Backpropagation → simple case

III

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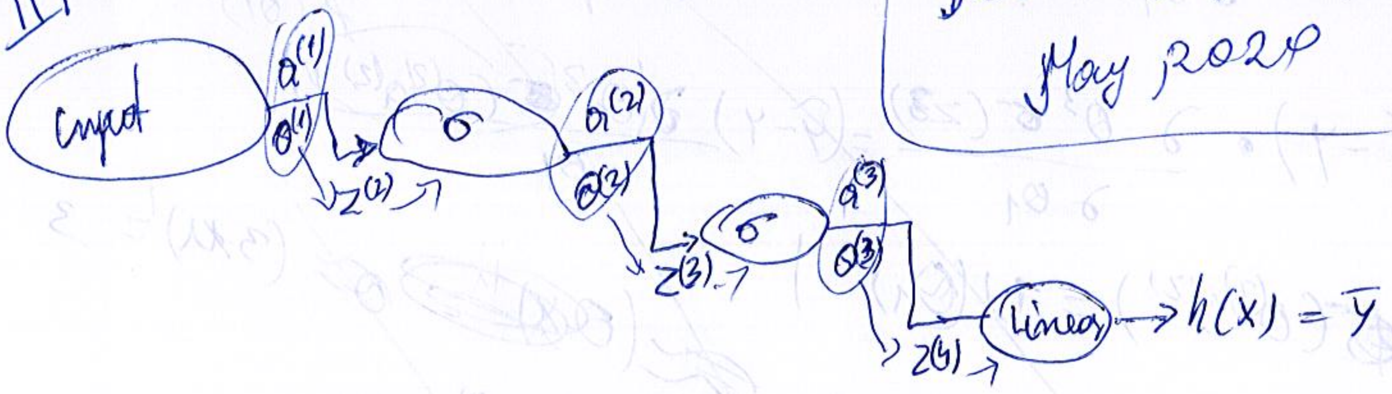


$$z^{(2)} = \theta^{1} a^{(1)}$$

$$a^{(2)} = \sigma(z^{(2)}) = \underline{\sigma(\theta^{1} a^{(1)})}$$

$$z^{(3)} = \theta^{(2)} \cdot a^{(2)} = \theta^{(2)} \sigma(z^{(2)}) = \underline{\theta^{(2)} \sigma(\theta^{(1)} a^{(1)})}$$

$$a^{(3)} = \sigma(z^{(3)}) = \sigma(\theta^{2} \sigma(z^{(2)})) = \sigma(\theta^{2} \sigma(\theta^{(1)} a^{(1)}))$$

$$z^{(4)} = \theta^{(3)} a^{(3)} = \theta^{(3)} \sigma(z^{(3)}) = \theta^{(3)} \cdot \sigma(\theta^{2} \sigma(z^{(2)})) = \underline{\theta^{(3)} \sigma(\theta^{(2)} \sigma(\theta^{(1)} a^{(1)}))}$$

$$a^{(4)} = g(z^{(4)}) = g(\theta^{(3)} a^{(3)}) = g(\theta^{(3)} \sigma^{(3)}) = \theta^{3} \sigma(\theta^{2} \sigma(z^{(2)})) =$$

$$= \theta^{3} \sigma(\theta^{2} \sigma(\theta^{(1)} a^{(1)}))$$

$$= h(x^{(i)}) = \tilde{y}^{(i)}$$

$$\frac{\partial J}{\partial \theta^{(3)}} = \frac{\partial}{\partial \theta^{(3)}} \frac{1}{2} (\tilde{y}^{(i)} - y)^2 = \frac{1}{2} \cdot 2 (\tilde{y}^{(i)} - y) \frac{\partial}{\partial \theta^{(3)}} (\underbrace{g(\theta^{(3)} a^{(3)})}_{\theta^{(3)} a^{(3)}} - y^{(i)}) =$$

$$= (\tilde{y}^{(i)} - y)\, a^{(3)}. \quad (1)$$

$$\frac{\partial J}{\partial \theta_2} = \frac{\partial}{\partial \theta_2} \frac{1}{2} (\tilde{y}^{(i)} - y)^2 = (\tilde{y}^{(i)} - y^{(i)}) \cdot \frac{\partial}{\partial \theta^{(2)}} (g(\theta^{3} a^{(3)}) - y^{(i)}) =$$

$$(\tilde{y} - y)\, \theta^{(3)} \frac{\partial a^{(3)}}{\partial \theta^{(2)}} = (\tilde{y} - y)\, \theta^{(3)} \cdot \frac{\partial \sigma(z^{(3)})}{\partial \theta^{(2)}} = (\tilde{y} - y)\, \theta^{(3)} \ldots$$

