Determination of the analytical solutions of the equations of vertical ballistic motion of bodies under quadratic drag and uniform gravitational field

September of 2024

1 Introduction

The aim of this paper is to present the analytical solutions for the time functions of vertical (one-dimensional) ballistic motion of bodies under a uniform gravitational field and quadratic drag. The deductions for each equation are thoroughly demonstrated and explained. Although a one-dimensional vertical movement is almost "utopian", the predicted results can serve as estimates for various applications, especially the prediction of trajectories in rocket modeling and experimental rocketry, scenarios which motivated this work. An example is shown, in which real experimental rocket flight data is compared with the predictions given by the equations.

2 Introduction to the physical situations

Consider an extended body of mass m, with a constant drag coefficient C_D, cross-sectional area (in relation to the direction of movement) A, which describes a rectilinear trajectory perpendicular to the ground, with velocity described by the vector $\vec{v}$, where a uniform gravitational field of acceleration g acts. In addition, consider that this body is immersed in a stationary fluid of specific mass ρ, in this case air, which generates a drag force defined by

$$\vec{F}_D = -\frac{1}{2}\rho A C_D |\vec{v}|^2 \hat{v}$$

Note that the air resistance force vector is in the opposite direction to the velocity.
Furthermore, the weight force acting on the body is described by

$$\vec{F}_g = -mg\hat{\jmath}$$

Where $\hat{\jmath}$ symbolizes the unit vector perpendicular to the ground, while the notation $\hat{\imath}$ refers to the parallel unit vector.

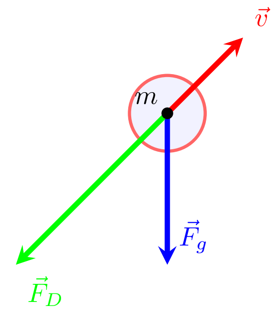


Figure 1: ilustration of a body with any velocity under the imposed conditions

As the velocity vector is always perpendicular, only the $\hat{j}$ component is relevant for analyzing situations.

There are therefore two possible ballistic phases: velocity pointing upwards ($+\hat{j}$); and velocity pointing downwards ($-\hat{j}$)

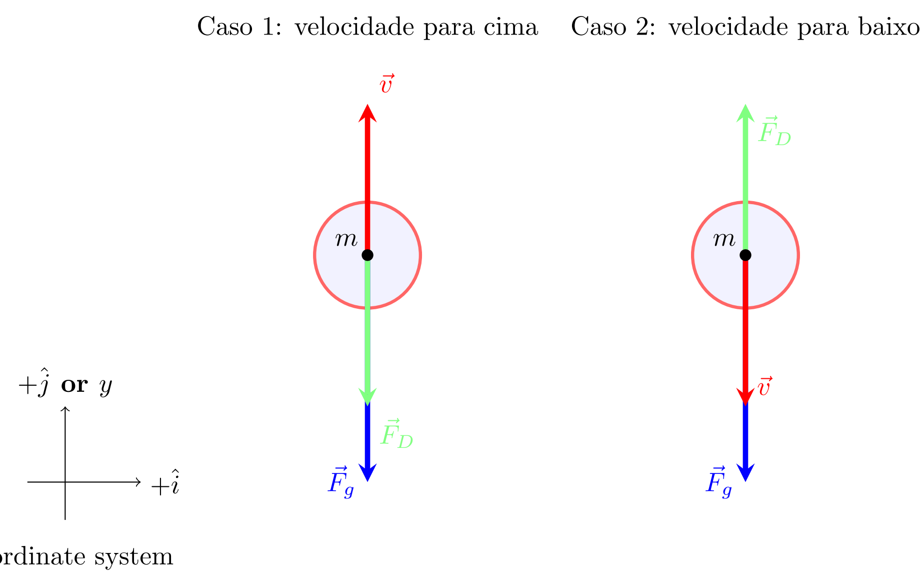


Figure 2: ilustration of the cases

In this way, Newton's second law gives $\vec{F}_R$ and $\vec{a}$ as the resulting force and acceleration:

$$\boxed{\vec{F}_R = \sum \vec{F}_i = \vec{F}_D + \vec{F}_g}$$

$$\Rightarrow \boxed{\vec{a} = \vec{g} - \kappa|\vec{v}|^2\hat{v}}$$

in which

$$\kappa = \frac{\rho A C_D}{2m}$$

is a constant defined for convenience.
Since the movement takes place in one dimension, it is convenient to use a scalar notation with the signs indicating the type of movement (accelerated/retarded, progressive/regressive, etc). Therefore, it is agreed that the position, represented here by y, is greater than or equal to zero, with the sub-index 0 and 1 indicating a magnitude at an initial and final moment, respectively.
Then we have equations 1 and 2 in scalar form:

$$\boxed{a = -g + \kappa v^2} \qquad \text{[eq. 1] [Descending ballistic phase]}$$

$$\boxed{a = -g - \kappa v^2} \quad \text{[eq. 2] [Ascending ballistic phase]}$$

in which we have that

$$a = \frac{dv}{dt}$$

where t represents the time variable.

3 Ascending ballistic phase

In this case, one works with equation 2, in which the weight force and the drag force work negatively for the displacement of the body.
The differential equation that results from this scenario is

$$\frac{dv}{dt} = -g - \kappa v^2$$

3.1 Time function of speed

Rearranging to separate the variables, we get

$$\frac{dv}{g + \kappa v^2} = -dt$$

Integrating both sides, we get

$$\frac{1}{g}\int_{v_0}^{v_1} \frac{dv}{1 + \frac{\kappa}{g}v^2} = -\int_{t_0}^{t_1} dt$$

By performing the following substitution

$$u^2 = \frac{\kappa}{g}v^2 \Rightarrow du = dv\sqrt{\frac{\kappa}{g}}$$

it can be written that

$$\frac{1}{g}\sqrt{\frac{g}{\kappa}}\int_{u_0}^{u_1} \frac{du}{1 + u^2} = -\Delta t$$

$$\frac{1}{\sqrt{\kappa g}}\int_{u_0}^{u_1} \frac{du}{1 + u^2} = -\Delta t$$

Now, focusing on the indefinite integral of the previous equation,

$$\int \frac{du}{1 + u^2}$$

A trigonometric substitution is carried out

$$\tan x = u \Rightarrow \sec^2(x)dx = du$$

$$\int \frac{\sec^2(x)dx}{1 + \tan^2(x)}$$

From the identity $\tan^2 x + 1 = \sec^2 x$, it follows that

$$= \int dx = x + C$$

Returning from x to the initial variable u

$$= \arctan(u) + C$$

Applying this indefinite integral to the previous equation, we get

$$\frac{1}{\sqrt{\kappa g}}\arctan(u)\bigg|_{u_0}^{u_1} = -\Delta t$$

$$\Rightarrow \frac{1}{\sqrt{\kappa g}}\arctan\left(v\sqrt{\frac{\kappa}{g}}\right)\bigg|_{v_0}^{v_1} = -\Delta t$$

$$\Rightarrow \Delta t \; = \frac{1}{\sqrt{\kappa g}}\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - \arctan\left((v_1\sqrt{\frac{\kappa}{g}}\right)\right]$$

In order to define a function $v(t)$, the conditions $t_0 = 0$ and $v_1 = v = v(t)$ are imposed.

$$t \; = \frac{1}{\sqrt{\kappa g}}\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - \arctan\left(v\sqrt{\frac{\kappa}{g}}\right)\right]$$

$$\Rightarrow \arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g} = \arctan\left(v\sqrt{\frac{\kappa}{g}}\right)$$

$$\boxed{\therefore v(t) = \sqrt{\frac{g}{\kappa}}\tan\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]}$$

3.2 Time function of position

It is assumed that the position over a time interval is determined by integrating the infinitesimals of y over that interval. Thus, writing in the form of a differential equation, we have

$$\frac{dy}{dt} = v$$

Separating the differentials,

$$dy = v(t)dt$$

$$dy = \sqrt{\frac{g}{\kappa}}\tan\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]dt$$

