GALVIN'S CONJECTURE AND WEAKLY PRECIPITOUS IDEALS

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ABSTRACT. We show that if κ is a Ramsey cardinal then every fine countably complete normal ideal concentrating on a set in V_κ (hence every normal ideal on ω_1) satisfies a weak form of precipitousness. As an application we show that Ramsey cardinals can be used instead Woodin cardinals and stationary tower forcing to push through Raghavan-Todorčević proof [13] of a longstanding conjecture of Galvin. In particular, if a Ramsey cardinal exists then for any uncountable set of reals X and any coloring c of $[X]^2$ with finitely many colors, there is a set $Y \subseteq X$ homeomorphic to the rationals such that c takes on at most two colors on $[Y]^2$. Finally, we use work of Donder and Levinsky [4] to show that this partition theorem for uncountable sets of reals can hold in a generic extension of Gödel's L, and therefore does not imply the existence of $0^\#$.

1. Introduction

1.1. **Context.** Our motivation is a recent result of Raghavan and Todorčević [13] from Ramsey theory. Assuming the existence of a Woodin cardinal, they establish a partition relation for uncountable sets of reals first conjectured by Galvin in the 1970s by proving

> If X is an uncountable set of reals and we color the (un-ordered) pairs from X with finitely many colors, then X contains a subset homeomorphic to the rationals on which at most two colors appear.

We will refer to the above statement as *Galvin's Conjecture*, although in Baumgartner's [1] the question is formulated only for the special case $X = \mathbb{R}$.

The value "two" in the phrase "at most two values" is critical: a classical construction of Sierpiński [19] provides a coloring $c : [\mathbb{R}]^2 \to \{0, 1\}$ that takes on both colors on any subset of $\mathbb{R}$ containing a subset order-isomorphic to the integers. Similarly, the restriction to uncountable sets of reals is necessary as well, as Baumgartner showed that there is a coloring of $[\mathbb{Q}]^2$ with countably many colors that takes on all values on any subset of $\mathbb{Q}$ homeomorphic to $\mathbb{Q}$. In contrast, Galvin showed earlier in unpublished work that for any coloring of $[\mathbb{Q}]^2$ with finitely many colors, we can find a set of rationals that is order-isomorphic (though not necessarily homeomorphic) to $\mathbb{Q}$ on which the coloring takes on at most two values. Thus, Galvin's Conjecture holds if we replace "homeomorphic to $\mathbb{Q}$" with the weaker requirement "order-isomorphic to $\mathbb{Q}$" , and Baumgartner's work shows that there is potentially a big difference between the two statements.

What about partitions of more general topological spaces? Unpublished work of Todorčević and Weiss shows that the result of Baumgartner mentioned above can

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be extended to the broader class of σ-discrete metric spaces: if X is a σ-discrete metric space, then there is a function $c : [X]^2 \to \omega$ that takes on every value on any subset of X homeomorphic to $\mathbb{Q}$. The authors of [13] were also able to extend their positive answer to Galvin's Conjecture to a much more extensive (and essentially optimal) class of topological spaces as long as there are enough large cardinals around.

Theorem 1 (Raghavan-Todorčević [13])**.** *If there is a proper class of Woodin cardinals (or a single strongly compact cardinal) then the following statements are equivalent for a metrizable space X*

- *X is not σ-discrete.*
- *Given $c : [X]^2 \to l$ with $l < \omega$ there is a $Y \subseteq X$ homeomorphic to Q on which c assumes at most two values.*

Under the same large cardinal assumptions, they show that the second statement in the above theorem holds for the broader class of regular spaces with a point countable base that are not left-separated.

1.2. **New Results.** Our goal in this paper is to reduce substantially the large cardinal assumptions needed to obtain the results from [13]. Woodin cardinals have large consistency strength (they imply the existence of inner models with measurable cardinals, for example), and the proofs from [13] rely on the fact that stationary tower forcing in the presence of Woodin cardinals gives rise to a generic elementary embedding $j : V \to M$ such that M is transitive and closed under ω-sequences. Larson's book [11] is a standard reference for this material.

We do this using some ideas originating in work of Shelah [17] on cardinal arithmetic. Our main tool is a concept – *weakly precipitous ideals* – formulated by Jech [9], but implicit in work of Shelah [17, 16] from the late 1970s and used by him to obtain extensions of the Galvin-Hajnal Theorem [8] in cardinal arithmetic. We show that below a Ramsey cardinal, essentially any normal ideal will be weakly precipitous (a result with independent interest whose exact formulation can be found in Corollary 4.3), and use these ideals to replace the use of stationary tower forcing in the original proof from [13]. The key point is that we can push through their argument in the scenario where the large cardinal gives rise to a generic elementary embedding into a target model with only a sufficiently large well-founded initial segment guaranteed to exist. Our main theorem is the following:

Theorem 2. *Suppose κ is a Ramsey cardinal. If $X \in V_\kappa$ is a regular topological space with a point-countable base that is not left-separated, then for any coloring c of $[X]^2$ with finitely many colors, there is a $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c takes on at most two values.*

Since any uncountable set of reals satisfies the assumption of the theorem, we obtain Galvin's Conjecture provided a Ramsey cardinal exists:

Corollary 1.1. If there is a Ramsey cardinal, then for any uncountable set of reals X and any coloring $c : [X]^2 \to l$ with $l < \omega$, there is a $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c assumes at most two values.

More generally, we get the same characterization of σ-discrete metrizable spaces.

Corollary 1.2. If κ is a Ramsey cardinal, then the following two statements are equivalent for a metrizable space X in V_κ:

- X is not σ-discrete.
- For any coloring of $[X]^2$ with finitely many colors, there is a $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which the coloring takes on at most two values.

In the final section of the paper, we examine the consistency strength of Galvin's Conjecture (for uncountable sets of reals) and prove that it does not imply the existence of $0^{\#}$. The proof works by showing Galvin's Conjecture holds if there is a notion of forcing giving rise to a generic elementary embedding with critical point ω_1 whose target model is well-founded out to the image of $\beth_2(\omega_1)^+$. Using work of Donder and Levinsky [4] we show that this can be arranged in a set-generic extension of L. This still requires a large cardinal assumption, but one which, in contrast to Ramsey cardinals, is compatible with the axiom $V = L$.

2. Background Material

Throughout this paper we will work with proper countably-complete fine normal ideals, as well as with their Rudin-Keisler projections. Our notation concerning these objects is standard and we generally follow Foreman [6]:

Definition 2.1. Suppose X is a set, $Z \subseteq \mathcal{P}(X)$, and J is an ideal on Z.

(1) J^* is the filter dual to J, so $Y \subseteq \mathcal{P}(X)$ is in J^* if and only if its complement is in J, and we say that J *concentrates on* Y.

(2) If Y is a J-positive subset of $\mathcal{P}(X)$ (that is, $Y \notin J$) then we define the *restriction* $J \restriction Y$ *of* J *to* Y by putting a set $A \subseteq Z$ into $J \restriction Y$ if and only if $A \cap Y \in J$. Note that $J \restriction Y$ concentrates on the set Y.

(3) J is *countably complete* if the union of countably many sets in J is again in J. If κ is a regular cardinal then J is $< \kappa$-complete if J is closed under unions of size less than κ. Thus, J is countably complete if and only if J is $< \aleph_1$-complete.

(4) Given $Y \subseteq \mathcal{P}(X)$, a function $f : Y \to X$ is *regressive* if $f(y) \in y$ for all $y \in Y$. The ideal J is *normal* if for any J-positive Y and regressive $f : Y \to X$ there is a J-positive $Z \subseteq Y$ and $x \in X$ such that $f(z) = x$ for all $z \in Z$.

(5) J is *fine* if for every $x \in X$ the set $\{z \in Z : x \in z\}$ is in J^*, that is, if J-almost every $z \in Z$ contains x.

