GALVIN'S CONJECTURE AND WEAKLY PRECIPITOUS IDEALS

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Abstract. We show that the Raghavan-Todorčević proof [8] of a longstanding conjecture of Galvin from Woodin cardinals can be carried out under weaker assumptions that follow from the existence of Ramsey cardinals, and that do not imply the existence of $0^\sharp$.

1. Introduction

Our starting point is recent work of Raghavan and Todorčević [8] where they use stationary tower forcing to show that the existence of a Woodin cardinal implies a partition relation for uncountable sets of reals first conjectured by Galvin (see, for example, page 182 of Baumgartner's [1]):

> If X is an uncountable set of reals and $c : [X]^2 \to l$ for some $l < \omega$, then X contains a subset homeomorphic to the rationals on which c assumes at most two values.

They were able to extend the partition relation to a much broader class (and essentially optimal) class of topological spaces. In this paper, we show how to obtain full results from much weaker large cardinal assumptions grounded in work of Shelah [10] and originally used to obtain bounds in cardinal arithmetic for strong limit cardinal fixed points of uncountable cofinality.

Definition 1.1. Suppose X and Y are topological spaces.

(1) For non-zero natural numbers k, l, and t, the symbol
$$X \to (Y)^k_{l,t}$$
means that for every set L of cardinality l and every coloring $c : [X]^k \to l$, there is a subsets Y' of X homeomorphic to Y an a set $T \subseteq L$ of cardinality t such that the range of $c \restriction [Y']^k$ is contained in T.

(2) For a natural number $k \leq 1$ the *k-dimensional Ramsey degree of Y inside X* (if it exists) is the least natural number $t \leq 1$ such that $X \to (Y)^k_{l,t}$ for all $l < \omega$.

Throughout this paper we will usually be working with proper countably-complete fine normal ideals on $[X]^\omega$ where X is topological space. Our notation concerning these objects is standard and we generally follow Foreman [4]:

Definition 1.2. Suppose X is a set, $Z \subseteq \mathcal{P}(X)$, and J is an ideal on Z.

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(1) J^* is the filter dual to J, so $Y \subseteq \mathcal{P}(X)$ is in J^* if and only if its complement is in J, and we say that J *concentrates on* Y.

(2) If Y is a J-positive subset of $\mathcal{P}(X)$ (that is, $Y \notin J$) then we define the *restriction* $J\restriction Y$ *of* J *to* Y by putting a set $A \subseteq Z$ into $J\restriction Y$ if and only if $A \cap Y \in J$. Note that $J\restriction Y$ concentrates on the set Y.

(3) J is *countably complete* if the union of countably many sets in J is again in J. If κ is a regular cardinal then J is $<\kappa$-complete if J is closed under unions of size less than κ. Thus, J is countably complete if and only if J is $<\aleph_1$-complete.

(4) Given $Y \subseteq [X]^\omega$, a function $f : Y \to X$ is *regressive* if $f(y) \in Y$ for all $y \in Y$. The ideal J is *normal* if for any J-positive Y and regressive $f : Y \to X$ there is a J-positive $Z \subseteq Y$ and $x \in X$ such that $f(z) = x$ for all $z \in Z$.

(5) J is *fine* if for every $x \in X$ the set $\{z \in Z : x \in z\}$ is in J^*, that is, if J-almost every $z \in Z$ contains x.

We are a little loose with our language concerning ideals of the form $J\restriction Y$: such an ideal can be viewed both as an ideal on Y and an ideal on Z,[1] but this sloppiness is harmless: what is important is that $J\restriction Y$ is the minimal ideal extending J that concentrates on Y.

As may be apparent from the above definition, we also use terms "J-almost every $z \in Z$" to mean that the set of such z is in J^*, and similarly "for a J-positive set of $z \in Z$" means that the set of such z is not in J. We also abbreviate these statements using quantifiers, so for example if ϕ is some formula then

$$(\forall^J z \in Z)\phi(z) \iff \{z \in Z : \phi(z)\} \in J^*,$$

and

$$(\exists^J z \in Z)\phi(z) \iff \{z \in Z : \phi(z)\} \notin J.$$

Countably complete normal ideals of the sort we consider satisfy an *a priori* stronger property that admits a more general notion of "regressive function". The facts that we use are listed in the following lemma and corollary.

Lemma 1.3. Suppose J is a countably complete normal ideal on a subset Z of $\mathcal{P}(X)$.

(1) If S is J-positive then any function $g : S \to [X]^{<\omega}$ that satisfies $g(z) \in [z]^{<\omega}$ is constant on a J-positive subset of S.

(2) If S is J-positive and $\mathfrak{A} = \langle A, (f_n : n < \omega)\rangle$ is an algebra[2] with $X \subseteq A$, then any function $g : S \to A$ with $g(z) \in \mathrm{cl}_{\mathfrak{A}}(z)$ is constant on a J-positive subset of S.

Proof. For (1), suppose $g : S \to [X]^{<\omega}$ with $g(z) \in [z]^{<\omega}$. Since J is countably complete, we may as well assume that $g(z)$ has fixed cardinality n for all $z \in S$. Since J is normal, we know that there is a J-positive $S_1 \subseteq S$ such that all $z \in S_1$ agree on the first element of $g(z)$. By repeating this n times, we arrive at a J-positive subset of S on which g is constant.

[1] Or even on $\mathcal{P}(X)$ for that matter.

[2] That is, A is a non-empty set equipped with countably many functions, with $f_n : [A]^{k_n} \to A$ for some $k_n < \omega$. If $Y \subseteq A$ then $\mathrm{cl}_{\mathfrak{A}}(Y)$ denotes the closure of Y under the functions of $\mathfrak{A}$.