$$= (\tilde{y} - y)\, \theta^{(3)} \cdot \sigma(z^{(3)})(1 - \sigma(z)) \frac{\partial}{\partial \theta_2} z^{(3)}$$

$$= (\tilde{y} - y)\, \theta^{(3)} \cdot \sigma(\theta^{(2)} a^{(2)})(1 - \sigma(\theta^{2} a^{(2)})) \cdot \frac{\partial(\theta^{(2)} a^{(2)})}{\partial \theta^{(2)}} = (\tilde{y} - y)\, \theta^{(3)} \sigma(\theta^{(2)} a^{(2)})(1 - \sigma(\theta^{(3)})) \cdot a^{(2)}$$

$$= (\tilde{y} - y)\, \theta^{(3)} \underbrace{\sigma'(\theta^{2} a^{(2)})}_{z^{(3)}} \cdot a^{(2)}$$

$$\frac{\partial J}{\partial \theta_1} = \frac{\partial \frac{1}{2}(\hat{y}-y)^2}{\partial \theta_1} = (\hat{y}-y)\cdot\frac{\partial\, g(\cancel{z}^{(4)})}{\partial \theta_1} = (\hat{y}-y)\frac{\partial(\theta^3 a^{(3)})}{\partial \theta_1} =$$

$f(\theta_1)$

$$(\hat{y}-y)\cdot\frac{\partial\, \theta^3\sigma(z^{3})}{\partial\theta_1} = (\hat{y}-y)\frac{\partial(\theta^3\sigma(\theta^{(2)}a^{(2)}))}{\partial\theta_1}$$

$$\sigma(\theta^{(2)}a^{(2)}) = U(\theta_1)$$

$(3x)' = 3$

$$= (\hat{y}-y)\frac{\partial(\theta^{(3)}\cdot U(\theta_1))}{\partial\theta_1} = (\hat{y}-y)\,\theta^{(3)}\frac{\partial U(\theta_1)}{\partial\theta_1} =$$

$$= (\hat{y}-y)\,\theta^{(3)}\frac{\partial\sigma(\theta^{(2)}a^{(2)})}{\partial\theta_1} = \sigma(z_2(\theta_1)) = (\hat{y}-y)\,\theta^3\frac{\partial\sigma(z(\theta_1))}{\partial\theta_1} =$$

$$= (\hat{y}-y)\,\theta^3\sigma(z^{(2)}(\theta_1))(1-\sigma(z^{(2)}(\theta_1))\frac{\partial z^{(2)}(\theta^{(1)})}{\partial\theta^{(1)}} =$$

$$= (\hat{y}-y)\,\theta^3\sigma(z^{(2)})(1-\sigma(z^{(2)})\frac{\partial(\theta^{(2)}a^{(2)})}{\partial\theta^{(1)}} =$$

$$= \boxed{(\hat{y}-y)\,\theta^{(3)}\cdot\sigma(z^{(2)})(1-\sigma(z^{(2)})\cdot\theta^{(2)}}\cdot\frac{\partial(a^{(2)})}{\partial\theta^{(1)}}) =$$

$$= (\hat{y}-y)\,\theta^3\cdot\sigma'(z^{(2)})\theta^{(2)}\cdot\frac{\partial\sigma(\theta_2 a_2)}{\partial\theta_1} =$$

$$h_\theta(x) = \theta_0 + \theta_1 x \qquad J(\theta_0, \theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

$$J(0,1) \qquad \Rightarrow \underline{h_\theta(x) = x} \quad \Rightarrow J(\theta_0,\theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(x^{(i)} - y^{(i)}\right)^2$$

$$\underline{J(0,1) = \frac{1}{8}\sum_{i=1}^{4}\left(x^{(i)} - y^{(i)}\right)^2} = \frac{1}{8}\left(1^2+1^2+1^2+1^2\right) = \frac{1}{2}$$

$$\underline{h_\theta(x) = -1 + 2x} \quad \Rightarrow h_\theta(6) = -1 + 2 \times 6 = 11$$

$$\begin{bmatrix} \theta^T x^{(1)} - y^{(1)} \\ \theta^T x^{(2)} - y^{(2)} \\ \vdots \\ \theta^T x^{(m)} - y^{(m)} \end{bmatrix} = \left(\theta^T \cancel{X} - y\right)$$

$$\left(\theta^T X - y\right)^T$$

$$\begin{bmatrix} \cdot & \end{bmatrix} \cdot$$

$\Theta_0 = 5; \quad \Theta_1 = -1; \quad \Theta_2 = 0$

$\cancel{h_\theta(x) = g(5 - x_1)}$

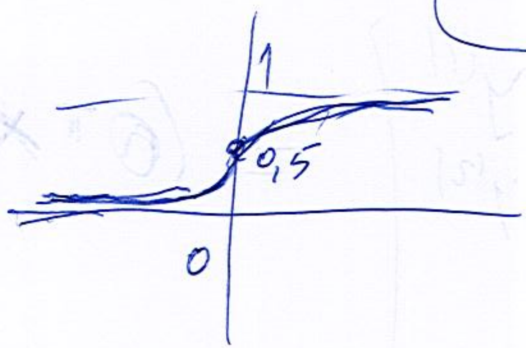
because

$h_\theta(x) = g(\Theta^T x)$ $\qquad \Theta^T x = 5 \cdot 1 - 1 x_1 + 0 \cdot x_2)$