We are deliberately loose with our language concerning ideals of the form $J \restriction Y$, as these can profitably be viewed as ideals with underlying set Y, or Z, or even $\mathcal{P}(X)$. We adopt whichever point of view is convenient at the time as the sloppiness is harmless: all that matters is that $J \restriction Y$ is the minimal ideal extending J that concentrates on Y, and the myriad variations differ only on a negligible set.

As may be apparent from the above definition, we also use terms "J-almost every $z \in Z$" to mean that the set of such z is in J^*, and similarly "for a J-positive set of $z \in Z$" means that the set of such z is not in J. We also abbreviate these statements using quantifiers, so for example if ϕ is some formula then

$$(\forall^J z \in Z)\phi(z) \iff \{z \in Z : \phi(z)\} \in J^*,$$

and

$$(\exists^J z \in Z)\phi(z) \iff \{z \in Z : \phi(z)\} \notin J.$$

We also remind the reader of basic properties of the *non-stationary ideal*. Again, we follow the conventions of Foreman [6].

Definition 2.2. If $F : [X]^{<\omega} \to X$ then C_F is defined to be the set of $x \in X$ that are closed under F. The *closed unbounded filter* on $\mathcal{P}(\kappa)$ is the filter generated by sets of the form C_F, and a subset Z of $\mathcal{P}(X)$ is stationary if for every such function F there is a $z \in Z$ that is closed under F. The non-stationary ideal consists of all subsets of $\mathcal{P}(X)$ that are disjoint to C_F for some F. If Z is a stationary subset of $\mathcal{P}(X)$ then we let NS_Z denote the ideal $NS \restriction Z$.

There are some standard facts about the non-stationary ideal that are very useful.

Lemma 2.3. With notation as above:

(1) A subset Z of $\mathcal{P}(X)$ is stationary if and only if for any sufficiently large regular χ and $x \in H(\chi)$, there is an $M \prec H(\chi)$ such that $x \in M$ and $M \cap X \in Z$.

(2) The non-stationary ideal is *normal*: if $Z \subseteq X$ is stationary and $F : Z \to X$ is regressive function (that is, $F(z) \in z$ for all $z \in Z$) then F is constant on a stationary subset of Z.

(3) The non-stationary ideal on Z is the minimal normal and fine ideal: if J is a normal fine countably complete ideal on a set $Z \subseteq \mathcal{P}(X)$ and $A \subseteq Z$ is non-stationary, then $A \in I$.

3. Partitions and Ideals

3.1. **Definitions.** This section will collect some definitions that are extracted from the original Raghavan-Todorčević proof of Galvin's Conjecture, and modified to fit with our framework of weakly precipitous ideals.

Definition 3.1. Suppose J is an ideal on a set X, and $c : [X]^2 \to l$ is a coloring of pairs from X with $l < \omega$.

(1) Given an $x \in X$, a J-positive subset A of X, and a color $i < l$, we say that x is i-large in A with respect to J if the set of $a \in A$ with $c(x, a) = i$ is J-positive.

(2) A pair $\langle A, B\rangle$ of J-positive sets is said to be *weakly $\langle i, j\rangle$-saturated over J* if the set of $a \in A$ that are i-large in B with respect to J is J-positive, and the set of $b \in B$ that are i-large in A with respect to J is J-positive, that is, if

$$(\exists^J a \in A)(\exists^J b \in B)\,[c(a, b) = i]\,, \tag{3.1}$$

and

$$(\exists^J b \in B)(\exists^J a \in A)\,[c(a, b) = j]\,. \tag{3.2}$$

(3) A pair $\langle A, B\rangle$ of J-positive sets is *$\langle i, j\rangle$-saturated over J* if for any J-positive $C \subseteq A$ and $D \subseteq B$, the pair $\langle C, D\rangle$ is weakly $\langle i, j\rangle$-saturated over J.

The following proposition collects some basic facts in this context.

Proposition 3.2. Suppose $c : [X]^2 \to l$ with $l < \omega$, and let A and B be J-positive subsets of X for some ideal J.

(1) There is a pair of colors $\langle i, j\rangle$ such that $\langle A, B\rangle$ is weakly $\langle i, j\rangle$-saturated over J.
(2) If $\langle A, B\rangle$ is $\langle i, j\rangle$-saturated over J, then so is $\langle C, D\rangle$ for any J-positive $C \subseteq A$ and $D \subseteq B$.
(3) If $\langle A, B\rangle$ is (weakly) $\langle i, j\rangle$-saturated over J, then $\langle A, B\rangle$ is (weakly) $\langle i, j\rangle$-saturated over any smaller ideal $I \subseteq J$.
(4) If $\langle A, B\rangle$ is $\langle i, j\rangle$-saturated over J, then J-almost every $\alpha \in A$ is i-large in B, and J-almost every $\beta \in B$ is j-large in A.

Proof. These are all fairly easy. For (1), the conclusion follows easily once we note that since J is an ideal and there are only finitely many colors available, then given x and a J-positive A there must be an i such that x is I-large in A. Part (2) follows immediately from the definition, and (3) follows as any J-positive subset of X is I-positive.

Statement (4) is proved by contradiction: if it fails then we can find J-positive $C \subseteq A$ such that no $x \in C$ is i-large in B, and a J-positive $D \subseteq B$ such that no element of D is j-large in A. But this is absurd, as the pair $\langle C, D\rangle$ contradicts statement (2). □

Lemma 3.3. If J is a ideal on X then there is an $\langle i, j\rangle$-saturated pair of J-positive sets over J for some pair of colors $\langle i, j\rangle$.

Proof. Suppose J were a counterexample. Then given any any pair $\langle S, T\rangle$ of J-positive subsets of $\tilde{X}$ and any pair of colors $\langle i, j\rangle$, we can find J-positive $A \subseteq S$ and $B \subseteq T$ such that one of (3.1) or (3.2) fails. Thus, by making successive extensions, we can run through all pairs of colors and arrive at a pair $\langle A, B\rangle$ of J-positive sets such that for *every* pair of colors $\langle i, j\rangle$ at least one of the two alternatives fails, but this contradicts the first part of Lemma 3.2. □

Lemma 3.4. If $\mathbb{J}$ is a non-empty set of ideals on X, then there is a $J \in \mathbb{J}$ and a pair of colors $\langle i, j\rangle$ such that for any extension I of J in $\mathbb{J}$ there is a pair that is $\langle i, j\rangle$-saturated over I.

Proof. Suppose this fails. Then given an ideal $J \in \mathbb{J}$ and pair of colors $\langle i, j\rangle$, we can extend J to an ideal $I \in \mathbb{J}$ over which there is no $\langle i, j\rangle$-saturated pair. By part (3) of Lemma 3.2, this remains true for any extension of the ideal I as well. Thus, by making successive extensions in $\mathbb{J}$ and running through the finitely many pairs of colors, we can arrive at an extension of J in $\mathbb{J}$ for which the previous lemma fails and this is a contradiction. □

Corollary 3.5. Suppose $\mathbb{J}$ is a non-empty set of ideals on X such that for any $I \in \mathbb{J}$ and I-positive $S \subseteq X$, there is an ideal in $\mathbb{J}$ extending $J \restriction S$. Then there is an ideal $J \in \mathbb{J}$ and a pair of colors $\langle i, j\rangle$ such that given any extension I of J in $\mathbb{J}$, any I-positive set S contains a pair of subsets $\langle A, B\rangle$ that are $\langle i, j\rangle$-saturated over I.

Roughly speaking, the ideal J has the property that for *any* extension I of J in $\mathbb{J}$, the $\langle i, j\rangle$-saturated pairs are dense in the I-positive sets.

3.2. **Bringing in Topology.** Our next goal is to assume that X carries a topology, and then isolate a set of circumstances that will enable the Raghavan-Todorčević construction to be carried out. The following is a modification of a key definition from their paper.