For (2), each $z \in S$ there is an $n < \omega$ such that $g(z) \in \operatorname{ran}(f_n \restriction [z]^{k_n})$. Since J is countably complete, we can stabilize this value of n on a J-positive set and thus we may assume that for some fixed $n < \omega$,

$$z \in S \Longrightarrow (\exists a \in [z]^{k_n}) \, [g(z) = f_n(a)] \, . \tag{1.1}$$

By (1), we can stabilize a on a J-positive subset T of S so if we let $\alpha = f_n(a)$ then

$$z \in T \Longrightarrow g(z) = f_n(a) = \alpha, \tag{1.2}$$

as required. □

Our conventions concerning elementary submodels of $H(\chi)$ are standard: we will be working with elementary submodels of expansions $\mathfrak{B}$ of the structure $\langle H(\xi), \in, <_\chi \rangle$ where χ is some sufficiently large regular cardinal, $H(\chi)$ is the collection of sets hereditarily of cardinality less than χ, and $<_\chi$ is some well-ordering of $H(\chi)$. The well-ordering is included in the language to allow us to define canonical Skolem functions, and so we can "skolemize" and code the structure $\mathfrak{B}$ into an algebra $\mathfrak{A}$ so that the closure $\mathrm{cl}_{\mathfrak{A}}(z)$ of a set z is just the Skolem hull $\mathrm{Sk}_{\mathfrak{B}}(z)$ of z in $\mathfrak{B}$. We will abuse notation slightly and not distinguish between $\mathfrak{A}$ and $\mathfrak{B}$: we just want the ability to talk about Skolem hulls of subsets of $H(\chi)$ to land at the following corollary:

Corollary 1.4. Let J be a countably complete normal ideal on $Z \subseteq \mathcal{P}(X)$, and suppose χ is some sufficiently large regular cardinal and $\mathfrak{A}$ is an expansion of $\langle H(\chi), \in, <_\chi \rangle$. If f is a function with domain a J-positive subset of Z satisfying $f(z) \in \mathrm{Sk}_{\mathfrak{A}}(z)$ then f is constant on a J-positive subset of S.

We also remind the reader of basic properties of the *non-stationary ideal.* Again, we follow the conventions of Foreman [4].

Definition 1.5. If $F : [X]^{<\omega} \to X$ then C_F is defined to be the set of $x \in X$ that are closed under F. The *closed unbounded filter* on $\mathcal{P}(\kappa)$ is the filter generated by sets of the form C_F, and a subset Z of $\mathcal{P}(X)$ is stationary if for every such function F there is a $z \in Z$ that is closed under F. The non-stationary ideal consists of all subsets of $\mathcal{P}(X)$ that are disjoint to C_F for some F. If Z is a stationary subset of $\mathcal{P}(X)$ then we let NS_Z denote the ideal $NS \restriction Z$.

There are some standard facts about the non-stationary ideal that are very useful.

Lemma 1.6. With notation as above:

(1) A subset Z of $\mathcal{P}(X)$ is stationary if and only if for any sufficiently large regular χ and $x \in H(\chi)$, there is an $M \prec H(\chi)$ such that $x \in M$ and $M \cap X \in Z$.

(2) The non-stationary ideal is *normal*: if $Z \subseteq X$ is stationary and $F : Z \to X$ is regressive function (that is, $F(z) \in z$ for all $z \in Z$) then F is constant on a stationary subset of Z.

(3) The non-stationary ideal on Z is the minimal normal and fine ideal: if J is a normal fine countably complete ideal on a set $Z \subseteq \mathcal{P}(X)$ and $A \subseteq Z$ is non-stationary, then $A \in I$.

2. Weakly precipitous families of ideals

We are going to be working with variants of a game first considered by Shelah [10] (see also Jech [5]), but rooted in earlier ideas of Magidor [7]. Our context is as follows:

- Z is a subset of $\mathcal{P}(X)$ for some X, and
- $\mathbb{J}$ is a non-empty collection of proper ideals concentrating on Z with the following closure property: if $J \in \mathbb{J}$ and $A \in J^+$, then there is an $I \in \mathbb{J}$ that extends J and concentrates on A. In this situation, we just say that $\mathbb{J}$ is *stable under restrictions* or *restriction stable.* If we have the stronger property that $J \restriction A \in \mathbb{J}$ for every $J \in \mathbb{J}$ and $A \in J^+$, then we say that $\mathbb{J}$ is *restriction closed* or *closed under restrictions.*

Definition 2.1. If $\mathbb{J}$ is restriction-stable family of ideals concentrating on subset Z of $\mathcal{P}(X)$, we define a game $\partial(\mathbb{J})$ of length ω between two players Empty and Non-empty as follows: The first move in the game involves Empty choosing an ideal J_0 from $\mathbb{J}$ and a set $A_0 \in J_0^*$. Given A_n and J_n, Empty chooses a J_n-positive $A_{n+1} \subseteq A_n$ and Non-empty chooses an ideal $J_{n+1} \in \mathbb{J}$ that extends J_n and concentrates on A_{n+1}. In the end, Empty wins if $\bigcap_{n<\omega} A_n = \emptyset$.

Our assumption that $\mathbb{J}$ is stable under restrictions is just to ensure that both players always have legal moves available in the game. Notice that a run of the game produces a decreasing sequence $\langle A_n : n < \omega\rangle$ of J_0-positive subsets of Z, and an increasing sequence $\langle J_n : n < \omega\rangle$ of ideals from $\mathbb{J}$.

Definition 2.2. A set $\mathbb{J}$ of ideals on some set $Z \subseteq \mathcal{P}(X)$ is *weakly precipitous* if it is stable under restrictions and Empty does not have a winning strategy in the game $\partial(\mathbb{J})$. We extend this terminology and say that an ideal J on Z is weakly precipitous if J is a member of some weakly precipitous family of ideals $\mathbb{J}$.

