1/

$K = \mathbb{R} \wedge K = \mathbb{C}$ (in fact K is any field), E | $(E, K, \oplus, \odot)$ it is a vectorial space when followed 10 axioms are satisfied

$\oplus : E \times E \longrightarrow E$ $\qquad \odot \; K \times E \longrightarrow E$

$\Updownarrow$

$\oplus : E \times E \ni (a, b) \longmapsto a \oplus b = c \in E$ 1) Closure of $\oplus$

$\odot : K \times E \ni (\lambda, a) \longmapsto \lambda \odot a \in E$ 2) Closure of $\odot$

3) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for any $x, y, z \in E$

4) $\exists! \Theta \in E, \forall x \in E \quad \Theta \oplus E = E \oplus \Theta = E$

5) $\forall x \in E, \exists! y \in E \qquad x \oplus y = y \oplus x = \Theta$

6) $\forall x \in E, \forall y \in E \qquad x \oplus y = y \oplus x$ commutative

1, 3, 4, 5 group $(G, \oplus)$

1, 3, 4, 5, 6 abbelian group $(G, \oplus)$

7) $\exists 1 \in E\ (\forall x \in E)\ 1 \odot x = x \odot 1 = x$

8) $\forall x, y \in E, \forall \lambda \in K \quad \lambda \odot (x \oplus y) = \lambda \odot x \oplus \lambda \odot y$ association of $\odot$ toward $\oplus$

9) $\forall \lambda_1, \lambda_2 \in K, \forall x \in E \quad (\lambda_1 \cdot \lambda_2) \odot x = \lambda_2 \odot (\lambda_1 \odot x)$

10) $\forall \lambda_1, \lambda_2 \in K, \forall x \in E \quad (\lambda_1 + \lambda_2) \odot x = \lambda_1 \odot x \oplus \lambda_2 \odot x$

$1 \rightarrow$ Neutral element of $(E, K, \oplus, \odot)$

$\Theta \rightarrow$ zero element of $(E, K, \oplus, \odot)$

we adnotate that y such that $x \oplus y = y \oplus x = \Theta$

$-x$

Let $\odot = \cdot$ and $\oplus = +$ $\quad x - y \overset{\text{Def}}{=} x + (-y)$

a) $0x = \Theta$ b) $\Theta \lambda = \Theta$ c) $(-1)x = -x$ d) $(x - y) + y = x$

<u>Proof</u>: a) $0 \cdot x = (1-1)x = 1 \odot x - 1 \odot x = x - x = \Theta$

or $0 \cdot x = 0x + \Theta = 0 \cdot x + \cancel{x} - x = 0 \cdot x + 1 \cdot x + (-1) \cdot x$

$= (0 + 1) x + (-1) \cdot x = 1 \cdot x + (-1) x = \cancel{\quad} x = \cancel{x - x} = \Theta$

$= x + (-x) = \Theta$

7) Again a) $0X = \Theta$

$0X \overset{(4)}{=} 0X + \Theta \overset{(5)}{=} 0X + X + (-X) \overset{7}{=} 0X + 1 \cdot X + (-X)$

$\overset{10}{=} (0+1) \cdot X + (-X) \overset{7}{=} 1 \cdot X + (-X) = X + (-X) = \Theta$

b) $\lambda\Theta = \Theta$: $\lambda\Theta \overset{?}{=} \lambda(X + (-X)) = \lambda X + \lambda(-X) = \lambda X + (-\lambda X) = \Theta$

$\lambda\Theta \overset{a)}{=} \lambda(0 \cdot \Theta)$ we proofed at a) $\forall X \in E$ $0 \cdot X = \Theta$

$\Rightarrow 0 \cdot \Theta = \Theta$ that is why $\lambda\Theta = \lambda(0 \cdot \Theta) \overset{9}{=} (\lambda \cdot 0) \cdot \Theta =$

$= 0 \cdot \Theta \overset{a)}{=} \Theta$

c) $\underline{(-1)X = -X}$ $\quad y + X = \Theta \Leftrightarrow y = -X$

$y = (-1)X$

$\underbrace{(-1) \cdot X}_{y} + X = (-1) \cdot X + X \overset{(7)}{=} (-1)X + 1 \cdot X \overset{(10)}{=} (-1+1) \cdot X$