Integrating both sides of the equations over the range of position and time, we write

$$\int_{y_0}^{y_1} dy = \sqrt{\frac{g}{\kappa}}\int_{t_0}^{t_1}\tan\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]dt$$

$$\Rightarrow \Delta y = -\sqrt{\frac{g}{\kappa}}\frac{1}{\sqrt{\kappa g}}\ln\left[\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right)\right]\bigg|_{t_0}^{t_1}$$

Since

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{du}{u} = \ln[\sec(x)],\; u = \cos(x)$$

Manipulating algebraically,

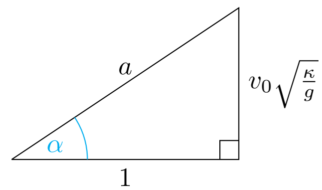
$$\Delta y = \kappa^{-1}\ln\left[\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t_0\sqrt{\kappa g}\right)\right] - \kappa^{-1}\ln\left[\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t_1\sqrt{\kappa g}\right)\right]$$

$$\Delta y = \kappa^{-1}\ln\left[\frac{\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t_0\sqrt{\kappa g}\right)}{\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t_1\sqrt{\kappa g}\right)}\right]$$

Letting $y_1 = y = y(t)$, $t_0 = 0$ e $t_1 = t$, we obtain

$$y(t) = y_0 + \kappa^{-1}\ln\left[\frac{\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right)\right)}{\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right)}\right]$$

To determine the value of $\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right)\right)$, consider the following triangle:



Then take an angle α whose tangent is $v_0\sqrt{\frac{\kappa}{g}}$, so that $\sec\alpha = \sec\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right)$. Thus, you can arbitrarily choose the two sides to satisfy this condition.

By the Pythagorean Theorem, the hypotenuse a is $\sqrt{1+\frac{v_0^2\kappa}{g}}$. Therefore, it is clear that the value of the secant of α must be $\sqrt{1+\frac{v_0^2\kappa}{g}}$.

Thus, with the expression found,

$$y(t) = y_0 + \kappa^{-1}\ln\left[\frac{\sqrt{1+\frac{v_0^2\kappa}{g}}}{\sec\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right)}\right]$$

$$\boxed{\therefore y(t) = y_0 + \kappa^{-1}\ln\left[\left(\cos\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right)\right) - t\sqrt{\kappa g}\right)\sqrt{1+\frac{v_0^2\kappa}{g}}\right]}$$

3.3 Time function of acceleration

When it comes to determining acceleration, just recall that acceleration is the ins -tantaneous rate of change in velocity. Thus, we write that

$$\frac{dv}{dt} = a$$

Differentiating v, using the definition, we have

$$a(t) = \frac{d}{dt}\left[\sqrt{\frac{g}{\kappa}}\tan\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]\right]$$

$$\Rightarrow a(t) = -\sqrt{\frac{g}{\kappa}}\sqrt{\kappa g}\sec^2\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]$$

$$\boxed{\therefore a(t) = -g\sec^2\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]}$$

3.4 Presence of a constant thrust force over a given time interval

In rocket modeling, it is of great interest to understand the dynamics of a body propelled by the expelling of mass at high speeds, using the principle of linear momentum conversion. However, as the mass flow rate is not constant, in addition to the body's own mass varying, it is very difficult to find a solution. The equation governing this scenario is given by

$$m(t)\frac{dv}{dt} = -m(t)g + \frac{dm}{dt}c - \frac{1}{2}\rho A C_D v^2$$

where c is the exhaust velocity of the propellant combustion gases (in module).

Using good empirical approximations, such as constant mass flow ($\dot{m}$ = const, in steady flow), we have

$$(m_0 - \dot{m}t)\,\frac{dv}{dt} = -\,(m_0 - \dot{m}t)\,g + \dot{m}c - \frac{1}{2}\rho A C_D v^2$$

Note that the derivatives of the solution now have variable coefficients. As briefly discussed in Lee (2014) [1], this equation will only have an analytical solution (apparently) when a factor q, defined as

$$q = \sqrt{\frac{\dot{m}c - mg}{\left(\frac{1}{2}A\rho C_D\right)}}$$

is constant. It is possible to rewrite this relationship as

$$\frac{Kq^2}{c} = \dot{m} - \frac{g}{c}m$$

with K representing the denominator inside the root. This ODE is a classic case where the solution is an exponential function (there are several ways to prove this!). Thus, the solution is

$$m(t) = e^{\frac{g}{c}t} - \frac{Kq^2}{g}$$

Therefore, when solving for the function of m with respect to time, given that q is a constant, one obtains an exponential function with behavior that is visibly at odds with observations of the behavior of mass in rocket models.

As the scope of this work is broader, attempts to solve the equation more completely will not be presented. However, an analytical solution, written using the Bessel Function and the Gamma Function, can be found using the following methods: transformation into a second-order ODE by the Riccati Equation; normalization (elimination of the first-order derivative); Möbius transformation [2] with equivalence of the invariant of the confluent hypergeometric function ${}_0F_1$; transformation from the form ${}_0F_1$ to the Bessel and Gamma function; and substitution of this last solution into the Riccati Equation. This deduction is extremely laborious and the final solution is extensive, drastically reducing the practicality of this form of the solution. A solution using this method was presented by Alves et al. (2021) [3].

Alternatively, in order to simplify the ODE and solve it easily, it is possible to assume two conditions: constant thrust (independent of mass or time) and constant mass. The mass used will have the numerical value of the arithmetic mean between the dry mass (without propellant) and the total mass (with propellant). For thrust, its numerical value will be the average thrust over the firing time interval. So, if $\overline{m}$ is the average mass and $\overline{E}$ is the average thrust, then

$$\overline{m} = \frac{m_{dry} + m_{total}}{2}$$

and

$$\overline{E} = \frac{1}{t_{burn}} \int_0^{t_{burn}} E(t)dt$$

Rewriting the equation of motion, we have

$$\overline{m}\frac{dv}{dt} = -\overline{m}g + \overline{E} - \frac{1}{2}\rho A C_D v^2$$

$$\Rightarrow \frac{dv}{dt} = -g + \frac{\overline{E}}{\overline{m}} - \kappa v^2$$

Adopting

$$K = \frac{\rho A C_D}{2\overline{m}}$$

to distinguish the κ parameter from the unpropelled phases. In addition, it is convenient to define an effective, or apparent, gravity factor as

$$G = -g + \frac{\overline{E}}{\overline{m}}$$

such that

$$\frac{dv}{dt} = G - Kv^2$$

This differential equation can be solved using the methods of the descending ballistic phase. Therefore, there will be a subsection (4.4) of the ballistic phase that will solve this equation in detail.

4 Descending ballistic phase

In this case, equation 1 is used, in which the weight force opposes the drag. It is also in this scenario that the concept of terminal velocity arises.

The differential equation that represents this situation is

$$\frac{dv}{dt} = \kappa v^2 - g$$

4.1 Time function of speed

By separating the variables, we have

$$\frac{dv}{\kappa v^2 - g} = dt$$

$$\Leftrightarrow \frac{1}{g}\frac{dv}{\frac{\kappa}{g}v^2 - 1} = dt$$

Integrating both sides and using $v(t_0) = v_0$ to indicate the velocity at the initial instant and $v(t_1) = v_1$ for the velocity at the final instant, we write

$$\frac{1}{g}\int_{v_0}^{v_1} \frac{dv}{\frac{\kappa}{g}v^2 - 1} = \int_{t_0}^{t_1} dt$$

$$\Rightarrow \frac{1}{g}\int_{v_0}^{v_1} \frac{dv}{\frac{\kappa}{g}v^2 - 1} = \Delta t$$

By substituting $u = v\sqrt{\frac{\kappa}{g}}$, we get

$$du = dv\sqrt{\frac{\kappa}{g}}$$

Thus,

$$\frac{1}{g}\sqrt{\frac{g}{\kappa}}\int_{u_0}^{u_1} \frac{du}{u^2 - 1} = \Delta t$$

recalling that $u_i = v_i\sqrt{\frac{\kappa}{g}}$

$$\Rightarrow \frac{1}{\sqrt{g\kappa}}\int_{u_0}^{u_1} \frac{du}{u^2 - 1} = \Delta t \qquad \text{[eq. 3]}$$