This game is reminiscent of the classical characterization of precipitous ideals due to Galvin: if we view ω_1 as a subset of $\mathcal{P}(\omega_1)$ in the usual way, then a normal ideal J on ω_1 is precipitous if and only if the family $\mathbb{J} = \{J \restriction A : A \notin J\}$ is weakly precipitous. Notice that this choice of $\mathbb{J}$ is the smallest suitably rich collection of ideals that contains J, so we have severely restricted the responses available to the non-empty player in $\partial(\mathbb{J})$. By working with larger collections of ideals, we open up more responses and theoretically make it more likely that he can defeat Empty.

Our use of the term *weakly precipitous* extends usage due originally to Jech [5], where a normal ideal J on ω_1 is defined to be weakly precipitous if and only if Empty does not win the version of our game that takes $\mathbb{J}$ to consist of all the ideals on ω_1 that extend J, and the initial move is required to be $J_0 = J$. This is easily seen to be equivalent to our usage defined above. Weakly precipitous ideals in this sense have also been considered by Shelah [10, 11] under the name of *almost nice* ideals (or almost nice filters, in the dual formulation).

Theorem 1. *Suppose X is a regular space with a point-countable base that is not left separated. If there is a weakly precipitous family of normal countably complete fine ideals concentrating on $\{M \in [X]^\omega : \overline{M} \setminus M \neq \emptyset\}$, then for any coloring c of $[X]^2$ with finitely many colors, there is a $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c takes on at most two values. Thus, $X \to (\mathbb{Q})^2_{l,2}$ whenever $1 \leq l < \omega$.*

Given such a space X, let $\mathcal{B}$ be a point-countable base for the topology, and define

$$\tilde{X} := \{M \in [X]^\omega : \overline{M} \setminus M \neq \emptyset\}. \tag{2.1}$$

By a result of Fleissner [3], the set $\tilde{X}$ is stationary[3] in $[X]^\omega$, and we choose for each $M \in \tilde{X}$ a point $x_M \in \overline{M} \setminus M$. Finally, for each $M \in \tilde{X}$ we let $\langle U_{M,n} : n < \omega\rangle$ enumerate those countably many $U \in \mathcal{B}$ with $x_M \in U$.

Convention: By *ideal on* $\tilde{X}$ we will always mean *normal fine countably complete proper ideal on* $\tilde{X}$.

Lemma 2.3. Suppose J is an ideal on $\tilde{X}$.

(1) For each $M \in \tilde{X}$, for J-almost all $N \in \tilde{X}$ we know $x_M \neq x_N$.

(2) Given a J-positive $S \subseteq \tilde{X}$, any function f on S with $f(M) \in \{U_{M,n} : n < \omega\}$ is constant on a J-positive subset of S.

(3) If S is J-positive then

$$(\forall^J M \in S)(\forall n < \omega)(\exists^J N \in S)\,[U_{N,i} = U_{M,i}]\,. \tag{2.2}$$

Proof. Part (1) follows because J is fine: given $M \in \tilde{X}$ we know that J-almost every $N \in \tilde{X}$ contains x_M and for such N we know $x_N \neq x_M$. To establish (2), it suffices to prove

$$(\forall^J M \in S)(\forall n < \omega)(\exists^J N \in S)\,[U_{N,n} = U_{M,n}] \tag{2.3}$$

as (2) follows easily. If (2.3) fails, then there is a particular $n < \omega$ and J-positive $S' \subseteq S$ such that

$$(\forall M \in S')(\forall^J N \in S)\,[U_{M,n} \neq U_{N,n}]\,. \tag{2.4}$$

Let χ be some sufficiently large regular cardinal, and let $\mathfrak{A}$ be a structure expanding $\langle H(\chi), \in, <_\chi, J, X, \mathcal{B}, S, ...\rangle$, where $\mathcal{B}$ is our point-countable base for X. Given $M \in \tilde{X}$ we know that $U_{M,n}$ is in $\mathrm{Sk}_{\mathfrak{A}}(M)$: since x_M is in $\overline{M}$ there must be a $y \in M$ such that $y \in U_{M,n}$. Since $\mathcal{B}$ is a point-countable base, there are only countably many sets in $\mathcal{B}$ that contain y (here is where point-countable is important!), we know each of them must be in $\mathrm{Sk}_{\mathfrak{A}}(M)$ as well.

An application of Corollary 1.4 to the function sending $M \in S'$ to $U_{M,n}$ gives us a single set U and J-positive $T \subseteq S' \subseteq S$ such that $U_{M,n} = U$ for all $M \in T$. But this is a contradiction: since T is a J-positive subset of S' we can by (2.4) find M and N in T with $U_{M,n} \neq U_{N,n}$. $\square$

Definition 2.4. Let J be an ideal on $\tilde{X}$, and let i and j be colors.

(1) We say $M \in \tilde{X}$ is i-large in B with respect to J if $\{N \in B : c(x_M, x_N) = i\}$ is J-positive.

(2) A pair $\langle A, B\rangle$ of J-positive sets is said to be *$\langle i, j\rangle$-saturated over* J if J-almost every $a \in A$ is i-large in B and J-almost every $b \in B$ is j-large in A, that is:

$$(\forall^J a \in A)(\exists^J b \in B)\,[c(a, b) = i]\,, \tag{2.5}$$

[3] He shows that a regular space X with a point-countable base is left-separated if and only $\{M \in [X]^\omega : \overline{M} \setminus M \neq \emptyset\}$ is non-stationary.

and

$$(\forall^J b \in B)(\exists^J a \in A)\,[c(a,b) = j]\,. \tag{2.6}$$

(3) A pair $\langle A, B\rangle$ of J-positive sets is *strongly* $\langle i, j\rangle$*-saturated over* J if *every* $a \in A$ is i-large in B and *every* $b \in B$ is j-large in A.

