On the time expressions for the pressurization of a combustion chamber with a solid, cylindrical grain propellant

October 21st, 2024

1 Abstract

The aim of this work is to describe the time functions of the pressure during pressurization until permanent conditions in a combustion chamber of a solid grain-propellant with a cylindrical cross-section, i.e. with a constant burning area.

2 Nomenclature

- T: temperature of a system [K]
- p: pressure of a system [Pa]
- ρ: specific mass[$\mathrm{kgm^{-3}}$]
- V: volume of a system [$\mathrm{m^3}$]
- γ: specifc heat ratio of a gas [adim.]
- R: gas constant of a specifc gas [$\mathrm{J \cdot kg^{-1} K^{-1}}$]
- $S.C.$: control surface
- $C.V.$: control volume
- a: multiplicative constant of Saint-Robert's law
- n: exponential constant of Saint-Robert's Law
- c: speed of sound in a gas [$\mathrm{ms^{-1}}$]
- A: area of a surface [$\mathrm{m^2}$]
- r: burn rate of propellant [$\mathrm{ms^{-1}}$]

2.1 Subindices

- 0: referring to the gas in the combustion chamber
- p: referring to propellant, in solid state
- SS: referring to a property in steady state
- a: referring to a property in ambient conditions
- t: referring to the throat of a convergent-divergent nozzle
- b: referring to the burning of propellant

3 Description of the system

Consider a control volume that encompasses the inhibition ("burning front") and the throat of the propellant ("converging-diverging" profile), constituting a free volume in the combustion chamber. Within this control volume, the notations for the following properties of interest will be adopted:

- Combustion temperature T_0 [K];
- Combustion pressure p_0 [Pa]
- Gas constant of the propellant gas mixture R $[\mathrm{J} \cdot \mathrm{kg}^{-1}\mathrm{K}^{-1}]$
- Ratio of specific heats of propellant gas γ [adim.]

The value of the control volume will be denoted as V_0

3.1 Assumptions made about the control volume

- Ideal gas: the equation of state $p = \rho RT$ applies, where ρ refers to the specific mass of the gas.
- Conservation of mass: all propellant (by mass) burned becomes gas, which enters into a mass balance
- Pressurization occurs so much faster than the initial volume increase (which is caused by the burning of the grain-propellant) so that $\frac{dV_0}{dt} = 0$ is assumed
- During pressurization, the combustion temperature and the gas constant are constant.

Adopting the above assumptions, we start from the equation of conservation of mass

$$\boxed{\frac{\partial}{\partial t}\iiint_{V.C.} \rho d\cancel{V} + \iint_{S.C.} \rho \vec{u} \cdot d\vec{A} = 0}$$

where $d\cancel{V}$ is an infinitesimal volume, $d\vec{A}$ is a vector with the magnitude of an infinitesimal area and oriented perpendicular to and away from the area. Thus, in the $C.V.$ of interest, we have

$$\frac{\partial}{\partial t}\iiint_{V.C.} \rho_0 d\cancel{V} + \iint_{S.C.} \rho \vec{u} \cdot d\vec{A} = 0$$

Using

$$\rho_0 = \frac{p_0}{RT_0}$$

One gets

$$\frac{\partial}{\partial t}\iiint_{V.C.} \frac{p_0}{RT_0} d\cancel{V} + \iint_{S.C.} \rho \vec{u} \cdot d\vec{A} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\frac{p_0}{RT_0}\iiint_{V.C.} d\cancel{V}\right] + \iint_{S.C.} \rho \vec{u} \cdot d\vec{A} = 0$$

Since the specific mass of the gas does not depend on the infinitesimal volume.

$$\Rightarrow \frac{\partial}{\partial t}\left[\frac{p_0}{RT_0} V_0\right] + \iint_{S.C.} \rho \vec{u} \cdot d\vec{A} = 0 \tag{3.1}$$

The second term in the equation refers to mass flows. As the only areas with flow are the combustion front and the throat, then,

$$\iint_{S.C.} \rho \vec{u} \cdot d\vec{A} = \iint_{A_b} \rho_p (-1) r dA + \iint_{A_t} \rho_0 u dA$$

$$= -\rho_p r A_b + \rho_0 v_t A_t$$

where v_t is the velocity at the throat, and r is the rate, or speed, of burning, in meters per second, which is established by Saint-Robert's empirical law:

$$\boxed{r = ap_0^n}$$

where a and n are experimentally determined constants and have varying units (and dimensions).