$= 0 \cdot X \overset{a)}{=} \Theta$

d) $(X - y) + y = X$

$(X - y) + y = (X \oplus (-y)) \oplus y \overset{3}{=} X \oplus ((-y) \oplus y) = X \oplus \Theta = X$

$(X-y) + y = X + (-1) + 1 \cdot y \overset{(10)}{=} X + ((-1)+1) \cdot y$

or

$(X-y) + y \overset{c)}{=} X + (-1) \cdot y + y \overset{7}{=}$

$= X + 0 \cdot y \overset{a)}{=} X + \Theta \overset{4)}{=} X$ //

$(E, K, \oplus, \odot) \to$ vec space
$A \subset E$
$(A, K, \oplus, \odot) \to$ vec space
$\overset{Def}{\Longleftrightarrow}$ A is subspace of E.

better said $(A, K, \oplus, \odot)$ is subspace of space $(E, K, \oplus, \odot)$.

<u>Theorem</u>: If $(E, K, \oplus, \odot)$ is vec. space and $A \subset E$ and $\forall X \in A$ $\lambda X \in A$ and $(\forall X, y \in A)$ $X \oplus y \in A \Rightarrow (A, K, \oplus, \odot)$ is subspace of vectorial space $(E, K, \oplus, \odot)$

Examples of vectorial spaces

$K = R$ $E = R$ $(R, R, +, \cdot)$ is vectorial spaces we check if axioms are satisfied

1. $x, y \in R \Rightarrow x + y \in R$, $\lambda x \in R$, $x + y = y + x$

$0 + x = 0$, $x + 0 = 0 (= 0$ of $R)$

$(x+y)+z = x+(y+z)$, $a(bx) = (ab)x$

~~$ab\, x = a(bx)$~~ $(a+b)x = ax + bx$

$(R^n, R, +, \cdot)$ is vectorial space

$x \in R$ $(x = x_1, \ldots, x_n)$

$-x = (-x_1, \ldots, -x_n)$

$R^n \ni \Theta = (0, 0, \ldots, 0)$

$x, y \in R^n$; $x + y =$

a) $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

b) $\lambda x = (\lambda x_1, \lambda x_2, \ldots \lambda x_n)$

Proof: check 10 axioms of vectorial space

Proof of subspace Theorem: given that $A \subset E$ and $(E, K, \oplus, \odot)$ is a space

$\lambda x \in A$ and $x \oplus y \in A$

1. $x, y \in A \Rightarrow x \oplus y \in A$ closure

~~$x \oplus y$~~

$x, y, z \in A \Rightarrow x, y, z \in E$ this

$(x+y)+z = x+(y+z)$ thus this axiom hold in A

$x + y = y + x$?

$x, y \in E \Rightarrow x+y = y+x \in A \Rightarrow$

~~$x, y \in A \Rightarrow x + y = z \in A$~~

~~$y, x \in A \Rightarrow y + x = z_1 \in A$~~

~~$x + y \in E$ A $y + x \in E$~~ $\Rightarrow x + y = y + x$

4)

E = set of functions from A to R $\quad f: A \to R$

$$\begin{cases} \forall f, g \in E \quad f+g: \text{def } (f+g)(x) = f(x) + g(x), \forall x \in A \\ \lambda \in R \quad \lambda f: (\lambda f)(x) = \lambda f(x), \forall x \in A \\ \theta = \theta(x) = 0 \quad \forall x \in A \\ -f \quad (-f)(x) = -f(x), \forall x \in A \end{cases}$$

$(E, R, +, \cdot) \to$ vectorial spaces

$G \subset E$, where G is set of bounded function from A to R

| def f is bounded $\Leftrightarrow f(A)$ set is bounded |

~~$G \subset E$, $f+g$~~

Theorem: f, g bounded $\Rightarrow f+g$ bounded and also λf is bounded.