Proposition 2.5. Let J be a ideal on $\tilde{X}$.

(1) Given J-positive sets S and T there is a pair of colors $\langle i, j\rangle$ and J-positive sets $A \subseteq S$ and $B \subseteq T$ such that both (2.5) and (2.6) hold.
(2) If $\langle S, T\rangle$ is a pair of J-positive sets that are $\langle i, j\rangle$-saturated over J, then any pair of J-positive sets $\langle A, B\rangle$ with $A \subseteq S$ and $B \subseteq T$ is $\langle i, j\rangle$-saturated over J.
(3) If I is another ideal on $\tilde{X}$ that extends J, then any pair $\langle A, B\rangle$ of I-positive sets that are $\langle i, j\rangle$-saturated over I is also $\langle i, j\rangle$-saturated over J.
(4) If $\langle A, B\rangle$ are $\langle i, j\rangle$-saturated over J then there is a set $C \in J^*$ such that $\langle A \cap C, B \cap C\rangle$ is strongly $\langle i, j\rangle$-saturated over J.

Lemma 2.6. If J is a ideal on $\tilde{X}$ then there is an $\langle i, j\rangle$-saturated pair of J-positive sets over J for some pair of colors $\langle i, j\rangle$.

Proof. Suppose J were a counterexample. Then given any any pair $\langle S, T\rangle$ of J-positive subsets of $\tilde{X}$ and any pair of colors $\langle i, j\rangle$, we can find J-positive $A \subseteq S$ and $B \subseteq T$ such that one of (2.5) or (2.6) fails. Thus, by making successive extensions, we can run through all pairs of colors and arrive at a pair $\langle A, B\rangle$ of J-positive sets such that for *every* pair of colors $\langle i, j\rangle$ at least one of the two alternatives fails, but this contradicts the first part of Lemma 2.5. □

Lemma 2.7. If $\mathbb{J}$ is a non-empty set of ideals on $\tilde{X}$, then there is a $J \in \mathbb{J}$ and a pair of colors $\langle i, j\rangle$ such that for any extension I of J in $\mathbb{J}$ there is a pair that is $\langle i, j\rangle$-saturated over I.

Proof. Suppose this fails. Then given an ideal $J \in \mathbb{J}$ and pair of colors $\langle i, j\rangle$, we can extend J to an ideal $I \in \mathbb{J}$ over which there is no $\langle i, j\rangle$-saturated pair. By part (3) of Lemma 2.5, this remains true for any extension of the ideal I as well. Thus, if we are given $J \in \mathbb{J}$ by making successive extensions in $\mathbb{J}$ and running through the finitely many pairs of colors, we arrive at an ideal $I \in \mathbb{J}$ for which the previous lemma fails and this is a contradiction. □

Corollary 2.8. If $\mathbb{J}$ is a restriction-stable set of ideals on $\tilde{X}$, then there is a $J \in \mathbb{J}$ and pair of colors $\langle i, j\rangle$ such that for any extension of I of J in $\mathbb{J}$, for any I-positive S there are I-positive subsets A and B of S that are $\langle i, j\rangle$-saturated over I.

Proof. Let J and $\langle i, j\rangle$ be as in the conclusion of the previous lemma. Given an extension I of J in $\mathbb{J}$ and an I-positive set S, we know that I has an extension I' in $\mathbb{J}$ that concentrates on S. If $\langle A, B\rangle$ is $\langle i, j\rangle$-saturated over I' then $\langle A \cap S, B \cap S\rangle$ is a pair of I-positive subsets of S that is $\langle i, j\rangle$-saturated over I. □

Roughly speaking, the ideal J has the property that given any extension I of J in $\mathbb{J}$, the $\langle i, j\rangle$-saturated pairs are dense in the I-positive sets.

The next lemma is a technical one, but it will drive an argument by induction that we use to prove the theorem.

Lemma 2.9. Suppose J is an ideal such that the $\langle i,j\rangle$-saturated sets are dense in the J-positive sets. Given any $n<\omega$ and J-positive S, there are A, B, and U such that

(1) A and B are subsets of S that are strongly $\langle i,j\rangle$-saturated over J, and
(2) $U=U_{M,n}$ for all $M\in A\cup B$.

Proof. By Lemma 2.3 we can find a J-positive $T\subseteq S$ and open $U\in\mathcal{B}$ such that

$$M\in T\Longrightarrow U_{M,n}=U. \tag{2.7}$$

By our assumption on J, we can find J-positive subsets of T that are $\langle i,j\rangle$-saturated, and by removing a small set from each we can assure that they are strongly $\langle i,j\rangle$-saturated over J. □

The following definition is a reformulation of a critical concept from [8].

Definition 2.10. Let J be an ideal concentrating on $\tilde{X}$.

(1) Given a J-positive set S, we say that $M\in\tilde{X}$ is *an $\langle i,j\rangle$-winner in S over J* if there is a sequence $\langle T_n:n<\omega\rangle$ of J-positive subsets of S such that for each $n<\omega$
   - any open neighborhood of x_M contains T_n for all but finitely many $n<\omega$,
   - $N\in T_n\Longrightarrow x_M\neq x_N$,
   - the pair $\langle T_n,T_k\rangle$ is $\langle i,j\rangle$-saturated over J for each $k<n$, and
   - $c(x_M,x_N)=i$ for each $N\in\bigcup_{n<\omega}T_n$.

(2) We say that J is *an $\langle i,j\rangle$-winning ideal* if every J-positive set S contains an $\langle i,j\rangle$-winner in S over J.