Thus, using the equation 3.1, we have

$$\frac{\partial}{\partial t}\left[\frac{p_0}{RT_0}V_0\right] - \rho_p r A_b + \rho_0 v_t A_t = 0$$

It is known that, in an ideal case, the mass flow rate in the throat is given by

$$\dot{m}_t = \frac{\Gamma}{c} A_t p_0$$

in which

$$\Gamma = \gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

and

$$c = \sqrt{\gamma R T_0}$$

Since $\rho_0 v_t A_t = \dot{m}_t$, we are left with

$$\frac{\partial}{\partial t}\left[\frac{p_0}{RT_0}V_0\right] - \rho_p r A_b + \frac{\Gamma}{c} A_t p_0 = 0$$

Since R and the combustion temperature do not vary temporally, then,

$$\frac{1}{RT_0}\frac{d}{dt}\left[p_0 V_0\right] - \rho_p r A_b + \frac{\Gamma}{c} A_t p_0 = 0$$

$$\Rightarrow \frac{1}{RT_0}\frac{d}{dt}\left[p_0 V_0\right] = \rho_p r A_b - \frac{\Gamma}{c} A_t p_0$$

It is assumed that the pressurization is so fast that the burning speed of the propellant offers irrelevant additional space. In other words, during this transient regime (of pressurization), we take

$$\frac{dV_0}{dt} = 0$$

Hence,

$$\frac{V_0}{RT_0}\frac{dp_0}{dt} = \rho_p r A_b - \frac{\Gamma}{c} A_t p_0$$

In another notation, we have

$$\frac{V_0}{RT_0}\dot{p}_0 = \rho_p r A_b - \frac{\Gamma}{c} A_t p_0$$

Since $r = ap_0^n$, then

$$\boxed{\frac{V_0}{RT_0}\dot{p}_0 = \rho_p a p_0^n A_b - \frac{\Gamma}{c} A_t p_0} \tag{3.2}$$

4 Solution of the time function of pressure during the pressurization phase

It can be seen that the "state" expression found corresponds to a non-linear ordinary differential equation. More precisely, it is the Bernoulli differential equation, written generically as

$$y'(x) + P(x)y(x) = Q(x)y^n(x)$$

One way to solve it is by using the linearization method offered by Leibniz (1695), which uses variable substitution

$$u(x) = y(x)^{1-n}$$

Where $n \neq 0$ nor $n \neq 1$ (else, these equations are linear). For these cases, the solutions are presented in Appendix A.

Returning to the 3.2 equation, it can be rearranged as follows:

$$\dot{p_0} = \left(-\frac{RT_0}{V_0}\frac{\Gamma}{c}A_t \right) p_0 + \left(\frac{RT_0}{V_0}\rho_p A_b a \right) p_0^n$$

For the sake of convenience, the following notations will be used for the coefficients in the above equation:

$$C_0 = -\frac{RT_0}{V_0}\frac{\Gamma}{c}A_t$$

$$C_1 = \frac{RT_0}{V_0}\rho_p A_b a$$

Resulting in

$$\dot{p_0} = C_0 p_0 + C_1 p_0^n$$

Starting with Leibniz's substitution, the following substitution is made:

$$u(t) = p_0^{1-n}(t)$$

Differentiating both sides with respect to t, we get

$$\frac{du}{dt} = (1-n)p_0^{-n}\frac{dp_0}{dt}$$

$$\Leftrightarrow \frac{dp_0}{dt} = \frac{1}{1-n}p_0^n \frac{du}{dt}$$

Therefore, we have

$$\frac{1}{1-n}p_0^n \dot{u} = C_0 p_0 + C_1 p_0^n$$

Dividing the above equation by p_0^n gives us

$$\frac{1}{1-n}\dot{u} = C_0 p_0^{1-n} + C_1$$

From the original substitution $u = p_0^{1-n}$, then

$$\frac{1}{1-n}\dot{u} = C_0 u + C_1$$

Rearranging,

$$\frac{1}{1-n}\dot{u} - C_0 u - C_1 = 0 \tag{4.1}$$

The equation above is a linear ODE and can easily be solved by various methods. Here, it will be considered that the solution of this equation is of the exponential form, as is widely known, but it is possible to prove this fact with other more rigid methods (such as Lapace transforms, for example).

