Galvin’s Conjecture and Weakly Precipitous Ideals

CMO-BIRS Set-theoretic Topology Workshop

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Sierpiński and Friends

Ramsey's Theorem

Theorem (Ramsey 1930)

If $[\omega]^k$ is partitioned into finitely many pieces, then one of the pieces contains a set of the form $[H]^k$ for some infinite H.

Terminology

- $[X]^k$ is the set of (unordered) subsets of X of cardinality k.
- $c : [X]^k \to \ell$ is a *coloring* of $[H]^k$ with ℓ colors.
- So given a coloring c of $[\omega]^2$ in finitely many colors, there is in infinite *homogeneous* or *monochromatic* subset H of ω.

Sierpiński's Coloring

Theorem (Sierpiński 1933)

There is a function $c : [\mathbb{R}]^2 \to \{0, 1\}$ such that c assumes both colors on any uncountable subset of $\mathbb{R}$.

- Fix a well-ordering $\prec$ of $\mathbb{R}$ and set $c(a, b) = 0$ if and only if $<$ and $\prec$ agree on the pair $\{a, b\}$.
- What do homogeneous sets look like?

And so...

Theorem (Sierpiński 1933)

There is a function $c : [\mathbb{R}]^2 \to \{0, 1\}$ such that c assumes both colors on any set that contains an order-theoretic copy of $\mathbb{Z}$.

- So Ramsey's Theorem doesn't immediately generalize to colorings of uncountable sets.
- There is a natural coloring of pairs of reals that doesn't admit "complicated" homogeneous sets.

Overarching Question

Is this as bad as it gets?

Todorčević and Minimal Walks

Theorem (Todorčević 1987)

$\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$.

- There is a coloring of pairs of countable ordinals with uncountably many colors such that each color occurs on every uncountable subset of ω_1.

Shelah and “Was Sierpinski Right? I”

Theorem (Shelah 1988 [Sh:276])

$2^{\aleph_0} \to [\aleph_1]^2_3$ *is consistent relative to large cardinals.*

- It is consistent that for every coloring of pairs of reals with finitely many colors, there is an uncountable H on which the coloring assumes at most two values.
- Original proof used an ω_1-Erdős cardinal. Improvements lowered this to somewhere in the Mahlo hierarchy.
- The resulting model has $2^{\aleph_0}$ relatively large (a fixed point of the $\aleph$-sequence).
- Unclear if either of these proof features is required for the result (although CH must fail).

What about $\mathbb{Q}$?

1. Sierpiński's coloring gives us $c : [\mathbb{Q}]^2 \to \{0, 1\}$ with no homogeneous set order-isomorphic to $\mathbb{Q}$.
2. Galvin (unpublished) proved that for every $c : [\mathbb{Q}]^2 \to \ell$ with $\ell < \omega$ there is a set A order-isomorphic to $\mathbb{Q}$ on which c takes at most 2 values.
3. Baumgartner (1986) found $c : [\mathbb{Q}]^2 \to \omega$ that assumes every color on $[A]^2$ whenever $A \subseteq \mathbb{Q}$ is *homeomorphic* to $\mathbb{Q}$.
4. Todorčević and Weiss (unpublished) showed that if X is a σ-discrete metric space then there is a $c : [X]^2 \to \omega$ that takes on every color on $[A]^2$ whenever $A \subseteq X$ is homeomorphic to $\mathbb{Q}$.

Galvin's Conjecture

Galvin's Conjecture

If $c : [\mathbb{R}]^2 \to \ell$ with $\ell < \omega$ then there is a set $A \subseteq \mathbb{R}$ homeomorphic to $\mathbb{Q}$ on which c takes on at most two values.

The Raghavan-Todorčević Partition Theorem

Theorem (Raghavan-Todorčević 2020)

Assuming large cardinals, Galvin's Conjecture is true. Moreover, in the presence of (enough) large (enough) cardinals, if X is a non-σ-discrete metric space then for any $c : [X]^2 \to \ell$ with $\ell < \omega$ there is a set $A \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c assumes at most two values.

- Note that this is a direct implication and not just a relative consistency result.
- It is a much stronger result than Galvin's Conjecture.

