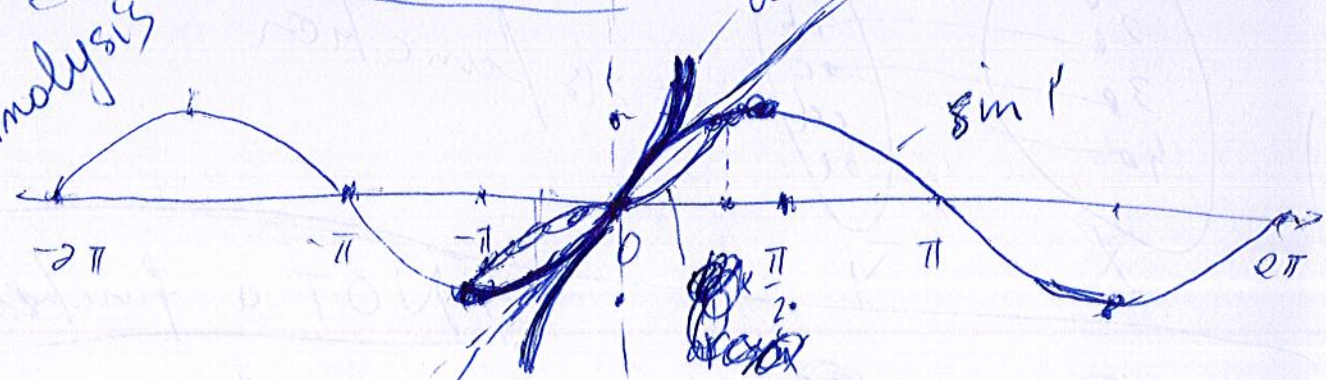
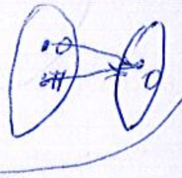
Iodian Mishtolen 2019

Math Analysis



$(R, [-1,1], \sin x) \Rightarrow$ na but no inject (not 1-1)

$\Rightarrow \sin(0) = \sin(\pi) = 0$



$\neq$ $([-\pi, \pi], [-1,1] \sin) \to$ no 1-1

$\neq$ $([0, 2\pi], [-1,1] \sin) \to$ no 1-1

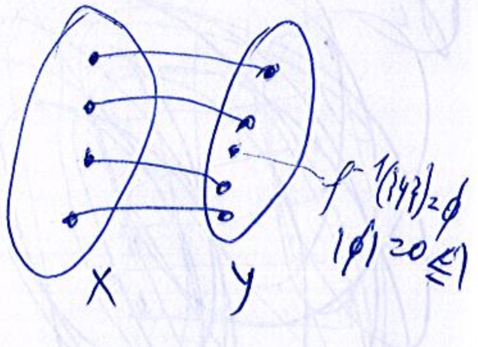
f ~~is~~ bijectio $\Leftrightarrow$ f is invertable

Def $f: X \to Y$ $A \in X$; $f(A) = \{y : \exists x \in A, f(x) = y\}$ $y = f(x)$

$\Leftrightarrow f(A) = \{f(x) : x \in A\}$, $B \subset Y$ $f^{-1}(B) = \{x \in X : f(x) \in B\}$

$f^{-1}(y) = \{x \in X : f(x) = y\}$

$\forall y \in Y$ $|f^{-1}(\{y\})| \leq 1 \Rightarrow f$ is syrgation



if $\forall y \in Y$ $|f^{-1}(\{y\})| = 1 \Rightarrow f$

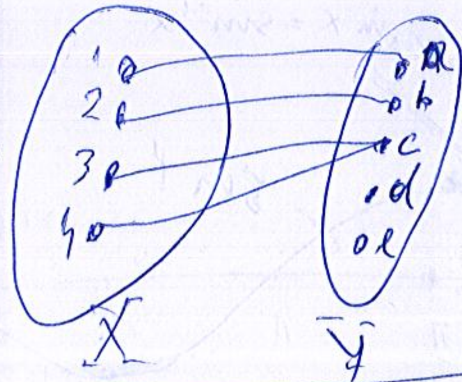
Let for $\forall y \in Y$ $|f^{-1}(\{y\})| = 1$ $f^{-1}: Y \to X$

