// Theorem 1.1

$$\sup \lambda A = \begin{cases} \lambda \sup A, \ \lambda \geq 0 & a) \\ \lambda \inf A, \ \lambda < 0 & b) \end{cases}$$

Proof a: $\forall x \in A$ $\lambda x \leq \sup \lambda A \Leftrightarrow \boxed{x < \frac{\sup \lambda A}{\lambda}} \Longrightarrow \frac{\sup \lambda A}{\lambda}$ is an upper bound of A. And we know that $\sup A$ is the smallest upper bound; hence,

$\boxed{\sup A \leq \frac{\sup \lambda A}{\lambda}}$ (1) $\xLeftrightarrow{\lambda > 0}$ $\boxed{\lambda \sup A \leq \sup \lambda A}$ (1')

$\forall x \in A$: $x \leq \sup A \xrightarrow{\lambda > 0} \boxed{\lambda x \leq \lambda \sup A} \Rightarrow \lambda \sup A$ is an upperbound for λA. And we know that suprem λA is the smallest upper-bound; thus,

$\boxed{\sup \lambda A \leq \lambda \sup A}$ (2)

(1) $\wedge$ (2) $\Rightarrow$ $\sup \lambda A \leq \lambda \sup A \leq \sup \lambda A \Rightarrow$

$\lambda \sup A = \sup \lambda A$ //

b) $\forall x \in A$: $\lambda x \leq \sup \lambda A \xRightarrow{\lambda < 0} x \geq \frac{\sup \lambda A}{\lambda} \Longrightarrow \frac{\sup \lambda A}{\lambda}$ is an lower bound of A

but $\inf A$ is the smallest lowerbound $\Rightarrow$ ~~$\sup \lambda A \leq$~~ $\boxed{\inf A \geq \frac{\sup \lambda A}{\lambda}}$ (3) $\xLeftrightarrow{\lambda < 0}$ $\boxed{\lambda \inf A \leq \sup \lambda A}$ (3')

$\forall x \in A$: $x \geq \inf A \xRightarrow{\lambda < 0} \lambda x \leq \lambda \inf A \Longrightarrow \lambda \inf A$ is an upper bound of λA. We know that $\sup \lambda A$ is the smallest one thus: $\boxed{\sup \lambda A \leq \lambda \inf A}$ (4)

(3') $\wedge$ (4) $\Rightarrow$ $\sup \lambda A \leq \lambda \inf A \leq \sup \lambda A \Rightarrow$

$\boxed{\lambda \inf A = \sup \lambda A}$ for $\lambda < 0$ //

$\lambda = 0$ $\sup 0 \cdot A = 0 \cdot \sup A$ //

Theorem 1.2 $\quad \inf \lambda A = \begin{cases} \lambda \inf A, & \lambda \geq 0 \quad a) \\ \lambda \sup A, & \lambda < 0 \quad b) \end{cases}$

a) • $\forall x \in A$ $\lambda x \geq \inf \lambda A \xrightarrow{\lambda > 0} \underbrace{x \geq \frac{\inf \lambda A}{\lambda}}_{} \Longrightarrow \frac{\inf \lambda A}{\lambda}$ is a lower bound of A (because). We know that inf of a set is the ~~smallest~~ biggest lowerbound, hence $\inf A \geq \frac{\inf \lambda A}{\lambda} \xrightarrow[(\lambda)]{\lambda > 0} \boxed{\lambda \inf A \geq \inf \lambda A}$ I

• $\forall x \in A$ $x \geq \inf A \xrightarrow{\lambda > 0} \lambda x \geq \lambda \inf A \Rightarrow \lambda \inf A$ is a lower bound of λA. And we know that $\inf \lambda A$ is the biggest lower bound; thus, $\boxed{\inf \lambda A \geq \lambda \inf A}$ II. From I ^ II $\Rightarrow$

$\inf \lambda A \geq \lambda \inf A \geq \inf \lambda A \Longleftrightarrow \boxed{\inf \lambda A = \lambda \inf A}$ //