Large Cardinals

- If X is a non-σ-discrete metric space and $X \in V_\delta$ with δ a Woodin cardinal, then every partition of $[X]^2$ into finitely many colors reduces to at most two colors on a subspace homeomorphic to $\mathbb{Q}$.
- Proof uses a game associated with the stationary tower forcing, related to the generic elementary embeddings you can obtain from Woodin cardinals.
- If there is a proper class of Woodin cardinals then the result holds for all non-σ-discrete metric spaces. The same conclusion holds if there is a strongly compact cardinal.
- For partitions of uncountable sets of reals, the result also follows from the existence of a precipitous ideal on ω_1.

After the preceding slide, the natural question should spring to your minds:

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Question

Hey, buddy. Isn’t this supposed to be a topology workshop?

“Isn’t this a topology workshop?”

Ok

Yes. You're right.

Things I will not define:

- Woodin cardinal
- Strongly compact cardinal
- Stationary tower forcing

The Raghavan-Todorčević Construction (Remix)

Set-up

1. X is a set of reals of cardinality $\aleph_1$, identified with ω_1 via some bijection.
2. $c : [X]^2 \to \ell$ colors $[X]^2$ with finitely many colors.
3. J will refer to a normal ideal on ω_1 (can think of as non-stationary for now).

Definition

Let $\langle i, j \rangle$ be a pair of colors. A pair $\langle A, B \rangle$ of J-positive sets is *weakly $\langle i, j \rangle$-saturated over J* if

$$(\exists^J \alpha \in A)(\exists^J \beta \in B)\,[c(\alpha, \beta) = i]$$

and

$$(\exists^J \beta \in B)(\exists^J \alpha \in A)\,[c(\alpha, \beta) = j]\,.$$

Observation 1

If J is an ideal then for any J-positive sets A and B there is a pair of colors $\langle i, j\rangle$ for which $\langle A, B\rangle$ is weakly $\langle i, j\rangle$-saturated over J.

- For every $\alpha \in A$ there must be an $i(\alpha) < \ell$ such that $\{\beta \in B : c(\alpha, \beta) = i(\alpha)\} \in J^+$, and so there is a single i such that $i = i(\alpha)$ for a J-positive subset of A. Identify j in the analogous manner.

Saturation

Definition

Let $\langle i, j\rangle$ be a pair of colors. A pair $\langle A, B\rangle$ of J-positive sets is *$\langle i, j\rangle$-saturated over J* if for every J-positive $C \subseteq A$ and $D \subseteq B$ the pair $\langle i, j\rangle$ is weakly $\langle i, j\rangle$-saturated over J.

- If $I \supseteq J$ then a pair that is $\langle i, j\rangle$-saturated over I is also $\langle i, j\rangle$-saturated over J, as I-positive sets are also J-positive.
- If $\langle A, B\rangle$ is $\langle i, j\rangle$-saturated over J then so is $\langle C, D\rangle$ for any J-positive $C \subseteq A$ and $D \subseteq B$.

Goal

Saturation Lemma

There is a normal ideal J on ω_1 and a pair of colors $\langle i, j \rangle$ such that for any normal $I \supseteq J$ there is an $\langle i, j \rangle$-saturated pair over I.

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Notes:

- Thus, if I is any normal ideal extending J then any I-positive Y contains an $\langle i, j \rangle$-saturated pair over I.

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Notes:

- Thus, if I is any normal ideal extending J then any I-positive Y contains an $\langle i, j\rangle$-saturated pair over I.
- To see this, apply the lemma to the normal ideal $I \upharpoonright Y$ and obtain A and B. The $\langle A \cap Y, B \cap Y\rangle$ is $\langle i, j\rangle$-saturated over I.

Saturation Lemma

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Notes:

- Thus, if I is any normal ideal extending J then any I-positive Y contains an $\langle i, j\rangle$-saturated pair over I.
- To see this, apply the lemma to the normal ideal $I \restriction Y$ and obtain A and B. The $\langle A \cap Y, B \cap Y\rangle$ is $\langle i, j\rangle$-saturated over I.
- Note that an $\langle i, j\rangle$ saturated pair over I is also $\langle i, j\rangle$-saturated over J.