Definition 3.6. Suppose X is a topological space, $c : [X]^2 \to l$ for some $l < \omega$, and J is an ideal on X.

(1) Given a J-positive set S we say that a point $x \in X$ is an $\langle i, j\rangle$-winner in S over J if there is a sequence $\langle T_n : n < \omega\rangle$ of J-positive subsets of $S \setminus \{x\}$ such that
- any open neighborhood of x contains all but finitely many of the sets T_n,
- $\langle T_n, T_k\rangle$ is $\langle i, j\rangle$-saturated over J when $k < n$, and
- $c(x, y) = i$ for each $y \in \bigcup_{n<\omega} T_n$.

(2) We say that J is an $\langle i, j\rangle$-winning ideal if every J-positive set S contains an $\langle i, j\rangle$-winner in S over J. We say that J wins for c if J is an $\langle i, j\rangle$-winning ideal for some pair of colors $\langle i, j\rangle$

The above definition is actually not as technical as it may seem: if we declare a sequence $\langle T_n : n < \omega\rangle$ of J-positive sets to be $\langle i, j\rangle$-saturated if $\langle T_n, T_k\rangle$ is $\langle i, j\rangle$ saturated whenever $k < n$, then our definition says that x is an $\langle i, j\rangle$ winner in S if the set $T = \{y \in S : c(x, y) = i\}$ is J-positive and we can find an $\langle i, j\rangle$-saturated sequence of subsets of T converging to x.

Note as well that if S and T are equal modulo J, then a point x will be an $\langle i, j\rangle$ winner in S if and only if it is an $\langle i, j\rangle$-winner in T. Thus, J is an $\langle i, j\rangle$-winning ideal if and only if every J-positive set S contains a J-positive set of $\langle i, j\rangle$-winners in S over J. Given this observation, we are ready to push through the Raghavan-Todorčević construction in a slightly more general formulation than in [13].

Theorem 3 (The Raghavan-Todorčević Construction)**.** *Suppose X is a topological space and c colors $[X]^2$ with finitely many colors. If there is a countably complete ideal on X that wins for c, then X has a countable subspace Y without isolated points on which c takes on at most two colors.*

Proof. Suppose X and $c : [X]^2 \to l$ are given, and fix an ideal J and pair of colors $\langle i, j\rangle$ such that every J-positive set contains a J-positive set of $\langle i, j\rangle$-winners.[1] We build Y using a construction with ω stages.

At a stage n we will be working with a downward-closed subtree Q_n of ${}^{<\omega}\omega$ of finite height. This tree Q_n is partitioned into *leaves* L_n and *branching nodes* B_n with the usual meaning. Our construction will ensure that B_n is finite during the construction (so only finitely many elements of Q_n will have a successor), but the nodes will be fully branching: if $\sigma \in B_n$ then $\sigma^\frown\langle m\rangle$ is in Q_n for every $m < \omega$. Finally, we will be making use of the lexicographic order $<_{\text{lex}}$ on ${}^{<\omega}\omega$: if σ and τ are finite sequences and neither is an initial segment of the other, then $\sigma <_{\text{lex}} \tau$ means that $\sigma(i) < \tau(i)$ for the least i where the sequences differ.

Turning now to the construction, we assume at the end of stage n that our construction has provided us with the following situation:

(a) Each branching node σ of Q_n has been assigned a point x_σ and these various points are distinct.

(b) Each leaf τ of Q_n is assigned a J-positive set $T(\tau, n) \subseteq X$, and for each $y \in T(\tau, n)$ and $\sigma \in B_n$ we have $x_\sigma \neq y$. We think of $T(\tau, n)$ as the current candidates for being chosen as x_τ.

[1] Since J is now fixed, we will eliminate the phrase "over J", and just speak of $\langle i, j\rangle$-winners.

(c) For $\sigma \in B_n$, if we let $S(\sigma) = \{m < \omega : \sigma^\frown\langle m\rangle \in L_n\}$, then
- $S(\sigma)$ contains all sufficiently large $m < \omega$,
- $c(x_\sigma, y) = i$ for all $y \in \bigcup_{m \in S(\sigma)} T(\sigma^\frown\langle m\rangle, n)$, and
- The sequence of sets $\langle T(\sigma^\frown\langle m\rangle, n) : m \in S(\sigma)\rangle$ converges to x_σ.

(d) If σ and τ are distinct leaves in Q_n with $\sigma <_{\mathrm{lex}} \tau$, then the pair of sets $\langle T(\tau, n), T(\sigma, n)\rangle$ is $\langle i, j\rangle$-saturated over J.

Since the above is technical, we will attempt to give a description of the situation to guide intuition. At the end of stage $n > 0$ we will have put finitely many points into Y: these are the points x_σ for $\sigma \in B_n$. Our overarching goal is to make sure that when we add a new point x to Y, we ensure that $c(x, y)$ is either i or j for all the points y that were added previously. We will do this by using the lexicographic ordering to organize the construction: when it comes time to choose x_σ, we look back at the $\tau \in {}^{<\omega}\omega$ for which we have already selected x_τ, and choose x_σ so that for these τ we have

$$c(x_\sigma, x_\tau) = \begin{cases} i & \text{if } \sigma \text{ is a proper extension of } \tau \text{ or } \sigma <_{\mathrm{lex}} \tau, \\ j & \text{if } \tau <_{\mathrm{lex}} \sigma. \end{cases}$$

The difficulty is building in enough flexibility to ensure that such a choice is possible while still taking care of the topological requirements on Y.

To see how we make sure Y has no isolated points, suppose $\sigma \in {}^{<\omega}\omega$ and x_σ has been assigned. If we let $\tau_m = \sigma^\frown\langle m\rangle$ for $m < \omega$, then we are going to ensure that the sequence $\langle x_{\tau_m} : m < \omega\rangle$ converges to x_σ. The need to do this while also making sure we respect the demands of the preceding paragraph explains why "$\langle i, j\rangle$-winner" is defined as it is: we guarantee that x_σ is an $\langle i, j\rangle$-winner and this will be exactly what is needed to keep the construction alive: the sets assigned to leaves extending σ will witness that x_σ is an $\langle i, j\rangle$-winner.

Turning now to the details, start by letting Q_0 consist of the empty sequence $\langle\ \rangle$ and just define $T(\langle\ \rangle, 0) = X$. As we move through stage n to stage $n+1$, we need to accomplish several tasks. We assume that some bookkeeping procedure hands us a leaf $\sigma(n) \in L_n$, and to obtain Q_{n+1} we will add to Q_n all immediate successors of $\sigma(n)$ in ${}^{<\omega}\omega$. Since this transforms $\sigma(n)$ into a branching node in Q_{n+1}, we will need to assign to it some point $x_{\sigma(n)}$ from $T(\sigma(n), n)$. We will choose $x_{\sigma(n)}$ carefully as it must satisfy many constraints, and after making that decision we will need to shrink the sets $T(\tau, n)$ for $\tau \neq \sigma(n)$ in B_n to obtain $T(\tau, n+1)$, and finally assign sets $T(\sigma(n)^\frown\langle m\rangle, n+1)$ to all the new leaves we have created.

Since our bookkeeping identifies $\sigma(k)$ as we prepare for stage $k+1$, we will just write x_k instead of $x_{\sigma(k)}$ when discussing the assignment of points. Thus, to progress to stage $n+1$ we will need to choose $x_n \in T(\sigma(n), n)$ by paying attention to how it connects to the sets $T(\tau, n)$ for other leaves $\tau \neq \sigma(n)$ in L_n. More precisely, we want to arrange it so that if $\tau \neq \sigma$ is another leaf of Q_n we have:

(e) if $\tau <_{\mathrm{lex}} \sigma(n)$ then $\{y \in T(\tau, n) : c(x_n, y) = i\}$ is J-positive, and

(f) if $\sigma(n) <_{\mathrm{lex}} \tau$ then $\{y \in T(\tau, n) : c(x_n, x_y) = j\}$ is J-positive.