$f: A \to R \quad g: A \to R \quad f(A) =: B, \; g(A) =: D$

$f(x) \in B \wedge g(x) \in D \Rightarrow$

$f(x) + g(x) \in B + D = C.$

$(f+g)(x) = C$

$\lambda f(x) \in \lambda B = E \quad$ As C, E are bounded

thus $f+g, \lambda f$ are bound

$E \doteq$ set of any $f : A \longrightarrow R \quad A \subseteq R$

$G =$ set of bounded $g : A \longrightarrow R$

So $G \subset E$ and from previous theorem

$f+g, \lambda f \in G \xrightarrow{\text{Theorem of subspaces}} (G, R, \oplus, \odot)$ is a vectorial space being subspace of $(E, R, \oplus, \odot)$

Def $\boxed{\sup_{x \in A} f(x) \overset{\text{Def}}{=} \sup f(A)}$

Theorem: $\boxed{\sup_{x \in A} |\lambda f(x)| = |\lambda| \sup_{x \in A} |f(x)|}$, if f is a bounded function.

Proof: $g(x) = |f(x)|$

$\sup_{x \in A} g(x) = \sup g(A) = \sup_{x \in A} |f(x)| = 0$

$\sup_{x \in A} |\lambda f(x)| = \cancel{|\lambda| \sup |f(x)|} = \sup_{x \in A} |\lambda| |f(x)| =$

$\sup_{x \in A} |\lambda| g(x) = \sup |\lambda| g(A) = |\lambda| \sup g(A) =$

$= |\lambda| \sup |f(x)|$

~~Theorem~~ $\cancel{\max_{k=1..n} |\lambda x_k|} = \quad \max_{k=1..n} |\lambda x_k| = |\lambda| \max_{k=1..n} |x_k|$

$f : \{x_1, \dots x_n\} \longrightarrow \{x_1, \dots x_n\} \quad A = \{x_1, \dots, x_n\}$

$f : A \to A \quad f(x) = x \quad f(x_k) = x_k \quad \forall k \in \{1, \dots n\}$ ~~$k=1..n$~~

$|\lambda x_k| = |\lambda| |x_k| = |\lambda| |f(x)| = |\lambda f(x_k)|$

$\max_{k=1..n} |\lambda x_k| = \max_{k=1..n} |\lambda f(x)| = \max \{|\lambda x_1|, \dots |\lambda x_n|\} = \cancel{\sup f(A)} = \cancel{\sup f(x)} = \sup |\lambda f(x)|$

6) so $\max_{k=1..n} |\lambda x_k| = \sup_{x \in A} |\lambda f(x)| \Longrightarrow |\lambda| \sup_{x \in A} |f(x)|$ $\Leftarrow$

$\max_{k=1..n} |x_k| = \sup |f(A)| = \sup_{x \in A} |f(x)|$

$\max_{k:1..n} |\lambda x_k| = \lambda \max_{k:1..n} |x_k| = |\lambda| \sup_{x \in A} |f(x)| =$

$\boxed{\lambda \max_{k:1..n} |x_k|}$

$(E, K, \oplus, \odot) \to$ Vect space

$(E, K, \oplus, \odot, (\cdot,\cdot)) \to$ Inner product space.

$(\cdot,\cdot)$ $\to$ is called inner product

$(\cdot,\cdot) : E \times E \ni (x,y) \longmapsto el \in K$

1) $\overline{(x,y)} = (y,x), \ \forall x,y \in E$

2) $(x+y, z) = (x,z) + (y,z), \ \forall x,y,z \in E$

3) $(\lambda x, y) = \lambda (x,y), \ \forall x,y \in E, \ \lambda \in K$

4) $(x,x) \in \mathbb{R}, \ (x,x) \geq 0, \ (x,x) = 0 \Longleftrightarrow x = \Theta$

$\boxed{x = \Theta \vee y = \Theta \Longrightarrow (x,y) = 0}$ $\not\Leftarrow$

$(\Theta, y) = (0 \cdot \Theta, y) = 0 \cdot (\Theta, y) = 0 \ (1)$

$(x, \Theta) = \overline{(\Theta, x)} \overset{(1)}{=} \overline{0} = 0$

~~$(x,y) = 0 \not\Longrightarrow x = \Theta \wedge y = \Theta$~~

$\boxed{(x,y) \leq \sqrt{(x,x)}\sqrt{(y,y)}}$ ~~Cauchy~~ Cauchy - Bunyakovsky ~~Schwarz~~ Schwarz