$\Rightarrow f^{-1}$ is ~~function Prove it~~

Proof $\boxed{f \text{ is a function} \stackrel{\text{def}}{\Leftrightarrow} \forall x \in X \ \exists! y \ f(x) = y}$

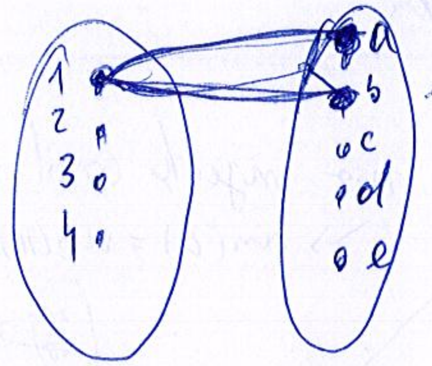
Proven late

②

i) 

$\to$ is function

$\to$ $\boxed{\text{NOT a function}}$

ii) 

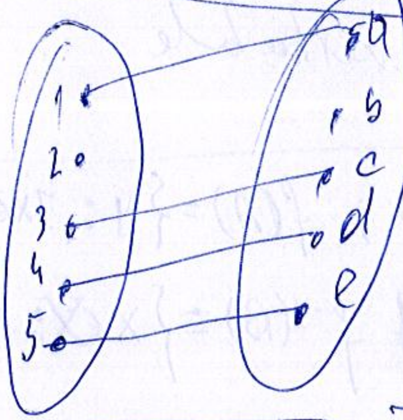
$f(\{1\}) = \{a, b\}$

$|f(\{1\})| = 2$ or

~~or~~

$\exists x = 1 \in X : \nexists! \, y \in Y \; f(x) = y$

because $\exists \; y = a, \exists \; y = b$

iii) 

$\to$ $\boxed{\text{NOT a function}}$

$\exists x=2$: there is no $y \in Y$ such that $f(2) = y$

$\boxed{\exists x = 2 \; \forall y \in Y \; f(x) \neq y}$

$\iff \exists x = 2, \nexists! \, y \in Y : f(x) = y$

$\exists x = 2 \in X, \nexists \, y \in Y : f(x) = y$

$\Downarrow$

$\exists x = 2 \in X, \nexists! \, y \in Y : f(x) = y$

or

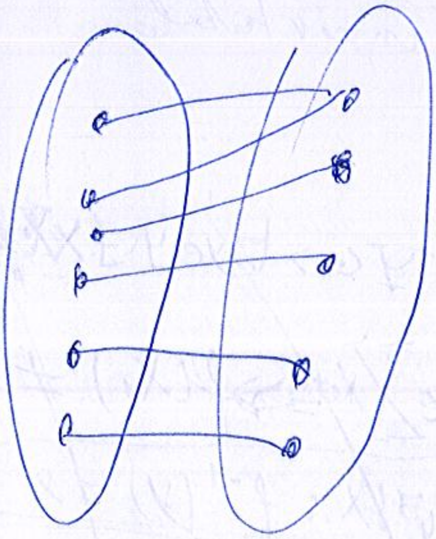
$\exists x = 2, \forall y \in Y : f(x) \neq y$

$\Downarrow$

$\exists x = 2, \nexists! \, y \in Y : f(x) = y$

$\boxed{f(X) = Y \overset{Def}{\iff} f \text{ is syrjective}}$

(3)