Simultaneously, consider the following indefinite integral I:

$$I = \int \frac{du}{u^2 - 1}$$

It can be written that

$$\Rightarrow I = \int \frac{du}{(u-1)(u+1)}$$

Then, separating by partial fractions

$$\Rightarrow I = \int \frac{A}{(u-1)}du + \int \frac{B}{(u+1)}du$$

Such that A, B are real and independent constants of u. This generates the following equation, so that the sum of these fractions is $\frac{1}{u^2-1}$:

$$A(u+1) + B(u-1) = 1$$

$$\Rightarrow u(A+B) + (A-B) = 1$$

Implying that

$$A + B = 0u$$

and

$$A - B = 1$$

Solving the system, we get $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Thus, we write I as

$$I = \frac{1}{2}\int \frac{du}{(u-1)} - \frac{1}{2}\int \frac{du}{(u+1)}$$

Knowing that

$$\int \frac{dx}{x} = \ln(x) + C$$

where C is any constant, then

$$I = \frac{1}{2}\ln(u-1) - \frac{1}{2}\ln(u+1)$$

$$\Rightarrow I = \frac{1}{2}\ln\left(\frac{u-1}{u+1}\right) + C$$

$$\Rightarrow I = -\frac{1}{2}\ln\left(\frac{u+1}{u-1}\right) + C$$

Note that $|u| \leq 1$. The reason behind this can be explained by the terminal velocity of a falling body. Clearly, it is never possible for a body to develop a drag force greater than its weight. This is due to the very nature of drag: a dissipative force (negative work) that must always oppose movement, which would not be the case if it were greater than weight. In fact, there is a limit case (in "infinite" time), demonstrated later, in which the speed approaches a horizontal asymptote. This case will occur when the velocity is such that the drag equals the weight, preventing the system from accelerating further to increase the drag. We can write v_t as the terminal velocity,

$$g = \kappa v_t^2$$

$$\Rightarrow |v_t| = \sqrt{\frac{g}{\kappa}}$$

As $u = v\sqrt{\frac{\kappa}{g}}$, the largest value will be $u = v_t\sqrt{\frac{\kappa}{g}} = 1$ The reason for analyzing this limit case is the substitution that will now be made. We know from hyperbolic geometry that

$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \text{ se } |x| \leq 1$$

Thus,

$$I = -\operatorname{artanh}(u) + C$$

Using I in eq. 3, we get

$$-\frac{1}{\sqrt{g\kappa}}\operatorname{artanh}(u)\Big|_{u_0}^{u_1} = \Delta t$$

$$\Rightarrow -\frac{1}{\sqrt{g\kappa}}\operatorname{artanh}\left(v\sqrt{\frac{\kappa}{g}}\right)\Bigg|_{v_0}^{v_1} = \Delta t$$

$$\therefore \Delta t = \frac{1}{\sqrt{g\kappa}}\left[\operatorname{artanh}\left(v_0\sqrt{\frac{\kappa}{g}}\right) - \operatorname{artanh}\left(v_1\sqrt{\frac{\kappa}{g}}\right)\right]$$

Choosing $t_0 = 0$, $t_1 = t$ and $v_0 = 0$, you can write the inverse function of the fall speed from rest

$$t = -\frac{1}{\sqrt{g\kappa}} \operatorname{artanh}\left(v\sqrt{\frac{\kappa}{g}}\right)$$

Solving for the explicit $v(t)$,

$$\boxed{v(t) = -\sqrt{\frac{g}{\kappa}} \tanh\left(t\sqrt{\kappa g}\right)} \qquad \text{[eq. 4]}$$

To demonstrate that the terminal velocity is a limit case (at infinity only), consider the value of this limit as t tends to infinity:

$$\begin{aligned}
v_t &= \lim_{t\to\infty} v(t) \\
&= -\sqrt{\frac{g}{\kappa}} \lim_{t\to\infty} \tanh\left(t\sqrt{\kappa g}\right) \\
&= -\sqrt{\frac{g}{\kappa}} \lim_{t\to\infty} \frac{e^{2t\sqrt{\kappa g}} - 1}{e^{2t\sqrt{\kappa g}} + 1} \\
&= -\sqrt{\frac{g}{\kappa}} \lim_{t\to\infty} 1 \\
&\boxed{= -\sqrt{\frac{g}{\kappa}}}
\end{aligned}$$

As the limit of convergence is the value of the function (which represents the ordinates), the graph has a horizontal asymptote of value $\sqrt{\frac{g}{\kappa}}$

One way to get an intuition of how fast the body is approaching terminal velocity is to determine the time until $v = 0.99v_t$. Equating eq. 4 to $0.99v_t$, we get that

$$0.99v_t = -\sqrt{\frac{g}{\kappa}} \tanh\left(t\sqrt{\kappa g}\right)$$

$$\begin{aligned}
\Rightarrow -0.99\sqrt{\frac{g}{\kappa}} &= -\sqrt{\frac{g}{\kappa}} \tanh\left(t\sqrt{\kappa g}\right) \\
\Rightarrow 0.99 &= \tanh\left(t\sqrt{\kappa g}\right) \\
\therefore t &= \frac{1}{\sqrt{\kappa g}} \operatorname{artanh}(0.99)
\end{aligned}$$

4.2 Time function of position

In order to determine the position function, it is sufficient to integrate the velocity function with respect to time, since $v = \frac{dy}{dt}$ Thus, from eq. 4,

$$\begin{aligned}
v(t)dt &= -\sqrt{\frac{g}{\kappa}} \tanh\left(t\sqrt{\kappa g}\right) dt \\
\Rightarrow \int_{y_0}^{y_1} dy &= -\sqrt{\frac{g}{\kappa}} \int_{t_0}^{t_1} \tanh\left(t\sqrt{\kappa g}\right) dt \\
\Rightarrow \Delta y &= -\sqrt{\frac{g}{\kappa}} \int_{t_0}^{t_1} \tanh\left(t\sqrt{\kappa g}\right) dt
\end{aligned}$$

Since $\int \tanh(x)dx = \ln(\cosh(x)) + C$, with C being any constant, then

$$\Delta y = -\sqrt{\frac{g}{\kappa}} \ln\left(\cosh\left(t\sqrt{\kappa g}\right)\right)\Big|_{t_0}^{t_1}$$

$$\Rightarrow \Delta y = \sqrt{\frac{g}{\kappa}} \ln\left(\cosh\left(t_0\sqrt{\kappa g}\right)\right) - \sqrt{\frac{g}{\kappa}} \ln\left(\cosh\left(t_1\sqrt{\kappa g}\right)\right)$$

Taking $t_0 = 0$, $y_1 = y$ and $t_1 = t$, one can find the explicit function $y(t)$

$$y(t) = y_0 - \sqrt{\frac{g}{\kappa}} \ln\left(\cosh\left(t\sqrt{\kappa g}\right)\right) \frac{1}{\sqrt{\kappa g}}$$

$$\boxed{\therefore y(t) = y_0 - \kappa^{-1} \ln\left(\cosh\left(t\sqrt{\kappa g}\right)\right)}$$

4.3 Time function of acceleration

To determine the acceleration a as a function of time, simply derive the function $v(t)$ with respect to t. Thus, from eq. 4, we have

$$a(t) = \frac{dv}{dt} = -\sqrt{\frac{g}{\kappa}} \operatorname{sech}^2(t\sqrt{\kappa g})\sqrt{\kappa g}$$

$$\boxed{\therefore a(t) = -g \operatorname{sech}^2(t\sqrt{\kappa g})}$$

4.4 Solution of the boost phase

Initially, we have

$$\frac{dv}{dt} = G - Kv^2$$

The structure of the differential equation is (almost) identical to that seen above, with only one member having the opposite sign. In this way, this sign will be carried through the analogous process, resulting in the same equations of motion (of descent) with the sign changed, since the multiplicative constant -1 "comes out" of the integrals and derivatives. Therefore, the equations of motion for the boost phase, with constant thrust, are as follows:

Aceleration:

$$a(t) = G \operatorname{sech}^2(t\sqrt{KG})$$

Velocity:

$$v(t) = \sqrt{\frac{G}{\kappa}} \tanh\left(t\sqrt{KG}\right)$$

Position (height):

$$y(t) = y_0 + K^{-1} \ln\left(\cosh\left(t\sqrt{KG}\right)\right)$$

It is clear that the domain of the three functions is the interval $[0, t_{burn}]$

5 Conclusion

In conclusion, by formulating and solving the differential equations that represent the ascending and descending ballistic phases, unidimensional and with drag, the following equations of motion (position, velocity and acceleration) as a function of time could be determined using differential and integral calculus. The inverse functions of these were also deduced and are in the previous sections.

