Lagrange Multipliers: Removing some confusions related to some geometric interpretations. Part I

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1 Introduction

Sometimes, when we deal with a complex task we have to recall some simple concepts, for which we may have the assumption that we knew everything about them until we come across some of their applications not necessarily complex but different from situations we have met priorly; and then surprisingly, what was easy and completely understandable by us starts to be associated with doubts.

The aim of this article is not to repeat lessons about Lagrange Multipliers, but to remove confusion caused by a missed step or wrongly illustrated concept, which I have come across often in many tutorials of both types of video and text, that creates an obstacle for people like me who do not get some conclusions for granted without having the proof or/and feeling its correctness.

As our process of understanding usually involves confusion, we do not have to blame ourselves for why we do not understand clearly the explanation. The lack of explanation clarity may be often caused because the authors assume acknowledgment of the problematic step for themselves and skip it while providing their explanation or because the authors themselves reproduced unconsciously that segment of the article from other sources.

As a result of such deficiencies in the explanation, some problems look extremely difficult before knowing how to tackle them. Fortunately, they look extremely easy after providing an understandable explanation. I hope this article will clear up some of the geometric-related confusion while grasping Lagrange Multipliers.

2 What is wrongly illustrated when explaining Lagrange Multipliers?

Please have a look at the following picture. While the authors mention the gradient, they associate the explanation with a hill and an arrow like in the picture below. In fact, this arrow shows the direction of the object/man walking up the hill to reach the top. But caution! This arrow does not correspond with the gradient which is in fact a 2-dimensional vector and not a 3-dimensional vector. We will show later what the gradient looks like.