• $\lambda = 0$. $\forall x$ $\underline{\inf 0A = \inf\{0, \ldots 0\} = 0}$ ^ $\underline{0 \inf A = 0}$

$\inf 0A = 0 \inf A = 0$

b) $\forall x \in A$ $\boxed{\lambda < 0}$ • $\lambda x \geq \inf \lambda A \xrightarrow{\lambda < 0} x \leq \frac{\inf \lambda A}{\lambda}$

$\Rightarrow \frac{\inf \lambda A}{\lambda}$ is an upper bound for A. And we know that $\sup A$ is the smallest upper bound, thus, $\sup A \leq \frac{\inf \lambda A}{\lambda}$

$\xrightarrow{\lambda} \boxed{\lambda \sup A \geq \inf \lambda A}$ (III)

• $x \leq \sup A \xrightarrow{\lambda < 0} \lambda x \geq \lambda \sup A \Rightarrow \lambda \sup A$ is a lower bound for λA. $\boxed{\inf \lambda A \geq \lambda \sup A}$ (IV)

From (III) ^ (IV) $\boxed{\inf \lambda A = \lambda \sup A}$ //

$A, B \subset R \ , \ \lambda \in R$

$A+B = \{z \in R : \exists x \in A \wedge \exists y \in B \ \ z = x+y\}$

$A-B = \{z \in R : (\exists x \in A)(\exists y \in R) \ z = x-y\}$

$\lambda A = \{z \in R : (\exists x \in A) \ z = \lambda A\}$

$A, B \subset R, \lambda \in R$

$A+B = \{z \in R : (\exists x \in A)(\exists y \in B) \ z = x+y\}$

$A-B = \{z \in R : (\exists x \in A)(\exists y \in B) \ z = x-y\}$

$\lambda A = \{z \in R : (\exists x \in A) \ z = \lambda x\}$

Theorem

A, B bounded $\Longrightarrow A+B, \ A-B, \ \lambda A$ are bounded

A bounded $\Leftrightarrow (\exists P_1 \in R_+)(\forall x \in A) \ x \leq |P|$

B bounded $\Leftrightarrow (\exists P_2 \in R_+)(\forall y \in B) \ y \leq |P|$

- $z \in A+B \Rightarrow (\exists x \in A) \wedge (y \in B) \ z = x+y$

$|x| \leq P_1 \wedge |y| \leq P_2$

$~~x+y \leq P_1 + P_2~~ = P$

$\underset{|z|}{x+y} \ |x+y| \leq |x| + |y| \leq P$ i.e $|z| \leq P$, DOE

$z \in A-B \qquad z = x-y$

- $|z| = |x-y| \leq |x+(-y)| \leq |x| + |y| \leq P_1 + P_2 = P$

$|z| \leq P$ DOE

- $|z| = |\lambda x| = |\lambda||x| \leq |\lambda| P_1 = P \qquad |z| \leq P$ DOE

// $A \subset R$

Upper border M of the set $A \Leftrightarrow \forall x \in A: x \leq M$

- A is limited from up $\Leftrightarrow (\exists M \in R)(\forall x \in A)$ $x \leq M$
- A is limited from bottom ~~has an lower border~~ $\Leftrightarrow (\exists m \in R)(\forall x \in A)$ $x \geq m$
- A is limited $\Leftrightarrow (\exists P \in R_+)(\forall x \in A)$ $|x| \leq P$ $(\Leftrightarrow -P \leq x \leq P)$

~~$A \subset R$ is limited~~

~~Upper bound of A~~

A has an upper bound $\Leftrightarrow (\exists M \in R)(\forall x \in A)$ $x \leq M$

A has a lower Bound $\Leftrightarrow (\exists m \in R)(\forall x \in A)$ $x \geq m$

A is bounded $\Leftrightarrow$ A has an upper bound and a lower Bound.