5.1 Ascent phase

- **Initial differential equation:** $a = -g - \kappa v^2 \Rightarrow \ddot{y} = -g - \kappa\left(\dot{y}\right)^2$
- **Time function of acceleration**: $a(t) = -g\sec^2\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]$
- **Time function of speed:** $v(t) = \sqrt{\frac{g}{\kappa}} \tan\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right]$
- **Time function of position:** $y(t) = y_0 + \kappa^{-1}\ln\left[\left(\cos\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t\sqrt{\kappa g}\right)\right)\sqrt{1 + \frac{v_0^2\kappa}{g}}\right]$

5.2 Descent phase

- **Initial differential equation:** $a = -g + \kappa v^2 \Rightarrow \ddot{y} = -g + \kappa \left(\dot{y}\right)^2$
- **Time function of acceleration**: $a(t) = -g\,\mathrm{sech}^2(t\sqrt{\kappa g})$
- **Time function of speed:** $v(t) = -\sqrt{\frac{g}{\kappa}}\tanh\left(t\sqrt{\kappa g}\right)$
- **Time function of position:** $y(t) = y_0 - \kappa^{-1}\ln\left(\cosh\left(t\sqrt{\kappa g}\right)\right)$

5.3 Some comments on the study of these scenarios

There are many situations in which these equations can be useful, especially if the aim is to predict trajectories (or estimates). However, it is necessary to be very careful with the trigonometric functions present in the equations, since when solving the equations, multiple answers can be discovered.

In general, it is interesting to draw the graphs to visualize them. Algebraically, when faced with a trigonometric equation, either the solution that matters is the trivial one or the first determination. Testing values to find absurdities is essential.

In studies of this type of movement, it is critical that the instant of time at which there is a switch between the ascending and descending phase is immediately determined. This will restrict the domains of the functions.

6 Example: Flight LAE-140/v2 (16/6/2024)

For example, we will use data from the launch of the LAE-140/v2 experimental rocket, with a B4-0 class engine, single stage, equipped with a MicroPeak altimeter (MP-81), no ejection charge and designed to fly for 10 seconds.

Since there is some uncertainty about the engine's characteristics, experimentally collected data from a static test of engines of the same model was used. The averages of thrust, burn time and propellant mass were chosen for the calculations of the propelled phase.

The drag coefficient was obtained from the GFCS (Carl Sagan Rocketry Club, in english) CD 2.1 software (UFPR) and the geometry of the mini-rocket. The specific mass of the air was calculated using the Ideal Law of Gases, a pressure of 914.8 hPa, a temperature of 24 °C and an ideal gas constant of 286.9 J/kgK. The other parameters were taken directly from the LAE-140/v2 launch session report.

6.1 Parameters used

1. Gravitational acceleration (in Curitiba): $g = 9.7876ms^{-1}$
2. Drag coefficient: $C_D = 0.91$
3. Air specific mass $\rho = 1.07305$ kg m^{-3}
4. Dry mass (without propellant, measured post-flight): $m_{dry} = 4.4675 \cdot 10^{-2}$ kg
5. Propellant Mass (expelled, measured post-flight): $m_{propellant} = 4.752 \cdot 10^{-3}$ kg
6. Reference cross-sectional area: $A = \frac{\pi}{4} \cdot \left(20.06 \cdot 10^{-3}\right)^2 \approx 3.15102 \cdot 10^{-4}$ m^2

6.2 Ascent Phase

6.2.1 Predicted boost phase

Since, a priori, it is impossible to know the exact thrust, burn time and propellant mass data, an estimate is made.

The mass of the rocket with the engine (total mass) was $m_{total} = 49{,}427$ g, while the mass of the total engine was 16,961 g. The expected propellant mass (based on tests of other engines of the same model by the GFCS) was 4.57g to 5.29g. The average of this propellant mass is $m_{propellant} = 4.864$g. Therefore, the average mass during the propelled phase is

$$\overline{m} = \frac{2m_{dry} + m_{propellant}}{2} = 4.7051 \cdot 10^{-2}\text{kg}$$

Next, set the K parameter,

$$K = \frac{\rho A C_D}{2\overline{m}} \approx 3.15102 \cdot 10^{-4} m^{-1}$$

According to previously performed static tests of engines of the same model, the average burning time, t_{burn}, is 0.786s and the average thrust, $\overline{E}$, is 4.49 N

Thus, the effective gravity is calculated to be

$$G = -g + \frac{\overline{E}}{\overline{m}} \approx 8.56408 \cdot 10^{1}\text{m s}^{-2}$$

Using the equations previously deduced, the speed at the end of the chiema should be

$$v_{burn} = v(t_{burn}) = \sqrt{\frac{G}{K}} \tanh\left(t_{burn}\sqrt{KG}\right)$$

$$\therefore v_{burn} \approx 63.6830 m/s$$

The height at the end of the burn should be 0 meters, given that the initial height was 0 meters,

$$y_{burn} = y(t_{burn}) = K^{-1} \ln\left(\cosh\left(t_{burn}\sqrt{KG}\right)\right)$$

$$\therefore y_{burn} = 25.7249 m$$

6.2.2 Graph (I)

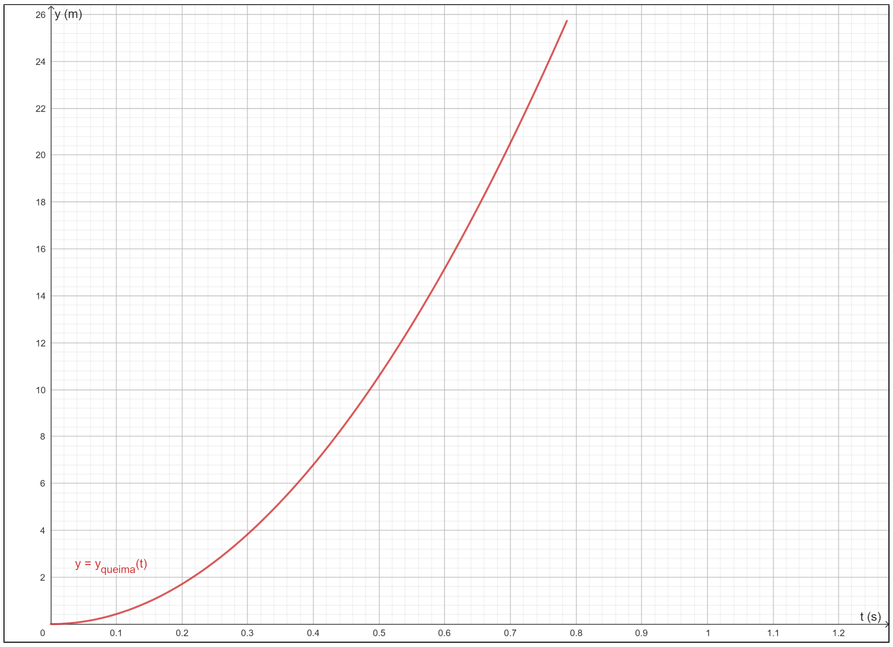


Figure 3: Height X Time graph in the boost phase ($t_{burn} \geq t \geq 0$)

