1) TODIAN MISHTAKU Aug 2022

I $E \to$ experiment $\Omega \to$ sample space = set of all possible outcomes

$\mathcal{S} = \{ s : s \subset \Omega \}$ i.e. $s \in \mathcal{S} \iff s \subset \Omega$; s is called EVENT (we can use ω instead s)

Example: Roll a six-sided die $\to$ Experiment

$\Omega = \{1, 2, 3, 4, 5, 6\}$; $\mathcal{S} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$
$\emptyset, \Omega \in \mathcal{S}$

~~$\mathcal{S} = \{\emptyset, \Omega, \{1\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,3,4\} \ldots\}$~~

PROBABILITY LAW assigns a numerical value $P(s)$ to any event $s \in \mathcal{S}$ (i.e. $s \subset \Omega$)
(or let say to any event $\omega_s \subset \Omega$

$P : \mathcal{S} \to R^n$ (or $P : \mathcal{S} \to [0,1]$, ~~...~~

I.E $P(\,)$ is defined in $\mathcal{S}$

$\boxed{P(\,) \text{ is a function of events of } \Omega}$

$P(\omega) = 1/6$ if $\omega = \{1\}$ and each of events is equally likely

$P(\{1\}) = \frac{1}{6}$

While PMF $P_X(\,)$ is a function of a random variable $X : \Omega \to M$ (where M is a measurable set)

$X(\omega) = X(\{1\}) = 1 \in M$ (with $\omega = 1$)

$X(\{2\}) = 2 \in M$

$P_X(\,) : X \to [0,1]$

$P_X[X(\{1\})] =$ smth

(2)

$X : \Omega \longrightarrow M$ (M is a measurable set)

Experiment two independent rolls of 6-sided unbiased die

$\Omega = \{\{1\}, \{2\} \dots \{6\}, \{1,2\}, \{1,3\}, \dots \{2,4\} \dots \{6,6\}\}$

$\omega = \{2\} \in \Omega$

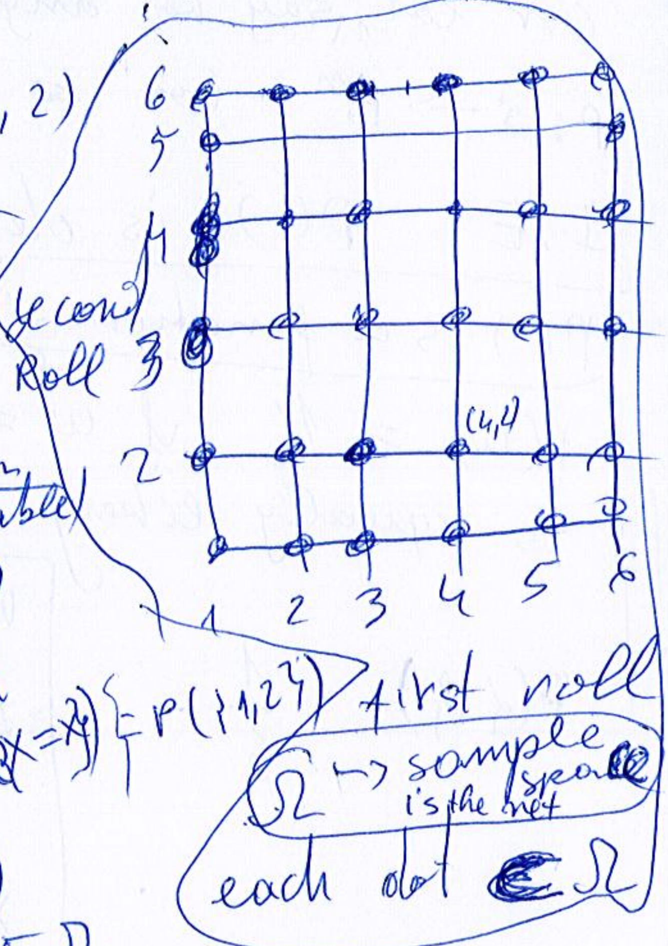
$P(\omega = \{2\}) = P(\omega) = P(\{2\}) = \frac{1}{36}$

$X(\omega) = ?$ ~~$X(\{2\})$~~

Def $\begin{cases} X : \Omega \to R^2 \\ X(\{a,b\}) = (a,b) \text{ where } a,b \in \{1,2,\dots 6\} \end{cases}$

ex. $X(\{3,2\}) = (3,2)$

PMF P_X is defined as of function of X random variable



$P_X(X = (1,2)) = P(\omega = \{1,2\})$

~~$P_X(X = x) = p$~~

~~$P_X(x) = P(\{X = x\})$~~ shortcut // $P(X = x) = P(\{1,2\})$

$P_X(x) = P(\omega : X(\omega) = x) = P(\omega)$
$x \in Val(X)$ $\omega \in \Omega$

PMF $P_X : X \to M$

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$P_X(X=x) = \mathbb{P}(\{X=x\})$ $\quad \{X=x\} = \{\omega \in \Omega : X(\Omega) = x\}$