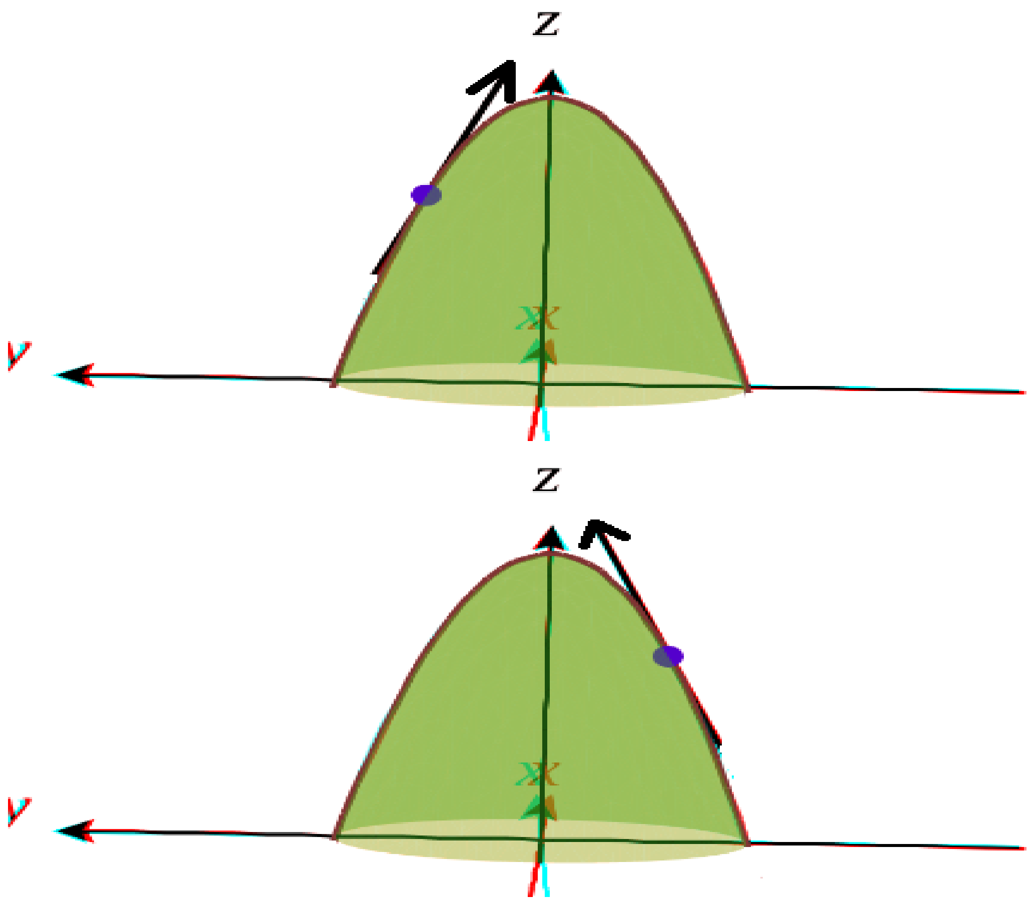

Figure 1: The black arrow is not the gradient. It just shows the direction to be followed while walking over a physical object (hill) in order to reach the top.

The following illustrations are showing tangent lines (when we take derivatives of z=f(x,y) with respect to y with fixed x) to the location represented by the blue point.

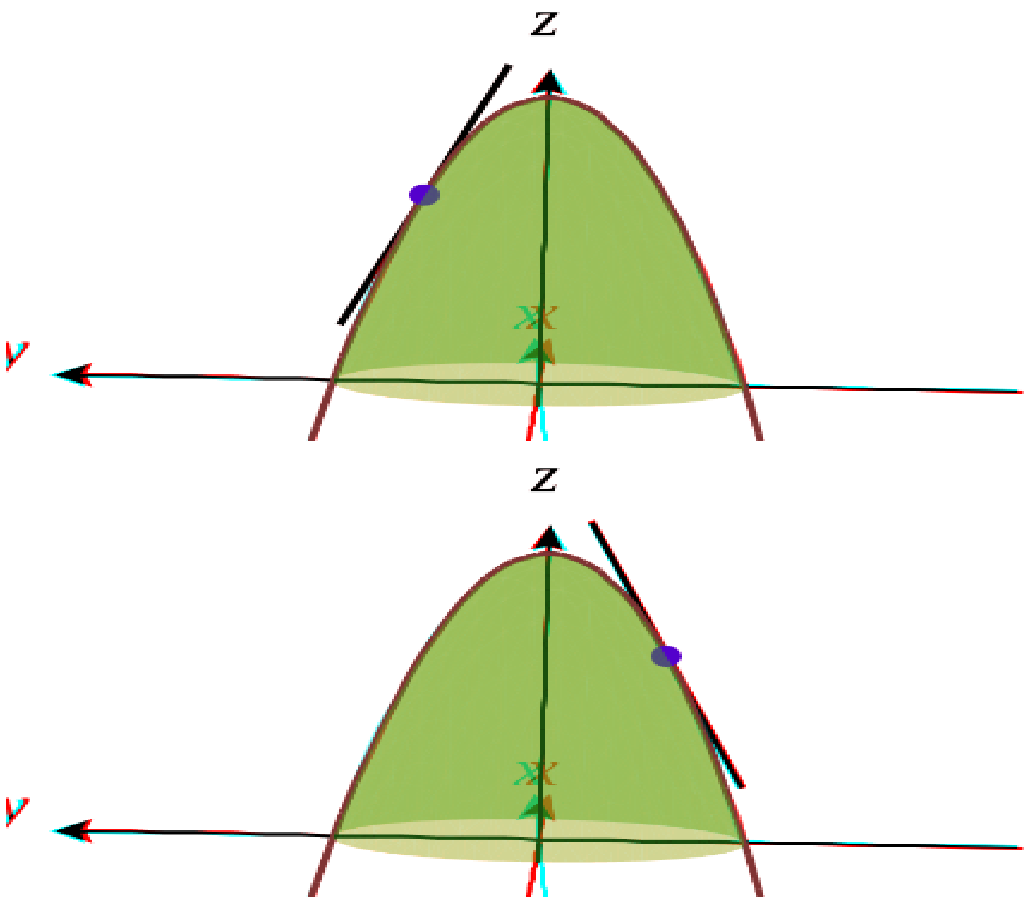

Figure 2: The following illustrations are showing tangent lines (when we take derivatives of z=f(x,y) with respect to y with fixed x) to the location represented by the blue point.