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The shortest correct proof

Theorem 1: f is biject $\Rightarrow$ f is invert | $f: X \to f(X)$

proof: Let's $g: f(X) \to X$ be a mapping which sends $y \in f(X)$ to x from X whenever $f(x) = y$ (1)

a) g is a function? From surjectivity of f $\Rightarrow$

$\forall y \in f(X), \exists x \in X: (f(x) = y) \xLeftrightarrow{\text{From def of } g} (g(y) = x)$

- $\forall y \in f(X), \exists x \in X: g(y) = x$ (2)

- $\forall y_1, y_2 \in Y: (g(y_1) \neq g(y_2)) \xLeftrightarrow{\text{from}(1)} (x_1 \neq x_2) \xRightarrow{f \text{ is injective}} (\underbrace{f(x_1)}_{y_1} \neq \underbrace{f(x_2)}_{y_2})$

$\forall y_1, y_2 \in Y: g(y_1) \neq g(y_2) \Rightarrow y_1 \neq y_2$ (3)

C.P.S. From def of g: $g(y_1) = x_1 \wedge g(y_2) = x_2 \wedge f(x_1) = y_1 \wedge f(x_2) = y_2$

$(2) \wedge (3) \Rightarrow g$ is well-defined function

Let $x \in X$, $y \in f(X)$ and $\boxed{g(y) = x}$ (4) $\wedge$ $\boxed{f(x) = y}$ (5)

- $g \circ f(x) = g(f(x)) \overset{(5)}{=} g(y) \overset{(4)}{=} x$

i.e. $\boxed{g \circ f(x) = x}$

- $f \circ g(y) = f(g(y)) \xrightarrow{(4)} f(x) \xrightarrow{(5)} y$

$\boxed{f \circ g(y) = y}$

i.e. $g \circ f = I_X \wedge f \circ g = I_Y$ (6)

$(2) \wedge (3) \wedge (6) \Rightarrow g$ is inversion of f

Theorem 2: f is invertable $\Rightarrow$ f is bijective

$g = f^{-1}$ such that $g(y) = x \Leftrightarrow f(x) = y$. $g: f(X) \to y$

- $\forall x_1, x_2 \in X: \boxed{x_1 \neq x_2} \xLeftrightarrow{g(y_1) = x_1,\ g(y_2) = x_2} g(y_1) \neq g(y_2) \xrightarrow{g \text{ is a funct}} y_1 \neq y_2 \Leftrightarrow$

$\xLeftrightarrow{y_1 = f(x_1),\ y_2 = f(x_2)} f(x_1) \neq f(x_2) \rightarrow f$ is injective (7)

- $\forall y \in f(X), \exists x \in X: g(y) = x$ (because g is function) $\Leftrightarrow$

so $\forall y \in f(X), \exists x \in X: f(x) = y \Rightarrow f$ is surjective (8)