Remember this

Saturation Lemma

There is a normal ideal J on ω_1 and a pair of colors $\langle i, j\rangle$ such that FOR ANY NORMAL $I \supseteq J$ there is an $\langle i, j\rangle$-saturated pair over I.

- Save this in your back pocket because we'll need it later.

Saturation Lemma (cont.)

Observation 2

For any normal ideal J there is a $\langle i, j\rangle$-saturated pair over J for SOME choice of colors $\langle i, j\rangle$.

- What does it mean for a pair of J-positive sets Y and Z to fail at being $\langle i, j\rangle$-saturated over J?
- We can find J-positive $A \subseteq Y$ and $B \subseteq Z$ such that $\langle A, B\rangle$ is not weakly $\langle i, j\rangle$-saturated over J.

Saturation Lemma (cont.)

One of two things must happen:

EITHER

$$(\forall^J \alpha \in A)(\forall^J \beta \in B)\,[c(\alpha,\beta) \neq i]\,,$$

OR

$$(\forall^J \beta \in B)(\forall^J \alpha \in A)\,[c(\alpha,\beta) \neq j]\,,$$

and note that this situation persists if we pass to J-positive subsets of A and B as well.

Saturation Lemma (cont.)

SO:

- If a pair of J-positive sets $\langle Y, Z\rangle$ fails to be $\langle i, j\rangle$-saturated over J, we can find J-positive $A \subseteq Y$ and $B \subseteq Z$ such that no J-positive refinement of $\langle A, B\rangle$ is weakly $\langle i, j\rangle$-saturated over J.
- What happens if we iterate this while moving through a list of all pairs of colors?
- We must run into a pair of colors $\langle i, j\rangle$ for which there is an $\langle i, j\rangle$-saturated pair over J: otherwise, we arrive at a pair of J-positive sets that fails to be weakly $\langle i, j\rangle$-saturated over J for any choice of $\langle i, j\rangle$.

Saturation Lemma (cont.)

Saturation Lemma

There is a normal ideal J on ω_1 and a pair of colors $\langle i, j\rangle$ such that for any normal $I \supseteq J$ there is an $\langle i, j\rangle$-saturated pair over I.

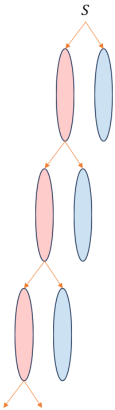
Suppose this fails.

- Then for every normal ideal J and pair of colors $\langle i, j\rangle$, there is a normal $I \supseteq J$ for which there is no $\langle i, j\rangle$-saturated pair over I.
- The situation is preserved if we take further extensions of I as well, so we can repeat the process and run through all pairs of colors.
- We arrive at a normal ideal such that for any choice of $\langle i, j\rangle$ there is no $\langle i, j\rangle$-saturated pair. Contradiction.

Fix such a normal ideal J and corresponding pair of colors $\langle i, j \rangle$.

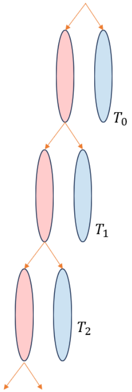
- For any normal $I \supseteq J$ and I-positive Y we can find I-positive subsets A and B of Y such that $\langle A, B \rangle$ is $\langle i, j \rangle$-saturated over I.
- The $\langle i, j \rangle$-saturated pairs are dense in $\mathcal{P}(\omega_1)/I$ for any normal ideal I extending J.
- This is the pair of colors for which we aim to find a corresponding topological copy of $\mathbb{Q}$.

Pictures



Imagine taking a J-positive set S and repeatedly splitting into $\langle i, j\rangle$-saturated pairs of J-positive sets "on the left".

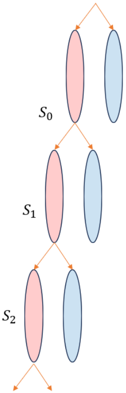
Pictures



Imagine taking a J-positive set and repeatedly splitting into $\langle i, j\rangle$-saturated pairs of J-positive sets “on the left”.

If $k < n$ then $\langle T_n, T_k\rangle$ is $\langle i, j\rangle$-saturated over J.