Claiming that

$$u(t) = Ae^{Bt} + C$$

We get

$$\dot{u}(t) = ABe^{Bt}$$

Substituting in the equation 4.1, we write

$$\frac{1}{1-n}ABe^{Bt} - C_0 Ae^{Bt} - C_0 C - C_1 = 0$$

As all the constants are independent of t, to maintain the homogeneity of the equation (in the left member), it must be true that

$$-C_0C - C_1 = 0 \Leftrightarrow C = -\frac{C_1}{C_0}$$

Implying that

$$\frac{1}{1-n}ABe^{Bt} - C_0Ae^{Bt} = 0$$

$$\Rightarrow \frac{1}{1-n}AB - C_0A = 0$$

Since the exponential will never be zero. Because A being zero makes the solution constant, it can be assumed that $A \neq 0$, implying

$$\frac{1}{1-n}B - C_0 = 0$$

Thus

$$B = (1-n)C_0$$

The solution to u is, therefore,

$$u(t) = Ae^{(1-n)C_0t} - \frac{C_1}{C_0}$$

To find A, you just need to know a boundary condition. It comes from substitution,

$$u(0) = p_0^{1-n}(0)$$

Since the pressure at the initial moment is the ambient pressure, then,

$$u(0) = p_a^{1-n}$$

Where p_a is the ambient pressure. Thus,

$$Ae^{(1-n)C_0\cdot 0} - \frac{C_1}{C_0} = p_a^{1-n}$$

$$\Rightarrow A - \frac{C_1}{C_0} = p_a^{1-n}$$

$$\Rightarrow A = p_a^{1-n} + \frac{C_1}{C_0}$$

The final solution for $u(t)$ becomes

$$u(t) = \left(p_a^{1-n} + \frac{C_1}{C_0}\right)e^{(1-n)C_0t} - \frac{C_1}{C_0}$$

Which results in the solution of $p_0(t)$:

$$p_0(t) = \left[\left(p_a^{1-n} + \frac{C_1}{C_0}\right)e^{(1-n)C_0t} - \frac{C_1}{C_0}\right]^{\frac{1}{1-n}}$$

By rewriting the constants C_0 and C_1 in the solution, we finally have that

$$p_0(t) = \left[\left(p_a^{1-n} - \frac{\rho_p A_b ac}{\Gamma A_t}\right)\exp\left((n-1)\left(\frac{RT_0\Gamma A_t}{V_0c}\right)t\right) + \frac{\rho_p A_b ac}{\Gamma A_t}\right]^{\frac{1}{1-n}}$$

Appendix B briefly discusses pressure in the permanent regime. It is deduced that this pressure is obtained by

$$p_{0,SS} = \left(\frac{\rho_p acK}{\Gamma}\right)^{\frac{1}{1-n}}$$

in which $K = \frac{A_b}{A_t}$ Therefore, it can be written that

$$\boxed{p_0(t) = \left[\left(p_a^{1-n} - p_{0,SS}^{1-n}\right) \exp\left((n-1)\left(\frac{RT_0\Gamma A_t}{V_0 c}\right)t\right) + p_{0,SS}^{1-n}\right]^{\frac{1}{1-n}}} \tag{4.2}$$