To do this, we need only look back to our working assumption (d). Give $\tau \neq \sigma_n$ in L_n, if $\sigma_n <_{\mathrm{lex}} \tau$ we know $\langle T(\tau, n), T(\sigma_n, n)\rangle$ is $\langle i, j\rangle$-saturated, so

$$(\forall^J x \in T(\sigma(n), n))(\exists^J y \in T(\tau, n))\,[c(x, y) = j]\,. \tag{3.3}$$

For similar reasons, if $\tau <_{\text{lex}} \sigma_n$ we have

$$(\forall^J x \in T(\sigma(n), n))(\exists^J y \in T(\tau, n))\,[c(x, y) = i]\,. \tag{3.4}$$

Since J is countably complete, it follows that almost every $x \in T(\sigma(n), n)$ will satisfy the requirements (e) and (f). Since the set of $\langle i, j\rangle$-winners in $T(\sigma(n), n)$ over $T(\sigma(n), n)$ is a J-positive subset of $T(\sigma(n), n)$, we can choose one that satisfies both (e) and (f) and declare that this will be x_n, the point assigned to the sequence $\sigma(n)$. We then define

$$T(\tau, n+1) = \{y \in T(\tau, n) : y \neq x_{n+1} \text{ and } c(x_{n+1}, y) = k\} \tag{3.5}$$

where $k = i$ if $\tau <_{\text{lex}} \sigma(n)$ and $k = j$ if $\sigma(n) <_{\text{lex}} \tau$. Notice that this choice ensures $T(\tau, n+1)$ is a J-large subset of $T(\tau, n)$, and guarantees that whenever our bookkeeping process finally hands us τ at some future stage, our available choices will be a subset of $T(\tau, n+1)$, and therefore we will connect to x_n in the proper way no matter which one is selected.

We still need to pay attention to the new leaves of Q_{n+1} and define J-large sets $T(\sigma(n)^\frown\langle m\rangle, n+1)$ for each $m < \omega$, but this where we use the fact that x_n is an $\langle i, j\rangle$-winner in $T(\sigma(n), n)$. Thus, there is a sequence $\langle T_m : m < \omega\rangle$ of J-positive subsets of $T(\sigma(n), n)$ such that for each $m < \omega$

- any open neighborhood of x_n contains T_m for all but finitely many $m < \omega$,
- $y \in T_m \Longrightarrow x_n \neq y$
- if $k < m$ then $\langle T_m, T_k\rangle$ is $\langle i, j\rangle$-saturated over J, and
- $c(x_n, y) = i$ for all $y \in \bigcup_{m<\omega} T_m$.

Now we define

$$T(\sigma(n)^\frown\langle m\rangle, n+1) = T_m, \tag{3.6}$$

and this leaves us in position to move on to the next stage of the construction.

This construction produces a countable set $Y = \{x_n : n < \omega\}$. Given a finite sequence σ from ${}^{<\omega}\omega$ there is a stage n for which $\sigma(n) = \sigma$. At that moment, we choose x_n and the sets $T(\tau, n+1)$ for the immediate successors τ of σ, and made sure that every neighborhood of x_n contains $T(\sigma^\frown\langle m\rangle, n+1)$ for all sufficiently large $m < \omega$. But this ensures that the points assigned to the successors of σ by our construction will converge to x_n, and hence Y is dense in itself.

To finish, we need only verify the range of $c \restriction [Y]^2$ is contained in $\{i, j\}$. This follows from our construction, though: if $n < k$ then x_k will be an element of $T(\sigma(k), k) \subseteq T(\sigma(k), n+1)$ and we have arranged every element y in $T(\sigma_k, n+1)$ satisfies $c(x_n, y) \in \{i, j\}$. More specifically, if $n < k < \omega$ then

- $c(x_n, x_k) = i$ if σ_n is an initial segment of σ_k.
- $c(x_n, x_k) = j$ if $\sigma_n <_{\text{lex}} \sigma_k$ and they are incomparable in ${}^{<\omega}\omega$.
- $c(x_n, x_k) = i$ if $\sigma_k <_{\text{lex}} \sigma_n$ and they are incomparable in ${}^{<\omega}\omega$.

□

Notice that if X is regular and first countable, then the space Y produced by the above construction is homeomorphic to $\mathbb{Q}$: regular second countable spaces are metrizable by Urysohn's Metrization theorem, and countable metrizable spaces without isolated points are homeomorphic to the rationals by an old result of Sierpiński.

4. Weakly precipitous ideals

In this section, we show that below a Ramsey cardinal, every normal fine countably complete ideal will automatically satisfy a weak version of precipitousness. We remind the reader of the characterization of precipitous ideals due to Galvin, Jech, and Magidor [7]:

Definition 4.1. Suppose J is an ideal on a set Z.

- The infinite game $\Game^{\text{prec}}(J)$ involves two players, Empty and Non-empty , who alternate in choosing J-positive subsets of Z, building a decreasing sequence $\langle A_n : n < \omega\rangle$. In the end Empty is declared the winner if $\bigcap_{n<\omega} A_n = \emptyset$.
- The ideal J is *precipitous* if and only if Empty does not have a winning strategy in the game $\Game^{\text{prec}}(J)$.

Precipitous ideals have been studied extensively in set theory primarily because they give rise to nice generic elementary embeddings added by relatively small notions of forcing: J is precipitous if and only if the generic ultrapower obtained after forcing with $\mathcal{P}(Z)/J$ is well-founded. It is consistent (relative to the existence of a measurable cardinal) that the non-stationary ideal on ω_1 is precipitous, and the existence of a precipitous ideal implies the existence of a measurable cardinal in an inner model. Thus, precipitous ideals are a prototypical example of the way in which small cardinals can exhibit large cardinal phenomena.

Returning to the main thread of our discussion, the game $\Game^{\text{prec}}(J)$ can be viewed as a contest between two players who are working to build an ideal extending J, with each player restricted to adding a single set at a time that must respect the earlier choices. Thus, at a point in the game the players will have built an ideal J_n (with $J_n = J \restriction A_n$), and the choice of A_{n+1} then gives us $J_{n+1} = J \restriction A_{n+1}$. Of course, this can be dualized to think of the players building a proper filter that extends J^*, and at a given stage the active player is allowed to put one new set into the filter.

How might we make this game easier for Non-empty to win? In the game $\Game^{\text{prec}}(J)$, once Empty selects A_n (and therefore J_n), Non-empty must extend to J_{n+1} by adding a single set to J_n. Thus, one obvious way of making things easier for him is to loosen the restrictions on choosing J_{n+1}: playing a larger ideal will put more constraints on Empty 's future moves (as they must be J_{n+1}-positive) and thus make the game potentially easier for Non-empty . This leads us to the following game first formulated by Jech in [9]:

Definition 4.2. Suppose J is an ideal on a set Z.

(1) The game $\Game^{\text{Wprec}}(J)$ is a game of length ω between two players Empty and Non-empty defined as follows:
   - During stage n, Empty will be choosing a subset A_n of Z, while Non-empty will be choosing an ideal J_n containing $Z \setminus A_n$.
   - The game begins with Empty selecting a J-positive subset A_0 of Z and Non-empty selected an ideal J_0 on Z extending $J \restriction A_0$.
   - Given J_n, Empty chooses a J_n-positive set A_{n+1}, and Non-empty responds with an ideal J_{n+1} on Z that extends $J_n \restriction A_{n+1}$.
   - After ω stages, Empty wins if and only if $\bigcap_{n<\omega} A_n = \emptyset$.

(2) The ideal J is *weakly precipitous* if Empty does not have a winning strategy in the game $\Game^{\mathrm{Wprec}}(J)$.