$= 5 - x_1$

$h_\theta(x) = g(5 - x_1)$ $\boxed{= \frac{1}{1 + e^{-\Theta^T x}} = \frac{1}{1 + e^{x_1 - 5}}}$

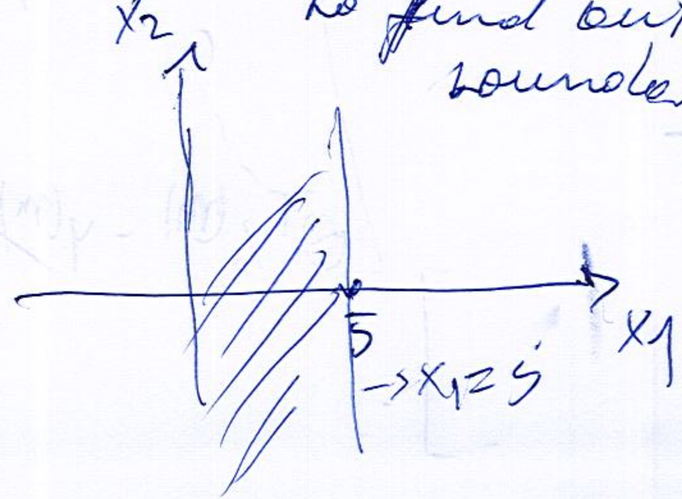
$\Theta^T x \geq 0$



$\Downarrow$

$5 - x_1 \geq 0 \iff x_1 < 5$

just for curiosity
we do not need
to find out decision
boundary



$$\boxed{\delta_j^L = \frac{\partial C}{\partial a_j^L}\sigma'(z_j^L)} \quad (1)$$

$$C = \frac{1}{2}(y_j - a_j^L)^2 \quad (3)$$

$$\boxed{\frac{\partial C}{\partial a_j^L} = (a_j^L - y_j)} \quad (4)$$

$$\boxed{\delta^L = \nabla_a C \odot \sigma'(z^L)} \quad (2)$$

$$\nabla_a C = \begin{bmatrix} \vdots \\ \frac{\partial C}{\partial a_j^L} \\ \vdots \end{bmatrix} \quad (5)$$

$$\sigma'(z^L) = \begin{bmatrix} \vdots \\ \sigma'(z_j^L) \\ \vdots \end{bmatrix}$$

$$\boxed{\nabla_a C = (a^L - y)} \quad (6)$$

$$\boxed{\delta^L = (a^L - y) \odot \sigma'(z^L)} \quad (7)$$

?

$$\nabla_a C = w^{l+1}\cdot \delta^{l+1}$$

$$\cancel{\nabla_a C = w^{l+1}\cdot \nabla_a C \odot \sigma'(z^{l+1})}$$

$$\boxed{\delta^l = (w^{l+1}\cdot \delta^{l+1}) \odot \sigma'(z^l)} \quad (8)$$

$$\boxed{\frac{\partial C}{\partial w_{jk}^l} = a_k^{l-1}\cdot \delta_j^l} = a_k^{l-1}\frac{\partial C}{\partial a_j^l}\sigma'(z_j^l) = \boxed{\underbrace{\frac{\partial C}{\partial a_j^l}\sigma'(z_j^l)}_{\delta_j^l}\cdot a_k^{l-1}}$$