While the real vectors representing gradients for the corresponding situations are presented below. The gradient of a function of two arguments is a 2-dimensional vector parallel with xOy plane with the start at our location on the hill represented by the blue dot.

$$\nabla f(x, y) = \frac{\partial f}{\partial x} * \vec{i} + \frac{\partial f}{\partial y} * \vec{j}$$

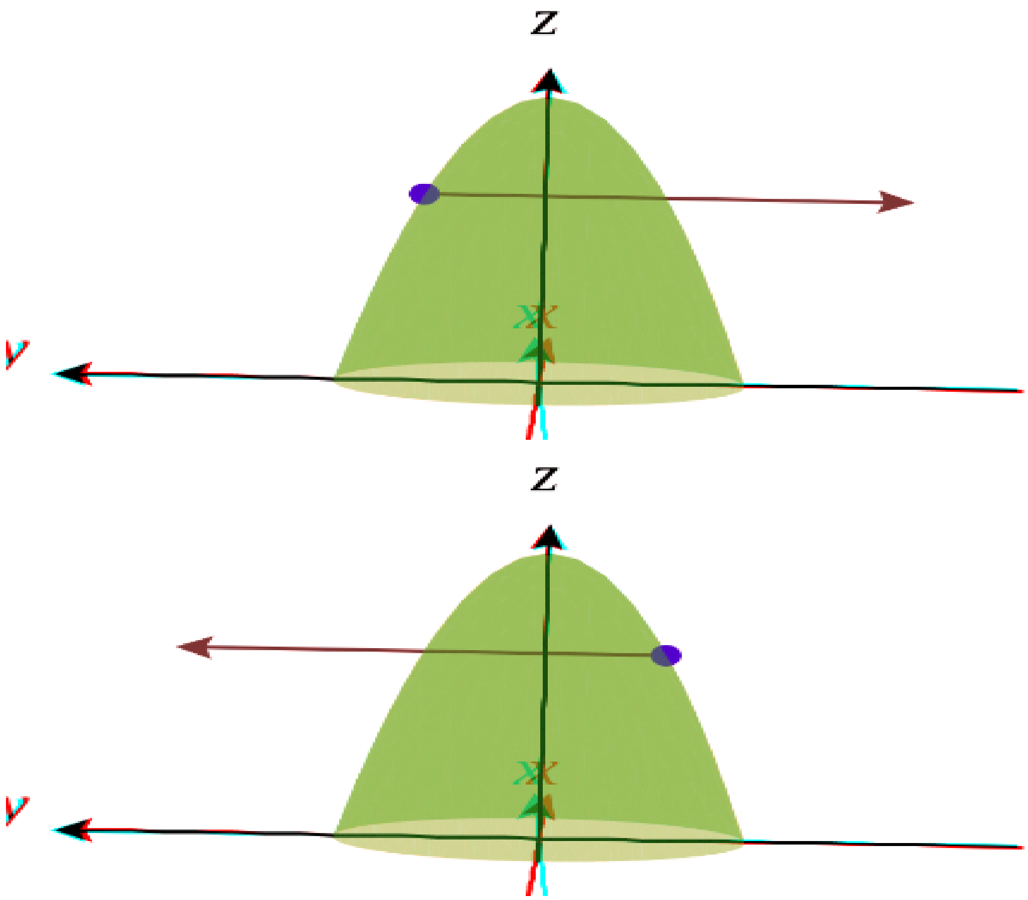

Figure 3: The black arrow represents the real gradient which is a 2-dimensional vector parallel with xOy plane with the start at our location on the hill represented by the blue dot.

