(1) Topological Spaces

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Theorem 1.1 → finite or non-finite (infinite) i.e. finite or infinite

a) Any sum of open sets is an open set.

b) Any limited (finite) intersection of open sets is open set.

Proof.

(E, ρ) $\quad \mathcal{T}$ a collection of open sets on E,

$\alpha \in \mathcal{A}$ $\quad \bigcup_{\alpha \in \mathcal{A}} A_\alpha$ → is an open set

$$\mathcal{A} = \{\alpha : A_\alpha \in E \wedge A_\alpha \text{ is open set}\}$$

A_α is open set $\Leftrightarrow A = \bigcup_{x \in A} B(x, r)$ for some $r > 0$

$$\bigcup_{\alpha \in \mathcal{A}} A_\alpha = \bigcup_{\alpha \in \mathcal{A}} \left(\bigcup_{x \in A_\alpha} B(x, r) \right) = \bigcup_{x \in \bigcup A_\alpha} B(x, r)$$

those $S = \bigcup A_\alpha = \bigcup_{x \in S} B(x, r) \Rightarrow S$ is open

II way

A_α is open $\Rightarrow \forall x \in A_\alpha \Rightarrow \exists\, B(x, r) \subset A_\alpha$

$x \in \bigcup_{\alpha \in \mathcal{A}} A_\alpha \Rightarrow (\exists \alpha \in \mathcal{A})\ x \in A_\alpha \xrightarrow{A_\alpha \text{ is open}} (\exists r > 0) B(x, r) \subset A_\alpha$

$B(x, r) \subset A_\alpha \Rightarrow B(x, r) \subset \bigcup A_\alpha$

So $(\forall x \in \bigcup_{\alpha \in \mathcal{A}} A_\alpha)(\exists r > 0)\ B(x, r) \subset \bigcup_{\alpha \in \mathcal{A}} A_\alpha \Leftrightarrow$

$\bigcup_{\alpha \in \mathcal{A}} A$ is open set $\square$

3) Task (Caution!)

Not any intersection of open sets is open set.

We showed that finite intersection it is open set.

Let's show that infinite intersection may not be open set.

$A = (a - \frac{1}{n}, a + \frac{1}{n})$

Let's show that $\bigcap_{n \in N} (a - \frac{1}{n}, a + \frac{1}{n}) = \{a\}$

First, a show that $\{a\} \in \bigcap (a - \frac{1}{n}, a + \frac{1}{n})$

Second $\bigcap (a - \frac{1}{n}, a + \frac{1}{n}) \in \{a\}$

I) $a - \frac{1}{n} \leq a \leq a + \frac{1}{n}$ as $\frac{1}{n} > 0$

thus $\left[(\forall n \in N) \quad a - \frac{1}{n} \leq a \leq a + \frac{1}{n}\right] \Longrightarrow$

$\{a\} \in (a - \frac{1}{n}, a + \frac{1}{n})$.

II $x \in (a - \frac{1}{n}, a + \frac{1}{n}) \quad |x - a| < \frac{1}{n}$

~~It was showed before (lemma)~~ if $(\forall n \in N) \; \rho(x, a) < \frac{1}{n}$

~~thus~~ $x = a$

$(\forall \varepsilon > 0) \; \rho(x, a) \leq \varepsilon \Rightarrow x = a$ ($\forall \varepsilon$ we pick $n = \frac{1}{\varepsilon}$)

(5) Theorem 2 $(E, \rho) \to$ m. space

Remember: $A \subset E$, A is open $\Leftrightarrow E \setminus A$ is closed (i)
A is closed $\Leftrightarrow E \setminus A$ is open (ii)

a) Any intersection of closed sets is closed.

b) Finite sum of closed sets is closed.

Proof:

a) $\{G_\alpha ; \alpha \in A\} \to$ a collection (family) of closed sets

$I = \bigcap_{\alpha \in A} G_\alpha$ $\quad E \setminus I = E \setminus \bigcap_{\alpha \in A} G_\alpha = \bigcup_{\alpha \in A} E \setminus G_\alpha$ (1)

$(\forall \alpha \in A)$ G_α is closed $\Rightarrow E \setminus G_\alpha$ is open set $\Rightarrow$

$\xRightarrow{\text{Theorem 1}} \bigcup_{\alpha \in A}(E \setminus G_\alpha)$ is open set $\xLeftrightarrow{(1)} E \setminus \bigcap_{\alpha \in A} G_\alpha$ is open set $\Rightarrow$

$\xRightarrow{\text{definition of closed set (ii)}} \bigcap_{\alpha \in A} G_\alpha$ is closed set. □

P.S. Any intersection means that it can be infinite intersection

b) $S = \bigcup_{i=1}^{n} G_i$, $\forall i \in \{1, \dots, n\}$ G_i is closed set. (2)

$E \setminus S = E \setminus \left(\bigcup_{i=1}^{n} G_i\right) = \bigcap_{i=1}^{n} (E \setminus G_i)$ (3). From (2) $\Rightarrow E \setminus G_i$ is open set $\forall i = 1 \dots n$ $\xRightarrow{\text{Theorem 1}} \bigcap_{i=1}^{n}(E \setminus G_i)$ is open set $\xLeftrightarrow{(3)} E \setminus \bigcup_{i=1}^{n} G_i$ is open set

$\Leftrightarrow \bigcup_{i=1}^{n} G_i = S$ is closed set. □

Caution!!! Not any sum of closed sets is closed. For instance infinity sums of closed sets do not have to be closed set as well.