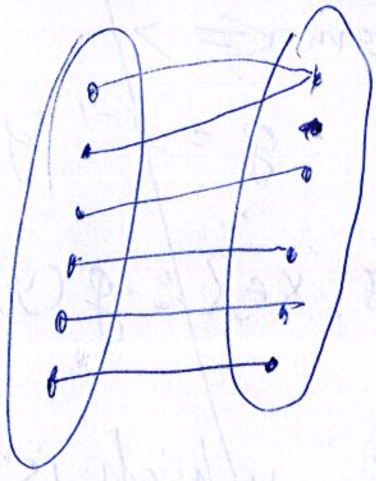


- It is a Functio
- but Not Injection (1-1

⇓

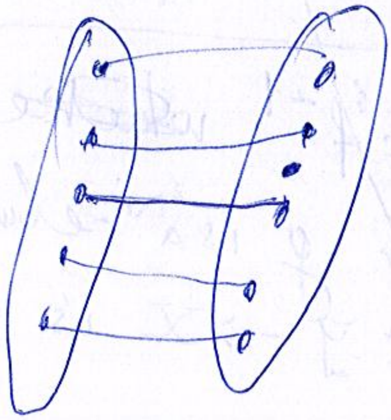
Not bijectio

- It is Syrjection



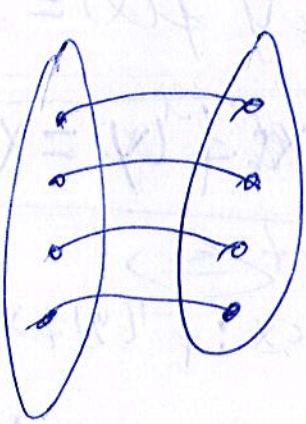
- It is a Functio
- Not Injection (i.e. 1-1)
- Not Syrjection

} Not bijection



- It is a Functio
- Not Syrjetion (no)
- It is Injectio (i.e 1-1)

} => NOT Bijed



- It is a funct
- It is Syrjctio

(1) f is bijection $\Leftrightarrow$ f is invertable

Proof: $\Rightarrow$

f is bijection $\Leftrightarrow$ a) $f(X) = Y \Leftrightarrow \forall y \in Y\ \exists x\ f(x) = y$

b) $x_1, x_2 \in X\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

? a) $\wedge$ b) $\Rightarrow$ c) $\boxed{\text{i.e. } \forall y_{\in Y} \exists! x : f^{-1}(y) = x}$

P.S. b') $\boxed{\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$

f is invertable $\Leftrightarrow$ f^{-1} is function $\Leftrightarrow$

$f^{-1} : Y \rightarrow X \quad g = f^{-1} \quad g : Y \rightarrow X$

Fref funct def $\Leftrightarrow$ $\forall y \in Y\ \exists! x \in X : g(y) = x$

$\Rightarrow \forall y \in Y\ \exists! x : f^{-1}(y) = x$ which is c)

Thus f^{-1} is a function

$\Leftarrow$ f is invertible $\Leftrightarrow \exists g = f^{-1}$ ~~where~~

$g : Y \rightarrow X$ ~~which~~ wher g is a relation

Let's show this relati $g : Y \rightarrow X$ is a function.

Proof: $\boxed{f \text{ is funct} \Leftrightarrow \forall x \in X, \exists! y \in Y\ f(x) = y}$ d)

f is invertible $\Rightarrow$ $\boxed{\forall y \in Y, \exists x : f^{-1}(y) = x}$ e)

Let's assume that f^{-1} is not a funct $\Leftrightarrow$

$\exists y \in Y, \nexists x \in X : f^{-1}(y) = x$ (i.e. $\exists y \in Y\ \forall x \in X : f^{-1}(y) \neq x$)) f)

Or ~~$\exists y_1, y_2 \in Y : f^{-1}(y_1) = f^{-1}(y_2)$~~

$\exists y \in Y, \exists x_1, x_2\ \ x_1 \neq x_2 \wedge f^{-1}(y) = \{x_1, x_2\} \Leftrightarrow \exists y \in Y, \exists x_1, x_2 : f(y) = x_1 \wedge f(y) = x_2 \Rightarrow x_1 \neq x_2$