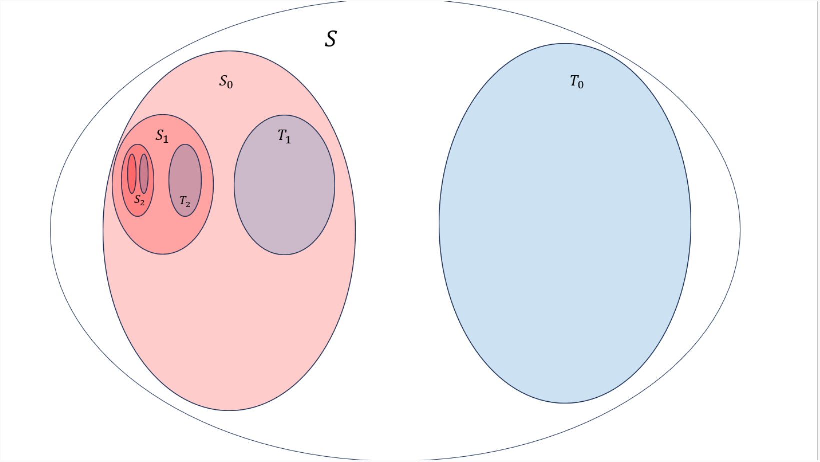
Pictures



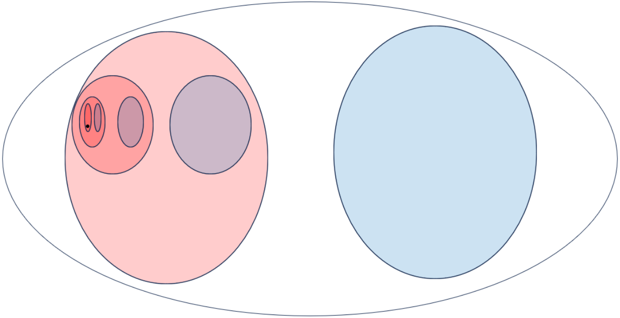
Imagine taking a J-positive set and repeatedly splitting into $\langle i, j\rangle$-saturated pairs of J-positive sets "on the left".

If we can ensure that the "left" sets have smaller and smaller diameter, we end up with the following sort of picture.

Pictures

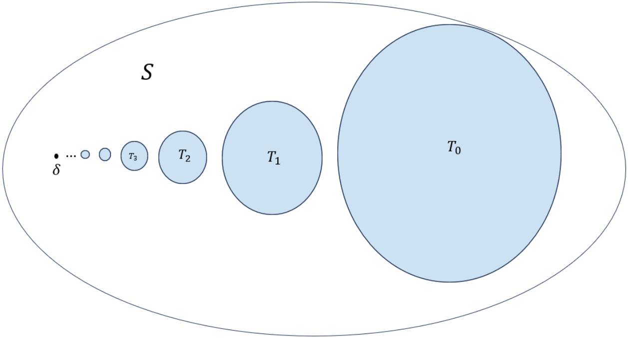


Pictures



If we can also somehow ensure that $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$ then we end up with the following sort of picture.

Pictures



The sets T_n are converging to a point $\delta \in \bigcap_{n=1}^{\infty} S_n \in S$, and $k < n$ implies $\langle T_n, T_k \rangle$ is $\langle i, j \rangle$-saturated over J.

Moreover, we can arrange $c(\delta, \epsilon) = i$ for all $\epsilon \in \bigcup_{n=1}^{\infty} T_n$.

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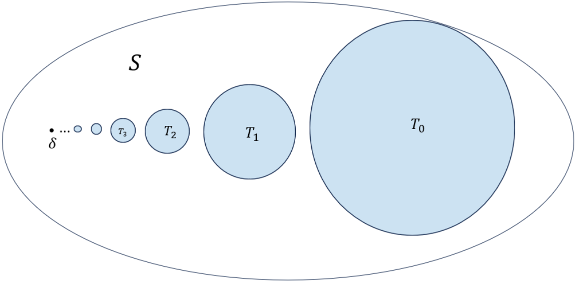

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Moreover, we can arrange $c(\delta, \epsilon) = i$ for all $\epsilon \in \bigcup_{n=1}^{\infty} T_n$.

This configuration is described as "δ is an $\langle i, j \rangle$-winner over J in S".