$(7) \wedge (8) \Rightarrow f$ is bijective

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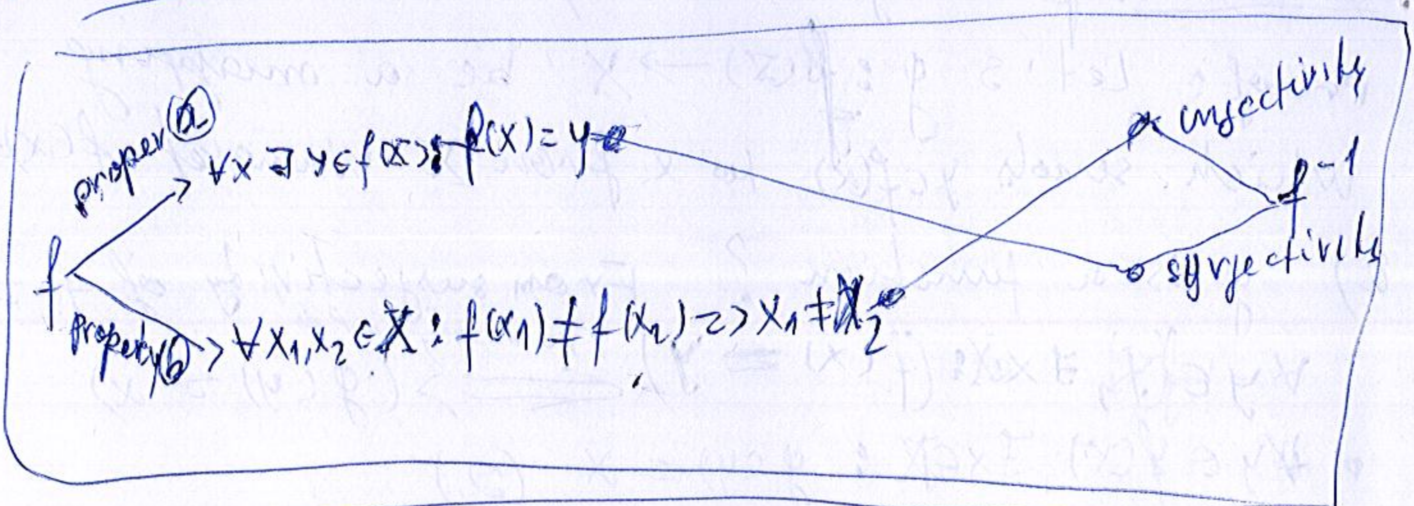
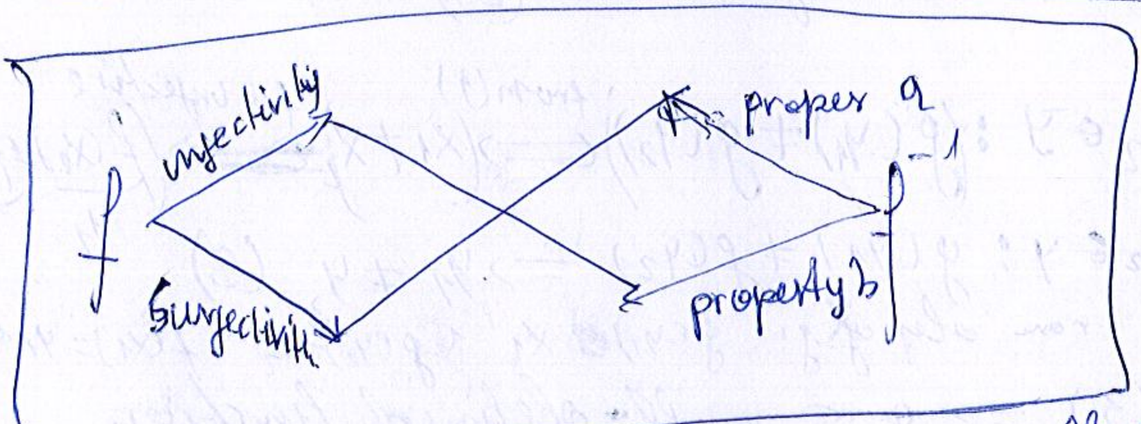
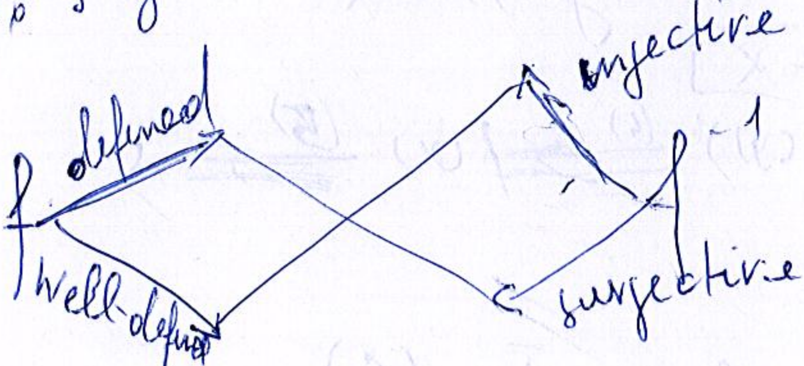
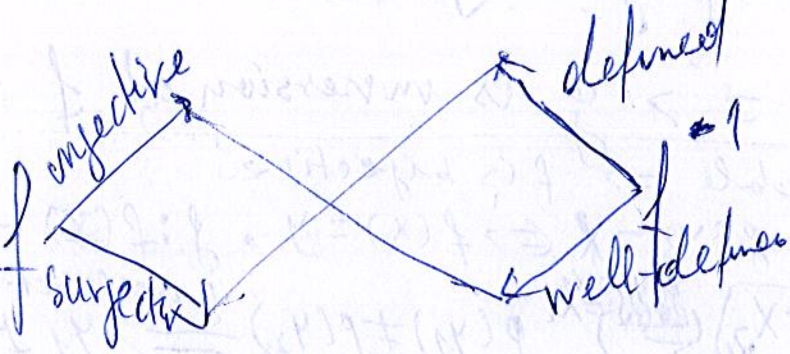
Notes





