t, we know m is legal for t_β and therefore $t_{\beta+1}$ was selected so that

$$p_{2(\beta+1)} \Vdash \text{“ } t_{\beta+1} + m_\beta \notin \dot{S}.\text{”}$$

Hence

$$q \Vdash \text{“ } t + m \notin \dot{S}\text{”}$$

as required. □

7. Building many examples

Our goal in this section is to prove that if $\mathbf{V} = \mathbf{L}$ then there is a minimal non-σ-scattered linear order of cardinality κ^+ for every infinite cardinal κ. We break this into two pieces, first showing that the existence of such an order of cardinality κ^+ follows from a combinatorial principle we denote $\boxtimes^{+\epsilon}_\kappa$, and then show that this combinatorial principle holds for every infinite cardinal if $\mathbf{V} = \mathbf{L}$.

Definition 7.1. The principle $\boxtimes^{+\epsilon}_\kappa$ asserts the existence of a sequence

$$\langle (C_\delta, X_\delta, f_\delta) : \delta < \kappa^+\rangle$$

such that:

(1) for limit $\delta < \kappa^+$ the set C_δ is a closed unbounded subset of δ of order-type at most κ,
(2) $X_\delta \subseteq \delta$ for all $\delta < \kappa^+$,
(3) $f_\delta : C_\delta \to \delta$ for all $\delta < \kappa^+$,
(4) if $\alpha \in \mathrm{acc}(C_\delta)$ then:
  - $C_\alpha = C_\delta \cap \alpha$,
  - $X_\alpha = X_\delta \cap \alpha$,

(5) for every subset $X \subseteq \kappa^+$ and every club $C \subseteq \kappa^+$ there is a limit ordinal $\delta \in C$ such that:
- $C_\delta \subseteq C$,
- $X \cap \delta = X_\delta$,
- if $\alpha \in \operatorname{acc}(C_\delta)$ then $f_\alpha = f_\delta \restriction C_\alpha$, and
- f_δ maps C_δ onto δ (so $\operatorname{otp}(C_\delta)$ must be κ).

The above principle is a strengthening of the combinatorial statement $\boxdiamond_\kappa$ first introduced by Gray [9] in his dissertation, and first appearing in the literature in work of Abraham, Shelah, and Solovay [3]. We will discuss $\boxdiamond_\kappa$ and other related principles later, relying on more recent work by Rinot and Schindler [26]. For now, we just state that it is the functions f_δ and their associated properties that motivate the use of "$+\epsilon$" in our notation, as these functions are not present in the formulation of $\boxdiamond_\kappa$.

Theorem 7.2. *If κ is an infinite cardinal for which $\boxdiamond_\kappa^{+\epsilon}$ holds, then there is a $T \subseteq \mathbb{S}_\kappa$ which is ϱ_2-coherent and full. Consequently, there is a minimal non-σ-scattered linear ordering of cardinality κ^+ which is moreover κ^+-Countryman.*

Proof. Let $\langle (C_\delta, X_\delta, f_\delta) : \delta < \kappa^+\rangle$ be a $\boxdiamond_\kappa^{+\epsilon}$-sequence, and fix an enumeration $\langle m_\delta : \delta < \kappa^+\rangle$ of all ϱ_2-modifiers, subject to the conditions that $\operatorname{ht}(m_\delta) < \delta$ and that each modifier appears in the enumeration unboundedly often.

We need to give a little attention to how we use our $\boxdiamond_\kappa^{+\epsilon}$-sequence to guess κ^+-trees. This will be done in a completely straightforward way, but at one point in the proof the specificity will be convenient. Since $\diamondsuit(\kappa^+)$ is a consequence of $\boxdiamond_\kappa^{+\epsilon}$, we know $\kappa^\kappa = \kappa^+$ and so we can fix an enumeration $\langle \sigma_\alpha : \alpha < \kappa^+\rangle$ of $\mathbb{S}_\kappa$ in order-type κ^+. Given any κ^+-tree $S \subseteq \mathbb{S}_\kappa$ we can code S with a set $X \subseteq \kappa^+$ by setting

$$X := \{\alpha < \kappa^+ : \sigma_\alpha \in S\}.$$

What we need to observe is that if we do this, then there will be a closed unbounded set of $\delta < \kappa^+$ for which

$$S_{<\delta} = \{\sigma_\alpha : \alpha \in X \cap \delta\}. \tag{7.1}$$

This observation will help us later when we try to apply $\boxdiamond_\kappa^{+\epsilon}$.