The Main Idea

The Raghavan-Todorčević Construction

If there is a normal ideal J on X such that every J-positive set S contains a J-positive set of $\langle i, j \rangle$-winners in S over J, then there is a set $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c takes on only values i or j.

- The issue is being able to arrange that certain infinite decreasing sequences of J-positive sets have non-empty intersection.
- This is where the large cardinals come into the picture.

The Main Idea

The Raghavan-Todorčević Construction (Remix)

If there is a normal ideal J on X such that every J-positive set S contains a J-positive set of $\langle i, j\rangle$-winners in S over J, then there is a set $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c takes on only values i or j.

- A normal precipitous ideal on ω_1 can work.
- If there is a Woodin cardinal, then the non-stationary ideal and any of its restrictions work.

The Raghavan-Todorčević Theorem

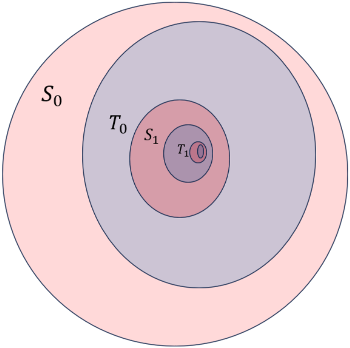
Theorem (Raghavan-Todorčević 2020)

If there is a Woodin cardinal then for any uncountable $X \subseteq \mathbb{R}$ and $c : [X]^2 \to \ell$ with $\ell < \omega$ there is a set $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c takes on at most two colors.

- As we have discussed, they proved a much more general result.

Weakly Precipitous Ideals

Precipitous Ideals



Empty player chooses J-positive S_n

Nonempty player responds with J-positive T_n

Together they build a decreasing sequence of J-positive sets, so $T_{n+1} \subseteq S_{n+1} \subseteq T_n \subseteq S_n$

Empty wins if $\bigcap_{n=0}^{\infty} S_n = \emptyset$.

J is *precipitous* if Empty does not have a winning strategy in the game.

Comments

- J is precipitous if and only if the generic ultrapower resulting from forcing with $\mathcal{P}(\omega_1)/J$ is well-founded.
- The definition I gave is a characterization due to Galvin.
- Precipitous ideals are equiconsistent with measurable cardinals. Their existence can be thought of as a weak form of measurability that can hold at smaller cardinals.
- We can replace ω_1 by other sets and structures in the definition.

Building a Filter

- Given the normal ideal J, think about the game as two players building filter extending J^*, the filter dual to J.

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Building a Filter

- Given the normal ideal J, think about the game as two players building filter extending J^*, the filter dual to J.
- When a move in the game is made, they are selecting the set they want to put into the filter, and must respect earlier choices.
- Both players can influence the construction of the filter one set at a time, and at each stage the resulting approximation is normal.
- Empty can always ensure that the resulting sequence of sets has empty intersection if and only if the ideal J fails to be precipitous.

Modifying the game

Question of the day

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- Thus, it would be harder for Empty to have a winning strategy as her moves are more constrained.

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What if we were to give the Nonempty player more power?

- The game becomes easier for Nonempty, as he can impose more restrictions on Empty by putting more than just one set into the filter.
- Thus, it would be harder for Empty to have a winning strategy as her moves are more constrained.
- What might this look like?

Weakly Precipitous Game

Let J be a normal ideal on ω_1. The game $\Game(J)$ is a contest between players Empty and Nonempty.

- At a stage $n+1$ Empty will be handed a normal ideal J_n and will choose a J_n-positive set A_{n+1}. Nonempty responds by selecting a normal ideal extending $J_n \restriction A_{n+1}$. To start, A_0 is a J-positive set.
- Empty wins if and only if $\bigcap_{n<\omega} A_n = \emptyset$.

The Modified Game

Weakly Precipitous Game

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- Empty wins if and only if $\bigcap_{n<\omega} A_n = \emptyset$.

- A normal ideal J is *weakly precipitous* if Empty does not have a winning strategy in $\Game(J)$.

History

History

- These ideals were first used by Shelah in the late 70s to extend the Galvin-Hajnal theorem in cardinal arithmetic to fixed points of uncountable cofinality. This work was not published until the mid 80s. He called such ideals *almost nice*.