Notice that a run of the game produces a decreasing sequence $\langle A_n : n < \omega\rangle$ of J-positive subsets of Z, and an increasing sequence $\langle J_n : n < \omega\rangle$ of ideals on $Z \subseteq \mathcal{P}(X)$. Our use of the term *weakly precipitous* reflects the original usage Jech [9], but these ideals were considered earlier by Shelah [17, 16] in an equivalent (dual) formulation under the name of *almost nice* filters. Jech [9] proved that ideals dual to Shelah's almost nice filters are weakly precipitous in the above sense, and gave the first published account of Shelah's proof (which appeared much later in his paper [17]) that such ideals exist in the presence of a Ramsey cardinal.

Theorem 4. *Suppose Z is a stationary subset of $P(X)$ for some set X, and let $f : Z \to A$ for some set A. If there is a cardinal κ such that*

$$z \in Z \Longrightarrow |z \cap \kappa| > \beth_2(|A|) \tag{4.1}$$

then the Rudin-Keisler projection $J = f[\mathrm{NS}\restriction Z]$ of the non-stationary ideal on Z is a weakly precipitous ideal on A.

Proof. Assume by way of contradiction that Empty has a winning strategy in $\Game^{\mathrm{Wprec}}(J)$. As we play the game $\Game^{\mathrm{Wprec}}(J)$, our plan is for Non-empty to build a decreasing sequence $\langle Z_n : n < \omega\}$ of specially selected stationary subsets of $Z = Z_0$, making sure that his move at stage n satisfies

$$J_n = f[\mathrm{NS}\restriction Z_n]. \tag{4.2}$$

We start with $J_0 = J$. For each $a \in A$ let T_a consist of all finite sequences of odd length consisting of partial plays $\langle A_0, J_0, \dots, A_{n-1}, J_{n-1}, A_n\rangle$ in the game in which Empty is using her winning strategy, but for which a has not yet been eliminated (that is, $a \in A_n$). Since Empty is using a winning strategy we know that T_a forms a well-founded tree, and this induces a ranking function on the elements of T_a in the usual way. Given $\sigma \in T_a$, we let $\mathrm{rk}_a(\sigma)$ denote the rank of σ in T_a. Since Empty is playing subsets of A and Non-empty is choosing ideals on A, there are at most $\beth_2(|A|)$ possible moves for each player, and so $|T_a| \leq \beth_2(|A|)$. Thus, we know

$$\mathrm{rk}_a(\sigma) < \beth_2(|A|)^+ \tag{4.3}$$

for any $a \in A$ and $\sigma \in T_a$.

Now assume we have been playing a game in which Empty is using her strategy, and we have arrived at the sequence $\sigma = \langle A_0, J_0, \dots, A_n, J_n, A_{n+1}\rangle$. Since Empty is using her winning strategy, for each $a \in A_{n+1}$ the sequence σ is in the tree T_a and therefore has been assigned corresponding rank $\mathrm{rk}_a(\sigma) < \beth_2(|Z|)^+$. Putting this together gives us a function $\rho_{n+1} : A_{n+1} \to \beth_2(|Z|)^+$ defined by

$$\rho_{n+1}(\alpha) = \mathrm{rk}_\alpha(\sigma). \tag{4.4}$$

At the same time, we know that on the side the Non-empty player has been building a decreasing sequence $\langle Z_i : i \leq n\rangle$ of stationary sets, and making sure that his play J_i is the projection $f[\mathrm{NS}\restriction Z_i]$. He takes the set A_{n+1} and defines

$$Y_{n+1} = f^{-1}[A_{n+1}] \cap Z_n. \tag{4.5}$$

Since A_{n+1} is J_n-positive and $J_n = f[\mathrm{NS}\restriction Z_n]$, we know that Y_{n+1} is a stationary subset of Z_n.

What comes next is an idea that traces back (at least) to Magidor [12]. We define a function $\tilde{\rho}_{n+1}$ mapping Y_{n+1} to κ as follows: given $z \in Y_{n+1}$ we define $\tilde{\rho}_{n+1}(z)$ to be the $\rho_{n+1}(f(z))^{\text{th}}$ element of the increasing enumeration of $z \cap \kappa$.

This makes sense because of our assumption (4.1): for any $z \in Z$ we know that $z \cap \kappa$ has order-type greater than $\rho_{n+1}(f(z))$, and so $\tilde{\rho}_{n+1}(z)$ will be a well-defined element of z. Moreover, since $\tilde{\rho}_{n+1}$ is regressive on the stationary set Y_{n+1}, there must be a stationary $Z_{n+1} \subseteq Y_{n+1}$ on which $\tilde{\rho}_{n+1}$ is constant, say with value γ_{n+1}. The ideal $\pi[\mathrm{NS} \restriction Z_{n+1}]$ is a legal move for Non-empty and so this is what he plays as his move J_{n+1}.

We obtain a contradiction by showing that the sequence of ordinals γ_n is strictly decreasing. First, note that $\rho_{n+1}(a) < \rho_n(a)$ for all $a \in A_{n+1}$ by the way ranking functions work: for each $a \in A_{n+1}$ the partial play σ_{n+1} is a proper extension of σ_n in the well-founded tree T_a. Given $z \in Z_{n+1}$ we know $f(z)$ is an element of A_{n+1} and so $\rho_{n+1}(f(z))$ is less than $\rho_n(f(z))$. But given our definitions of $\tilde{\rho}_{n+1}$ and $\tilde{\rho}_n$, we conclude

$$\gamma_{n+1} = \tilde{\rho}_{n+1}(z) < \tilde{\rho}_n(z) = \gamma_n, \tag{4.6}$$

as required. This contradiction means that Empty cannot have a winning strategy in $\Game^{\mathrm{Wprec}}(J)$, and we are done. □

As a corollary, we see that below a Ramsey cardinal many ideals are automatically weakly precipitous.

Corollary 4.3. If κ is a Ramsey cardinal, then any normal fine σ-complete ideal on a set in V_κ is weakly precipitous. In particular, if there is a Ramsey cardinal then every normal ideal on ω_1 is weakly precipitous.

Proof. The proof uses the fact that a Ramsey cardinal κ is *completely Jónsson*, which means that κ is strongly inaccessible and for any stationary $Z' \subseteq \mathcal{P}(X')$ in V_κ, the set

$$\tilde{Z} := \{\tilde{z} \subseteq V_\kappa : \tilde{z} \cap X \in Z' \text{ and } |\tilde{z} \cap \kappa| = \kappa\} \tag{4.7}$$

is stationary.[2] Note that in the above situation, the non-stationary ideal on Z' is the Rudin-Keisler image of the non-stationary ideal on $\tilde{Z}$ under the natural projection sending $\tilde{z}$ to $\tilde{z} \cap X'$.

Now suppose J is a normal fine σ-complete ideal on $Z \subseteq \mathcal{P}(X)$ in V_κ. By a theorem of Burke [2] (or really, the version of Burke's result presented by Foreman in [6]) we can in V_κ find sets Z' and X' such that

- Z' is a stationary subset of $\mathcal{P}(X')$,
- $z' \in Z' \Longrightarrow z' \cap X' \in Z'$, and
- $J = \pi'[\mathrm{NS} \restriction Z']$, where $\pi'(z') = z' \cap X$.

Thus, we can represent J as the Rudin-Keisler image of the non-stationary ideal restricted to some stationary set under the natural projection.

Now we use the fact that κ is completely Jónsson: applying the definition we find a $\tilde{Z}$ satisfying (4.7) via the natural map $\tilde{\pi}(\tilde{z}) = \tilde{z} \cap Z'$. The composition $\pi = \pi' \circ \tilde{\pi}$ shows that ideal J satisfies the assumptions of Theorem 4, and hence J must be weakly precipitous. □

[2]See Larson [11] for the relevance of completely Jónnson cardinals to stationary tower forcing, and for a proof (Remark 2.3.3) that Ramsey cardinals have this property.