!! My agreement defined = proper a; well-defined = prop b





In some books well-defined means both defined and well-defined

(1) Task $(g \circ f)^{-1} \overset{?}{=} f^{-1} \circ g^{-1}$

Proof: $f: X \to Y$; $g: f(X) \to X$

$x \in X$ $\quad g \circ f(x) = g(f(x))$

$(g \circ f)^{-1}(x) = g^{-1}(f(x))$ $\quad f^{-1}(x) = (f(x))^{-1}$

$f(x) = y \Leftrightarrow g(y) = x$ $\quad g^{-1}(y)$

$f^{-1} \circ g^{-1}(y)$

$f: X \to f(X)$ | $g: A \to B$

$f^{-1}: f(X) \to X$ | $g^{-1}: B \to A$

$f(x) = y \Leftrightarrow f^{-1}(y) = x$ | $g(a) = b \Leftrightarrow g^{-1}(b) = a$

$f^{-1} \circ g^{-1}(b) = ?$ $\quad g \circ f(x) = g(f(x)) = g(y)$

$f \circ g(a) = f(g(a))$ $\quad g(a) \in X$

$g: f(X) \to X$

$g^{-1}: X \to f(X)$

from $g \circ f \Rightarrow Y \subset A$
from $f^{-1} \circ g^{-1} \Rightarrow A \subset Y$
$\Rightarrow A = Y (= f(X))$
$f \circ g \Rightarrow B \subset X$
$g^{-1} \circ f^{-1} \Rightarrow X \subset B$

$f^{-1} \circ g^{-1} = f^{-1}(g^{-1}(y)) = f^{-1}(X)$

$a = y \Rightarrow g(y) = x$ ~~$f \circ g = f(g(a) = f(x) = y$~~

$\begin{cases} x, y \\ g(y) = x \\ f(x) = y \\ f \circ g = y \end{cases}$

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$g \circ f(x) = g(f(x)) = g(y) = b$

$$\begin{cases} x; & f(x) = y \quad (1) \iff f^{-1}(y) = x \quad (4) \\ y; & g(y) = b \quad (2) \iff g^{-1}(b) = y \quad (5) \\ b; & g(f(x)) = g(y) = b \quad (3) \end{cases}$$

$g \circ f^{-1}$

$g \circ f: x \to y \to b$

$g \circ f: X \to B$

$(g \circ f)^{-1}: B \to X$

Let see at point b the value of $f^{-1} \circ g^{-1}$

let $c = b$

$f^{-1} \circ g^{-1} = f^{-1}(g^{-1}(b))$ ~~Let c~~

(2) $g(y) = b \overset{(5)}{\Rightarrow} \boxed{g^{-1}(b) = y}$

We know that $f(x) \overset{(1)}{=} y$ thus $f^{-1}(y) = x$

$f^{-1} \circ g^{-1}(b) = f^{-1}(g^{-1}(b)) = f^{-1}(y) = x$

$$\left.\begin{array}{l} f^{-1} \circ g^{-1}(b) = x \\ g \circ f(x) = b \end{array}\right\} \Rightarrow f^{-1} \circ g^{-1} = (g \circ f)^{-1}$$