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- Jech named these ideals *weakly precipitous* in the early 80s.
- Shelah observed that if there is a Ramsey cardinal then there is a weakly precipitous normal ideal on ω_1.
- Donder and Levinsky investigated this and related concepts in 1989, and showed that weakly precipitous normal ideals on ω_1 can exist in generic extensions of L.

Saturation Lemma

There is a normal ideal J on ω_1 and a pair of colors $\langle i, j \rangle$ such that FOR ANY NORMAL $I \supseteq J$ there is an $\langle i, j \rangle$-saturated pair over I.

The Raghavan-Todorčević Construction (Remix)

If there is a normal ideal J on X such that every J-positive set S contains a J-positive set of $\langle i, j \rangle$-winners in S over J, then there is a set $Y \subseteq X$ homeomorphic to $\mathbb{Q}$ on which c takes on only values i or j.

New stuff

First Ingredient

If J is a weakly precipitous normal ideal satisfying the conclusion of the Saturation Lemma, then every J-positive set S contains a J-positive set of $\langle i, j\rangle$-winners in S over J.

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If there is a Ramsey cardinal κ then EVERY normal ideal on ω_1 is weakly precipitous.

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First Ingredient

If J is a weakly precipitous normal ideal satisfying the conclusion of the Saturation Lemma, then every J-positive set S contains a J-positive set of $\langle i, j \rangle$-winners in S over J.

Second Ingredient

If there is a Ramsey cardinal κ then EVERY normal ideal on ω_1 is weakly precipitous.

Theorem (TE 2023)

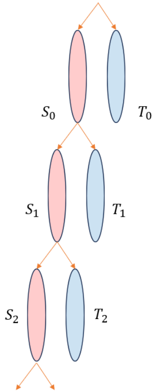
If there is a Ramsey cardinal, then for any coloring $c : [X]^2 \to \ell$ of an uncountable set of reals with finitely many colors there is a set $Y \subseteq X$ homeomorphic to the rationals on which the coloring takes on at most two colors.

First Ingredient

If J is a weakly precipitous normal ideal satisfying the conclusion of the Saturation Lemma, then every J-positive set S contains a J-positive set of $\langle i, j\rangle$-winners in S over J.

- So for EVERY normal ideal $I \supseteq J$ an I-positive set contains an $\langle i, j\rangle$-saturated pair over I.

Play the Game



Go back to our picture of repeatedly splitting into $\langle i, j\rangle$-saturated pairs.

This time, we play the weakly precipitous game with J.

When Empty moves, she knows S_n is in J^*_{n+1} and her strategy is to select the left piece of a certain $\langle i, j\rangle$-saturated pair over J_{n+1}.

This strategy does not win, and this is how we can produce the $\langle i, j\rangle$-winners we need to run the Raghavan-Todorčević construction.

Definition

κ is a Ramsey cardinal if $\kappa \to (\kappa)^{<\omega}_2$, that is for every $F : [\kappa]^{<\omega} \to \{0,1\}$ there is a set H of cardinality κ such that $F \restriction [H]^n$ is constant for each $n < \omega$.

- This is equivalent to $\kappa \to (\kappa)^{<\omega}_\theta$ for every $\theta < \kappa$ (so we are allowed to use θ colors).

What we use

Representation Lemma

If κ is a Ramsey cardinal and J is a normal ideal on ω_1 then there is a stationary $\tilde{X} \subseteq [\kappa]^\kappa$ such that

- $X \in \tilde{X} \implies X \cap \omega_1 < \omega_1$, and
- $J = \pi[NS \restriction \tilde{X}]$ where $\pi : \tilde{X} \to \omega_1$ is $\pi(X) = X \cap \omega_1$.

- $\tilde{X} = \{N \cap \kappa : N \prec V_\kappa, |N| = \kappa$ and $J \in N$ and $N \cap \omega_1 \in \bigcap(N \cap J^*)\}$
- So $\delta = N \cap \omega_1$ is not a member of any set in $N \cap J$.
- Main point: For J-almost every δ the Skolem hull of δ in (an expansion of) V_κ is a (countable) model that can be inflated (via indiscernibles) to a model N of size κ with $N \cap \kappa \in \tilde{X}$

Lifting

Key Idea (Magidor?)