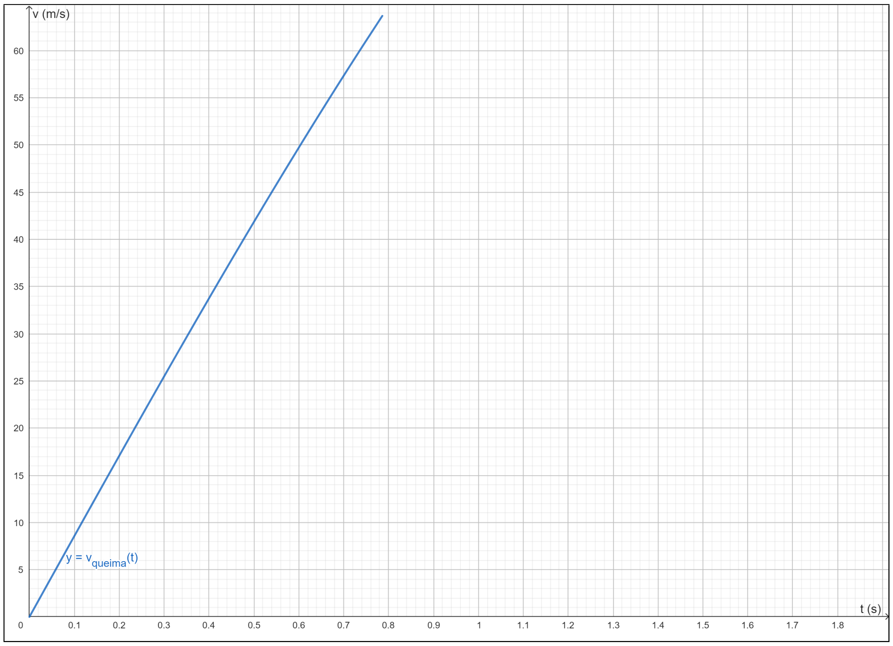


Figure 4: Velocity X Time in the boost phase ($t_{burn} \geq t \geq 0$)

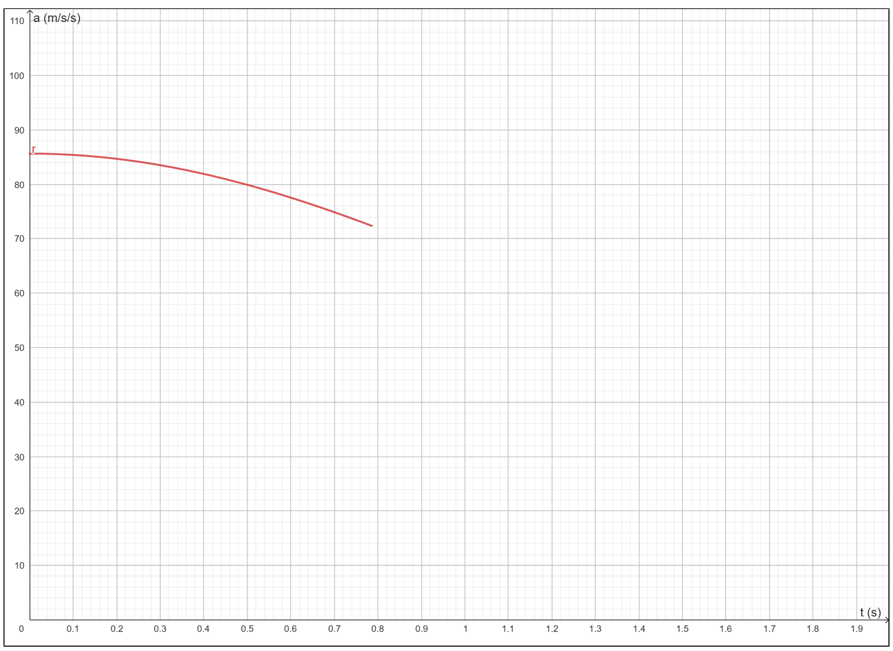


Figure 5: Acceleration X Time graph in the boost phase ($t_{burning} \geq t \geq 0$)

6.2.3 Defining κ

Since the mass after burning phase is the dry mass, then the parameter κ will be used. Now, the actual letter kappa will be used in the notation. Given that. Dado que

$$m = m_{dry} = 44.675g$$

Then,

$$\kappa = \frac{\rho A C_D}{2m} \therefore \kappa = 3.44364 \cdot 10^{-4} m^{-1}$$

6.2.4 Time to Apogee

As the apogee is the instant when the trajectory changes direction, the velocity at this point is zero. Mathematically, we write

$$v(t') = \sqrt{\frac{g}{\kappa}} \tan\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t'\sqrt{\kappa g}\right] = 0$$

$$\Rightarrow \tan\left[\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t'\sqrt{\kappa g}\right] = 0$$

It is clear that the first solution with $k = 0$ of the set of solutions $\pi \cdot k$; $k \in \mathbb{Z}$ corresponds to the value sought, since it is the first time that the function (velocity) equals zero from the origin towards the positive reals.

$$\Rightarrow \arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t'\sqrt{\kappa g} = 0$$

$$\Rightarrow t' = \frac{1}{\sqrt{\kappa g}} \arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right)$$

$$\therefore t' = 4.75958s$$

Where $v_0 = v_{burn}$. The time until apogee is $t_{apogee} = t' + t_{burn}$, since t' represents the time between the end of burning and apogee. Therefore,

$$t_{apogee} = 5.54558s$$

6.2.5 Height of apogee

To find the height of the apogee, simply insert the (predicted) apogee time into the position equation, considering the initial height as that at the end of the (predicted) burn. Thus,

$$y(t_{apogee}) = y_0 + \kappa^{-1} \ln\left[\left(\cos\left(\arctan\left(v_0\sqrt{\frac{\kappa}{g}}\right) - t_{apogee}\sqrt{\kappa g}\right)\right)\sqrt{1 + \frac{v_0^2\kappa}{g}}\right]$$

$$\Rightarrow y_{apogee} = y(t_{apogee}) = 25.72495 + 128.73115$$

$$\therefore y_{apogee} = 154.45610m$$

6.2.6 Graph (II)

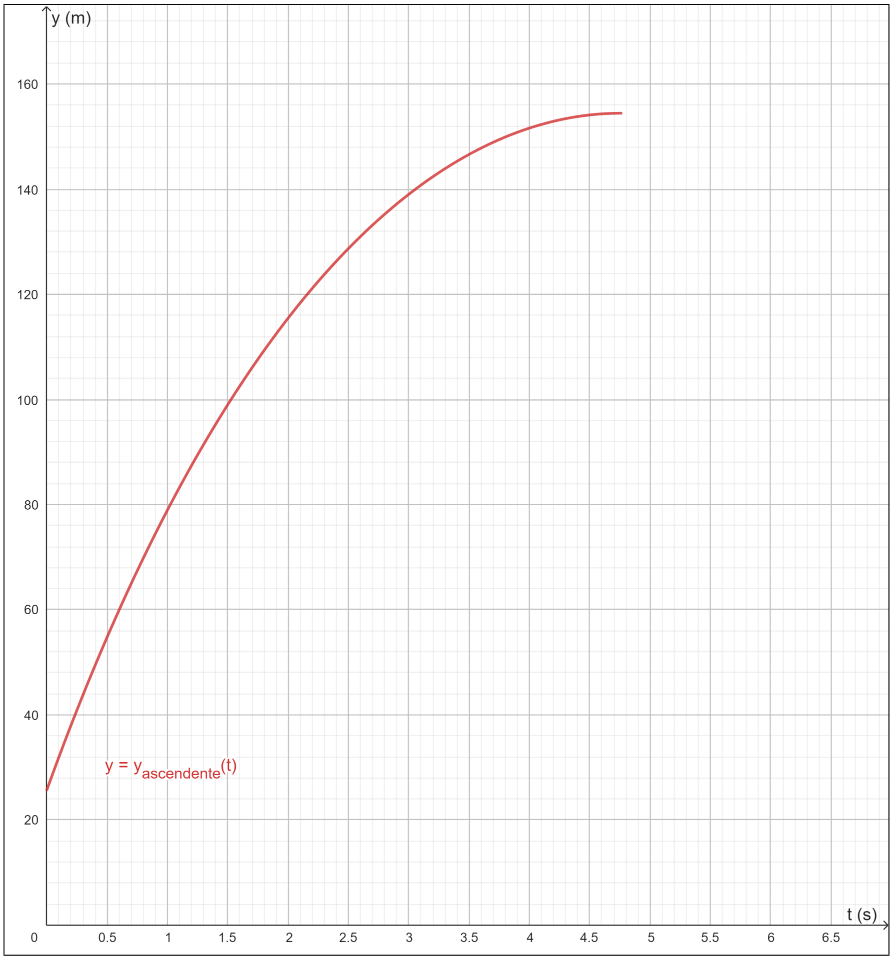


Figure 6: Height X Time graph in the ascending phase ($t' \geq t \geq 0$)