Now that we have the preceding notation in hand, our proof of Theorem 1 now splits into two pieces:

① If there is a weakly precipitous family $\mathbb{J}$ of ideals concentrating on $\tilde{X}$, then for any coloring $c:[X]^2\to l$ there is a pair of colors $\langle i,j\rangle$ such that some $J\in\mathbb{J}$ is an $\langle i,j\rangle$-winning ideal.

② If $c:[X]^2\to l$ and there is a $\langle i,j\rangle$-winning ideal on $\tilde{X}$, then there is a $Y\subseteq X$ homeomorphic to $\mathbb{Q}$ such that $\operatorname{ran}(c\restriction[Y]^2)\subseteq\{i,j\}$.

The following lemma takes care of ① for us:

Lemma 2.11. If there is a weakly precipitous family $\mathbb{J}$ of ideals concentrating on $\tilde{X}$, then for any coloring $c:[X]^2\to l$ there is a pair of colors $\langle i,j\rangle$ such that some $J\in\mathbb{J}$ is an $\langle i,j\rangle$-winning ideal.

Proof. By our earlier work, there is an ideal $J\in\mathbb{J}$ and a pair of colors $\langle i,j\rangle$ such that whenever I is an extension of J in $\mathbb{J}$, any I-positive set contains a pair of I-positive sets that is $\langle i,j\rangle$-saturated over I (hence over J). Given such a J and a J-positive set S, we show that there is an $\langle i,j\rangle$-winner in S over J. This is sufficient, as $\mathbb{J}$ will contain an ideal extending $J\restriction A$.

We now describe a strategy for Empty in the game $\Game(\mathbb{J})$ designed to produce such a point. We start in the obvious way: Empty sets J_0 to be some extension of $J\restriction A$ in $\mathbb{J}$ and $A_0=A$.

Stage n+1: At this point, Empty will have A_n and J_n at her disposal. Since J_n extends J, we apply Lemma 2.9 to find an open set U_n and a pair $\langle A_{n+1}, B_{n+1}\rangle$ such that

- $\langle A_{n+1}, B_{n+1}\rangle$ are J_n-positive subsets of A_n that are strongly $\langle i,j\rangle$-saturated over J_n, and
- U_n is an open set with $U_{M,n} = U_n$ for all $M \in A_{n+1} \cup B_{n+1}$.

Empty now plays A_{n+1} as her next move.

Since this strategy does not win, there is some run of the game where $\bigcap_{n<\omega} A_n$ is non-empty, and we show that any $M \in \bigcap_{n<\omega} A_n$ is an $\langle i,j\rangle$-winner.

Note that since x_M is in A_{m+1} we know that $U_{M,n} = U_n$ for all n, and our construction guarantees

$$A_{n+1} \cup B_{n+1} \subseteq \bigcap_{i\leq n} U_{M,i}. \tag{2.8}$$

Thus, any open neighborhood of x_M will contain $A_n \cup B_n$ for all but finitely many n.

Given $n < \omega$ we know that $\langle A_{n+1}, B_{n+1}\rangle$ is strongly $\langle i,j\rangle$-saturated over J_n (hence J), and this tells us that x_M is i-large in B_{n+1} over J. Thus, if we define

$$T_n := \{N \in B_{n+1} : x_M \neq x_N \text{ and } c(x_M, x_N) = i\}, \tag{2.9}$$

we know that T_n is J-positive, and by the previous paragraph any open neighborhood of x_M will contain all but finitely many of the sets T_n.

Thus, we are left with verifying that $\langle T_n, T_k\rangle$ is $\langle i,j\rangle$-saturated over J whenever $k < n$. Our construction guarantees that the pair $\langle A_{k+1}, B_{k+1}\rangle$ is $\langle i,j\rangle$-saturated over J. But T_n is a J-positive subset of A_{k+1} and T_k is a J-positive subset of B_{k+1}, so $\langle T_n, T_k\rangle$ is also $\langle i,j\rangle$-saturated over J. □

To establish 2, we just implement the Raghavan-Todorčević construction:

Lemma 2.12. If $c : [X]^2 \to l$ and there is a $\langle i,j\rangle$-winning ideal J on $\tilde{X}$, then there is a $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ such that $\operatorname{ran}(c \restriction [Y]^2) \subseteq \{i,j\}$.

Proof. Suppose X and $c : [X]^2 \to l$ are given, and fix an ideal $J \in \mathbb{J}$ and pair of colors $\langle i,j\rangle$ such that every J-positive set contains a J-positive set of $\langle i,j\rangle$-winners.[4] We build the desired copy of $\mathbb{Q}$ in X through an inductive construction.

The construction will have length ω. At a stage n we will be working with a downward-closed subtree Q_n of ${}^{<\omega}\omega$ of finite height. This tree Q_n is partitioned into *leaves* L_n and *branching nodes* B_n with the usual meaning. We will make sure that B_n is finite during the construction (so only finitely many elements of Q_n will have a successor), but the nodes will be fully branching: if $\sigma \in B_n$ then $\sigma^\frown\langle m\rangle$ is in Q_n for every $m < \omega$.

We assume that our prior work provides us with the following:

(a) Each branching node σ of Q_n has been assigned $M(\sigma) \in \tilde{X}$, and the corresponding points $x_{M(\sigma)}$ are distinct.

(b) Each leaf τ of Q_n is assigned a J-positive $T(\tau, n) \subseteq \tilde{X}$, and for each $N \in T(\tau, n)$ and $\sigma \in B_n$ we have $x_N \neq x_{M(\sigma)}$. We think of $T(\tau, n)$ as the current candidates for being chosen as M_τ.

[4] Since J is now fixed, we will eliminate the phrase "over J", and just speak of $\langle i,j\rangle$-winners.