The tree is built via a construction of length κ^+, and we build a sequence $\langle t_\alpha : \alpha < \kappa^+\rangle$ of elements of $\mathbb{S}_\kappa$ with $\operatorname{top}(t_\alpha) = \alpha$ that further satisfy

$$t_\beta \equiv_\kappa t_\alpha \restriction \beta + 1$$

whenever $\beta < \alpha < \kappa^+$. At a typical stage α of our construction, we will have available the sequence $\langle t_\beta : \beta < \alpha\rangle$ (hence we will know $T_{<\alpha}$) and will need to produce a suitable t_α with domain $\alpha + 1$.

The particular choice of t_α will matter only in cases where α is a limit ordinal, because if α is a successor ordinal $\gamma + 1$ then we set

$$t_\alpha := t_\gamma{}^\frown\langle 0\rangle.$$

At a limit stage α of our construction, we commit to building a $t_\alpha \in \mathbb{S}_\kappa$ which corresponds to a cofinal branch through $T_{<\alpha}$ and satisfies the following two conditions:

$$\beta \in \operatorname{acc}(C_\alpha) \Longrightarrow t_\beta \subseteq t_\alpha, \tag{7.2}$$

and

$$\operatorname{acc}(C_\alpha) \subseteq t_\alpha^{-1}(\{0\}) \subseteq \operatorname{acc}(C_\alpha) \cup \{\beta+1 : \beta \in \operatorname{nacc}(C_\alpha)\}. \tag{7.3}$$

Notice that this last condition will guarantee that the set of $\beta < \alpha$ for which $t_\alpha(\beta) = 0$ will have order-type at most κ. Since α is a limit ordinal, membership of t_α to $\mathbb{S}_\kappa$ requires a condition along the lines of (7.3) to allow us to define $t_\alpha(\alpha) = 0$.

We have no freedom if $\operatorname{acc}(C_\alpha)$ happens to be unbounded in α, as (7.2) will force us to define

$$t_\alpha := \Big(\bigcup_{\beta \in \operatorname{acc}(C_\alpha)} t_\beta \Big)^\frown \langle 0 \rangle,$$

and this will be an element of $\mathbb{S}_\kappa$ with the required properties. Thus, the only leeway in our construction occurs when the set $\operatorname{acc}(C_\alpha)$ is bounded below α, and whatever substantive action we take must occur at these stages.

Suppose then that our construction has arrived at a limit ordinal α for which $\gamma := \sup(\operatorname{acc}(C_\alpha))$ is less than α. In such a situation, we know that $C_\alpha \backslash \gamma + 1$ must have order-type ω, so we can list it in increasing order as $\langle \alpha_n : n < \omega \rangle$. When we choose $t_\alpha \in \mathbb{S}_\kappa$, we will want to make sure that it satisfies the following *structural requirements*:

- $\operatorname{top}(t_\alpha) = \alpha$,
- $t_\gamma \subseteq t_\alpha$
- $t_\alpha \upharpoonright \beta + 1 \in T_{<\alpha}$ for all $\beta < \alpha$, and
- there is an $m < \omega$ such that
$$t_\alpha^{-1}(\{0\}) \cap (\gamma, \alpha) = \{\alpha_n + 1 : m \leqslant n < \omega\}.$$

As long as t_α satisfies these requirements, our construction can proceed. They are not difficult to arrange: if s is any 1-extension of t_γ in $T_{<\alpha}$ at all, then we can extend s to a suitable t_α by means of a capped sequence of length ω whose tops consist of the ordinals $\alpha_n + 1$ for $m \leqslant n < \omega$.

Our work at stage α will depend on the set X_α presented to us by the $\diamondsuit_\kappa^{+\epsilon}$-sequence. Let us agree to call α an *active* stage if the following three criteria are satisfied:

- X_α codes an unbounded subtree Y_α of $T_{<\alpha}$,
- Y_α does not contain a frozen cone of $T_{<\alpha}$, and
- there is a $\xi \in C_\gamma$ for which $t_\gamma + m_{f_\gamma(\xi)}$ is in Y_α.