Note that in contrast to precipitous ideals, the class of weakly precipitous ideals is closed under Rudin-Keisler projections:

Proposition 4.4. If I is a weakly precipitous ideal on Y and f maps Y to Z, then the ideal $J = f[I]$ is also weakly precipitous.

Proof. It suffices to assume Empty has a winning strategy in $\Game^{\mathrm{Wprec}}(J)$ and show this gives rise to one in $\Game^{\mathrm{Wprec}}(I)$. This is done in the natural way: she promises that her moves in the upstairs game will have the form $A_n = f^{-1}[B_n]$, where B_n is a move in the downstairs game dictated by her winning strategy. In details, her strategy upstairs results in partial plays $\langle A_0, I_0, \dots, A_n, I_n\rangle$ for which there is a corresponding run $\langle B_0, J_0, \dots, B_n, J_n\rangle$ in the downstairs game satisfying

- $J_k = f[I_k]$,
- $A_k = f^{-1}(B_k)$, and
- B_{k+1} is her strategy's response to $\langle B_0, J_0, \dots, B_k, J_k\rangle$ if $k < n$.

Since her downstairs strategy wins, it follows that $\bigcap_{n<\omega} A_n$ must be empty as well which results in a win for her. $\square$

We need to generalize the game $\Game^{\mathrm{Wprec}}$ slightly in order to take advantage of our work in the previous section. This will involve restricting $\mathsf{Non\text{-}empty}$ somewhat and require that he play ideals of a certain form.

Definition 4.5. Let $\mathbb{J}$ be a non-empty collection of ideals on some set X.

(1) $\mathbb{J}$ is *closed under restrictions* if given $J \in \mathbb{J}$ and $A \in J^+$, the ideal $J\restriction A$ is also in $\mathbb{J}$.

(2) $\mathbb{J}$ is *stable under restrictions* if given $J \in \mathbb{J}$ and $A \in J^+$, there is an extension of $J\restriction A$ in $\mathbb{J}$.

(3) If J is an ideal on X then the game $\Game^{\mathrm{Wprec}}_{\mathbb{J}}(J)$ is the version of the game $\Game^{\mathrm{Wprec}}(J)$ where $\mathsf{Non\text{-}empty}$ is restricted to choosing ideals from $\mathbb{J}$.

(4) We say $\mathbb{J}$ is a weakly precipitous family of ideals on X if Empty does not have a winning strategy in $\Game^{\mathrm{Wprec}}_{\mathbb{J}}(J)$ for any $J \in \mathbb{J}$.

Corollary 4.6. If κ is a Ramsey cardinal and $X \in V_\kappa$ then the collection $\tilde{\mathbb{J}}$ of normal countably-complete fine ideals on X is a restriction-closed weakly precipitous family of ideals. If $f : X \to Y$ then $\mathbb{J} = f[\tilde{\mathbb{J}}] = \{f[I] : I \in \tilde{\mathbb{J}}\}$ is a restriction-closed weakly precipitous family of countably complete fine ideals on Y.

Proof. This follows immediately from the proof of Theorem 4, Corollary 4.3, and Proposition 4.4. $\square$

We close this section with a remark that having suitable inner models containing Ramsey cardinals is often sufficient to conclude that relevant ideals are weakly precipitous. For example, if there is an inner model M of ZFC with, say, $V_{\omega+4} \subseteq M$ that contains a Ramsey cardinal, then the collection of normal ideals on ω_1 is weakly precipitous: if Empty has a winning strategy then this will be absolute between V and M as it involves only objects of small rank. This highlights why weakly precipitous ideals are useful in studying cardinal arithmetic: if the singular cardinals hypothesis fails (that is, cardinal arithmetic is "interesting") then we can generally find such inner models, the the weakly precipitous filters we obtain can be used to prove theorems. This is the way Shelah proceeds in [17, 16] and their continuations.

5. Finding Winning Ideals

Our goal in this section is to show that we can produce winning ideals from a combination of topological and set-theoretic assumptions. We remind the reader of the following definitions:

Definition 5.1. Let X be a topological space.

(1) X has a *point-countable base* if the topology on X has a base $\mathcal{B}$ of open sets with the property at each $x \in X$ is a member of at most countably many elements of $\mathcal{B}$.

(2) X is *left-separated* if there is a well-ordering of X such that each initial segment is a closed set.

Fact 1. Suppose X is a regular space with a point countable base that is not left-separated.

- X is first countable, and
- $\tilde{X} := \{Y \in [X]^\omega : \overline{Y} \setminus Y \neq \emptyset\}$ is stationary in X^ω

The first of these is immediate given the definition of point countable base, but the second is a theorem of Bill Fleissner [5].

Theorem 5. *Suppose κ is a Ramsey cardinal and $X \in V_\kappa$ is a regular topological space with a point countable base that is not left-separated. Then for any coloring c of $[X]^2$ with finitely many colors, there is a countably complete ideal on X that wins for c.*

We will prove this theorem after obtaining a series of lemmas. To start, let $c : [X]^2 \to l < \omega$ be a coloring, let $\mathcal{B}$ be a point-countable base for X, and let $\tilde{\mathbb{J}}$ be the collection of normal countably complete fine ideals on $\tilde{X} \subseteq \mathcal{P}(X)$. We know $\tilde{\mathbb{J}}$ is non-empty because $\tilde{X}$ is stationary, and since we are inside V_κ our work in the last section tells us that $\tilde{\mathbb{J}}$ is a weakly precipitous family.

Now we define a function $p : \tilde{X} \to X$ by letting $p(Y)$ be some element of $\overline{Y} \setminus Y$, and let $\mathbb{J}$ be the set of ideals of the form $J = p[\tilde{J}]$ for some $\tilde{J} \in \tilde{\mathbb{J}}$. By Corollary 4.6, $\mathbb{J}$ is a weakly precipitous family of countably complete ideals on X.

Lemma 5.2. Suppose $f : X \to \mathcal{B}$ be a neighborhood assignment for X using the open sets from $\mathcal{B}$. For any $J \in \mathbb{J}$, we can find a set $U \in \mathcal{B}$ such that $U = f(x)$ for a J-positive set of $x \in X$.

Proof. If the lemma fails, then there is an ideal $J \in \mathbb{J}$ such that for any $U \in \mathcal{B}$ the set of $x \in X$ with $U = f(x)$ is in J. Choose an ideal $\tilde{J} \in \tilde{\mathbb{J}}$ with $J = \pi[\tilde{J}]$, so $\tilde{J}$ is a normal countably complete fine ideal on $\tilde{X}$.

Given $Y \in \tilde{X}$ we know that $f(p(Y)) \cap Y$ is non-empty by the definition of p, and so we can define a regressive function g on $\tilde{X}$ by setting $g(Y)$ to be some element of $Y \cap f(p(Y))$. Since $\tilde{J}$ is a normal ideal on $\tilde{X}$, there is a single $y \in X$ and a $\tilde{J}$-positive set $\tilde{Y} \subseteq \tilde{X}$ on which g is constant with value y. Since $\mathcal{B}$ is point-countable and $f(p(Y))$ is a neighborhood of y in $\mathcal{B}$ for each $Y \in \tilde{Y}$, we can invoke the countable completeness of $\tilde{J}$ and assume that there is a $U \in \mathcal{B}$ such that $f(p(Y)) = U$ for all $Y \in \tilde{Y}$. This leads to a contradiction: our assumptions are that $J = p[\tilde{J}]$ and J-almost every $x \in X$ satisfies $f(x) \neq U$, but this implies $f(p(Y)) \neq U$ for $\tilde{J}$-almost every $Y \in \tilde{X}$. □

In fact, we get a little more.

Corollary 5.3. Suppose $f : X \to \mathcal{B}$ be a neighborhood assignment for X. For any $J \in \mathbb{J}$ and J-positive set S we can find a set $U \in \mathcal{B}$ such that $U = f(x)$ for a J-positive subset of S.