It can be observed that if $0 < n < 1$, the function $p_0(t)$ will have an asymptote at $y = p_{0,SS}$. This is verified by the following limit:

$$\lim_{t\to+\infty} \left[\left(p_a^{1-n} - p_{0,SS}^{1-n}\right) \exp\left((n-1)\left(\frac{RT_0\Gamma A_t}{V_0 c}\right)t\right) + p_{0,SS}^{1-n}\right]^{\frac{1}{1-n}} =$$

$$lim_{t\to+\infty} \left[\left(p_a^{1-n} - p_{0,SS}^{1-n}\right) \exp\left(-\infty\right) + p_{0,SS}^{1-n}\right]^{\frac{1}{1-n}} =$$

$$lim_{t\to+\infty} \left[p_{0,SS}^{1-n}\right]^{\frac{1}{1-n}} = p_{0,SS}$$

5 Time to pressurization

As seen in the previous section, the steady state pressure is an asymptote for the function found, so it is impossible to reach this state in a finite amount of time (a common problem in fluid studies). However, it is possible to find out the time needed to reach a fraction of the permanent regime pressure.

It is common to use a fraction$\alpha = 0.99$, but a generic expression will be deduced so that anyone can compare different possible values.

One can write the equality between the pressure of the pressurization regime and the fraction of this in the permanent regime as

$$\left[\left(p_a^{1-n} - p_{0,SS}^{1-n}\right) \exp\left((n-1)\left(\frac{RT_0\Gamma A_t}{V_0 c}\right)t_{pr}\right) + p_{0,SS}^{1-n}\right]^{\frac{1}{1-n}} = \alpha p_{0,SS}$$

Rearranging to find the pressurization time t_{pr}

$$\Rightarrow \left(p_a^{1-n} - p_{0,SS}^{1-n}\right) \exp\left((n-1)\left(\frac{RT_0\Gamma A_t}{V_0 c}\right)t_{pr}\right) + p_{0,SS}^{1-n} = \alpha^{1-n} p_{0,SS}^{1-n}$$

$$\Rightarrow \left(p_a^{1-n} - p_{0,SS}^{1-n}\right) \exp\left((n-1)\left(\frac{RT_0\Gamma A_t}{V_0 c}\right)t_{pr}\right) = (\alpha^{1-n} - 1) p_{0,SS}^{1-n}$$

$$\Rightarrow (n-1)\left(\frac{RT_0\Gamma A_t}{V_0 c}\right)t_{pr} = \ln\left[(\alpha^{1-n} - 1)\frac{p_{0,SS}^{1-n}}{p_a^{1-n} - p_{0,SS}^{1-n}}\right]$$

$$\boxed{t_{pr} = \frac{1}{(n-1)}\left(\frac{V_0 c}{RT_0\Gamma A_t}\right)\ln\left[(\alpha^{1-n} - 1)\frac{p_{0,SS}^{1-n}}{p_a^{1-n} - p_{0,SS}^{1-n}}\right]} \tag{5.1}$$

It is important to note that the fraction is raised to $1 - n$, which can easily be forgotten in a manual calculation.

6 Apendix A (solutions with n = 0 and n = 1)

The following equation is used as a basis:

$$\frac{V_0}{RT_0}\dot{p_0} = \rho_p a p_0^n A_b - \frac{\Gamma}{c} A_t p_0 \tag{6.1}$$

6.1 Case 1: n = 0

From the equation 6.1, we write

$$\frac{V_0}{RT_0}\dot{p_0} = \rho_p a p_0^0 A_b - \frac{\Gamma}{c}A_t p_0$$

$$\Rightarrow \frac{V_0}{RT_0}\dot{p_0} + \frac{\Gamma}{c}A_t p_0 - \rho_p a A_b = 0$$

As this is a linear ODE with constant coefficients, it is solved by taking the following as the solution

$$p_0 = Ae^{Bt} + C \tag{6.2}$$

Substituting, one gets

$$\frac{V_0}{RT_0}ABe^{Bt} + \frac{\Gamma}{c}A_t Ae^{Bt} + C\frac{\Gamma}{c}A_t - \rho_p a A_b = 0$$