If α is an active stage, then let $\zeta \in C_\gamma$ be the least ξ as above. We say that this ζ is *targeted for action* at stage α, and our task will be to find an extension t_α of t_γ that satisfies all the structural requirements with the additional property that

$$(t_\alpha + m_{f_\gamma(\zeta)}) \upharpoonright \alpha \text{ is not a cofinal branch through } Y_\alpha. \tag{7.4}$$

If on the other hand α is not an active stage, then we can simply let $t_\alpha \in \mathbb{S}_\kappa$ be any extension of t_γ that satisfies the structural requirements.

Suppose now that α is an active stage, and $\zeta \in C_\gamma$ is the corresponding target. It suffices to produce a 1-extension s of t_γ in $T_{<\alpha}$ with the property that $s + m_{f_\gamma(\zeta)}$ is not in Y_α, as such an s can be extended to the t_α we need. To do this, let N be the norm of the modifier $m_{f_\gamma(\zeta)}$. Since Y_α does not contain a frozen cone of $T_{<\alpha}$, we know that $t_\gamma + m_{f_\gamma(\zeta)}$ has an N-extension t in $T_{<\alpha}$ that is not in Y_α. By definition, the modifier $-m_{f_\gamma(\zeta)}$ will be legal for t, and

$$s := t - m_\zeta$$

will be a 1-extension of t_γ in $T_{<\alpha}$ of the sort we seek, and therefore we can find t_α which satisfies (7.4) in addition to the structural requirements. This completes stage α.

Why does this construction succeed? We let T be the κ^+-tree determined by our sequence $\langle t_\alpha : \alpha < \kappa^+\rangle$, so that level α of T will consist of all the legal modifications of t_α. Our task is to show that any unbounded subtree of T contains a frozen cone, so assume by way of contradiction that $S \subseteq T$ is a counterexample, and let $X \subseteq \kappa^+$ code S.

There is a closed unbounded set E of ordinals $\delta < \kappa^+$ satisfying the following two statements:

- if s is in $S_{<\delta}$ and $n < \omega$, then s has an n-extension in $T_{<\delta}$ that is not in S;
- if $\delta \in E$ then $S_{<\delta}$ is coded by $X \cap \delta$.

Notice that this last is where we use the property of our coding mechanism discussed in the context of (7.1).

If χ is some sufficiently large regular cardinal, we can find an elementary submodel M of $H(\chi)$ of cardinality κ that contains S, T, and E such that:

- $M \cap \kappa^+ = \delta < \kappa^+$,
- $C_\delta \subseteq E$,
- $X_\delta = X \cap \delta$, and
- f_δ maps C_δ onto δ.

This can be achieved because of the properties of our $\boxdiamond^{+\epsilon}_\kappa$-sequence: note that the definition implies that there will be a stationary set of δ satisfying the last three requirements above, hence we can find δ satisfying the first.

Since S contains an element from level δ of T, there is at least one legal modification of t_δ in S. Since δ is a limit ordinal, we may assume that the relevant modifier m satisfies $\mathrm{ht}(m) < \delta$, and hence the modifier m will be in the model M and therefore will appear before stage δ in our enumeration of $\mathbb{S}_\kappa$.

Since the function f_δ maps C_δ onto δ, the modifier m guarantees that there is some least $\zeta \in C_\delta$ for which

$$t_\delta + m_{f_\delta(\zeta)} \upharpoonright \delta \text{ is a cofinal branch through } S_{<\delta}. \tag{7.5}$$

Now turn our focus to the way our construction proceeds through the stages indexed by $\mathrm{acc}(C_\delta)$. Suppose now that α is in $\mathrm{acc}(C_\delta)$. By the coherence of our $\boxdiamond^{+\epsilon}_\kappa$-sequence, we know

$$X_\alpha = X_\delta \cap \alpha = X \cap \alpha$$

and since α is also in E, we conclude that X_α codes $S_{<\alpha}$. We also know that $S_{<\alpha}$ does not contain a frozen cone of $T_{<\alpha}$, as this fact will reflect to α by our choice of E. Thus, any $\alpha \in \mathrm{acc}(C_\delta)$ will satisfy the first two requirements needed to be an active stage of our construction.