$\sup A$ is the ~~least up~~ smallest upper bound $\Leftrightarrow$

1) $\forall x \in A$: $x \leq \sup A$

2) $\forall \varepsilon > 0$: $x + \varepsilon \geq \sup A$ $(\Leftrightarrow x \geq \sup A - \varepsilon)$

~~$\sup A$ is the~~ $\operatorname{Inf} A$ is the biggest lower bound $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} 1) \ \forall x \in A: x \geq \inf A \\ 2) \ \forall \varepsilon > 0: \ x - \varepsilon \leq \inf A \ (\Leftrightarrow x \leq \inf A + \varepsilon) \end{cases}$$

Theorem $\boxed{A \text{ is bounded} \Leftrightarrow A \text{ has upper bound and lower bound}}$

A is bounded $\Leftrightarrow (\exists P \in R_+)(\forall x \in A)$ $|x| \leq P$ (1)

$\Rightarrow$

(1) $\Leftrightarrow$ $-P \leq x \leq P$ let $m = -P$; $M = P$

The $(\exists m = -P)(\forall x \in A)$ $x \geq m$ —

$\Leftarrow$ $P = \max(|m|, |M|)$

$-|m|$ m P

$\boxed{m \leq x \leq M}$

1) $x \leq M \leq P$

$m \leq |m| \leq P$

$x \geq m$

$-m \geq -|m| \geq -P$

$x \geq -m \geq -|m| \geq -P$ $\Rightarrow x \geq -P$ (2)

(1) ∧ (2)

$-P \leq x \leq P$

$$\sup \lambda A = \begin{cases} \lambda \sup A, & \lambda \ge 0 \\ \lambda \inf A, & \lambda < 0 \end{cases}$$

$\lambda = 0$ $\quad \sup 0 = 0 \cdot \sup A = 0$

$x \in A \quad \boxed{x \le \sup A \xRightarrow{\lambda \ge 0} \lambda x \le \lambda \sup A}$

$\Leftarrow x \le \frac{\sup A}{\lambda} \quad \forall x \in A \Rightarrow \frac{\sup A}{\lambda}$ is an upper bound of A

$\sup A$ is the smallest upper bound thus

$\sup A \le \frac{\sup A}{\lambda} \Leftarrow \lambda \sup A \le \sup A$

$\lambda \sup A$ is an upper bound of λA

but sup of set is the smalles up bound

thus $\sup \lambda A \le \lambda \sup A \Rightarrow$

$x \quad \boxed{\sup A \ge \frac{\sup \lambda A}{\lambda}}$ (1) $\Rightarrow \frac{\sup \lambda A}{\lambda}$ is one

$\lambda x \le \sup \lambda A \Rightarrow x \le \frac{\sup \lambda A}{\lambda}$

$\Rightarrow \boxed{\sup A \le \frac{\sup \lambda A}{\lambda}}$ (2)

(1) $\le \sup A$

$\Rightarrow \frac{\sup \lambda A}{\lambda} = \sup A \Leftarrow$

$\sup \lambda A = \lambda \sup A$

Theorem

1) $\operatorname{Inf}(A+B) \geq \operatorname{Inf} A + \operatorname{Inf} B$

2) $\sup(A+B) \leq \sup A + \sup B$

Proof $z \in A+B$ $\exists x \in A \wedge \exists y \in B$ $z = x+y$

$x \in A$ $x \geq \inf A$ $\Rightarrow x+y \geq \inf A + \inf B$

$y \in B$ $y \geq \inf B$ $z \geq \inf A + \inf B$

$(\forall z \in A+B)\, z \geq \inf A + \inf B \Rightarrow$

$\inf A + \inf B$ is a lower bound of $A+B$

But inf of a given set is alway $\geq$ any lower bound thus $\boxed{\inf(A+B) \geq \inf A + \inf B}$

i.e. $\inf C \geq$ any lower Bound of C

inf is the biggest lower bound

2) Analog $\forall x \leq \sup A$, $\forall y \leq \sup B$ $\Rightarrow x+y \leq \sup A + \sup B$

$z < \sup A + \sup B$ - an upper bound of $A+B$

But sup C is the smalles upper bound

Thus $\boxed{\sup(A+B) \leq \sup A + \sup B}$