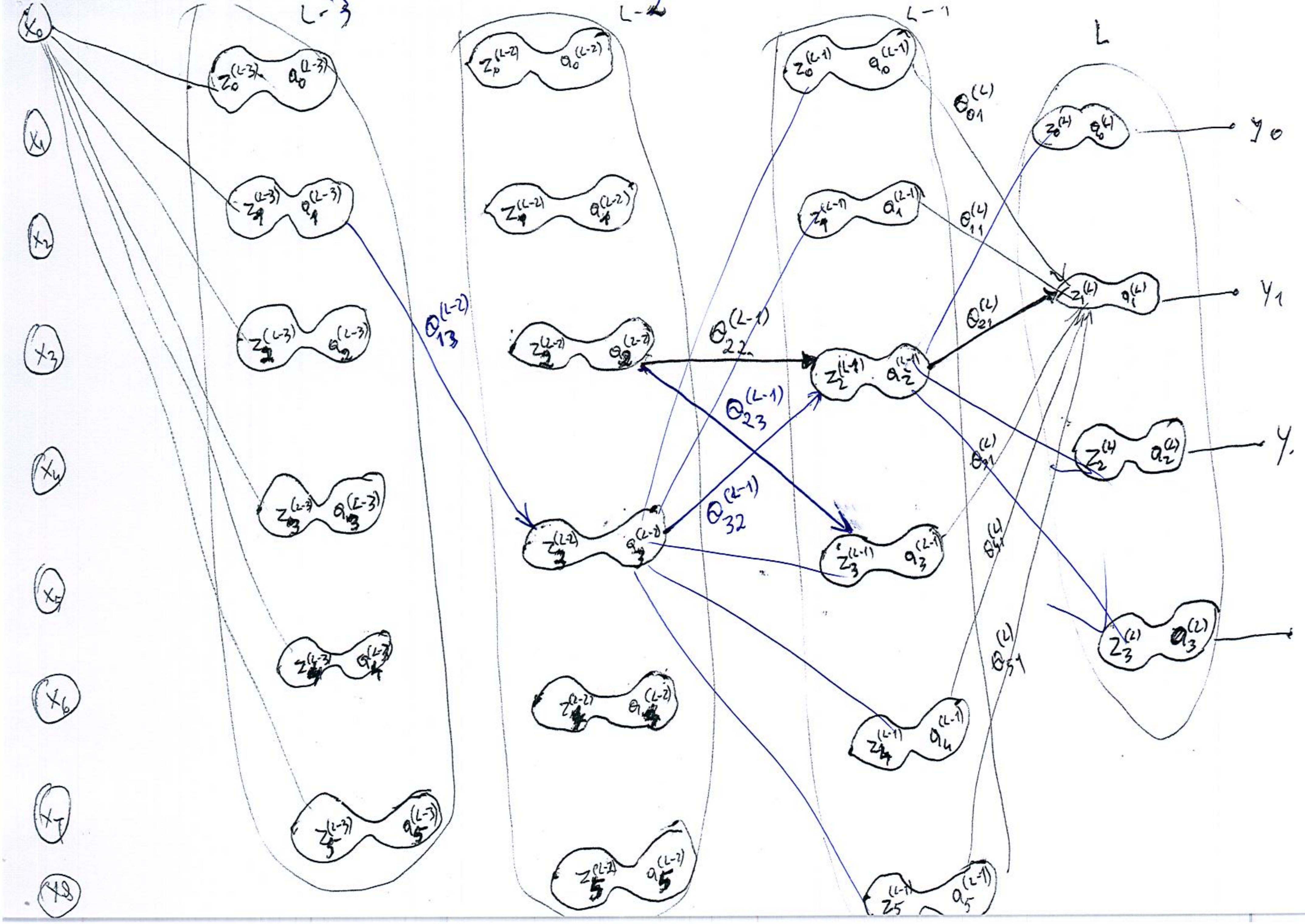
$$\frac{\partial C}{\partial w} = a_{in}\cdot \delta_{out}$$

