

Theorem 1.1

$$\sup \lambda A = \begin{cases} \lambda \sup A, & \lambda \geq 0 \\ \lambda \inf A, & \lambda < 0 \end{cases} \quad \begin{matrix} a) \\ b) \end{matrix}$$

Proof a: $\forall x \in A, \lambda x \leq \sup \lambda A \Leftrightarrow \boxed{x \leq \frac{\sup \lambda A}{\lambda}} \Rightarrow \frac{\sup \lambda A}{\lambda}$

is an upper bound of A . And we know that $\sup A$ is the smallest upper bound; hence,

$$\boxed{\sup A \leq \frac{\sup \lambda A}{\lambda}} \quad (1) \xLeftrightarrow{\lambda > 0} \boxed{\lambda \sup A \leq \sup \lambda A} \quad (1')$$

$\forall x \in A: x \leq \sup A \xrightarrow{\lambda > 0} \boxed{\lambda x \leq \lambda \sup A} \Rightarrow \lambda \sup A$ is an upper bound for λA . And we know that $\sup \lambda A$ is the smallest upper bound; thus,

$$\boxed{\sup \lambda A \leq \lambda \sup A} \quad (2)$$

$$(1) \wedge (2) \Rightarrow \sup \lambda A \leq \lambda \sup A \leq \frac{\sup \lambda A}{\lambda} \Rightarrow$$

$$\lambda \sup A = \sup \lambda A //$$

b) $\forall x \in A: \lambda x \leq \sup \lambda A \Rightarrow x \leq \frac{\sup \lambda A}{\lambda}$ $\xrightarrow{\sup \lambda A \text{ is an lower bound of } A}$

but $\inf A$ is the smallest lower bound $\Rightarrow \boxed{\inf A \geq \frac{\sup \lambda A}{\lambda}} \quad (3)$

$$\boxed{\lambda \inf A \leq \sup \lambda A} \quad (3')$$

$\forall x \in A: x \geq \inf A \xrightarrow{\lambda < 0} \lambda x \leq \lambda \inf A \Rightarrow \lambda \inf A$ is an upper bound of λA . We know that $\sup \lambda A$ is the smallest one thus: $\boxed{\sup \lambda A \leq \lambda \inf A} \quad (4)$

$$(3') \wedge (4) \Rightarrow \sup \lambda A \leq \lambda \inf A \leq \sup \lambda A \Rightarrow$$

$$\boxed{\lambda \inf A = \sup \lambda A} \text{ for } \lambda < 0 //$$

$$\lambda = 0 \quad \sup 0 \cdot A = 0 \cdot \sup A //$$

Theorem 4.2

$$\inf \lambda A = \begin{cases} \lambda \inf A, & \lambda \geq 0 \\ \lambda \sup A, & \lambda < 0 \end{cases} \quad \begin{matrix} a) \\ b) \end{matrix}$$

a) $\forall x \in A$ $\lambda x \geq \inf \lambda A \xrightarrow{\lambda \geq 0} \textcircled{\lambda x} \geq \inf \lambda A \Rightarrow \inf \lambda A$ is a lower bound of λA (closure). We know that \inf of a set is the ~~smallest~~ biggest lower bound, hence $\inf A \geq \inf \frac{\lambda A}{\lambda} \xrightarrow{\lambda > 0} \boxed{\lambda \inf A \geq \inf \lambda A}$ I

$\forall x \in A$ $x \geq \inf A \xrightarrow{\lambda > 0} \textcircled{\lambda x} \geq \lambda \inf A \Rightarrow \lambda \inf A$ is a lower bound of λA . And we know that $\inf \lambda A$ is the biggest lower bound; thus, $\boxed{\inf \lambda A \geq \lambda \inf A}$ II. From I & II \Rightarrow

$$\inf \lambda A \geq \lambda \inf A \geq \inf \lambda A \Leftrightarrow \boxed{\inf \lambda A = \lambda \inf A}$$