3 Lagrange Multipliers: A function to maximize subject to a constraint function

As it may be notable, the gradient (green arrow) of function f represented by the green surface is parallel to the gradient (red arrow) of the constraint function represented by the red surface. Both gradients are, of course, parallel to the plane xOy, even when they are not parallel with each other as they both lie on xOy plane or any plane parallel to it as we have explained earlier.

$$\nabla f(x,y) = \nabla g(x,y)$$

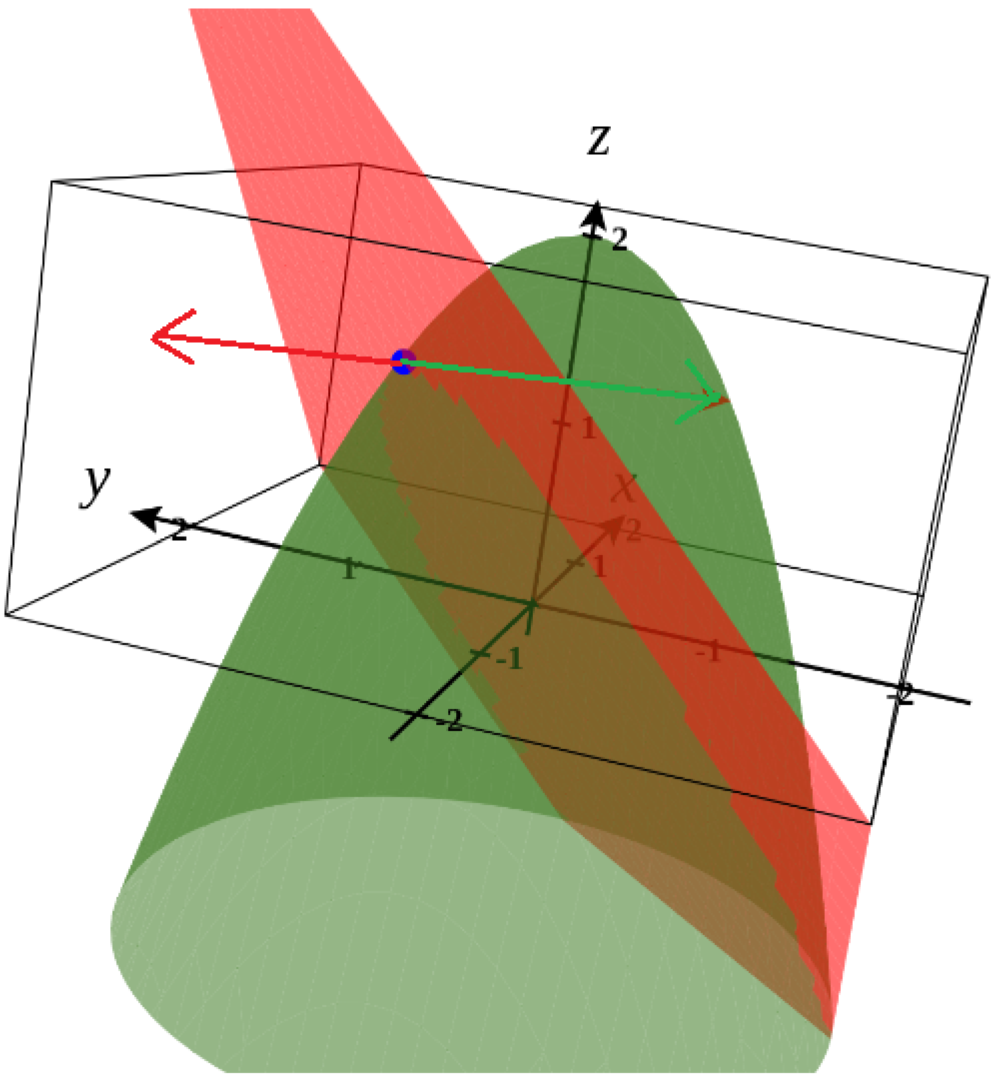

Figure 4: As it may be notable, the gradient (green arrow) of function f represented by the green surface is parallel to the gradient (red arrow) of the constraint function g represented by the red surface. The starting point of these vectors is the required solution.