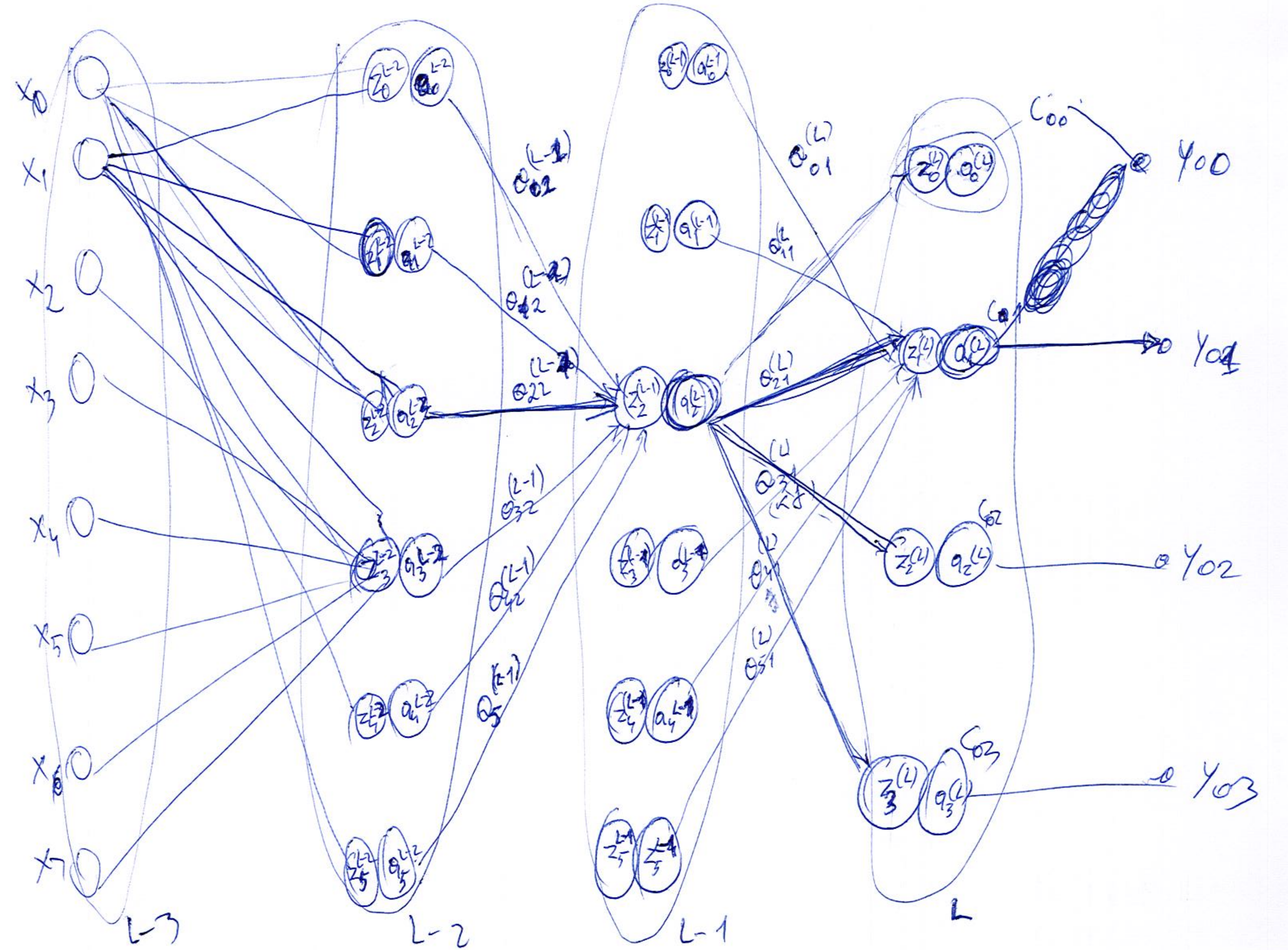
$$\bigcirc \xrightarrow{\overbrace{a_{in} \times \delta_{out}}^{\frac{\partial C}{\partial w}}} \bigcirc$$

$$\frac{\partial C}{\partial w_{jk}^l} = \underbrace{\frac{\partial C}{\partial a_j^{(l)}}\cdot \frac{\partial a_j^{(l)}}{\partial z_j^{(l)}}}_{= \delta_j^l = \frac{\partial C}{\partial z_j^l}} \cdot \frac{\partial z_j^{(l)}}{\partial w_{jk}^l}$$