(5) $f: X \to f(X)$ is bijective $\Rightarrow f$ is invertabl

$\boxed{f \text{ is a function}}$ (from X onto $f(X)$) $\Longleftrightarrow$

$$\left.\begin{cases} \forall x \in X, \exists y \in f(X): f(x) = y & (1) \\ \forall x_1, x_2 \in X: f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2 & (2) \end{cases}\right\} \text{well-defined function}$$

$\boxed{\text{Bijectivity}}$

$$\begin{cases} \forall y \in f(X), \exists x \in X: f(x) = y & (3) \text{ surjectivity} \\ \forall x_1, x_2 \in X: x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) & (4) \text{ injectivity} \end{cases}$$

$\boxed{\text{Invertibility}}$

Definiti
~~mapping~~ $g: f(X) \to X$ such that $g \circ f(x) = \underset{\in X}{x} \wedge f \circ g(y) = \underset{\in Y}{y}$
is called inverted function of f. (5)

Proof

Let's show that f is invertable $\Leftrightarrow \exists g$ which is inversion of f

Let's g be a mapping $g: f(X) \to X$ such that

$\forall x \in X, \forall y \in f(X) \quad f(x) = y \Leftrightarrow g(y) = x$ (6)

Let's show g is a well-defined function

a) from (3) $\forall y \in f(X), \exists x \in X: f(x) = y \Rightarrow$

$\forall y \in f(X), \exists x \in X: g(y) = x$ ✓

b) $\forall y_1, y_2 \in f(X) \xleftrightarrow[\exists x_2, g(y_2) = x_2]{\exists x_1, g(y_1) = x_1} g(y_1) \neq g(y_2) \Longleftrightarrow$

$x_1 \neq x_2 \xrightarrow{(4)} f(x_1) \neq f(x_2) \xLeftrightarrow{f(x_1) = y_1,\ f(x_2) = y_2 \text{ from the } g \text{ definition}} y_1 \neq y_2$

i.e. $\forall y_1, y_2 \in f(X): g(y_1) \neq g(y_2) \Rightarrow y_1 \neq y_2$

a) $\wedge$ b) $\Rightarrow g$ is well defined.

c) we must show that $g \circ f(x) = x$ for $\forall x \in X \wedge$

$f \circ g(y) = y$.

6) c) continue

$g\circ f(x) = ?$ Let $f(x) = y \in f(X)$ then by

definition of g: $g(y) = x$. (7)

Thus $g\circ f(x) = g(f(x)) = g(y) \overset{(6)}{=} x$

$f\circ g(y) = f(g(y)) \overset{7}{=} f(x) = y$

i.e $\begin{cases} g\circ f(x) = x \\ f\circ g(y) = y \end{cases}$

a) ∧ b) ∧ c) $\Longrightarrow$ g is inverted function of f $\Leftrightarrow$

f is invertable

just for fun, we do not need, because f is a given function

$\boxed{\text{Theorem 2}}$

f is invertable $\Rightarrow$ f is byjective

Proof: f is invertable $\Rightarrow \exists$ funct $g: f(X) \to X$

Such that $\underline{g(y) = x \text{ when } f(x) = y}$ (8) and

$\underline{g\circ f(x) = x}$ (9) and $\underline{f\circ g(y) = y}$ (10)

a) is f a funct?

~~$\forall y \in f(X), \exists x \in X : g(y) = x$~~

$\forall x \in X, \exists y \in f(X) : g(y) = x$ (11) from surjectivity

of g. (11) $\xleftrightarrow{\text{from } g \text{ definition (8)}}$ $\boxed{\forall x \in X, \exists y \in f(X) : f(x) = y}$ (12)

$\forall y_1, y_2 \in f(X)$: $y_1 \neq y_2 \overset{(4) \text{ injectivity prop}}{\Longrightarrow} g(y_1) \neq g(y_2)$