We now show that all sufficiently large elements of $\mathrm{nacc}(\mathrm{acc}(C_\delta))$ will be active stages of our construction. More specifically, if $\alpha \in \mathrm{acc}(C_\delta)$ and

$$\zeta < \gamma := \sup(\mathrm{acc}(C_\delta) \cap \alpha) < \alpha,$$

then α will satisfy the third requirement of being an active stage of our construction. To see this, note that since we are working with a $\boxdiamond^{+\epsilon}_\kappa$-sequence we have

$$\zeta \in C_\gamma = C_\delta \cap \gamma, \tag{7.6}$$

and

$$f_\gamma(\zeta) = f_\delta(\zeta). \tag{7.7}$$

Since $t_\delta + m_{f_\delta(\zeta)} \restriction \delta$ is a cofinal branch through through $S_{<\delta}$, we know

$$t_\gamma + m_{f_\gamma(\zeta)} \restriction \gamma \text{ is a cofinal branch through } S_{<\gamma},$$

and therefore α must be an active stage of the construction.

Said another way, we have shown that all sufficiently large $\alpha \in \operatorname{nacc}(\operatorname{acc}(C_\delta))$ are active stages. This is enough to get a contradiction: since $\operatorname{otp}(C_\delta) = \kappa$ we know

$$\operatorname{otp}(\operatorname{nacc}(\operatorname{acc}(C_\delta)) \backslash (\gamma + 1)) = \kappa,$$

and our construction guarantees that once an ordinal has been targeted at such a stage α, it will never be targeted again at any future stage from $\operatorname{acc}(C_\delta)$. Thus, we must eventually arrive at an active stage $\alpha \in C_\delta$ where ζ will be targeted for action, but the choice of t_α then contradicts (7.5). We conclude that S must contain a frozen cone, and the theorem is established. $\square$

We still need to prove that $\Diamond\!\!\!\!\square_\kappa^{+\epsilon}$ holds for every infinite cardinal in $\mathbf{L}$. We will approach this indirectly by showing that $\Diamond\!\!\!\!\square_\kappa^{+\epsilon}$ follows from a principle $\Diamond\!\!\!\!\square_\kappa^*$ introduced by Rinot and Schindler [26], and then quoting some of their work. With this in mind, here is the definition of $\Diamond\!\!\!\!\square_\kappa^*$.

Definition 7.3. The principle $\Diamond\!\!\!\!\square_\kappa^*$ asserts the existence of a sequence $\langle (C_\delta^*, \mathcal{X}_\delta^*, f_\delta^*) : \delta < \kappa^+ \rangle$ such that:

(1) C_δ^* is closed unbounded in δ of order-type at most κ;
(2) $\mathcal{X}_\delta^*$ is a subset of $\mathcal{P}(\delta)$ of cardinality at most κ;
(3) $f_\delta^* : C_\delta^* \to \mathcal{X}_\delta^*$ is a function that is a surjection whenever $\operatorname{otp}(C_\delta^*) = \kappa$;
(4) if $\alpha \in \operatorname{acc}(C_\delta^*)$, then:
   - $C_\alpha^* = C_\delta^* \cap \alpha$;
   - $f_\alpha^*(\beta) = f_\delta^*(\beta) \cap \alpha$ for all $\beta \in C_\alpha^*$;
(5) for every $X \subseteq \kappa^+$ and club $C \subseteq \kappa^+$ the set of $\alpha < \kappa^+$ for which $C_\alpha^* \subseteq^* C$ and $X \cap \alpha \in \mathcal{X}_\alpha^*$ contains a club;
(6) the set of $\delta < \kappa^+$ for which $\operatorname{otp}(C_\delta^*) = \kappa$ is stationary.

Theorem 7.4. *$\Diamond\!\!\!\!\square_\kappa^*$ implies $\Diamond\!\!\!\!\square_\kappa^{+\epsilon}$ for any infinite cardinal κ.*

Proof. Our proof is a modification of one from [26] , where the authors prove that $\Diamond\!\!\!\!\square_\kappa^*$ implies $\Diamond\!\!\!\!\square_\kappa$. The difference is that we need to keep track of more information to make sure we achieve the "$+\epsilon$."