Example: $\bigcup [\frac{1}{n}, 1 - \frac{1}{n}] = (0, 1)$ (4)

Proof on the next page $\to$

Definition

(1) $E \to$ any set, $\mathcal{T}$ a collection of sets on E such that

1) $\emptyset, E \in \mathcal{T}$

2) $(\forall \alpha \in A)[A_\alpha \in \mathcal{T}] \Rightarrow \left[\bigcup_{\alpha \in A} A_\alpha \in \mathcal{T}\right]$

3) $\left[A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3} \ldots A_{\alpha_m} \in \mathcal{T}\right] \Rightarrow \left[\bigcap_{i=1..m} A_{\alpha_i} \in \mathcal{T}\right]$

Then $(E, \mathcal{T})$ is a topological space

where sets of $\mathcal{T}$ are called open sets in $(E, \mathcal{T})$

$G \in \mathcal{T} \Rightarrow E \setminus G$ is closed set

Theorem

(E, ρ) is a metric space and $\mathcal{T}$ is the collection of all open sets on E then $(E, \mathcal{T})$ is topological space generated from (E, ρ).

Proof:

As we showe $\emptyset, E$ are open sets (in fact they are both open and clo[sed]) those 1) $\emptyset, E \in \mathcal{T}$

2) $(\forall \alpha \in A)$ A_α is open set $\overset{A_\alpha \in \mathcal{T}\ \forall \alpha \in \Lambda}{\Rightarrow}$ any sum, thus, $\bigcup_{\alpha \in A} A_\alpha$ is open set as well thus $\bigcup_{\alpha \in \Lambda} A_\alpha \in \mathcal{T}$

3) $A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}$ are open sets $\Rightarrow$ thus $A_{\alpha_i} \in \mathcal{T}\ \forall i = 1..n$

$\Rightarrow \bigcap_{i=1..n} A_{\alpha_i} \in \mathcal{T}$ as they are a finite intersection of open sets thus $\in \mathcal{T}$

③ 1) $\bigcup_{\alpha \in A} A_\alpha = \bigcup_{\alpha \in A} (G_\alpha \cap A) = \left(\bigcup_{\alpha \in A} G_\alpha\right) \cap A$

$H = \bigcup_{\alpha \in A} G_\alpha$ H is sum of open sets thus H is open set, hence $H \in \mathcal{T}$

i.e. $\bigcup_{\alpha \in A} G_\alpha \in \mathcal{T} \Rightarrow H \cap A \in \mathcal{T}_A$ i.e. $(\bigcup G_\alpha) \cap A \in \mathcal{T}_A$

$\Rightarrow \boxed{\bigcup_{\alpha \in A} A_\alpha \in \mathcal{T}_A}$ □

3) $A_{\alpha_i} \in \mathcal{T}_A \Leftrightarrow A_\alpha = G_{\alpha_i} \cap A$ where $G_\alpha \in \mathcal{T}$.

$\bigcap_{i=1..n} A_{\alpha_i} = \bigcap_{i=1..n} (G_{\alpha_i} \cap A) = \left(\bigcap_{i=1..n} G_{\alpha_i}\right) \cap A$

$G_{\alpha_i} \in \mathcal{T} \Rightarrow \bigcap_{i=1..n} G_{\alpha_i} \in \mathcal{T} \Rightarrow \left(\bigcap_{i=1..n} G_{\alpha_i}\right) \cap A \in \mathcal{T}_A$

$\Rightarrow \boxed{\bigcap_{i=1..n} A_{\alpha_i} \in \mathcal{T}_A}$ □

Theorem $(E, \rho) \to$ metric space

$A \subset E$, $(A, \rho) \to$ metric space submetric space of (E, ρ)

$(E, \mathcal{T}) \to$ topological space generated by metric space, where $\mathcal{T}$ is collection (family) of open sets on (E, ρ).

$(A, \mathcal{R}) \to$ topological space generated by metric space (A, ρ), $\mathcal{R}$ family of open sets on (A, ρ)

(5) So $H = \left(\bigcup_{x\in G} B(x,r)\right) \cap A = \bigcup_{x\in G} \left(B(x,r) \cap A\right) = \bigcup_{x\in G} B_A(x,r)$

$\Rightarrow$ H is an open set on (A,ρ) thus $H \in \mathcal{R}$ (i.e $(A,\mathcal{R})$)

We showed that: $\boxed{(\forall H)\ (H \in \mathcal{T}_A) \Rightarrow H \in \mathcal{R}}$ (a)

$(\Leftarrow)$ $[H \in \mathcal{R}]_{(1)} \overset{?}{\Rightarrow} [H \in \mathcal{T}_A]_{(2)} \Leftrightarrow [(\exists G \in \mathcal{T})\ H = G \cap A]_{(3)}$

Let H be a set from $\mathcal{R}$, thus set H is open on (A,ρ).

We must show that $H = G \cap A$ where $G \in \mathcal{T}$ i.e is an open set ~~from~~ on (E,ρ).

$H \in \mathcal{R} \Leftrightarrow$ is open on $(A,\rho) \Rightarrow H = \bigcup_{x\in H} B_A(x,r)$, for some $r>0$

$H = \bigcup_{x\in H} B_A(x,r) = \bigcup_{x\in H} (B(x,r) \cap A) = \left(\bigcup_{x\in H} B(x,r)\right) \cap A$

Let $G := \bigcup_{x\in H} B(x,r)$, G is sum of open balls.

Open balls are open set thus G as sum of open sets is an open set $\Rightarrow G \in \mathcal{T}$

$\left.\begin{matrix} H = G \cap A \\ G \in \mathcal{T} \end{matrix}\right\} \Rightarrow H \in \mathcal{T}_A$

Hence, we showed that:

$\left[\forall H\ (H \in \mathcal{R})\right] \Rightarrow [H \in \mathcal{T}_A]$ (b)

(a) $\wedge$ (b) $\Rightarrow$

$\Rightarrow \mathcal{T}_A = \mathcal{R}$