E.X. X = sum of outcomes of ~~two~~ subsequent rolls

$$\{X=8\} = \boxed{\{\{4,4\}, \{2,6\}, \{6,2\}, \{3,5\}, \{5,3\}\}} = A$$

$$\underline{\underline{A \subset \Omega}}$$

$$P_X(X=8) = P(\{X=x\}) \overset{\text{shortcut}}{=\!=} P(X=8) = P(A) = \frac{5}{36}$$

$P_X(X=8)$: function of random variable

$P(\{X=x\})$: function of events, function of Ω

$A, B \subset \Omega$ def of conditional prob

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}} \Rightarrow P(A \cap B) = P(A|B) \cdot P(B) \quad (1)$$

$P(B) \neq 0$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A) \cdot P(A) \quad (2)$$

$P(A) \neq 0$

$(1) \wedge (2) \Leftrightarrow P(A|B)P(B) = \mathbb{P}(B|A)P(A) \Leftrightarrow$

$$\boxed{P(A|B) = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{P(B)}}$$

Bay's formula

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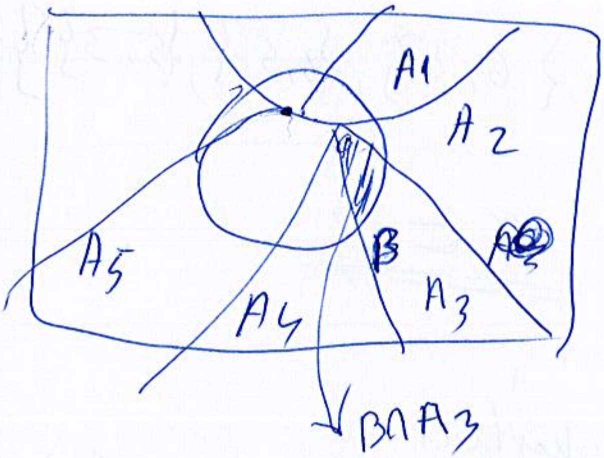
We have $p(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}$

$A_{1} \dots m$



Ω

$\begin{cases} A_i \cap A_j = \emptyset & \text{for any } 0 \le i, j \le m,\ i \ne j \\ \bigcup_{i=1}^{m} A_i = \Omega \end{cases}$

I.E $\{A_1, \dots A_m\}$ is a full partition of Ω

Then, $\Omega = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_m$

$\Omega = \bigcup_{i=1}^{m} A_i$

$\boxed{B = B \cap \Omega = B \cap \left(\bigcup_{i=1}^{m} A_i\right) = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \dots \cup (B \cap A_m)}$

$\boxed{B = \bigcup_{i=1}^{m} (B \cap A_i)}$ $\quad \forall\, i \ne j,\ i,j \in \{1..m\}$ $[A_i \cap A_j = \emptyset] \Rightarrow [(B \cap A_i) \cap (B \cap A_j) = \emptyset]$

disjoint $\quad$ disjoint

$\cancel{\bigcap_{i=1}^{m}(B \cap A_i) = \emptyset}$

Hence, from second axiom of probability

$\boxed{P\left(\bigcup_{i=1}^{m} (B \cap A_i)\right) = \sum_{i=1}^{m} P(B \cap A_i)}$

$P(B) = P\left(\bigcup_{i=1}^{m} (B \cap A_i)\right) = P(B \cap A_1) + P(B \cap A_2) + \dots + P(B \cap A$

$$P(B) = P(B \cap A_1) + \dots + P(B \cap A_n) \quad (\mathrm{I})$$

$$P(B \cap A_i) = P(B|A_i) \cdot P(A_i) \quad (\mathrm{II})$$

From conditional probability

$(\mathrm{I}) \wedge (\mathrm{II})$ $\boxed{P(B) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}$

~~Theo~~ Theorem of TOTAL PROBABILITY

BAY'S FORMULA (in a different version; using partition)

$P(\cancel{B|A}) = \cancel{\frac{P(A|B)\,P(B)}{P(A)}}$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} ;$$

$$\boxed{P(A|B) = \frac{P(B|A) \cdot P(A)}{\sum_{i=1}^{n} P(B|A_i) \cdot P(A_i)}}$$

A_i a full partition of B

Baye's rule

Def 1:

Events A, B are indipendent $\Longleftrightarrow P(A\cap B)=P(A)\cdot P(B)$

Theorem: $\left\{\begin{array}{l}\text{Events } A, B \text{ are independent} \\ P(B)>0\end{array}\right\} \Leftrightarrow P(A|B)=P(A)$

Why the condition $P(B)>0$?

From Conditional probab: $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$

Thus, if $P(B)=0 \Rightarrow P(A|B)$ is undefined.