$\forall x_1, x_2 \in X$: $f(x_1) \neq f(x_2) \xleftrightarrow{y_1 = f(x_1),\ y_2 = f(x_2)} y_1 \neq y_2 \overset{(4)}{\Rightarrow} g(y_1) \neq g(y_2)$

$\xleftrightarrow{g(y_1) = x_1,\ g(y_2) = x_2} x_1 \neq x_2$

i.e $\boxed{\forall x_1, x_2 \in X : f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2}$ (13)

(12) ∧ (13) $\Rightarrow$ $\underline{f \text{ is well defined}}$

1/ We want to show that f is surjective and

$\forall y_1, y_2 \in f(X)$ $[g(y_1) \neq g(y_2) \Rightarrow y_2 \neq y_1]$ injective

injective so that:

* $\forall y \in f(X), \exists x \in X : f(x) = y$ surjec

* $\forall x_1, x_2 \in X : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ injec

proof

g is a funct $\Rightarrow \forall y \in f(X), \exists x \in X : g(y) = x$

but that is $\Leftrightarrow f(x) = y$ thus

$\boxed{\forall y \in f(X), \exists x \in X : f(x) = y}$ (14) thus f is surjective (3)

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \xLeftrightarrow{\text{from def of } g} g(y_1) \neq g(y_2) \xrightarrow{g \text{ is a funct}}$$

$$y_1 \neq y_2 \xLeftrightarrow[y_2 = f(x_2)]{y_1 = f(x_1)} f(x_1) \neq f(x_2)$$

i.e. $\boxed{\forall x_1, x_2 \in X \Rightarrow f(x_1) \neq f(x_2)}$ (15) injective

$(14 \wedge 15) \Rightarrow$ f is bijective

(8) Shorter proofs:

f is bijective $\Rightarrow$ f is invertable

Proof:

Let define $g : f(X) \longrightarrow X$ such that $g(y) = x$ whenever $f(x) = y$ (1).

Is g a well defined function?

$\forall y \in f(X), \exists x \in X : f(x) = y$ (2) (from surjectivity of f)

$(2) \Leftrightarrow \boxed{\forall y \in f(X)\ \exists x \in X : g(y) = x}$ (3)

$\forall x_1, x_2 \in X :$ $\cancel{f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2}$ $x_1 \neq x_2 \xRightarrow{\text{from inject of } f} f(x_1) \neq f(x_2)$

$\Leftrightarrow g(y_1) \neq g(y_2)$

$\forall y_1, y_2 \in Y :$ $\cancel{g(y_1) \neq g(y_2)}$ $\boxed{y_1 \neq y_2} \xLeftarrow{f(x_1)=y_1,\ f(x_2)=y_2} f(x_1) \neq f(x_2)$

$\Rightarrow x_1 \neq x_2 \Leftrightarrow \boxed{g(y_1) \neq g(y_2)}$

i.e. $\boxed{\forall y_1, y_2 \in Y : y_1 \neq y_2 \Rightarrow g(y_1) \neq g(y_2)}$ (4)

$\forall y_1, y_2 \in f(X) : \boxed{g(y_1) \neq g(y_2)} \xLeftrightarrow{g(y_1)=x_1,\ g(y_2)=x_2} x_1 \neq x_2 \xRightarrow{\text{inje of } f}$

$f(x_1) \neq f(x_2) \xLeftrightarrow{\text{def of } g} \boxed{y_1 \neq y_2}$

i.e. $\boxed{\forall y_1, y_2 \in f(X) : y_1 \neq y_2 \Rightarrow g(y_1) \neq g(y_2)}$ (4)

$(3) \wedge (4)$ g is well defined

$g \circ f(\underline{x}) = g(f(x)) = g(y) = \underline{x}$ ✓ (5)

$\underline{f \circ g(\underline{y})} = f(g(y)) = f(x) = \underline{y}$ ✓ (6)

$(3) \wedge (4) \wedge (5) \wedge (6) \rightarrow g$ is inversa of f

$g \overset{\text{def}}{=} f^{-1}$