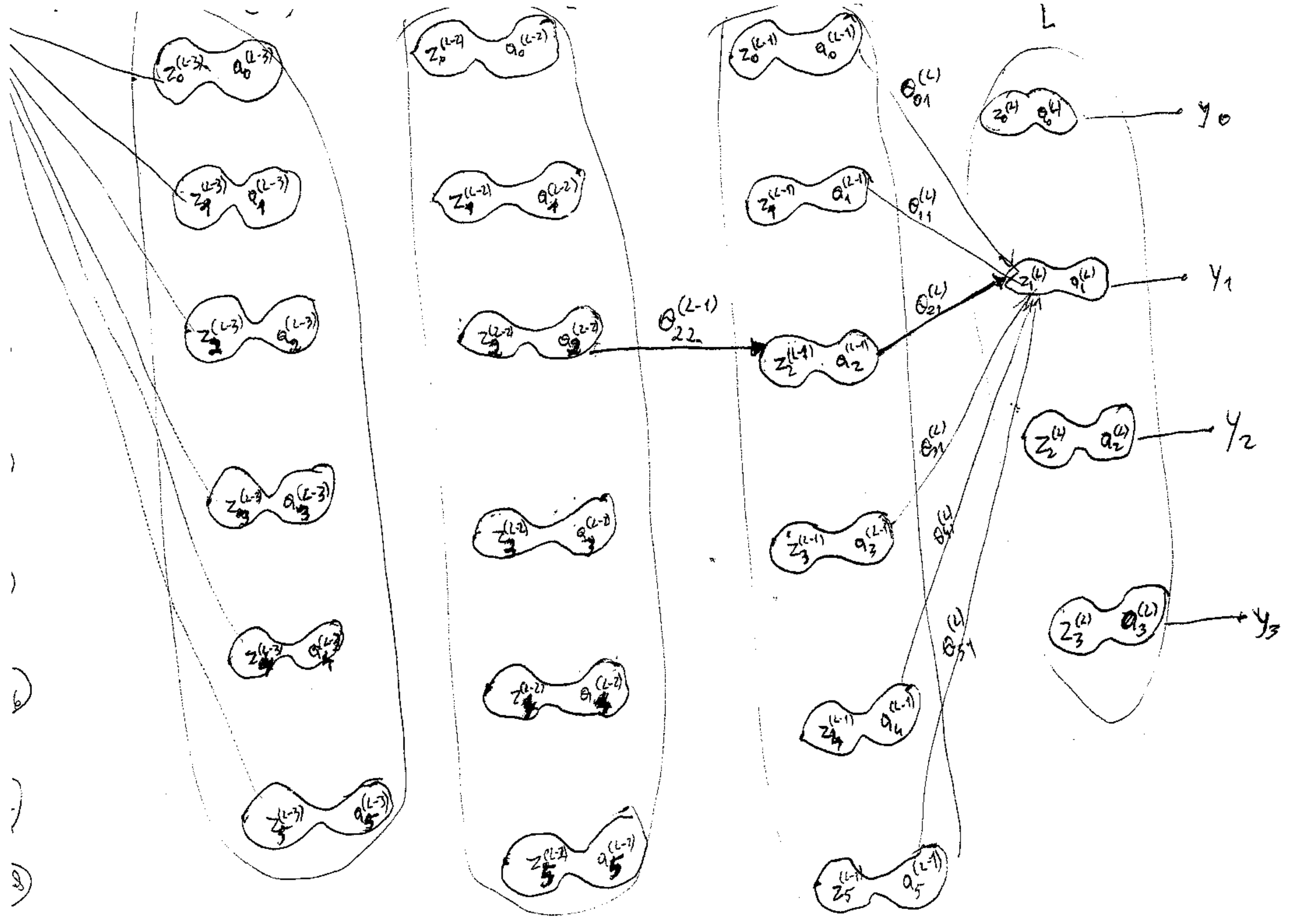
$$= \frac{\partial C}{\partial a_j^{(l)}}\cdot \sigma'(z_j^{(l)})\cdot a_k^{l-1}$$

$$\frac{\partial C}{z_j^{(l)}} = \frac{\partial C}{\partial a_j^{(l)}}\cdot \sigma'(z_j^{(l)}) \qquad dz = da_j^{(l)}\cdot \sigma'(z_j^{(l)})$$


L-3
L-2
L-1
L
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
$z_0^{(L-3)}$ $a_0^{(L-3)}$
$z_1^{(L-3)}$ $a_1^{(L-3)}$
$z_2^{(L-3)}$ $a_2^{(L-3)}$
$z_3^{(L-3)}$ $a_3^{(L-3)}$
$z_4^{(L-3)}$ $a_4^{(L-3)}$
$z_5^{(L-3)}$ $a_5^{(L-3)}$
$\theta_{13}^{(L-2)}$
$z_0^{(L-2)}$ $a_0^{(L-2)}$
$z_1^{(L-2)}$ $a_1^{(L-2)}$
$z_2^{(L-2)}$ $a_2^{(L-2)}$
$z_3^{(L-2)}$ $a_3^{(L-2)}$
$z_4^{(L-2)}$ $a_4^{(L-2)}$
$z_5^{(L-2)}$ $a_5^{(L-2)}$
$\theta_{22}^{(L-1)}$
$\theta_{23}^{(L-1)}$
$\theta_{32}^{(L-1)}$
$z_0^{(L-1)}$ $a_0^{(L-1)}$
$z_1^{(L-1)}$ $a_1^{(L-1)}$
$z_2^{(L-1)}$ $a_2^{(L-1)}$
$z_3^{(L-1)}$ $a_3^{(L-1)}$
$z_4^{(L-1)}$ $a_4^{(L-1)}$
$z_5^{(L-1)}$ $a_5^{(L-1)}$
$\theta_{01}^{(L)}$
$\theta_{11}^{(L)}$
$\theta_{21}^{(L)}$
$\theta_{31}^{(L)}$
$\theta_{41}^{(L)}$
$\theta_{51}^{(L)}$
$z_0^{(L)}$ $a_0^{(L)}$
$z_1^{(L)}$ $a_1^{(L)}$
$z_2^{(L)}$ $a_2^{(L)}$
$z_3^{(L)}$ $a_3^{(L)}$
y_0
y_1
$y_.$