Note! Disjoint $A,B \neq$ A and B are independent

A and B are disjoint $\Leftrightarrow A\cap B=\phi \Rightarrow \boxed{P(A\cap B)=0}$

IF $P(A)>0 \wedge P(B)>0 \Rightarrow \boxed{P(A)\cdot P(B)\neq 0}$

$\left\{\begin{array}{l}P(A)>0\\ P(B)>0\\ A\cap B=\phi\end{array}\right\} \Rightarrow \boxed{P(A\cap B)\neq P(A)\cdot P(B)}$

$\Updownarrow$

$\boxed{A \text{ and } B \text{ are NOT independent}}$

$\boxed{\left\{\begin{array}{l}P(A)>0\\ P(B)>0\\ A\cap B=\phi\end{array}\right\} \Rightarrow \boxed{A, B \text{ are dependent}}}$

not independent

Example $B=A^c \Rightarrow A\cap A^c=\phi$ $\quad P(A\cap B)=0$

and if $P(A), P(B)>0$

$\boxed{A \text{ and } A^c \text{ are dependent}}$

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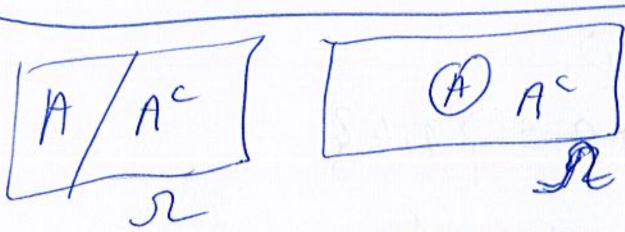
if $\begin{cases} P(A) > 0 \\ P(A^c) > 0 \end{cases} \Rightarrow \boxed{A, A^c \text{ are NOT independent}}$

$\Updownarrow$

$\boxed{A, A^c \text{ are dependent}}$

$\Updownarrow$

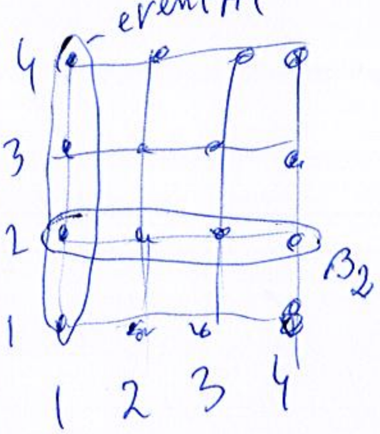
$\boxed{0 < P(A) < 1 \Rightarrow \boxed{A \text{ and } A^c \text{ are NOT independent}}}$



Example 1 Two successive rolls of fair 4-sided die

$A_i = \{\text{first roll} = i\}$ $B_j = \{\text{second roll} = j\}$

If all outcomes are equally likely, are A_i and B_j independent?



$|\Omega| = 16$ $\Omega = \{(1,1), (1,2) \ldots (4,4)\}$

$A_1 = \{(1,1); (1,2), (1,3), (1,4)\}$ $|A_1| = 4$

$P(A_1) = \frac{4}{16} = \frac{1}{4}$ (for any fixed i)

$P(B_2) = 4/16 = \frac{1}{4}$

$P(A_i \cap B_j) = P(\text{First roll is } i \text{ and second roll is } j)$
$= P(\text{the outcome is } (i,j))$

$P(A_i \cap B_j) = \frac{1}{16}$ for (concrete) fixed i, j

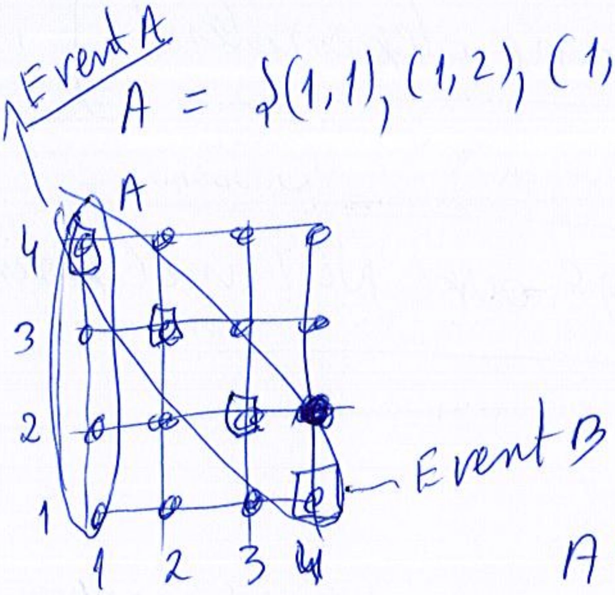
$P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} = P(A_i \cap A_j) \Leftrightarrow A_i, A_j$ are independent

(8)

b) Are the events

$A = \{ \text{1st roll is } 1 \}$ $\quad B = \{ \text{sum of the 2 rolls is } 5 \}$

$A = \{(1,1), (1,2), (1,3), (1,4)\}$



$B = \{(1,4), (2,3), (3,2), (4,1)\}$

$|A| = 4 \quad |B| = 4$

$A \cap B = \{1,4\}$

$|A \cap B| = \frac{1}{16}$

$P(A) = \frac{4}{16} = \frac{1}{4} \quad P(B) = \frac{4}{16} = \frac{1}{4} \quad P(A \cap B) = \frac{1}{16}$

$\boxed{P(A) \cdot P(B) = P(A \cap B)} \Rightarrow$ A, B are independent

CONDITIONAL INDEPENDENCE

Def:

Events A and B are independent given C

$\iff P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$ (1)

We say that A and B are (conditionally) independent

Theorem: $P(C) > 0$

Event A, B are conditionally independent given C)

$\iff P(A \mid C) = P(A \mid B \cap C)$

$\boxed{P(A \mid B \cap C) = P(A \mid C)}$

Proof: $P(A \cap B \cap C) = P(A \cap \underbrace{(B \cap C)}_{\text{event } Z}) = P(A \mid B \cap C) \cdot P(B \cap C)$