To keep the equation homogeneous, then

$$C\frac{\Gamma}{c}A_t - \rho_p a A_b = 0$$

$$\Rightarrow C = \frac{\rho_p a A_b c}{\Gamma A_t}$$

Thus,

$$\frac{V_0}{RT_0}ABe^{Bt} + \frac{\Gamma}{c}A_t Ae^{Bt} = 0$$

$$\Rightarrow \frac{V_0}{RT_0}B + \frac{\Gamma}{c}A_t = 0$$

Given that $A \neq 0$

Therefore,

$$B = -\frac{\Gamma RT_0}{V_0 c}A_t$$

Since $p_0(0) = p_a$, where p_a is the ambient pressure (initial instant), then,

$$p_a = A + C$$

$$\Rightarrow A = p_a - \frac{\rho_p a A_b c}{\Gamma A_t}$$

Appendix B shows that the pressure at steady state is

$$p_{0,SS} = \left(\frac{\rho_p a A_b c}{\Gamma A_t}\right)^{\frac{1}{1-n}}$$

That is,

$$A = p_a - p_{0,SS}^{1-n}$$

Referring back to the equation 6.2,

$$\boxed{p_0(t) = \left(p_a - p_{0,SS}^{1-n}\right)e^{-\frac{\Gamma RT_0}{V_0 c}A_t t} + p_{0,SS}^{1-n}}$$

The pressurization time t_{pr} up to a fraction α of the steady state pressure can be written as

$$\boxed{t_{pr} = -\frac{V_0 c}{\Gamma RT_0 A_t}\ln\left[(\alpha - 1)\frac{p_{0,SS}^{1-n}}{p_a - p_{0,SS}^{1-n}}\right]}$$

6.2 Case 2: n = 1

$$\frac{V_0}{RT_0}\dot{p_0} = \rho_p a p_0^1 A_b - \frac{\Gamma}{c} A_t p_0$$

$$\Rightarrow \frac{V_0}{RT_0}\frac{dp_0}{dt} = \left(\rho_p a A_b - \frac{\Gamma}{c} A_t\right) p_0$$

Separating the variables, we have

$$\frac{dp_0}{p_0} = \frac{RT_0}{V_0}\left(\rho_p a A_b - \frac{\Gamma}{c} A_t\right) dt$$

Integrating the equation,

$$\int_{p_a}^{p_0} \frac{dp_0}{p_0} = \int_0^t \frac{RT_0}{V_0}\left(\rho_p a A_b - \frac{\Gamma}{c} A_t\right) dt$$

$$\Rightarrow \ln\left(\frac{p_0}{p_a}\right) = \frac{RT_0}{V_0}\left(\rho_p a A_b - \frac{\Gamma}{c} A_t\right) t$$

Isolating the pressure $p_0(t)$, one gets

$$\boxed{p_0(t) = p_a \exp\left[\frac{RT_0}{V_0}\left(\rho_p a A_b - \frac{\Gamma}{c} A_t\right) t\right]}$$

As this function is non-asymptotic and has an increasing character, the time until pressurization t_{pr} of $p_{0,SS}$ is given by

$$\boxed{t_{pr} = \frac{V_0}{RT_0\left(\rho_p a A_b - \frac{\Gamma}{c} A_t\right)} \ln\left(\frac{p_{0,SS}}{p_a}\right)}$$

7 Apendix B (steady state pressure)

Initially, we have

$$\frac{1}{RT_0}\frac{d}{dt}[p_0 V_0] = \rho_p r A_b - \frac{\Gamma}{c} A_t p_0$$

In the permanent regime, the product $p_0 V_0$ is invariant, i.e,

$$0 = \rho_p r A_b - \frac{\Gamma}{c} A_t p_0$$

$$\Leftrightarrow \frac{\Gamma}{c} A_t p_0 = \rho_p r A_b$$

Since $r = a p_0^n$

$$\frac{\Gamma}{c} A_t p_0 = \rho_p a p_0^n A_b$$

$$\Rightarrow p_0^{1-n} = \frac{\rho_p a c A_b}{A_t \Gamma}$$

It is common to refer to the ratio between the areas A_b and A_t as K. It follows that

$$\boxed{p_0 = \left(\frac{\rho_p a c K}{\Gamma}\right)^{1-n}} \tag{7.1}$$