(c) If $\tau = \sigma^\frown\langle m\rangle$ is a leaf of Q_n then for each $N \in T(\tau, n)$ we have $c(x_{M(\sigma)}, x_N) = i$ and $x_N \in U_{M(\sigma),n}$.

(d) If σ and τ are distinct leaves in Q_n with $\sigma <_{\text{lex}} \tau$, then the pair $\langle T(\tau, n), T(\sigma, n)\rangle$ is $\langle i, j\rangle$-saturated over J.

We start by letting Q_0 consist of the empty sequence $\langle\ \rangle$ and $T(\langle\ \rangle, 0) = \tilde{X}$. As we move through stage n to stage $n+1$, we need to accomplish several tasks. We assume that some bookkeeping procedure hands us a leaf $\sigma_n \in L_n$, and to obtain Q_{n+1} we will add to Q_n all immediate successors of σ_n in ${}^{<\omega}\omega$. Since this transforms σ_n into a branching node in Q_{n+1}, we will need to assign to it some $M(\sigma_n)$ from $T(\sigma_n, n)$. We will choose $M(\sigma_n)$ carefully (as $x_{M(\sigma)}$ must satisfy many constraints), and after making that decision we will need to shrink the sets $T(\tau, n)$ for $\tau \neq \sigma_n$ in B_n to obtain $T(\tau, n+1)$, and assign sets $T(\sigma_n^\frown\langle m\rangle, n+1)$ to all the new leaves we have created.

Since our bookkeeping identifies σ_k as we prepare for stage $k+1$, we will write M_k and x_k instead of $M(\sigma_k)$ and $x_{M(\sigma_k)}$ when discussing the assignment of points. Thus, to progress to stage $n+1$ we will need to choose $M_n \in T(\sigma_n, n)$ by paying attention to how x_n connects to the sets $T(\tau, n)$ for other leaves $\tau \neq \sigma_n$ in L_n. More precisely, we want to arrange it so that if $\tau \neq \sigma$ is another leaf of Q_n we have:

(e) if $\tau <_{\text{lex}} \sigma_n$ then $\{N \in T(\tau, n) : c(x_n, x_N) = i\}$ is J-positive, and

(f) if $\sigma_n <_{\text{lex}} \tau$ then $\{N \in T(\tau, n) : c(x_n, x_N) = j\}$ is J-positive.

To do this, we need only look back to our working assumption (d). Give $\tau \neq \sigma_n$ in $L(n)$, if $\sigma_n <_{\text{lex}} \tau$ we know $\langle T(\tau, n), T(\sigma_n, n)\rangle$ is $\langle i, j\rangle$-saturated, so

$$(\forall^J M \in T(\sigma_n, n))(\exists^J N \in T(\tau, n))\,[c(x_M, x_N) = j]\,. \tag{2.10}$$

For similar reasons, if $\tau <_{\text{lex}} \sigma_n$ we have

$$(\forall^J M \in T(\sigma_n, n))(\exists^J N \in T(\tau, n))\,[c(x_M, x_N) = i] \tag{2.11}$$

Since J is countably complete, it follows that almost every $M \in T(\sigma_n, n)$ will satisfy the requirements (e) and (f). Since the set of $\langle i, j\rangle$-winners in $T(\sigma_n, n)$ over T is a J-positive subset of $T(\sigma_n, n)$, we can choose one that satisfies (e) and (f) and this choice determines both M_n and x_n. We then define

$$T(\tau, n+1) = \{N \in T(\tau, n) : x_N \neq x_{n+1} \text{ and } c(x_{n+1}, X_N) = k\} \tag{2.12}$$

where $k = i$ if $\tau <_{\text{lex}} \sigma_n$ and $k = j$ if $\sigma_n <_{\text{lex}} \tau$. Notice that this choice ensures $T(\tau, n+1)$ is a J-large subset of $T(\tau, n)$, and guarantees that whenever our bookkeeping process finally hands us τ at some future stage, our available choices will be a subset of $T(\tau, n+1)$, and therefore we will connect to x_n in the proper way no matter which one is selected.

We still need to pay attention to the new leaves of Q_{n+1} and define J-large sets $T(\sigma_n^\frown\langle m\rangle, n+1)$ for each $m < \omega$, but this where we use the fact that M_n is an $\langle i, j\rangle$-winner in $T(\sigma, n)$. Thus, there is a sequence $\langle T_m : m < \omega\rangle$ of J-positive subsets of $T(\sigma_n, n)$ such that for each $m < \omega$

- any open neighborhood of x_n contains T_m for all but finitely many $m < \omega$
- $N \in T_m \Longrightarrow x_n \neq x_N$
- if $k < m$ then $\langle T_m, T_k\rangle$ is $\langle i, j\rangle$-saturated over J, and
- $c(x_n, x_N) = i$ for all $N \in \bigcup_{m<\omega} T_m$.

Now we define

$$T(\delta_n^\frown\langle m\rangle, n+1) = T_m, \tag{2.13}$$

and this leaves us in position to move on to the next stage of the construction.

In the end, we will have produced a set $Y = \{x_n : n < \omega\}$. Our construction guarantees that Y is dense in itself (given $\sigma \in {}^{<\omega}\omega$, the points attached to the successors of σ converge to δ_σ). Thus, we need only verify the range of $c\restriction[Y]^2$ is contained in $\{i, j\}$. This follows from our construction, though: if $n < k$ then x_k will be an element of $T(\sigma_k, k) \subseteq T(\sigma_k, n+1)$ and we have arranged every element x in $T(\sigma_k, n+1)$ satisfies $c(x_n, x) \in \{i, j\}$. More specifically, if $n < k < \omega$ then

- $c(x_n, x_k) = i$ if σ_n is an initial segment of σ_k.
- $c(x_n, x_k) = j$ if $\sigma_n <_{\text{lex}} \sigma_k$ and they are incomparable in ${}^{<\omega}\omega$.
- $c(x_n, x_k) = i$ if $\sigma_k <_{\text{lex}} \sigma_n$ and they are incomparable in ${}^{<\omega}\omega$.