$= P(A \mid B \cap C) \cdot \underbrace{P(B \mid C) \cdot P(C)}_{P(B \cap C)}$

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \mid B \cap C) \cdot P(B \mid C) \cdot \cancel{P(C)}}{\cancel{P(C)}}$$

$P(C) > 0$

$$\boxed{P(A \cap B \mid C) = P(A \mid B \cap C) \cdot P(B \mid C)} \quad (2)$$

→ This is just a derivation of conditionality theorem for events A and $B \cap C$; and B and C

~~If A, B are cond. independent~~ ~~$P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$~~

From (1) and (2) ⟹ $\boxed{P(A \mid B \cap C) = P(A \mid C)}$ □ ∴

If $P(C) = 0$, then $P(A \cap B \mid C)$ is undefined from $\frac{P(A \cap B \cap C)}{P(C)}$

Analog r.v.

(Def): X, Y ind given A $\overset{\text{Def}}{\iff}$ $p_{X,Y \mid A}(x, y) = p_{X \mid A}(x) \cdot p_{Y \mid A}(y)$

Theorem: X, Y r.v. ind given A $\Rightarrow p_{X \mid Y, A}(x \mid y) = p_{X \mid A}(x)$ $\forall y$ $p_{Y \mid A}(y) > 0$

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Def

A, B conditionally independent (on C) $\Leftrightarrow$

$$\Leftrightarrow \quad P(A \cap B | C) = P(A|C) \cdot P(C)$$

Example (Ordinary) Independence ($P(A \cap B) = P(A)P(B)$)

does not imply conditional independence

$\big(P(A \cap B | C) = P(A|C) \cdot P(B|C)$

Consider two independent fair coin tosses in which all four possible outcomes are equally likely:

$$\Omega = \{HH, HT, TH, TT\}$$

$H_1 = \{\text{first toss is } H\}$ $H_2 = \{\text{2nd toss is } H\}$

$D = \{\text{the two tosses have different outcomes}\}$

H_1, H_2 are (ordinary) independent ← This is given from the task

$$P(H_1 \cap H_2 | D) \overset{?}{=} P(H_1|D) \cdot P(H_2|D)$$

$P(H_1|D) = ?$

$H_1 = \{HH, HT\}$; $H_2 = \{TH, HH\}$

$D = \{TH, HT\}$ $H_1 \cap D = \{HT\}$

$H_2 \cap D = \{TH\}$

$H_1 \cap H_2 = \{HH\}$

$$P(H_1|D) = \frac{P(H_1 \cap D)}{P(D)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$P(H_2|D) = \frac{P(H_2 \cap D)}{P(D)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$P(H_1 \cap H_2 | D) = 1/4$

$\frac{1}{4} = P(H_1|D) \cdot P(H_2|D) \neq P(H_1 \cap H_2 | D) = \frac{1}{2}$

(11) $P(H_1 \cap H_2 | D) = \frac{P(H_1 \cap H_2 \cap D)}{P(D)} = \frac{0}{P(D)\ 1/2} = \frac{0}{} = 0$ $H_1 \cap H_2 = \{HH\}$ $D = \{HT, TH\}$ $(H_1 \cap H_2) \cap D = \emptyset$

In the previous example we showed that although H_1, H_2 are (ordinary) independent i.e $P(H_1 \cap H_2) = P(H_1) \cdot P(H_2)$, they are not conditionally independent because

$$P(H_1|D) \cdot P(H_2|D) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq 0 = P(H_1 \cap H_2 | D)$$

COROLLARY

A, B (ordinary) independent events
$P(C) > 0$
$P(A|C) > 0$
$P(B|C) >$
$A \cap B \cap C = \emptyset$

$\Downarrow$

$\boxed{A, B \text{ are not conditionally independent}}$

because $0 > P(A|C)P(B|C) \neq P(A \cap B|C) = \frac{0}{a} = 0$

$$P(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{0}{P(C) \neq 0} = 0$$

$$0 = P(A \cap B | C) \neq P(A|C) \cdot P(B|C) \neq 0$$

(12) Conditionally independent, but not (ordinary) independent!

Example: Event $B = \{$picking up blue coin$\}$

we have ~~two~~ biased coins

$B^c = \{$picking up red coin$\}$

given

$$P(B) = P(B^c) = \frac{1}{2}$$

After picking up randomly blue or red coin we proceed with two independent tosses

Probability to have head ~~...~~ when blue coin is used is 0,99; in case of red 0,01

$P(H_i|B) = 0,99 \quad i=1,2 \quad | \quad P(H_i|B^c) = 0,01 \quad i=1,2$

Solution

$H_1 = \{$First toss outcome is $H\} = \{HT, HH\}$

$H_2 = \{$second toss outcome is $H\} = \{TH, HH\}$

$\Omega = \{TH, TT, HT, HH\}$

$H_1 \cap H_2 = \{HH\} \quad P(H_1 \cap H_2) = \frac{1}{4}$

$$\boxed{P(H_1 \cap H_2 | B) = P(H_1|B) \cdot P(H_2|B) = 0,99^2}$$

$\Rightarrow$ conditionally independent of H_1, H_2 given B

$P(H_1|B) = \frac{P(\ldots \cap B)}{\ldots} = 0,99$; $P(H_2|B) = 0,99$; $P(B) = \frac{1}{2}$ $P(B^c) = \frac{1}{2}$