Let $\langle (C_\delta^*, \mathcal{X}_\delta^*, f_\delta^*) : \delta < \kappa^+ \rangle$ be a $\Diamond\!\!\!\!\square_\kappa^*$-sequence, and let S be the stationary set of δ for which $\operatorname{otp}(C_\delta^*) = \kappa$. By Lemma 2.5 of [26], we may assume that for every club $D \subseteq \kappa^+$ there is a limit $\delta < \kappa^+$ satisfying

$$\operatorname{otp}(C_\delta^*) = \kappa \qquad \text{and} \qquad C_\delta^* \subseteq D.$$

Their proof involves building a $\Diamond\!\!\!\!\square_\kappa$-sequence $\langle (C_\alpha, X_\alpha) : \alpha < \kappa^+ \rangle$ from the above objects. The actual definition of the sets X_α need not concern us, but we will need to use that they have the additional property that for any closed unbounded $C \subseteq \kappa^+$ and any $X \subseteq \kappa^+$, there is a limit ordinal $\delta < \kappa^+$ such that

$$C_\delta^* \subseteq C, \qquad X \cap \delta = X_\delta, \qquad \text{and} \qquad f_\delta^* \text{ maps } C_\delta^* \text{ onto } \mathcal{X}_\delta^*. \tag{7.8}$$

The above is the content of their Claim 2.6.1, and this is all that we to know about the sequence $\langle X_\alpha : \alpha < \kappa^+ \rangle$.

Their construction of the sets C_α does require more of our attention. They build $\langle C_\alpha : \alpha < \kappa^+\rangle$ by defining a certain closed unbounded $E \subseteq \kappa^+$ and then letting

$$C_\alpha := \begin{cases} C^*_\alpha \cap E & \text{if } \alpha = \sup(C^*_\alpha \cap E), \\ C^*_\alpha \backslash \sup(C^*_\alpha \cap E) & \text{if } \sup(C^*_\alpha \cap E) < \alpha, \text{ and} \\ c_\alpha & \text{otherwise.} \end{cases} \tag{7.9}$$

What is c_α? Note that if we are in the "otherwise" case, then α has countable cofinality, and then we let c_α be some cofinal subset of α of order-type ω.

We need to use the above information together with properties of the original $\Diamond^+_\kappa$-sequence to define the functions f_δ needed for $\Diamond^{+\epsilon}_\kappa$. Given an ordinal $\alpha < \kappa^+$, we know that the set of $\delta < \kappa^+$ for which α is a member of $\mathcal{X}^*_\delta$ contains a club. Thus, there is a single closed unbounded $E^* \subseteq \kappa^+$ such that if $\delta \in E^*$, then $\delta \subseteq \mathcal{X}^*_\delta$.

We will use the club $\hat{E} := E \cap E^*$ to help us define the functions f_δ by focusing on those δ for which $C^*_\delta \subseteq \hat{E}$. Given such a δ, we know that C_δ will equal C^*_δ because of (7.9) and so we can define $f_\delta : C_\delta \to \delta$ by

$$f_\delta(\alpha) = \begin{cases} f^*_\delta(\alpha) & \text{if this an ordinal, and} \\ 0 & \text{otherwise.} \end{cases}$$

For δ outside of $\hat{E}$, we just let f_δ be the function with domain C_δ that is constant with value 0.

We now work towards showing $\langle (C_\delta, X_\delta, f_\delta) : \delta < \kappa^+\rangle$ is a $\Diamond^{+\epsilon}_\kappa$-sequence. A good deal of this is handled by Rinot and Schindler, as $\langle (C_\delta, X_\delta) : \delta < \kappa^+\rangle$ satisfies all conditions of Definition 7.1 that do not involve the functions f_δ. Thus, we need only establish that for any closed unbounded subset C of κ^+ and $X \subseteq \kappa^+$, there is a limit ordinal $\delta < \kappa^+$ that satisfies

$$\begin{aligned} & C_\delta \subseteq C, \\ & X \cap \delta = X_\delta, \\ & \alpha \in \operatorname{acc}(C_\delta) \Longrightarrow f_\alpha = f_\delta \restriction \alpha \text{ and,} \\ & f_\delta \text{ maps } C_\delta \text{ onto } \delta. \end{aligned} \tag{7.10}$$

Given C, find a limit ordinal δ satisfying (7.8) for our X and the closed unbounded set $C \cap \hat{E}$. The requirement that $X \cap \delta$ should equal X_δ is satisfied by such a δ as this is one of the characteristics required by (7.8). Similarly, we know that C^*_δ will equal C_δ because C^*_δ is a subset of $\hat{E}$, and so the first requirement of (7.10) will also be satisfied. The last requirement holds as well, because we know f^*_δ must map C_δ onto $\mathcal{X}^*_\delta$, and $\delta \subseteq \mathcal{X}^*_\delta$ for any $\delta \in \hat{E}$.