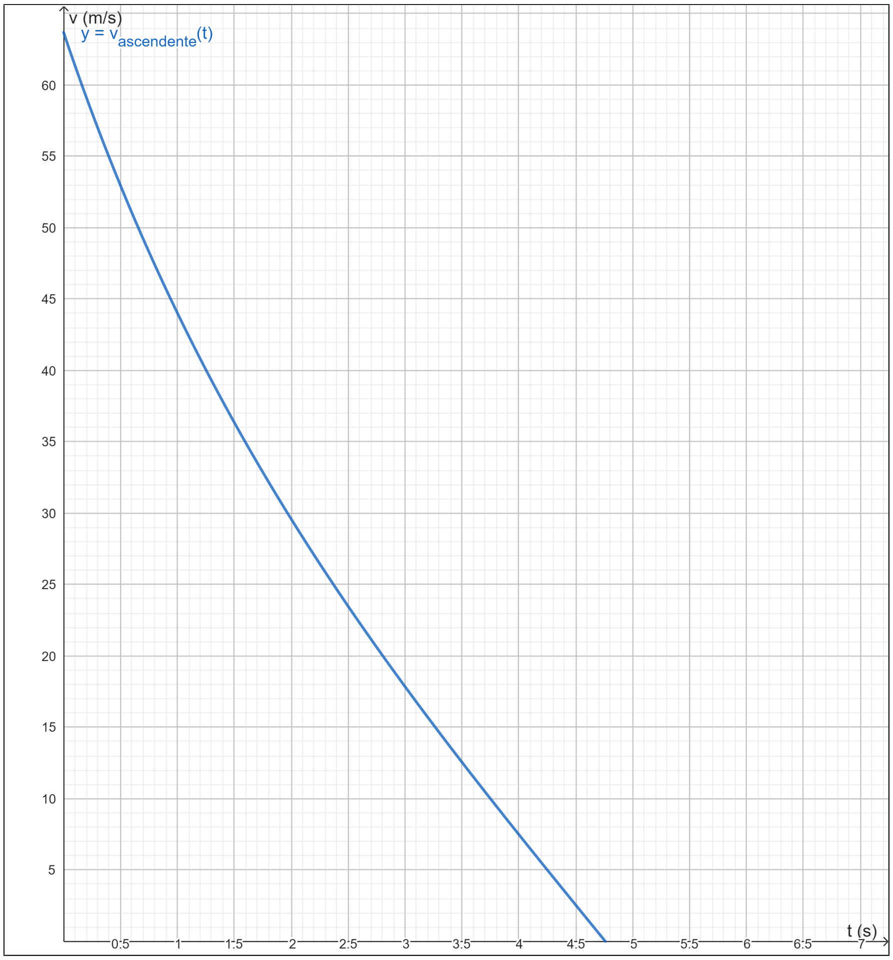


Figure 7: Speed X Time graph in the ascending phase ($t' \geq t \geq 0$)

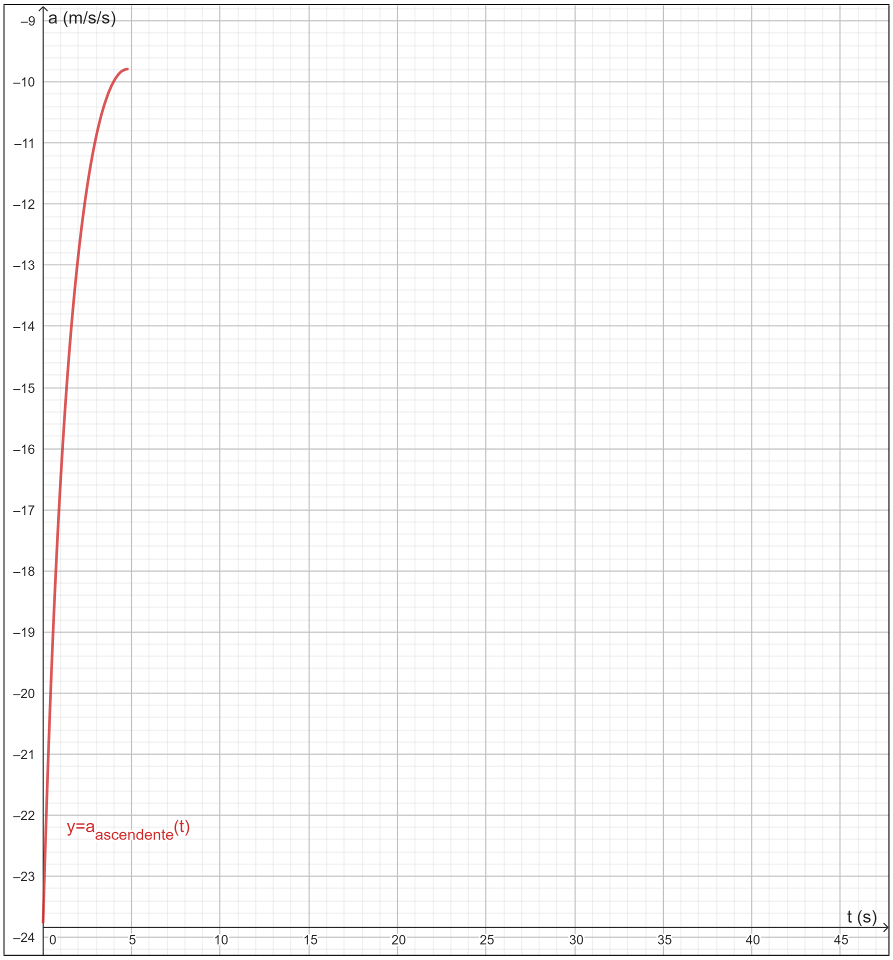


Figure 8: Acceleration X Time graph in the ascending phase ($t' \geq t \geq 0$)

6.3 Descent phase

6.3.1 Falling time

The fall time is determined by the root of the position function in the descending phase. Thus,

$$y(t_{fall}) = y_{apogee} - \kappa^{-1} \ln\left(\cosh\left(t_{fall}\sqrt{\kappa g}\right)\right) = 0$$

Isolating t_{fall}, it follows that

$$t_{fall} = \frac{1}{\sqrt{\kappa g}} \cosh^{-1}\left(e^{\kappa y_{apogee}}\right)$$

Substituting the values, we get

$$\therefore t_{fall} = 6.12668s$$

Therefore, the total flight time is

$$t_{total} = t_{apogee} + t_{fall} = 11.67226s$$

6.3.2 Impact velocity

The impact velocity is obtained from the value of the velocity function when the time is equal to t_{fall}. Hence, we have

$$v_{impact} = v(t_{fall}) = -\sqrt{\frac{g}{\kappa}} \tanh\left(t_{fall}\sqrt{\kappa g}\right)$$

$$\therefore |v_{impact}| = 43.14205 \text{ m s}^{-1}$$

6.3.3 Acceleration at impact

The acceleration at the moment of impact is calculated by the time function of the acceleration when the time is equal to t_{fall}. So,