$\bullet \lambda = 0$ $\forall x$ $\inf 0A = \inf \{0, \dots, 0\} = 0 \wedge \inf A = 0$
 $\inf 0A = 0 \inf A = 0$

b) $\forall x \in A$ $\textcircled{\lambda < 0}$ $\bullet \lambda x \geq \inf \lambda A \xrightarrow{\lambda < 0} \textcircled{\lambda x} \leq \inf \lambda A$
 $\Rightarrow \inf \lambda A$ is an upper bound for λA . And we know that $\sup A$ is the smallest upper bound, thus, $\sup A \leq \frac{\inf \lambda A}{\lambda}$
 $\xrightarrow{\lambda < 0} \boxed{\lambda \sup A \geq \inf \lambda A}$ (III)

$\bullet x \leq \sup A \xrightarrow{\lambda < 0} \lambda x \geq \lambda \sup A \Rightarrow \lambda \sup A$ is a lower bound for λA . $\boxed{\inf \lambda A \geq \lambda \sup A}$ (IV)

From (III) & (IV) $\boxed{\inf \lambda A = \lambda \sup A} //$

$$A, B \subset \mathbb{R}, \lambda \in \mathbb{R}$$

$$A+B = \{z \in \mathbb{R} : \exists x \in A \exists y \in B \cdot z = x+y\}$$

$$A-B = \{z \in \mathbb{R} : (\exists x \in A)(\exists y \in B) z = x-y\}$$

$$\lambda A = \{z \in \mathbb{R} : (\exists x \in A) z = \lambda x\}$$

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Theorem A, B bounded $\Rightarrow A+B, A-B, \lambda A$ are bounded

$$A \text{ bounded} \Leftrightarrow (\exists p_1 \in \mathbb{R}_+) (\forall x \in A) x \leq p_1$$

$$B \text{ bounded} \Leftrightarrow (\exists p_2 \in \mathbb{R}_+) (\forall y \in B) y \leq p_2$$

$$z \in A+B \Rightarrow (\exists x \in A) \wedge (y \in B) z = x+y$$

$$|x| \leq p_1 \quad \text{and} \quad |y| \leq p_2$$

$$\cancel{x \leq p_1} \quad \cancel{y \leq p_2}$$

$$\cancel{x+y \leq p_1+p_2 = p}$$

$$\cancel{x+y} \quad |x+y| \leq |x|+|y| \leq p \quad \text{i.e. } |z| \leq p \quad \text{d.o.e.}$$

$$z \in A-B \quad z = x-y$$

$$|z| = |x-y| \leq |x+(-y)| \leq |x|+|y| \leq p_1+p_2 = p$$

$$|z| \leq p \quad \text{d.o.e.}$$

$$|z| = |\lambda x| = |\lambda| |x| \leq |\lambda| p_1 = p \quad |z| \leq p \quad \text{d.o.e.}$$

// $\forall x \in R$ Upper border M of the set $A \Leftrightarrow \forall x \in A : x \leq M$

- A is limited from top $\Leftrightarrow (\exists M \in R)(\forall x \in A) x \leq M$
- A is limited from bottom $\Leftrightarrow (\exists m \in R)(\forall x \in A) x \geq m$
- A is limited $\Leftrightarrow (\exists P \in R_+)(\forall x \in A) |x| \leq P \Leftrightarrow -P \leq x \leq P$

~~$A \subset R$ is limited~~ $\left. \begin{array}{l} A \text{ has an upper bound } \Leftrightarrow \\ (\exists M \in R)(\forall x \in A) x \leq M \end{array} \right\}$

~~Upper bound of A~~ A has a lower Bound $\Leftrightarrow (\exists m \in R)(\forall x \in A) x \geq m$

A is bounded $\Leftrightarrow A$ has an upper bound and a lower Bound.