To establish the final requirement, we take advantage of the coherence properties of our original $\Diamond^*_\kappa$-sequence. In particular, we know that $C^*_\alpha = C^*_\delta \cap \alpha$ for any $\alpha \in \operatorname{acc}(C^*_\delta)$, which implies that C^*_α is a subset of $\hat{E}$ for every $\alpha \in \operatorname{acc}(C_\delta)$. But then C^*_α must equal C_α and f_α must be $f_\delta \restriction C_\alpha$ as well. Thus, our sequence $\langle (C_\delta, X_\delta, f_\delta) : \delta < \kappa^+\rangle$ is a $\Diamond^{+\epsilon}_\kappa$-sequence. $\square$

We finish now with the promised result for $\mathbf{L}$:

Corollary 7.5. If $\mathbf{V} = \mathbf{L}$ then for every infinite cardinal κ there is a minimal non-σ-scattered linear order of cardinality κ^+.

Proof. Rinot and Schindler establish that $\mathbf{V} = \mathbf{L}$ implies $\boxed{\Diamond}^*_\kappa$ for every infinite cardinal κ by way of Theorem 3.1 and Lemma 2.6 of [26]. Our Theorems 7.2 and 7.4 supply the rest of the proof. $\square$

8. Concluding remarks

We feel that it is likely that the methods of this paper can be adapted to show that in $\mathbf{L}$, there is a κ-Aronszajn line which is minimal with respect to being non-σ-scattered whenever κ is an uncountable regular cardinal which is not weakly compact. Presumably if κ is regular uncountable and not weakly compact, $\mathbf{L}$ satisfies a suitable principle $\boxed{\Diamond}^{+\epsilon}(\kappa)$, which in turn implies that there is a κ-Aronszajn tree $T \subseteq \omega^{<\kappa}$ with the following properties:

- T is ϱ_2-coherent and full;
- every subset of T of cardinality κ contains an antichain of cardinality κ;
- every subtree of T contains a frozen cone.

The arguments presented in this paper then show that the lexicographic ordering on any antichain in T of cardinality κ is minimal with respect to not being σ-scattered.

Galvin asked whether there is a minimal non-σ-scattered linear order with the additional property that every uncountable suborder contains a copy of ω_1—this is equivalent to being minimal with respect to not being a countable union of well orders (see [6, Problem 4]). As noted in the introduction, Ishiu and the third author have shown that a negative answer follows from PFA^+ [12] and Lamei Ramandi has shown that a positive answer is consistent [16]. It remains an open problem whether there are consistent examples of linear orders of cardinality greater than $\aleph_1$ which are minimal with respect to not being a countable union of well orders. Such orders necessarily are not κ-Aronszajn and hence the methods of this paper do not seem to shed much light on this problem. Todorcevic has shown that $\square_{\aleph_\omega}$ implies that there is a linear order of cardinality $\aleph_{\omega+1}$ of density $\aleph_\omega$ such that every suborder of cardinality $\aleph_\omega$ is a countable union of well orders [32, 7.6]. Note, however, that the construction of Dushnik and Miller [8] generalizes to show that if $2^\kappa = \kappa^+$, then there is no minimal linear order of cardinality κ^+ and density κ. Thus at least consistently, Todorcevic's example [32, 7.6] does not solve Galvin's problem; one would need an analog of Baumgartner's model [4] at the level of $\aleph_{\omega+1}$, which seems beyond the reach of current methods.

A minimal non-σ-scattered ordering of cardinality greater λ than $\aleph_1$ is a "non-reflecting" object, in the sense that it enjoys a property which is not enjoyed by any of its properly smaller suborderings. This phenomenon is ruled out by large cardinal assumptions. For instance if λ is weakly compact, then any non-σ-scattered order of cardinality λ has a non-σ-scattered suborder of smaller cardinality. Similarly if κ is supercompact, then any non-σ-scattered linear order has a non-σ-scattered suborder of cardinality less than κ. The proofs of these statements are routine modifications of arguments in [6, §7].

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