Proof. Since $\mathbb{J}$ is stable under restrictions, we can fix an $I \in \mathbb{J}$ extending $J \restriction S$. An application of Lemma 5.2 to the ideal I then gives us what we need. $\square$

By Corollary 3.5, we can fix an ideal $J \in \mathbb{J}$ and a pair of colors $\langle i, j\rangle$ such that for any ideal I extending J in $\mathbb{J}$, any I-positive set contains an $\langle i, j\rangle$-saturated pair of I-positive sets. This ideal J will be fixed for the remainder of this section, and we will show that it is an $\langle i, j\rangle$-winning ideal by connecting its properties back to the topology on X.

Definition 5.4. For $x \in X$, let $\langle U^x_n : n < \omega\rangle$ be a decreasing sequence of sets in U that form a neighborhood base for x in X.

Lemma 5.5. If I is an extension of J in $\mathbb{J}$, then for any $n < \omega$ and I-positive set S we can find I-positive subsets A and B of S and an open set $U \in \mathcal{B}$ such that

- $\langle A, B\rangle$ is an $\langle i, j\rangle$-saturated pair over I, and
- $U = U^x_n$ for all $x \in A \cup B$.

Proof. Corollary 5.3, S has an I-positive subset T on which the map sending x to U^x_n is constant, and this determines U. Since I extends J in $\mathbb{J}$, we know that T contains a pair of I-positive sets that are $\langle i, j\rangle$-saturated over I, and we are done. $\square$

Proof of Theorem 5. To finish the proof of Theorem 5, we show that our ideal J wins for $\langle i, j\rangle$. It suffices to prove that every J-positive set A contains an $\langle i, j\rangle$-winner over J, and this will be done using the properties we have established for J combined with the fact that $\mathbb{J}$ is a weakly precipitous family of ideals on X.

Let I be some extension of $J \restriction A$ in $\mathbb{J}$. We describe a strategy for Empty in the game $\Game^{\mathrm{Wprec}}_{\mathbb{J}}(I)$ designed to produce such a point. We start in the obvious way by choosing $A_0 = A$.

Stage n+1: At this point, Empty will have A_n and J_n at her disposal. Since J_n extends J, we apply Lemma 5.5 to find an open set U_n and a pair $\langle A_{n+1}, B_{n+1}\rangle$ such that

- $\langle A_{n+1}, B_{n+1}\rangle$ are J_n-positive subsets of A_n that are $\langle i, j\rangle$-saturated over J_n, and
- U_n is an open set with $U^x_n = U_n$ for all $x \in A_{n+1} \cup B_{n+1}$.

Empty now plays A_{n+1} as her next move.

Since this strategy does not win for Empty , there is some run of the game where $\bigcap_{n<\omega} A_n$ is non-empty, and we show that any $x \in \bigcap_{n<\omega} A_n$ is a $\langle i, j\rangle$-winner over J. Note that since x is in A_{m+1} we know that $U^x_n, = U_n$ for all n, and our construction guarantees

$$A_{n+1} \cup B_{n+1} \subseteq \bigcap_{i\leq n} U_i. \tag{5.1}$$

Thus, any open neighborhood of x will contain $A_n \cup B_n$ for all but finitely many n.

Given $n < \omega$ we know that $\langle A_{n+1}, B_{n+1}\rangle$ is $\langle i, j\rangle$-saturated over J_n (and hence over J), and this tells us that x is i-large in B_{n+1} over J. Thus, if we define

$$T_n := \{y \in B_{n+1} : x \neq y \text{ and } c(x, y) = i\}, \tag{5.2}$$

we know that T_n is J-positive, and by the previous paragraph any open neighborhood of x will contain all but finitely many of the sets T_n.

Thus, we are left with verifying that $\langle T_n, T_k\rangle$ is $\langle i, j\rangle$-saturated over J whenever $k < n$. Our construction guarantees that the pair $\langle A_{k+1}, B_{k+1}\rangle$ is $\langle i, j\rangle$-saturated over J. But T_n is a J-positive subset of A_{k+1} and T_k is a J-positive subset of B_{k+1}, so $\langle T_n, T_k\rangle$ is also $\langle i, j\rangle$-saturated over J, finishing the proof of Theorem 5. □

6. Galvin's Conjecture does not imply $0^{\#}$

If we look back at the proof of Theorem 5, it is clear that what makes the proof go through for the space X is the existence of a suitable collection of ideals on the stationary set $\tilde{X} \subseteq [X]^{\omega}$. If we are interested in obtaining the partition theorem only for colorings of uncountable sets of reals, then we can push through the proof assuming only the existence of a suitable collection of normal ideals on ω_1. The proof is outlined below.

Theorem 6. *If there is a weakly precipitous family of normal ideals on ω_1 that is stable under restrictions, then for any uncountable set of reals X and any coloring of $[X]^2$ with finitely many colors, there is a subset Y of X homeomorphic to $\mathbb{Q}$ on which the coloring takes on at most two values.*

Proof. If X is a set of reals of cardinal $\aleph_1$, then by using a bijection we can just assume that the underlying space of X is ω_1 and ω is dense in X. If we assume this, then $\tilde{X} := \{Y \in [X]^{\omega} : \overline{Y} \setminus Y \neq \emptyset\}$ contains each infinite countable ordinal α, and these are a closed unbounded subset of $[X]^{\omega}$. Thus, any normal ideal on $\tilde{X}$ is, via its restriction to the closed unbounded subset $\omega_1 \subseteq \mathcal{P}(\omega_1)$, really just a normal ideal on ω_1 in the usual sense.

□

The key to showing that Galvin's Conjecture may hold in a generic extension of the constructible universe L is some work of Donder and Levinski [4] from the late 1980s. They show that it is consistent for ω_1 to carry a weakly precipitous ideal in a forcing extension of L, but their proof actually shows that there is a family of ideals as in the assumptions of Theorem 6, and our contribution is to just point out how their results are enough to get what we need.

Definition 6.1. Suppose $\kappa < \lambda$ are cardinals. We say that κ is λ-*semi-precipitous* if there is a notion of forcing $\mathbb{P}$ such that every condition in $\mathbb{P}$ forces that $V[G]$ contains an elementary embedding $j : H(\lambda) \to M$ with critical point κ such that M is transitive. The cardinal κ is *semi-precipitous* if it is λ-semi-precipitous for all $\lambda > \kappa$.

Notice that the definition does not contain any assumptions about the cardinality of $\mathbb{P}$, and asking κ to be λ-semi-precipitous means we have a generic elementary embedding with critical point κ added by some notion of forcing, and the target model is well-founded out to $j(\lambda)$.

Theorem 7. *If ω_1 is $\beth_2(\omega_1)^{+}$-semi-precipitous, then there is a weakly precipitous family of normal ideals on ω_1 that is stable under restrictions.*

Proof. Let $\mathbb{P}$ be the notion of forcing giving rise to the generic elementary embedding j, and let $\dot{U}$ be a $\mathbb{P}$-name for the V-normal V-ultrafilter induced by j, so

$$\Vdash_{\mathbb{P}} \dot{U} = \{A \in \mathcal{P}^V(\omega_1) : \omega_1 \in j(A)\}. \tag{6.1}$$

Then for $p \in \mathbb{P}$ we let J_p be the ideal of subsets of ω_1 that p decides are *not* going to be in $\dot{U}$, so

$$J_p = \{A \subseteq \omega_1 : p \Vdash (\omega_1 \setminus A) \in \dot{U})\}. \tag{6.2}$$

Each J_p is a normal ideal on ω_1, and a set A is J_p positive if and only if some $q \leq p$ forces that $\check{A}$ is in $\dot{U}$. From this, it follows that $\mathbb{J} = \{J_p : p \in \mathbb{P}\}$ is a non-empty set of normal ideals on ω_1 that is stable under restrictions.