$P(H_1 \cap H_2) \overset{?}{=} P(H_1) \cdot P(H_2)$ | $P(H_1) = P(H_1|B) \cdot P(B) + P(H_1|B^c) \cdot P(B^c)$

$P(H_2) = P(H_2|B) \cdot P(B) + P(H_2|B^c) \cdot P(B^c)$

4/3

① $P(H_1) = P(H_1|B)\cdot P(B) + P(H_1|B^c)\cdot P(B^c) = 0{,}99\cdot\frac{1}{2} + 0{,}01\cdot\frac{1}{2} = \frac{1}{2}$

② $P(H_2) = P(H_2|B)\cdot P(B) + P(H_2|B^c)\cdot P(B^c) = 0{,}99\cdot\frac{1}{2} + 0{,}01\cdot\frac{1}{2} = \frac{1}{2}$

③ $P(H_1\cap H_2) = P(H_1\cap H_2|B)\cdot P(B) + P(H_1\cap H_2|B^c)\cdot P(B^c)$

$$= 0{,}99^2\cdot\frac{1}{2} + 0{,}01^2\cdot\frac{1}{2} \approx \frac{1}{2}$$

(Recall
④ $P(H_1\cap H_2|B) = P(H_1|B)\cdot P(H_2|B) = 0{,}99\cdot 0{,}99$

$\cancel{P(H_1\cap H_2|B^c) = P(H_1|B^c)\cdot P(H_2|B^c)}$

⑤ $P(H_1\cap H_2|B^c) = P(H_1|B^c)\cdot P(H_2|B^c) = 0{,}01\cdot 0{,}01$

Note:
For ①, ②, ③ we exploit Theorem of Total Probability
For ④, ⑤ we exploit conditional indepence

$P(H_1\cap H_2) \approx \frac{1}{2} \neq \frac{1}{4} = P(H_1)\cdot P(H_2)$

$P(H_1\cap H_2) \neq P(H_1)\cdot P(H_2) \Rightarrow H_1$ and H_2 are not independent although H_1 and H_2 are (conditionally) independent given B.

(14)

Independence for a set of events (collection)

$A_1, \dots, A_n$

Definition of Independence for several Events

Events $A_1, A_2, \dots, A_n$ are independent

$\iff P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$ for any $S \subset \{1, 2, \dots, n\}$ (for any subset S of $\{1, \dots, n\}$)

Pairwise independence is not enough for independence of the entire collection $\{A_i\}_{i=1}^{n}$

Example

$\Omega = \{TT, TH, HT, HH\}$

$H_1 = \{1^{st}$ toss is head$\} = \{HT, HH\}$

$H_2 = \{2^{st}$ toss is head$\} = \{TH, HH\}$

$D = \{$ the two tosses have different outcomes

$D = \{HT, TH\}$

$(H_1 \cap H_2) = \{HH\}$ $\quad H_1 \cap D = \{HT\}$ $\quad H_2 \cap D = \{TH\}$

$H_1 \cap H_2 \cap D = \emptyset$

$P(H_1 \cap H_2) = \frac{1}{4} = P(H_1) \cdot P(H_2) = \frac{1}{2} \cdot \frac{1}{2} \iff H_1, H_2$ are ind

$P(H_1 \cap D) = \frac{1}{4} = P(H_1) \cdot P(D) = \frac{1}{2} \cdot \frac{1}{2} \iff H_1, D$ are ind

$P(H_2 \cap D) = \frac{1}{4} = P(H_2) \cdot P(D) = \frac{1}{2} \cdot \frac{1}{2} \iff H_2, D$ are ind

$P(H_1 \cap H_2 \cap D) = 0 \neq P(H_1) \cdot P(H_2) \cdot P(D) = \frac{1}{8} \implies H_1, H_2, D$ are NOT ind

(15)

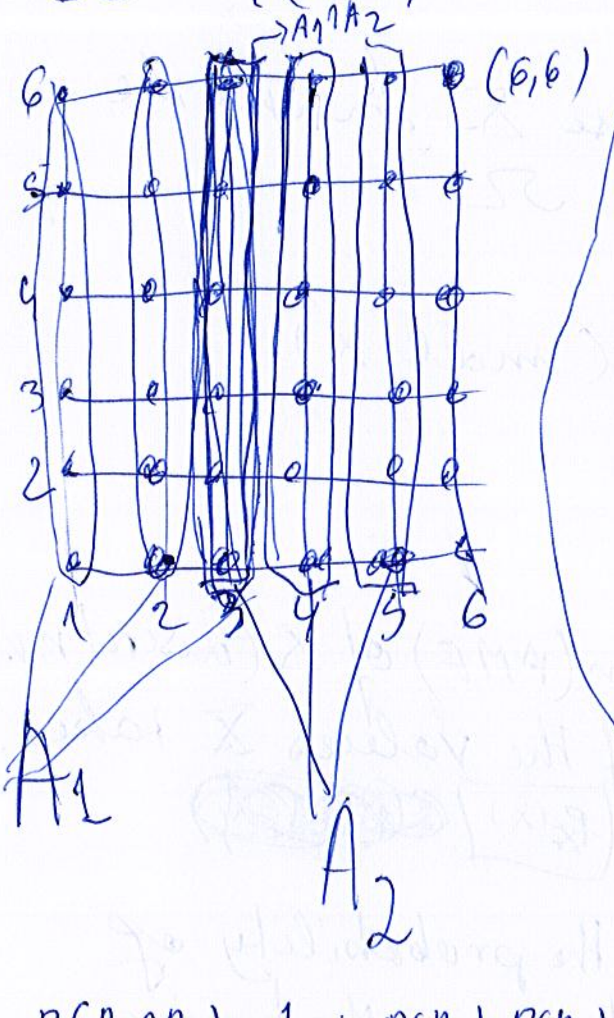
$$\boxed{\text{The equality } P(A_1\cap A_2\cap A_3) = P(A_1)\cdot P(A_2)\cdot P(A_3) \text{ is NOT enough for Independence}}$$