$\sup A$ is the ~~least~~ smallest upper bound \Leftrightarrow

1) $\forall x \in A : x \leq \sup A$

2) $\forall \varepsilon > 0 : x + \varepsilon \geq \sup A \Leftrightarrow x \geq \sup A - \varepsilon$

~~$\sup A$ is the~~ $\inf A$ is the biggest lower bound \Leftrightarrow

$\Leftrightarrow \left\{ \begin{array}{l} 1) \forall x \in A : x \geq \inf A \\ 2) \forall \varepsilon > 0 : x - \varepsilon \leq \inf A \end{array} \right. \Leftrightarrow x \leq \inf A + \varepsilon$

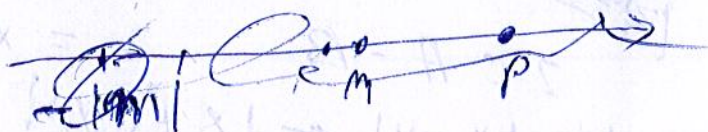
Theorem A is bounded $\Leftrightarrow A$ has upper bound and lower bound

$\Rightarrow A$ is bounded $\Leftrightarrow (\exists P \in R_+)(\forall x \in A) |x| \leq P$

(1) $\Leftrightarrow -P \leq x \leq P$ let $m = -P; M = P$

Then $(\exists m = -P)(\forall x \in A) x \geq m$

$\Leftrightarrow P = \max(|m|, |M|)$



$m \leq x \leq M$

$x \leq M \leq P$

$x \geq m$

$m \leq |m| \leq P$

$-m \geq -|m| \geq -P$

$x > -m \geq -|m| \geq -P$

$\Rightarrow x > -P/2, P \leq x \leq P$

$$\sup \lambda A = \begin{cases} \lambda \sup A, & \lambda \geq 0 \\ \lambda \inf A, & \lambda < 0 \end{cases}$$

$$\lambda = 0 \quad \sup 0 = 0 \quad \sup A \geq 0$$

$$x \in A \quad \boxed{x \leq \sup A \xrightarrow{\lambda \geq 0} \lambda x \leq \lambda \sup A}$$

$$\Leftrightarrow x \leq \sup A \quad \forall x \in A \Rightarrow \sup A \text{ is an upper bound of } A$$

$\sup A$ is the smallest upper bound thus

$$\sup A \leq \lambda \sup A \quad \Leftrightarrow \lambda \sup A \leq \sup A$$

$\lambda \sup A$ is an upper bound of λA

but \sup of set is the smallest upper bound

$$\text{thus } \sup \lambda A \leq \lambda \sup A \Rightarrow$$

$$x \in \boxed{\sup A \geq \frac{\sup \lambda A}{\lambda}} \quad (1) \Rightarrow \frac{\sup \lambda A}{\lambda} \text{ is one}$$

$$\lambda x \leq \sup \lambda A \Rightarrow x \leq \frac{\sup \lambda A}{\lambda}$$

$$\Rightarrow \boxed{\sup A \leq \frac{\sup \lambda A}{\lambda}} \quad (2) \leq \sup A$$

$$\Rightarrow \frac{\sup \lambda A}{\lambda} = \sup A \Rightarrow \sup \lambda A = \lambda \sup A$$

Theorem

$$1) \inf(A+B) \geq \inf A + \inf B$$

$$2) \sup(A+B) \leq \sup A + \sup B$$

Proof $z \in A+B \Rightarrow \exists x \in A \wedge \exists y \in B \quad z = x+y$

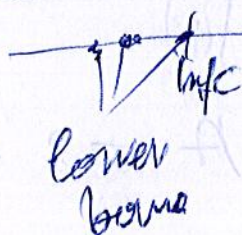
$$\begin{array}{lll} x \in A & x \geq \inf A & \Rightarrow x+y \geq \inf A + \inf B \\ y \in B & y \geq \inf B & z \geq \inf A + \inf B \end{array}$$

$$(\forall z \in A+B) z \geq \inf A + \inf B \Rightarrow$$

$\inf A + \inf B$ is a lower bound of $A+B$

But \inf of a given set is always \geq any lower bound. Thus $\boxed{\inf(A+B) \geq \inf A + \inf B}$

ii. $\inf C \geq$ any lower bound of C



$\boxed{\inf \text{ is the biggest lower bound}}$

$$2) \text{ Analog } \left. \begin{array}{l} \forall x \leq \sup A \\ \forall y \leq \sup B \end{array} \right\} \Rightarrow x+y \leq \sup A + \sup B$$

$$z < (\sup A + \sup B) \text{ - on upper bound of } A+B$$

But $\sup C$ is the smallest upper bound
Thus $\boxed{\sup(A+B) \leq \sup A + \sup B}$