The proof that this collection is weakly precipitous mirrors the proof of Theorem 4. Suppose Empty has a winning strategy in the game $\eth^{\mathrm{Wprec}}_{\mathbb{J}}(J_p)$ for some $p \in \mathbb{P}$, and we will get a contradiction by showing how $\mathsf{Non\text{-}empty}$ can use the notion of forcing $\mathbb{P}$ to defeat the alleged winning strategy.

Just as in the previous proof, for each $\alpha < \omega_1$ we let T_α be the tree consisting of all partial plays of odd length versus the winning strategy that have yet to eliminate α. Since Empty is playing a winning strategy, the tree T_α is well-founded, and we let $\mathrm{rk}_\alpha(\sigma)$ be the rank of σ in T_α. Given the constraints of the game, we know

$$|\,\mathrm{rk}_\alpha(\sigma)| \leq \beth_2(\omega_1). \tag{6.3}$$

Given a partial play σ of the game in which $\mathsf{Non\text{-}empty}$ must respond to A_{n+1}, we work with the rank function ρ_{n+1} mapping A_{n+1} to $\beth_2(\omega_1)^+$ given by

$$\rho_{n+1}(\alpha) = \mathrm{rk}_\alpha(\sigma). \tag{6.4}$$

As we play the game, $\mathsf{Non\text{-}empty}$ will be building a decreasing sequence of conditions $\langle p_n : n < \omega\rangle$ in $\mathbb{P}$ with $p_0 = p$ and a sequence $\langle \gamma_n : n < \omega\rangle$ of ordinals such that

$$p_n \Vdash \check{A}_n \in \dot{U}, \tag{6.5}$$

and

$$p_n \Vdash (\mathrm{otp}([\rho_n]_{\dot{U}}) = \check{\gamma}_n. \tag{6.6}$$

Note that if a condition forces A_n to be in $\dot{U}$, then the condition will force that $[\rho_n]_{\dot{U}}$ is an ordinal less than $\beth_2(\omega_1)^+$ in the generic ultrapower (as we are well-founded out that far), and so we can extend further to decide which ordinal this is.

So given a partial play σ in which Empty has been using her winning strategy, and for which $\mathsf{Non\text{-}empty}$ must respond to A_{n+1}, he extends p_n to p_{n+1} that forces A_{n+1} to be in $\dot{U}$ and also decides which ordinal corresponds to $[\rho_{n+1}]_{\dot{U}}$. Note that this ordinal is strictly less than γ_n because ρ_{n+1} is strictly below ρ_n on A_{n+1}. This is impossible, as otherwise we produce an infinite decreasing sequence of ordinals, and so Empty cannot have a winning strategy, and $\mathbb{J}$ is weakly precipitous. $\square$

The proof of Theorem 4 can be reframed using the above argument, working with a generic ultrapower "upstairs" on a suitable stationary set that projects down to ω_1, as the elementary embedding added will factor through an embedding with critical point ω_1. Returning to our discussion, the following theorem will now get us to our goal for the section:

Theorem 8 (Theorem 8 of Donder-Levinsky [4])**.** *If there is a semi-precipitous cardinal, then in L there is a notion of forcing $\mathbb{P}$ such that ω_1 is semi-precipitous in the extension $L[G]$.*

Proof. We briefly sketch their argument for the benefit of the reader, and proofs of the various assertions are found in [4]. A semi-precipitous cardinal κ in V is also semi-precipitous in L, and κ will be strongly inaccessible in L. If $\mathbb{P} = \mathrm{Coll}(\omega, \kappa)$ and let G be an L-generic filter on $\mathbb{P}$m, then in $[L[G]$ the cardinal κ is now ω_1, and an argument relying on the fact that $|\mathbb{P}| = \kappa$ shows that it will remain semi-precipitous. □

Corollary 6.2. If there is a semi-precipitous cardinal then Galvin's Conjecture can be forced to hold in a set-generic extension of L. Thus, Galvin's Conjecture does not imply the existence of $0^{\#}$.

7. Open Questions

7.1. **Large Cardinals.** This paper only partially addresses the most obvious open question arising from Raghavan and Todorčević's work, as we only succeed in lowering the large cardinals needed to obtain the result and do not manage to eliminate them. Thus, we ask:

Question 1. Does Galvin's Conjecture have large cardinal strength?

In a related vein, we ask the following:

Question 2. Suppose that every coloring of a non-σ-discrete metrizable space with finitely many colors reduces to two colors on a homeomorphic copy of $\mathbb{Q}$. Must $0^{\#}$ exist?

Presumably a class-generic extension of L along the lines of the Donder-Levinsky argument will do the job, but we have not checked.

In a very difficult paper, Shelah [18] shows that assuming the existence of a cardinal fairly low in the Mahlo hierarchy, one can build a model in which every coloring of $\mathbb{R}$ with finitely many colors reduces to two colors on some uncountable set. Thus, the weak version of Galvin's Conjecture where one simply colors all of $[\mathbb{R}]^2$ rather than just $[X]^2$ for an uncountable X can be forced using very mild large cardinal assumptions. Can we get the stronger version? More precisely:

Question 3. Assuming the large cardinal hypothesis of [18], can we force Galvin's Conjecture to hold?

7.2. **Higher Dimensions.** What about higher-dimensional colorings rather than just colorings of pairs? Here, the background restriction is the generalization of Sierpinski's example: There is a coloring of $[\mathbb{R}]^k$ with $k!(k-1)!$ colors that takes on every color on countable dense-in-itself subset of $\mathbb{R}$. Laver (unpublished) proved that for each $k \geq 1$ that there is a finite t_k such that for any coloring of $[\mathbb{Q}]^k$ into finite pieces, there is a $Y \subseteq Q$ order-isomorphic to $\mathbb{Q}$ on which the coloring takes at most t_k values.[3]

Raghavan and Todorčević have shown that there is no direct generalization of their proof of Galvin's Conjecture as as the size of the continuum suddenly becomes relevant. Their recent paper [14] shows the following:

[3]The exact value is $t_k = T_{2k-1}$, where $\tan(x) = \sum_{n=0}^{\infty} \frac{T_n}{n!} x^n$, as shown by Denis Devlin [3]. A proof of Devlin's result appears in the monograph [20] by Todorčević.

Theorem 9 (Raghavan-Todorčević)**.** *If $|\mathbb{R}| = \aleph_n$ then there is a coloring of $[\mathbb{R}]^{n+2}$ with countably many colors that assumes every value on every subset of $\mathbb{R}$ homemorphic to $\mathbb{Q}$. More generally, there is such a coloring of $[X]^2$ whenever X is a Hausdorff space of cardinality $\aleph_n$.*

This theorem is a broad generalization of Baumgartner's earlier theorem on colorings of $[\mathbb{Q}]^2$ from [1], and their proof relies on a set mapping theorem of Kuratowski [10]. On the other hand, Shelah [15] shows that higher-dimensional analogues of (a weak form of) Galvin's Conjecture are consistent relative to large cardinals. He proves that starting with an Erdős cardinal one can build a model of ZFC in which the continuum is (necessarily, given Theorem 9) greater than $\aleph_\omega$ and such that there is a function $h : \omega \to \omega$ such that for any coloring c of $[\mathbb{R}]^n$ with finitely many colors, there is an uncountable subset H on which the coloring assumes at most $h(n)$ values.[4] Since every such H contains a copy of the rationals, this gets us the consistency of a higher-dimensional form of Galvin's Conjecture for colorings of $[\mathbb{R}]^2$ at the price of having a very large continuum. There are many natural questions to ask, so we put forward only the following one as represenative:

Question 4. Is it consistent (relative to large cardinals) that $\aleph_1 < 2^{\aleph_0} < \aleph_\omega$ and for every $c : [\mathbb{R}]^3 \to l$ with $l < \omega$ there is a copy of $\mathbb{Q}$ on which c takes at most $12 = 3!2!$ colors?

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