Example: $A_1 = \{1^{st} \text{ roll is } 1, 2, 3\}$

$A_2 = \{1^{st} \text{ roll is } 3, 4, 5\}$

$A_3 = \{\text{the sum of the two rolls is } 9\}$

$\Omega = \{(1,1), (1,2) \dots (6,6)\}$



$A_1\cap A_2\cap A_3 = A_1\cap(A_2\cap A_3) = \{(3,6)\} \Rightarrow P(A_1\cap A_2\cap A_3) = \frac{1}{36}$

$P(A_1)\cdot P(A_2)\cdot P(A_3) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{36} = \frac{1}{36}$

$P(A_1\cap A_2) = P(\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\})$

$P(A_1\cap A_2) = \frac{6}{36}$

$P(A_1\cap A_3) = P(\{3,6\}, \cancel{\{4,5\}}, \cancel{\{5,\dots\}})$

$= P(\{3,6\}) = \frac{1}{36}$

$P(A_1) = P(A_2) = \frac{18}{36} = \frac{1}{2}$

$P(A_3) = P(\{(3,6),(4,5),(5,4),(6,3)\}) = \frac{4}{36}$

$P(A_2\cap A_3) = P(\{(3,6),(4,5),(5,4)\})$

$= \frac{3}{36} = \frac{1}{12}$

$P(A_1\cap A_2) = \frac{1}{6} \neq P(A_1)\cdot P(A_2) = \frac{1}{2}\cdot\frac{1}{2}$

$P(A_1\cap A_3) = \frac{1}{36} \neq P(A_1)\cdot P(A_3) = \frac{1}{2}\cdot\frac{4}{36} = \frac{1}{9}$

$\boxed{P(A_2\cap A_3) = \frac{1}{12} \neq P(A_2)\cdot P(A_3) = \frac{1}{2}\cdot\frac{1}{12}}$

(16) Recall

P is a function of $\mathbb{S} = \{ s : s \subset \Omega \}$

$P : \mathbb{S} \longrightarrow [0,1]$

~~Discrete~~

Random variable X is of function of Ω

$X : \Omega \longrightarrow M$ (M meassurable set)

$\Omega \to$ set of all possible outcomes

In case of discrete random variables

PMF $P_X(x)$ assigns a numerical value to each of $x \in Val(X)$ $\quad P_X(x) = P(X = x)$

$\boxed{\sum_{x \in Val(X)} P_X(x) = 1}$ Why? Because X maps the whole Ω

$\parallel$

$\sum_X P_X(x) = 1$ $\quad \boxed{\sum_x P_X(x) = 1}$ (small x)

(Differently saying...)

~~Probability mass (PMF)~~

The Probabilit Mass Function (PMF) of X (discret r.v.) Captures the probabilities of the values X takes. This function is denoted $\boxed{P_X(x)}$ ~~[illegible]~~

$P_X(x) = P(\{X = x\})$ i.e. $P_X(x)$ is the probability of event $\{X=x\}$ consisting to all outcomes that give rise to a value of X equal to x.

~~[illegible]~~

(17)

$p_X(x) = \mathbb{P}(\{X = x\})$, X a discrete random variable

$\{X = x\} = \{\omega \in \Omega : X(\omega) = x\}$

$p_X(x) = P\{\omega \in \Omega : X(\omega) = x\}$

Probability Mass Function ~~p_X~~ characterizes the discrete random variable X ~~by~~ through probabilities it gives values that X can take up.

~~The PMF, p_X,~~

$p_X(x)$ gives the probability of the event consisting of outcomes that give rise to a value of d.r.v X that equals to x.