Assume J, $\tilde{X}$, and π are as in the preceding slide. There is a mapping $f \mapsto \tilde{f}$ that takes a function $f : \omega_1 \to \kappa$ to a regressive $\tilde{f} : \tilde{X} \to \kappa$ such that $f <_J g \implies \tilde{f} <_{NS} \tilde{g}$

Main Lemma

Lemma (Main Lemma)

If there is a Ramsey cardinal then every normal ideal on ω_1 is weakly precipitous.

- In hindsight, this a version of an observation of Burke that in the presence of a Woodin cardinal, any normal filter on ω_1 can be generically extended to one with a well-founded V-ultrapower.
- Really: if there is a Ramsey cardinal κ then any normal filter on ω_1 can be generically extended to one with V-ultrapower well-founded out to κ.

Sketch of Proof

- Assume Empty has a winning strategy in $\Game(J)$.
- For each $\alpha < \omega_1$ let T_α consist of all finite sequences of odd length $\langle A_0, J_0, \ldots, J_n, A_{n+1}\rangle$ of partial plays where Empty is using her strategy and $\alpha \in A_{n+1}$.
- T_α forms a well-founded tree because the strategy wins, and so we have a ranking function of elements of T_α.
- $\mathsf{rk}_\alpha(\sigma)$ denotes the rank of $\sigma \in T_\alpha$.
- $\mathsf{rk}_\alpha(\sigma) < \beth_2(\omega_1)^+ < \kappa$ as $|T_\alpha| \leq \beth_2(\omega_1)$.

Nonempty can win

- Nonempty promises to build a decreasing sequence $\langle \mathcal{Z}_n : n < \omega \rangle$ of stationary subsets of $\tilde{X}$ on the side during the game.
- He promises that $J_{n+1} = \pi[NS \upharpoonright \mathcal{Z}_{n+1}]$ after selecting $\mathcal{Z}_{n+1}$. (Note that J_{n+1} is probably not going to generated over $J_n \upharpoonright A_{n+1}$ by a single set, so this is where we are taking advantage of the change in rules.)
- Suppose we arrive at $\sigma = \langle A_0, J_0, \ldots, J_n, A_{n+1} \rangle$ and Empty has been using her winning strategy.
    - For each $\alpha \in A_{n+1}$ the sequence σ is in T_α and has a corresponding rank $\mathsf{rk}_\alpha(\sigma)$.
    - This induces a function $\rho_{n+1} : A_{n+1} \to \beth_2(\omega_1)^+$ by $\rho_{n+1} = \mathsf{rk}_\alpha(\sigma)$.

Taking Stock

Situation

- $J_n = \pi[NS \restriction \mathcal{Z}_n]$ where $\mathcal{Z}_n$ is a stationary subset of $\mathcal{X}$.
- A_n is J_n-positive, so $\mathcal{Y}_n := \mathcal{Z}_n \cap \pi^{-1}[A_n] \subseteq \mathcal{Z}_n$ is stationary.
- $\rho_n : A_n \to \beth_2(\omega_1)^+ < \kappa$.
- $\rho_n(\alpha) < \rho_i(\alpha)$ for any $\alpha \in A_n$ and $i < n$ by the way ranks work.
- So $\langle \rho_i : i \leq n \rangle$ is $<_{J_n}$-decreasing.

Lifting the Function

Definition

Define $\tilde{\rho}_{n+1} : \mathcal{Y}_{n+1} \to \kappa$ by

$$\tilde{\rho}_{n+1}(Z) = \text{ the } \rho_{n+1}(\pi(Z))^{\mathrm{th}} \text{ element of the increasing enumeration of } Z.$$

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- $\pi(Z) \in A_{n+1}$ for any $Z \in \mathcal{Y}_{n+1} \subseteq \mathcal{X}$ by choice of $\mathcal{Y}_{n+1}$.
- $\tilde{\rho}_{n+1}(Z)$ is actually an element of Z for all $Z \in \mathcal{Y}_{n+1}$ because

$$\rho_n(\pi(Z)) < \beth_2(\omega_1)^+ < |Z| = \kappa.$$

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- $NS \restriction \mathcal{Y}_{n+1}$ is normal, therefore $\tilde{\rho}_{n+1}$ is constant on a stationary subset $\mathcal{Z}_{n+1}$ of $\mathcal{Y}_{n+1} \subseteq \mathcal{Z}_{n+1}$, say with value $\gamma_{n+1} < \kappa$.