4 Cases when the gradient of constraint function g does not exist

Recalling the gradient formula

$$\nabla f(x,y) = \frac{\partial f}{\partial x} * \vec{i} + \frac{\partial f}{\partial y} * \vec{j}$$

and as we the constraint is a plane with equation x=const, partial derivates of z=f(x,y_0) with respect to each of variables x,y will be zero at point (x,y_0)

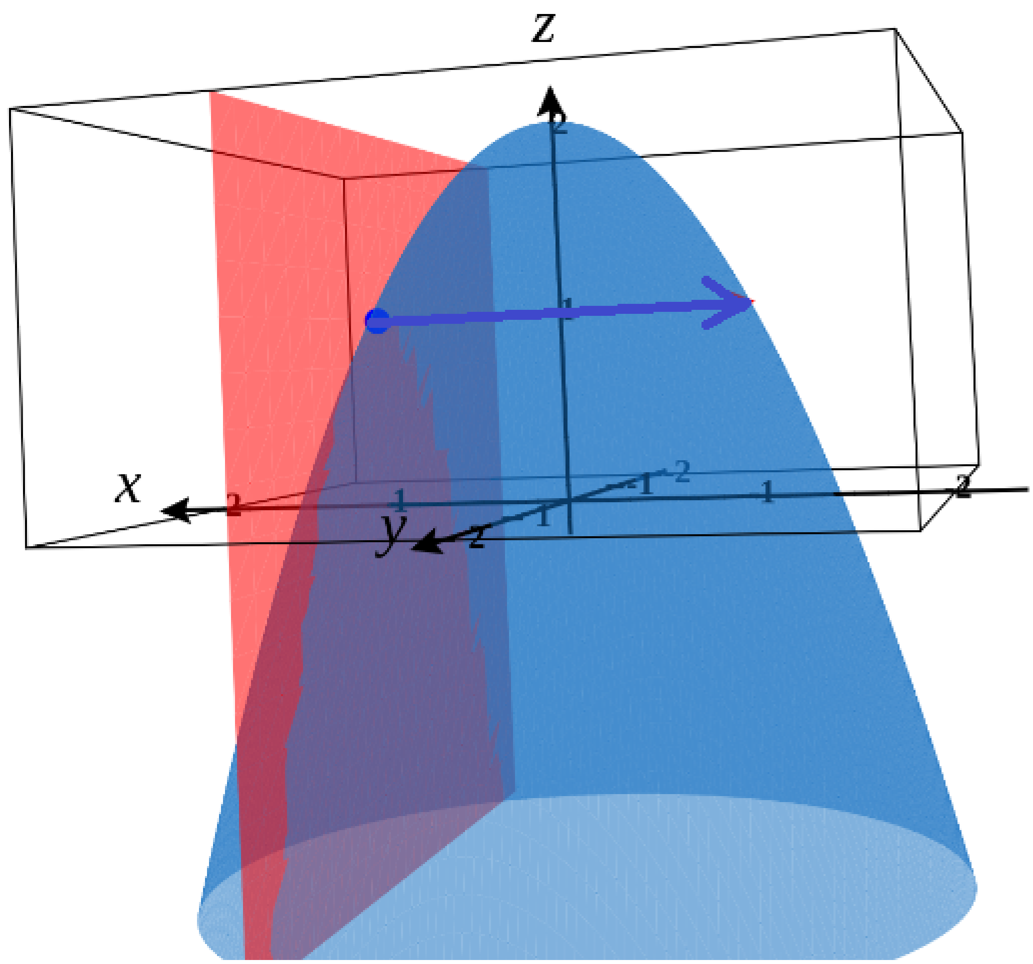

Figure 5: There exists only one gradient (blue arrow) - the gradient of the function f represented by the blue surface. The gradient of the function g is the zero vector. The starting point of the existing vector is the required solution.