x0
x1
x2
x3
x4
x5
x7
L-3
L-2
L-1
L
y00
y02
y03
c00
c02
c03


L
$z_0^{(L-3)}$ $a_0^{(L-3)}$
$z_1^{(L-3)}$ $a_1^{(L-3)}$
$z_2^{(L-3)}$ $a_2^{(L-3)}$
$z_3^{(L-3)}$ $a_3^{(L-3)}$
$z_4^{(L-3)}$ $a_4^{(L-3)}$
$z_5^{(L-3)}$ $a_5^{(L-3)}$
$z_0^{(L-2)}$ $a_0^{(L-2)}$
$z_1^{(L-2)}$ $a_1^{(L-2)}$
$z_2^{(L-2)}$ $a_2^{(L-2)}$
$z_3^{(L-2)}$ $a_3^{(L-2)}$
$z_4^{(L-2)}$ $a_4^{(L-2)}$
$z_5^{(L-2)}$ $a_5^{(L-2)}$
$\Theta_{22}^{(L-1)}$
$z_0^{(L-1)}$ $a_0^{(L-1)}$
$z_1^{(L-1)}$ $a_1^{(L-1)}$
$z_2^{(L-1)}$ $a_2^{(L-1)}$
$z_3^{(L-1)}$ $a_3^{(L-1)}$
$z_4^{(L-1)}$ $a_4^{(L-1)}$
$z_5^{(L-1)}$ $a_5^{(L-1)}$
$\Theta_{01}^{(L)}$
$\Theta_{11}^{(L)}$
$\Theta_{21}^{(L)}$
$\Theta_{31}^{(L)}$
$\Theta_{41}^{(L)}$
$\Theta_{51}^{(L)}$
$z_0^{(L)}$ $a_0^{(L)}$
$z_1^{(L)}$ $a_1^{(L)}$
$z_2^{(L)}$ $a_2^{(L)}$
$z_3^{(L)}$ $a_3^{(L)}$
y_0
y_1
y_2
y_3

$$\frac{\partial y}{\partial \theta_1} = \frac{\partial \frac{1}{2}\left(g(\theta^{(3)} a^{(3)}) - y\right)^2}{\partial \theta_1} = \left(g(\theta^3 a^{(3)}) - y\right) \frac{\partial g(\theta^{(3)} a^{(3)})}{\partial \theta_1} =$$

$$= (\hat{y} - y)\, \theta^{(3)} \frac{\partial a^{(3)}}{\partial \theta_1} = (\hat{y} - y) \cdot \theta^3 \frac{\partial \sigma(z^{(3)})}{\partial \theta_1} =$$

$$= (\hat{y} - y) \cdot \theta^3 \cdot \sigma(z^{(3)})(1 - \sigma(z^{(3)})) \frac{\partial z^{(3)}}{\partial \theta_1} =$$

$$= (\hat{y} - y) \cdot \theta^3 \cdot \sigma'(z^{(3)}) \cdot \frac{\partial(\theta^2 a^{(2)})}{\partial \theta^{(1)}} = (\hat{y} - y) \cdot \theta^3 \cdot \sigma'(z^{(3)}) \cdot \theta^2 \frac{\partial a^{(2)}}{\partial \theta^{(1)}}$$

$$= (\hat{y} - y) \cdot \theta^{(3)} \cdot \sigma'(z^{(3)}) \cdot \theta^2 \frac{\partial \sigma(z^{(2)})}{\partial \theta_1} = (\hat{y} - y) \cdot \theta^{(3)} \cdot \sigma'(z^{(3)}) \cdot \theta^2$$