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- Nonempty now selects $J_{n+1} = \pi[NS \restriction \mathcal{Z}_{n+1}]$ and play continues.

The Contradiction

- $\langle A_n : n < \omega \rangle$ is a decreasing sequence of subsets of ω_1.
- $\langle \mathcal{Z}_n : n < \omega \rangle$ is a decreasing sequence of stationary subsets of $[\kappa]^\kappa$ and $J_n = \pi[NS \restriction \mathcal{Z}_n]$

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- For each $n < \omega$ the sequence $\langle \rho_i : i \leq n \rangle$ is $<_{J_n}$-decreasing.
- For each $n < \omega$ the sequence $\langle \tilde{\rho}_i : i \leq n \rangle$ is $<_{NS \restriction \mathcal{Z}_n}$-decreasing.

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- For each $n < \omega$ and $\alpha \in A_n$ the sequence $\langle \rho_i(\alpha) : i \leq n \rangle$ is strictly decreasing because these are ranks in a well-founded tree.
- For each $n < \omega$ the sequence $\langle \rho_i : i \leq n \rangle$ is $<_{J_n}$-decreasing.
- For each $n < \omega$ the sequence $\langle \tilde{\rho}_i : i \leq n \rangle$ is $<_{NS \restriction \mathcal{Z}_n}$-decreasing.
- For $i \leq n$ the function $\tilde{\rho}_i$ is constant with value γ_i on $\mathcal{Z}_n$.

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- $\langle \mathcal{Z}_n : n < \omega \rangle$ is a decreasing sequence of stationary subsets of $[\kappa]^\kappa$ and $J_n = \pi[NS \restriction \mathcal{Z}_n]$
- For each $n < \omega$ and $\alpha \in A_n$ the sequence $\langle \rho_i(\alpha) : i \leq n \rangle$ is strictly decreasing because these are ranks in a well-founded tree.
- For each $n < \omega$ the sequence $\langle \rho_i : i \leq n \rangle$ is $<_{J_n}$-decreasing.
- For each $n < \omega$ the sequence $\langle \tilde{\rho}_i : i \leq n \rangle$ is $<_{NS \restriction \mathcal{Z}_n}$-decreasing.
- For $i \leq n$ the function $\tilde{\rho}_i$ is constant with value γ_i on $\mathcal{Z}_n$.
- Therefore $\langle \gamma_n : n < \omega \rangle$ is a strictly decreasing sequence of ordinals.

Summary of Proof

- Suppose X is an uncountable set of reals and $c : [X]^2 \to \ell$.
- There is a normal ideal J on ω_1 satisfying the conclusion of the Saturation Lemma for some pair of colors $\langle i, j\rangle$.
- If there is a Ramsey cardinal, this ideal J is weakly precipitous.
- The previous two statements imply every J-positive set S contains a J-positive set of $\langle i, j\rangle$-winners.
- The Raghavan-Todorčević construction then builds us a countably dense-in-itself $Y \subseteq X$ such that $\mathrm{ran}(c \restriction [Y]^2) \subseteq \{i, j\}$.

Not Appearing in this Talk

- The actual Raghavan-Todorčević construction, which is very elegant.
- The proof that every normal ideal on ω_1 is weakly precipitous requires only the existence of an inner model with a Ramsey cardinal that is correct about the set of normal ideals on ω_1. This follows from violations of the SCH above $\beth_2(\omega_1)$, for example.
- If κ is Ramsey and J is a normal ideal on a set $Z \subseteq \mathcal{P}(X)$ for some $X \in V_\kappa$ then J is weakly precipitous.
- Our argument obtains the full conclusion of the Raghavan-Todorčević theorem as well, using Ramsey cardinals instead of Woodin cardinals.
- Forcing with the stationary tower out through a Ramsey cardinal adds a generic elementary embedding whose target need not be well-founded, but it is still very well-behaved. It has λ-like elements for every λ of cofinality greater than $\beth_3(\omega_1)$ in V.

The End

Thank you for listening.