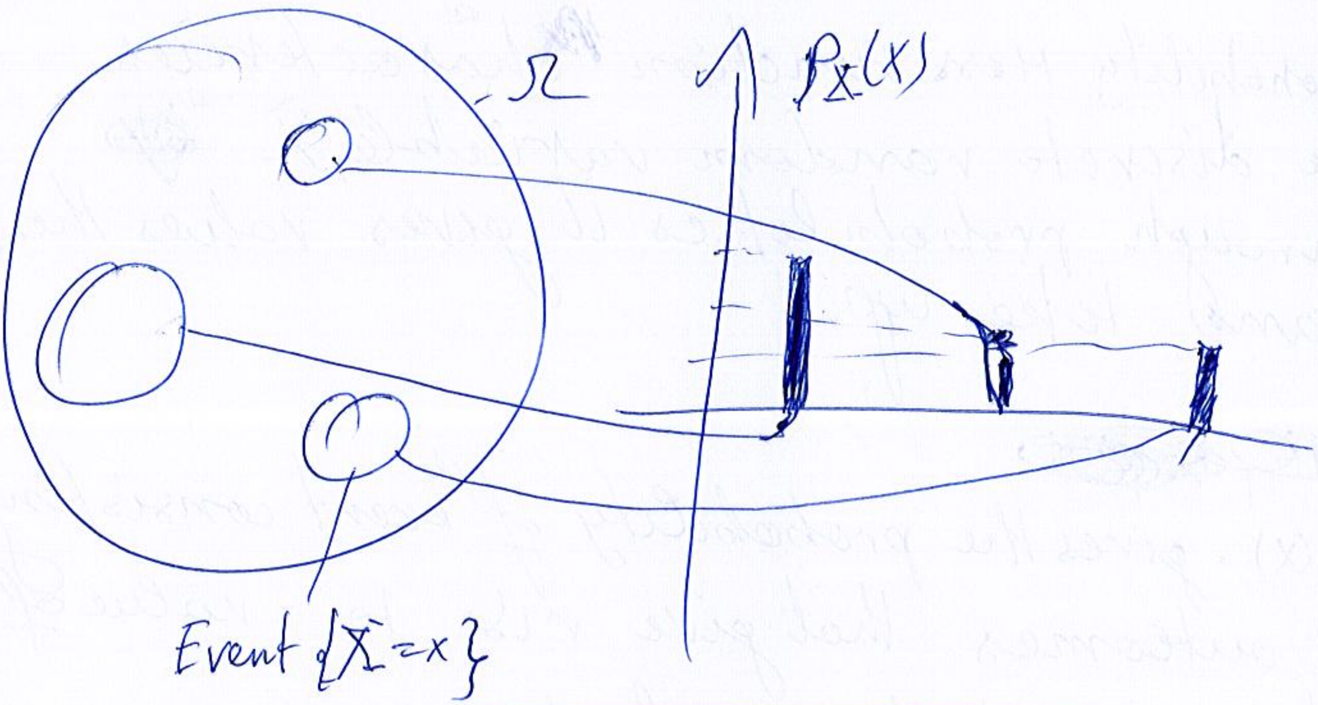
$p_X(x) = P(\{X = x\}) = P(\{\omega \in \Omega : X(\omega) = x\})$

$\boxed{\sum_x p_X(x) = 1}$ $\boxed{\sum_y p_Y(y) = 1}$ if $X : \Omega \to M$, $Y : \Omega \to M$

Random variables map everything from sample space to something thats why $\boxed{\sum_x p_X(x) = 1}$ $\boxed{\sum_y p_Y(y) = 1}$

(18) As x ranges over all values of X, the events $\{X=x\}$ are disjoint and form a partition of Ω.

$S \subset Val(X) \Rightarrow P(X \in S) = \sum_{x \in S} P_X(x)$



Binomial Random Variable

$p_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$p_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, 2 \ldots n$

$\sum_{k=0}^{n} P_X(k) = 1 \Rightarrow \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$

$Val(X) = \{0, 1, \ldots, n\}$

$\sum_x P_X(x) = 1 \quad x = k$

The geometric random variable

$X = 1, 2, \ldots$

$$0 < p < 1$$

$$P_X(k) = (1-p)^k p \qquad k = 1, 2, 3 \ldots$$

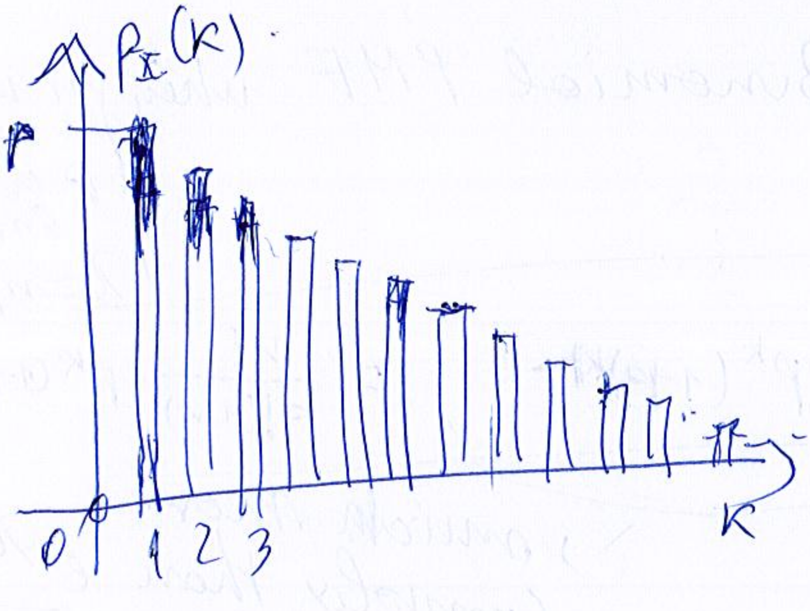
$$\sum_{x=1} P_X(x) = 1 \qquad \sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^k p = p \sum_{k=1}^{\infty} (1-p)^k$$