$\square$

3. Examples of weakly precipitous families

Our plan in this section is to show that weakly precipitous families of ideals exist under very mild large cardinal hypotheses. We will begin by showing that a suitable instance of Chang's Conjecture is sufficient to obtain such families on ω_1. Foreman's [4] is a wonderful resource for this information.

Recall that for cardinals κ and θ the notation $(\kappa, \omega_1) \twoheadrightarrow (\theta, <\omega_1)$ means that any structure $\mathfrak{A}$ in a countable language with universe κ has an elementary substructure $\mathfrak{B}$ of cardinality θ such that $\mathfrak{B} \cap \omega_1$ is countable. The following proposition contains some well-known facts originating in work of Rowbottom [9]. In addition to Foreman's [4], an interested reader can find more information in Theorem 8.1 from Kanamori's [6].

Proposition 3.1. Then the following are equivalent for any cardinal $\theta \leq \kappa$

(1) $\kappa \to [\theta]^{<\omega}_{\omega_1, <\omega_1}$.
(2) $(\kappa, \omega_1) \twoheadrightarrow (\theta, <\omega_1)$.
(3) $\{X \subseteq \kappa : |X| = \theta \text{ and } X \cap \omega_1 < \omega_1\}$ is stationary in $\mathcal{P}(\kappa)$.

We turn now to the specific instance of Chang's Conjecture we need for our argument to go through. The following argument has a folklore flavor, and traces back (at least) to early work of Magidor [7]. A variant of this argument is also used by Shelah [10] and Jech [5].

Theorem 2. *If there is a cardinal κ such that $(\kappa, \omega_1) \twoheadrightarrow (\beth_2(\omega_1)^+, <\omega_1)$, then for any stationary $S \subseteq \omega_1$ there is a weakly precipitous family of normal ideals on ω_1 that concentrates on S.*

Proof. Given such a cardinal κ, we let

$$\mathcal{X} := \{X \subseteq \kappa : |X| = \beth_2(\omega_1)^+ \text{ and } X \cap \omega_1 < \omega_1\} \tag{3.1}$$

be the stationary subset of $\mathcal{P}(\kappa)$ corresponding to this instance of Chang's Conjecture, and let $\pi : \mathcal{X} \to \omega_1$ be the natural projection defined by $\pi(X) = X \cap \omega_1$. The projection is our tool for transferring information between $\mathcal{X}$ and ω_1, particularly via the Rudin-Keisler ordering on ideals. In our situation, if $\mathcal{I}$ is an ideal on $\mathcal{X}$ then we do the standard thing and define the *projection* $\pi[\mathcal{I}]$ of $\mathcal{I}$ by

$$\pi[\mathcal{I}] = \{X \subseteq \omega_1 : \pi^{-1}[X] \in \mathcal{I}\}. \tag{3.2}$$

Our goal is to show that

$$\mathbb{J} = \{\pi[\mathrm{NS} \restriction \mathcal{Z}] : \mathcal{Z} \text{ a non-stationary subset of } \mathcal{X}\} \tag{3.3}$$

is a weakly precipitous family that contains ideals concentrating on any stationary $S \subseteq \omega_1$.

The second piece of this is easy: if Z is a stationary subset of ω_1 then $\mathcal{Z} = \pi^{-1}[Z]$ is stationary in $\mathcal{X}$, and $\pi[\mathrm{NS} \restriction \mathcal{Z}]$ is an ideal in $\mathbb{J}$ that concentrates on Z.

As for the first part, assume by way of contradiction that Empty has a winning strategy in $\Game(\mathbb{J})$, with $J = J_0$ her first move. Since $J_0 \in \mathbb{J}$ we can find a stationary $\mathcal{Z}_0$ with $J_0 = \pi[\mathrm{NS} \restriction \mathcal{Z}_0]$. As the game progresses, Non-empty will be building a decreasing sequence $\langle \mathcal{Z}_n : n < \omega\rangle$ of stationary subsets of $\mathcal{X}$ and making sure that his move J_n (for $n > 0$) is the projection of $\mathrm{NS} \restriction \mathcal{Z}_n$.

Now for each $\alpha < \omega_1$ let T_α consist of all finite sequences of odd length consisting of partial plays $\langle A_1, J_1, \dots, A_{n-1}, J_{n-1}, A_n\rangle$ in the game in which Empty is using her winning strategy, but for which α has not yet been eliminated (that is, $\alpha \in A_n$). Since Empty is using a winning strategy, we know that T_α forms a well-founded tree, and this induces a ranking function on the elements of T_α in the usual way. Given $\sigma \in T_\alpha$, we let $\mathrm{rk}_\alpha(\sigma)$ denote the rank of σ in T_α. Since Empty is playing subsets of ω_1 and Non-empty is choosing ideals on ω_1, there are at most $\beth_2(\omega_1)$ possible moves for each player and so $|T_\alpha| \leq \beth_2(\omega_1)$. Thus, we know

$$\mathrm{rk}_\alpha(\sigma) < \beth_2(\omega_1)^+ \tag{3.4}$$

for any $\alpha < \omega_1$ and $\sigma \in T_\alpha$.[5]