5 More examples

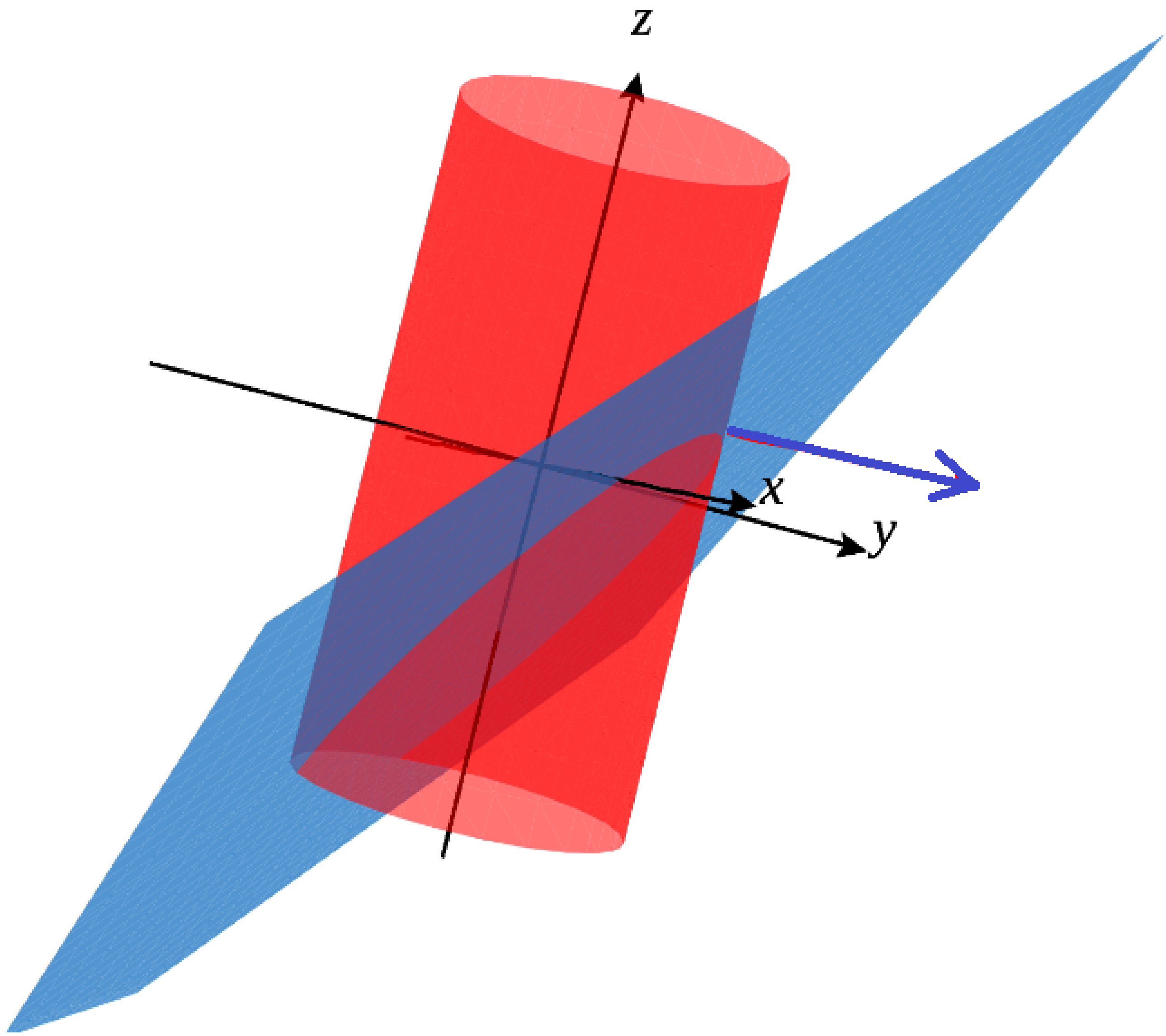

Figure 6: There does not exist at all a gradient of any point of the surface representing the function f. There exists only the gradient(blue arrow) of constraint function g represented by the blue plane. The starting point of this vector is the required solution.