(1) $x \in S$ is isolated point of $S \iff \exists r>0 \quad S \cap B(x,r) = \{x\}$

$\iff \exists r>0$ such that $B(x,r)$ contains only x.

$\iff \exists r>0 : B(x,r) \cap (S \setminus \{x\}) = \emptyset \iff \forall y \in S, y \neq x \Rightarrow y \notin B(x,r)$

$\iff \exists r>0 : \exists! \, x \in B(x,r)$ i.e. ~~$\forall y \neq x \; y \notin B$~~ $\{x \in B(x,r)$

(2) $x \in M$ is adherent point of $S \iff \forall r>0 \; S \cap B(x,r) \neq \emptyset$

$\iff \forall r>0 \quad B(x,r)$ contains at least one point from S

$\iff (\forall r>0)(\exists s \in S) \; s \in B(x,r)$ [$s \in S$ s may be equal to x]

(3) $x \in M$ is accumulated point of $S \iff \forall r>0 \; B(x,r) \cap S \setminus \{x\} \neq \emptyset$

$\iff (\forall r>0)(\exists s \in S \setminus \{x\}) \; s \in B(x,r)$

(must include a center

accumulated point is an adherent point (not vice versa always)

$|x| < a$

$x < a$

$-x \leq a$

$x > a$

$a \leq x \leq a$

$S = (x_n) = \{1 - \frac{1}{n}\}$

0 1

$\exists x = 1$ which is accumulated point $\iff$

$\exists r>0 : \forall y \in S, y \neq 1 \Rightarrow y \notin B(1,r)$

$\exists r>0 \quad \forall y = 1 - \frac{1}{n}, y \neq 1 \Rightarrow y \notin B(1,r) \iff$

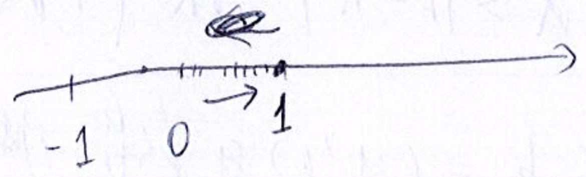
$\iff (1 - \frac{1}{n}) \notin B(1,r) = \{y : \rho(1,y) < r\}$

$\iff y \notin \{y : |y-1| < r\} \iff y \notin \{y : -r < y-1 < r\}$

$\exists r = 1 \iff \forall y \notin \{y : 1-r < y < 1+r\} = \{1 - \frac{1}{n} \; 1-r < 1 - \frac{1}{n} < 1+r\}$

$\frac{1}{n} > r \quad n > \frac{1}{r} \qquad -r < \frac{1}{n} < r \qquad \left|\frac{1}{n}\right| < r \quad \frac{1}{n} < r$

$$X_n = 1 \cancel{+} \frac{1}{n} = \left(\frac{n \cancel{+} 1}{n}\right)$$



$1 - \frac{1}{1} = 0$

$1 - \frac{1}{2} = \frac{1}{2}$

$1 - \frac{1}{3} = \frac{2}{3}$

$1 - \frac{1}{4} = \frac{3}{4}$

$1 - \frac{1}{1000\ldots} = 1$

$X \in R$ is acclum of S
if $\forall r > 0 \ \exists \ s \in S \setminus \{x\}$
such $s \in B(X, r)$

$B(x, r) \cap (S \setminus \{x\}) \neq \emptyset$

$X = 1$

$\exists - s \in B(X, r) \Leftrightarrow \{y : |x - y| < r\}$

$\forall r > 0$

$|x - y| < r \Leftrightarrow x - y < r$

$-(x - y) < r$

~~$y - x < r$~~

$x - y > -r$

$-r < x - y < r$

$\Downarrow$

$y - r < x < r + y$

$y - r < x < y + r$

~~$x - r < y < x + r$~~

$x - r < y < x + r$

Def 1 (E, ρ)

E is separable $\Leftrightarrow \exists A \subset E$ such that A is countable set and $E = \overline{A}$.

$$\Leftrightarrow \left[(\forall x \in E)(\exists (x_n) \subset A)\ x_n \to x \right] \quad (1)$$

Def 2

E is separable $\Leftrightarrow \left[\overline{(\forall x \in E)(\forall r > 0)\ B(x,r) \cap A \neq \emptyset} \right] \quad (2)$

Prove that def 1 $\Leftrightarrow$ def 2

$(1) \Rightarrow (2)$

$(\forall x \in E)(\exists (x_n) \subset A)\ x_n \to x \Leftrightarrow$

$(\forall x \in E)(\exists (x_n) \subset A) \{ (\forall \varepsilon > 0)(\exists n_0)(\forall n \geq n_0)\ \rho(x_n, x) \leq \varepsilon$

$$\begin{cases} (\forall x \in E)(\forall \varepsilon > 0)(\exists n_0)(\forall n \geq n_0) \left[\rho(x_n, x) \leq \varepsilon \right] \Rightarrow \\ \Rightarrow \begin{bmatrix} x_n \in B(x, \varepsilon) \\ x_n \in A \end{bmatrix} \Rightarrow B(x,\varepsilon) \cap A \neq \emptyset \end{cases}$$

$\Rightarrow (\forall x \in E)(\forall \varepsilon > 0)\ B(x, \varepsilon) \cap A \neq \emptyset$ $\square$ $\Rightarrow$

$\Leftarrow$ $(\forall x \in E)(\forall r > 0)\ B(x,r) \cap A \neq \emptyset \Rightarrow$ $(\forall x \in E)\ \exists (x_n) \subset A$ $x_n \to x$

$(\forall x \in E)(\forall n > 0\ r = \frac{1}{n})\ B(x, \frac{1}{n}) \cap A \neq \emptyset \Rightarrow$

$\Rightarrow (\forall x \in E)(\forall n)\ \exists\ x_n \in B(x, \frac{1}{n})$, $x_n \in A$ $\Rightarrow$ so we build $(x_n) \subset A$, $\rho(x_n, x) < \frac{1}{n} \to 0$