Our goal is to formulate a strategy for Non-empty that leads to a contradiction when we deploy it against Empty 's alleged winning strategy. Assume we have been playing a game in which Empty is using her strategy, and we have arrived at the sequence $\sigma = \langle A_0, J_0, \dots, A_n, J_n, A_{n+1}\rangle$. Since Empty is using her winning strategy, for each $\alpha \in A_{n+1}$ the sequence σ will be in the tree T_α and so has been assigned corresponding rank $\mathrm{rk}_\alpha(\sigma) < \beth_2(\omega_1)^+$. This induces a function $\rho_{n+1} : A_{n+1} \to \beth_2(\omega_1)^+$ defined by

$$\rho_{n+1}(\alpha) = \mathrm{rk}_\alpha(\sigma). \tag{3.5}$$

At the same time, we know that on the side the Non-empty player has been building a decreasing sequence $\langle \mathcal{Z}_i : i \leq n\rangle$ of stationary sets, and making sure to let J_i be the projection of $\mathrm{NS} \restriction \mathcal{Z}_i$. He takes the set A_{n+1} and defines

$$\mathcal{Y}_{n+1} = \pi^{-1}[A_{n+1}] \cap \mathcal{Z}_n. \tag{3.6}$$

Since A_{n+1} is J_n-positive and $J_n = \pi[\mathrm{NS} \restriction \mathcal{Z}_n]$, we know that $\mathcal{Y}_{n+1}$ is a stationary subset of $\mathcal{Z}_n$.

We define a function $\tilde{\rho}_{n+1}$ mapping $\mathcal{Y}_{n+1}$ to κ by setting

$$\tilde{\rho}_{n+1}(Z) = \text{ the } \rho_{n+1}(\delta_Z)^{\text{th}} \text{ element of the increasing enumeration of } Z. \tag{3.7}$$

Since each element of $\mathcal{X}$ has cardinality $\beth_2(\omega_1)^+$, it is clear that this definition makes sense and by its very definition the function is regressive on $\mathcal{Y}_{n+1}$. Thus, there is an ordinal γ_{n+1} and stationary $\mathcal{Z}_{n+1} \subseteq \mathcal{Y}_{n+1}$ such that $\tilde{\rho}_{n+1}$ is constant

[5] This is critical, and explains why $\beth_2(\omega_1)^+$ appears in the assumptions of Theorem **??**.

with value γ_{n+1} on $\mathcal{Z}_{n+1}$. The ideal $\pi[\mathrm{NS}\restriction\mathcal{Z}_{n+1}]$ is a legal move for Non-empty , and so this is what he plays as J_{n+1}.

This leads to a contradiction, because the sequence $\langle\gamma_n : 1 \leq n < \omega\rangle$ is strictly decreasing. To see why, note that for each $\alpha \in A_{n+1}$ the partial play σ_{n+1} is a proper extension of σ_n in T_α, and therefore it is assigned lower rank in T_α. Thus, $\rho_{n+1}(\alpha) < \rho_n(\alpha)$ for all $\alpha \in A_{n+1}$.

Now choose $X \in \mathcal{Z}_{n+1}$. By definition, we know $\delta_X = X \cap \omega_1$ is in $A_{n+1} \subseteq A_n$ and so

$$\rho_{n+1}(\delta_X) < \rho_n(\delta_X). \tag{3.8}$$

Since $\mathcal{Z}_{n+1}$ is a subset of $\mathcal{Z}_n$, we also know $\tilde{\rho}_n(X) = \gamma_n$ and $\tilde{\rho}_{n+1}(X) = \gamma_{n+1}$. But (3.8) means that the $\rho_{n+1}(\delta_X)^{\text{th}}$ element of the increasing enumeration of X comes before the $\rho_n(\delta_X)^{\text{th}}$ element, so we are forced to conclude $\gamma_{n+1} < \gamma_n$. Thus, Empty does not have a winning strategy in $\Game(\mathbb{J})$ and $\mathbb{J}$ is a weakly precipitous family of normal ideals on ω_1. $\square$

The above argument is very flexible, so for example we have the following:

Theorem 3. *Suppose κ is a Ramsey cardinal and $X \in V_\kappa$. Then there is a weakly precipitous family on $[X]^\omega$ that contains ideals extending $\mathrm{NS}\restriction\tilde{X}$ whenever $\tilde{X}$ is stationary in $[X]^\omega$.*

Proof. In this setting, we define

$$\mathcal{X} = \{Y \subseteq \kappa : |Y| = \kappa \text{ and } |X \cap Y| = \aleph_0\}. \tag{3.9}$$

This set is a stationary subset of $\mathcal{P}(\kappa)$ because κ is Ramsey, and we again have a natural projection $\pi : \mathcal{X} \to [X]^\omega$ defined by setting $\pi(Y) = X \cap Y$. Then

$$\mathbb{J} := \{\pi[\mathrm{NS}\restriction\mathcal{Z}] : \mathcal{Z} \text{ a stationary subset of } \mathcal{X}\} \tag{3.10}$$

is weakly precipitous by exactly the same argument. $\square$

The above argument generalizes to the setting where κ is Ramsey in an inner model M of ZFC, as long as M contains (say) $V_{\rho+4}$, where ρ is the rank of X, as a winning strategy for Empty in V would be in N. A similar fact concerning inner models of Woodin cardinals is noted by Raghavan and Todorčević [8], but inner models with Ramsey cardinals are easier to come by. Thus, for example, Galvin's conjecture for uncountable sets of reals holds if the Singular Cardinals Hypothesis fails. It is this fact that Shelah exploits in [10] to obtain ZFC bounds on cardinal exponentiation at strong limit singular fixed points of uncountable cofinality.

Finally, we point out that work of Donder and Levinsky [2] shows that we can get weakly precipitous families of ideals on ω_1 in a generic extension of L, and so Galvin's Conjecture for sets of reals does not imply that $0^\sharp$ exists, although it is still unknown whether this statement has large cardinal strength.

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