$$a_{impact} = a(t_{fall}) = -g\,\mathrm{sech}^2(t_{fall}\sqrt{\kappa g})$$

$$\therefore |a_{impact}| = 3.37817 \text{ m s}^{-2}$$

6.3.4 Graph (III)

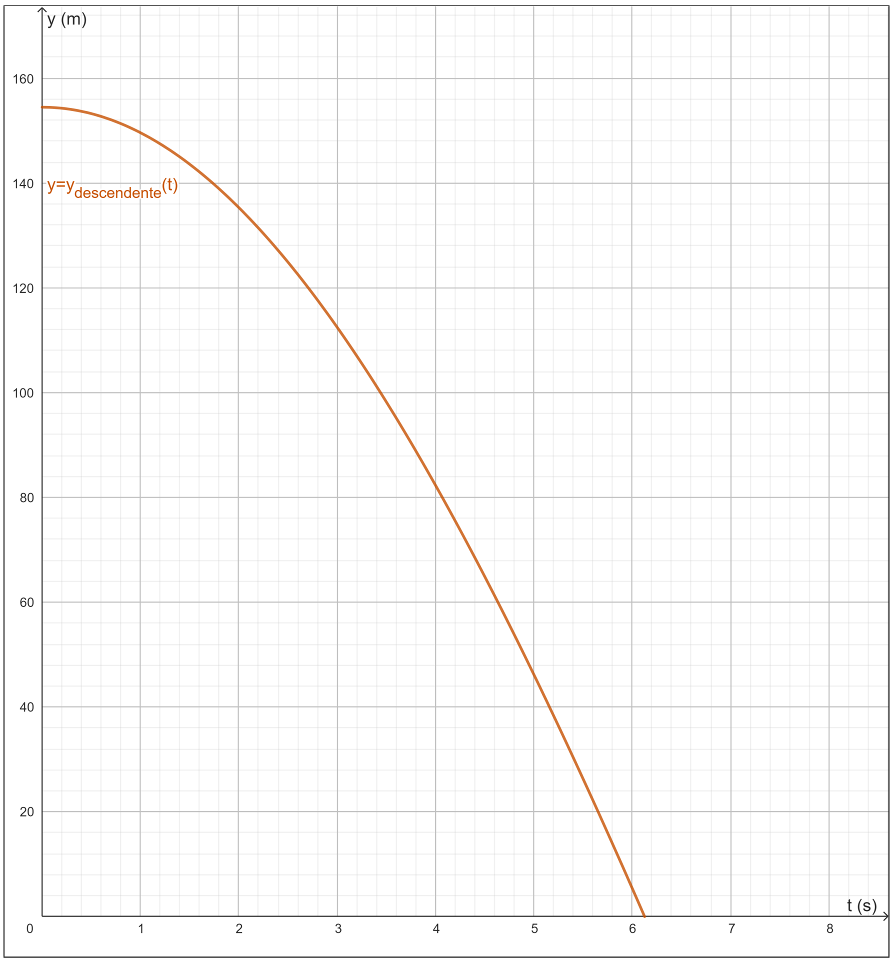


Figure 9: Height X Time graph in the descending phase ($t_{fall} \geq t \geq 0$)

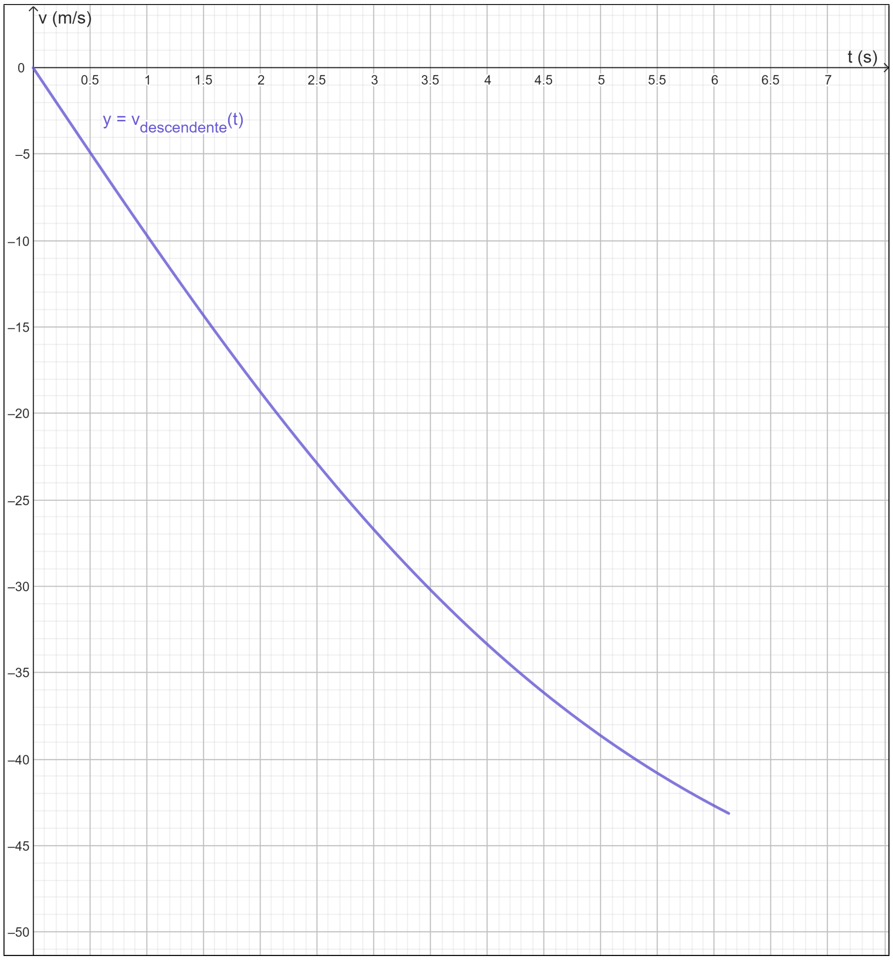


Figure 10: Velocity X Time graph in the descending phase ($t_{fall} \geq t \geq 0$)

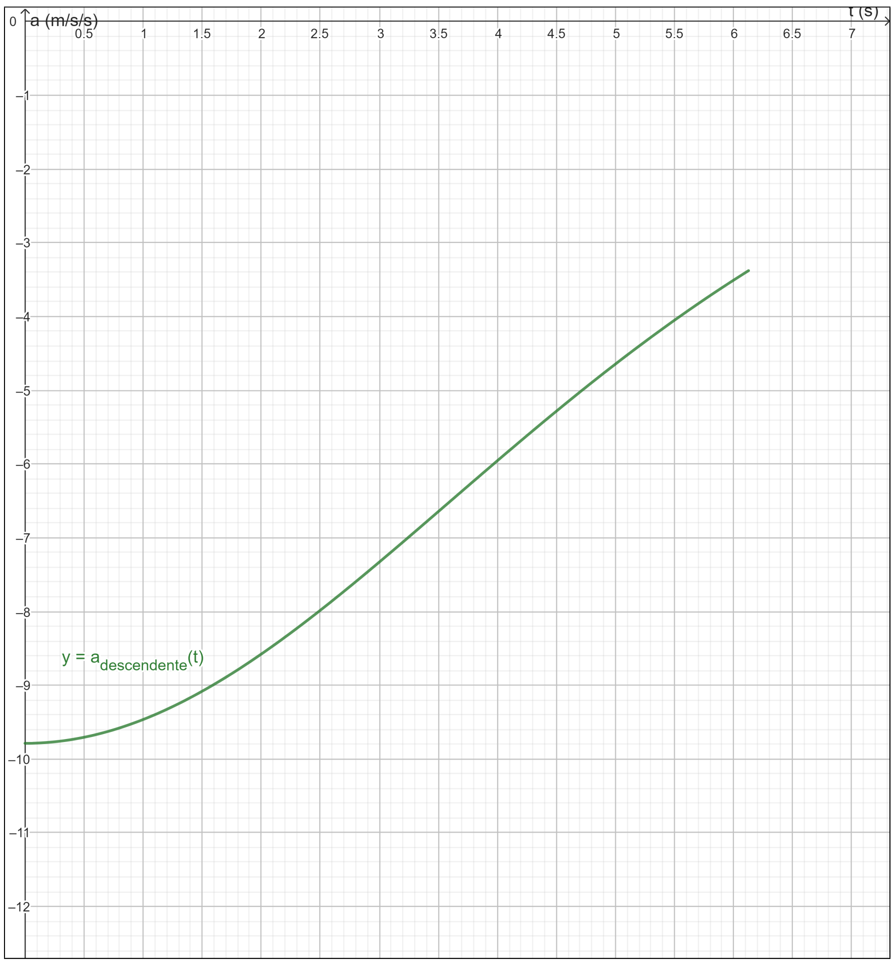


Figure 11: Acceleration X Time graph in the descending phase ($t_{fall} \geq t \geq 0$)

By combining the height graphs for each phase, adjusting their domains and translating them accordingly, it is possible to plot a single height graph for the entire flight:

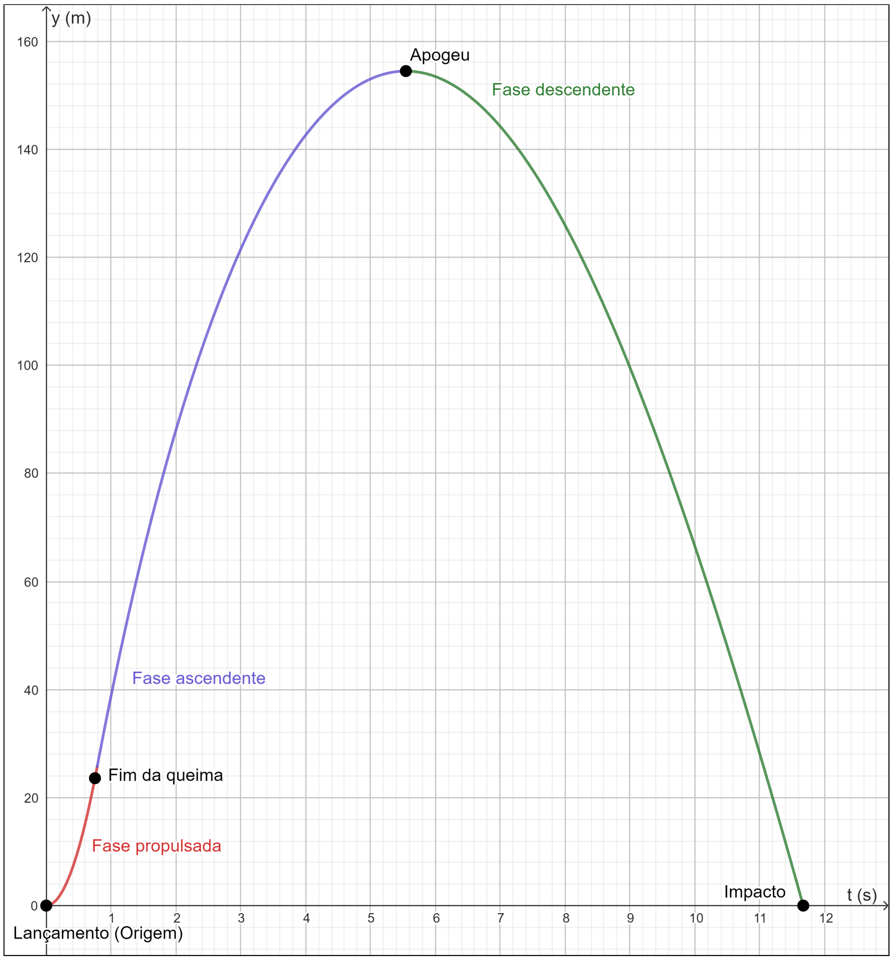


Figure 12: Height X Time graph of LAE-140-v2 ($t_{total} \geq t \geq 0$)

6.4 Comparing the flight data

Some data and graphs were extracted from altimeters or measured by launch participants. A table comparing the main measured and predicted data (in this work) has been constructed.

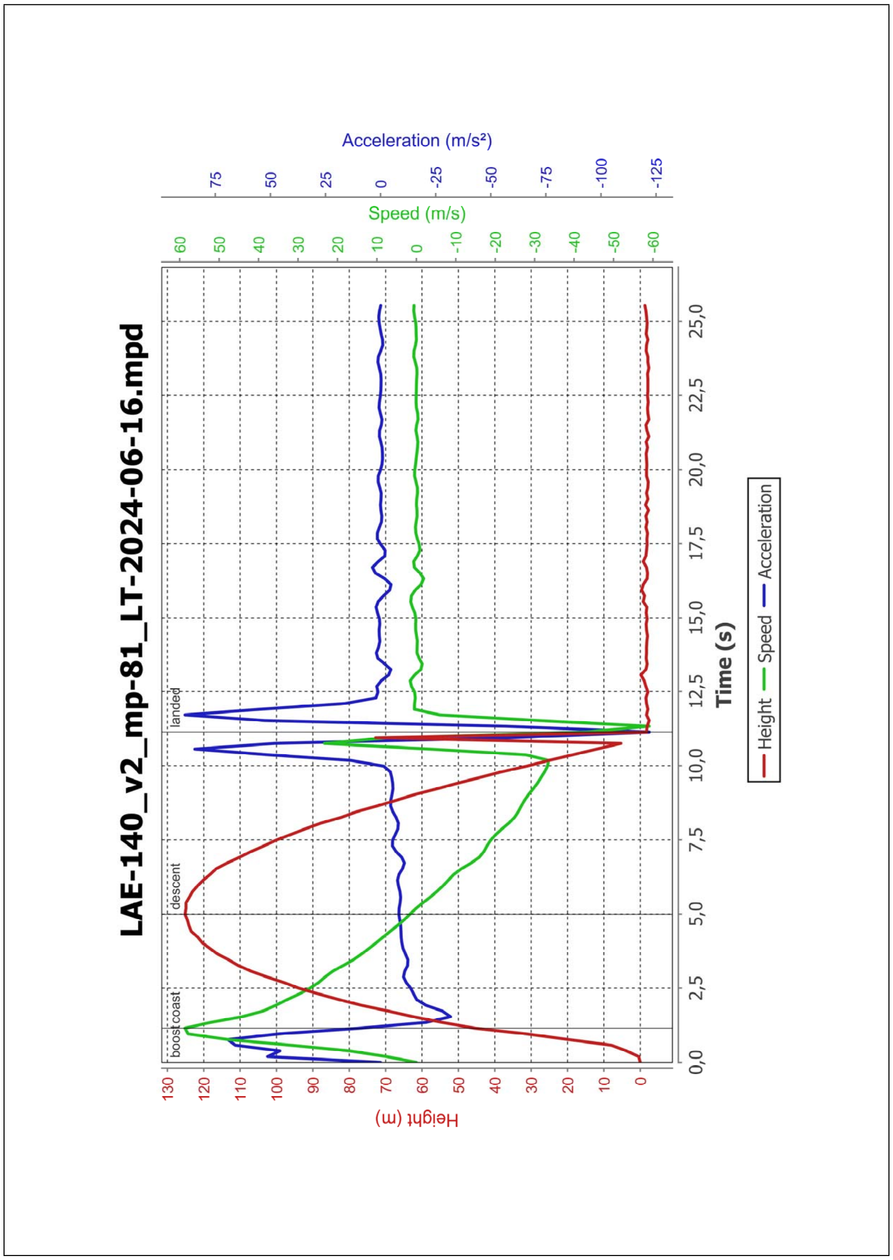


Figura 13: Gráfico com altura, velocidade e aceleração versus tempo

Dado	Predicted	Measured	Absolute difference	Relative Difference
t_{burn}	0.786 s	1.15 s	0.36 s	31%
v_{burn}	63.6830 m/s	58.67 m/s	5.01 m/s	8.5%
h_{burn}	25.7249 m	45.6 m	19.9 m	44%
t_{apogee}	5.54558 s	4.99 s	0.56 s	11%
y_{apogee}	154.45610 m	125.2 m	29.3 m	23%
t_{fall}	6.12668 s	5.95 s	0.18 s	3%
v_{impact}	43.14205 m/s	21.0 m/s	22.1 m/s	105%
t_{total}	11.67226 s	10.94 s	0.73 s	6.7%

It is important to note that the discrepancy in the firing time may have been due to the thrust curve (i.e. not constant), which contributes gradually less to the thrust towards the end of the firing time. Furthermore, it can be seen that the approximation assumed in the propelled phase had a relatively large variation, demonstrating the importance of calculating the boost phase accurately. Additionally, there is a large uncertainty in the data at the moment of landing due to the mechanical impact.

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[1] LEE, S.-H.; ALDREDGE, R. C. Analytic approach to determine optimal conditions for maximizing altitude of sounding rocket: Flight in standard atmosphere. v. 46, p. 374–385, 1 out. 2015.
[2] CHAN, L.; E.S. CHEB-TERRAB. Non-Liouvillian solutions for second order linear ODEs. arXiv (Cornell University), 1 jan. 2004. [3] ALVES, A. L.; BENTO, S. S.; MARCHI, C. H. Movimento Vertical de Minifoguetes: Equações de Trajetórias e Análises Gráficas. Revista Brasileira de Ensino de Física, v. 43, p. e20200479, 15 fev. 2021.