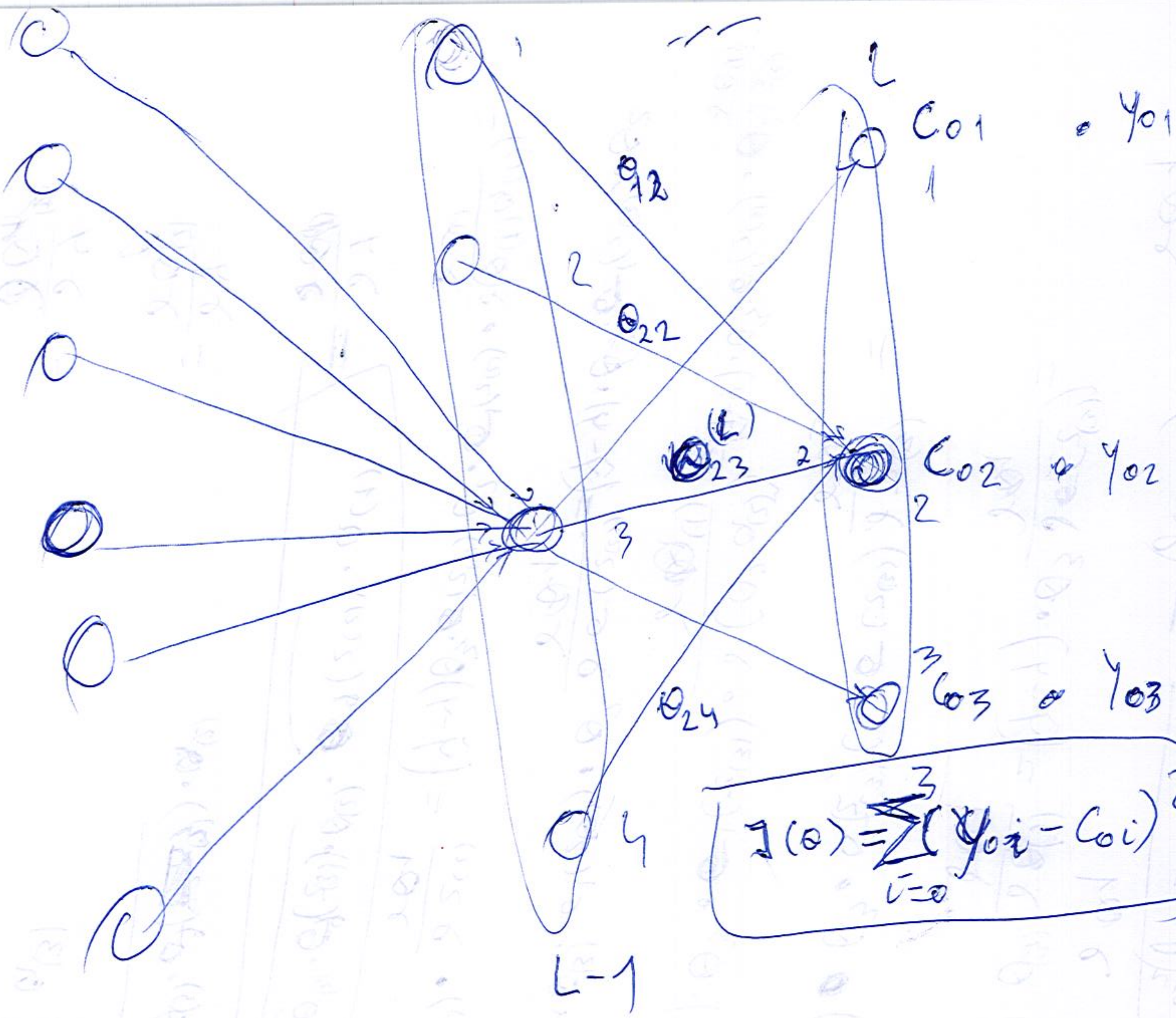
$$\cdot\, \sigma'(z^{(2)}) \cdot \frac{\partial z^{(2)}}{\partial \theta_1} = (\hat{y} - y)\theta^3 \cdot \sigma'(z^{(3)}) \cdot \theta^2 \cdot \sigma'(z^{(2)}) \cdot \frac{\partial(\theta^{(1)} a^{(1)})}{\partial \theta^1} =$$

$$\boxed{= (\hat{y} - y) \cdot \theta^{(3)} \cdot \sigma'(z^{(3)}) \cdot \theta^{(2)} \cdot \sigma'(z^{(2)}) \cdot a^{(1)}} = \frac{\partial y}{\partial \theta^{(1)}}$$

$$(\hat{y} - y) \cdot \theta^{(3)} \cdot \sigma'(z^{(3)}) \cdot a^{(2)} = \frac{\partial y}{\partial \theta^{(2)}}$$

$$(\hat{y} - y)\, a^{(3)} = \frac{\partial y}{\partial \theta^{(3)}}$$

$$\left.\begin{aligned}
\frac{\partial J}{\partial \theta^{(3)}} &= \delta^{(3)} \cdot a^{(3)} \\
\frac{\partial J}{\partial \theta^{(2)}} &= \overbrace{\delta\, \theta^3 \sigma'(z^{(3)})}^{\delta^{(2)}}\, a^{(2)} \\
\frac{\partial J}{\partial \theta^{(1)}} &= \delta^{(2)} \theta^{(2)} \sigma'(z^{(2)}) \cdot a^{(1)}
\end{aligned}\right\}
\quad
\begin{aligned}
\delta^{(3)} &= \hat{y} - y \\
\delta^{(2)} &= \delta^{(3)} \cdot \theta^{(3)} \cdot \sigma'(z^{(3)}) \\
\delta^{(1)} &= \delta^{(2)} \cdot \theta^{(2)} \cdot \sigma'(z^{(2)})
\end{aligned}$$



$$\boxed{J(\theta) = \sum_{i=0}^{3} (y_{oi} - C_{oi})^2}$$

$$z_2^{(L)} = \sum_{k=1}^{l_{L-1}=4} \theta_{2k}^{l} a^{l-1}$$

$$C_{o2} = a_2^{l}(z_2^{(2)})$$

$$\frac{\partial J(\theta)}{\theta_{23}} =$$