$0 < p < 1 \Rightarrow 0 < q = 1 - p < 1$

$$\left\{ \begin{array}{l} q^k \rightarrow \text{geometric progression} \quad 0 < q < 1 \\ \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \end{array} \right.$$

$$p \sum_{k=1}^{\infty} (1-p)^k = p \cdot \frac{1}{1-(1-p)} = \frac{p}{p} = 1 \quad \therefore$$

I.e. $\boxed{\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^k p = 1}$



The Poisson random variable

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} =$$

$$\sum_{k=0}^{\infty} p_X(k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\overbrace{\left(1+\frac{\lambda}{\cancel{1}}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\dots\right)}^{e^{\lambda}}$$

$$= e^{-\lambda}\cdot e^{\lambda} = 1$$

$$e^{\lambda} = 1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\dots+$$

$$e^{x} = 1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\dots \qquad e^{x} = \sum_{x=0}^{\infty}\frac{\lambda^x}{x!}, \; x=1 \dots$$

$$e^{x} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!} \qquad \boxed{e^{x} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}}$$

Poisson $\overset{\text{PMF}}{\approx}$ Binomial PMF when n very large, p very small, $\lambda = np$

$$\boxed{e^{-\lambda}\frac{\lambda^k}{k!} \approx \binom{n}{k}p^k(1-p)^{n-k}} = \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$

, much more complex than $e^{-\lambda}\frac{\lambda^k}{k!}$

(21)

If Y is a function of X ~~and~~ and x is a random variable, then Y is also a random variable, since it provide values for any outcome from Ω.

$$\boxed{Y = g(X)}$$

$$P_Y(y) = \sum_{x:\{x: g(x) = y\}} P_X(x)$$

$$\boxed{P_Y(y) = \sum_{\{x \mid g(x) = y\}} P_X(x)}$$

Expectation $E[X]$ (center of gravity, mea, etc)

$$\boxed{E[X] = \sum_x x P_X(x)}$$

$$Var(X) = E[(X - E(X))^2] \rightarrow \text{dispersion } \text{~~from mean~~} \text{ around its mean}$$

Standard deviati $\sigma_X = \sqrt{Var(X)}$.

$$\boxed{\text{Theorem: } E[g(X)] = \sum_x g(x) P_X(x)}$$

Theorem $X : \Omega \to M$

$$\boxed{Y = g(X) \implies E[g(X)] = \sum_x g(x) P_X(x)}$$

Proof:

$$Y = g(X)$$

$$E[Y] = \sum_y y \cdot P_Y(y) \qquad P_Y(y) = \sum_{\{x \mid g(x) = y\}} P_X(x)$$

Hence $E[Y] = \sum_y y \cdot \sum_{\{x \mid g(x)=y\}} P_X(x) = \sum_y \sum_{\{x \mid g(x)=y\}} y P_X(x) =$

$$= \underbrace{\sum_y \sum_{\{x \mid g(x)=y\}}}_{\text{capture all } x\text{'s}} g(x) P_X(x) = \boxed{\sum_x g(x) P_X(x)}$$

~~Corollary~~

$$\mathrm{Var}(X) = \boxed{E[(X - E[X])^2]} = \sum_x g(x) P_X(x) = \sum_x (x - E[X])^2 P_X(x)$$

$$g(X) = (X - E[X])^2$$

$$\boxed{E[X^n] = \sum_x x^n P_X(x)}$$ nth moment

we use for calculat

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2 - 2X E[X] + E[X]^2] =$$

$$= E[X^2] - 2E[X] \cdot E[E[X]] + E[X]^2 = E[X^2] - 2(E[X])^2 + E[X]^2$$

$$= \boxed{E[X^2] - (E[X])^2}$$

Second way of proving $E[(X-E[X])^2] = E[X^2]-E[X]^2$

$$E[(X-E[X])^2] = \sum_x (x-E[X])^2 p_X(x) =$$

$$\sum_x (x^2 - 2xE[X] + E[X]^2)\, p_X(x) = \underbrace{\sum_x x^2 p_X(x)}_{E[X^2]} - 2\sum_x x E[X]\cdot p_X(x) \quad \sum_x E[X]^2 p_X(x)$$

$$= E[X^2] - 2E[X] \underbrace{\sum_x x\cdot p_X(x)}_{E[X]} + \sum_x E[X]^2 p_X(x) =$$

constant

$$= E[X^2] - 2E[X]\cdot E[X] + E[X]^2 \sum_x p_X(x) = 1$$

$$= E[X^2] - 2E[X]^2 + E[X]^2 \cdot 1 = \boxed{E[X^2] - E[X]^2}$$

Corollary

$$\boxed{Var(X)} \overset{Def}{=} \boxed{E[(X-E[X])^2]} = \boxed{\sum_x (x-E[X])^2 p_X(x)} =$$

$$= \boxed{E[X^2] - E[X]^2}$$ Corollary

$$\underline{Y = aX + b} \qquad E[Y] = aE[X] + b$$

$$Var(Y) = E[(Y-E[Y])^2] = E[(aX+b - E[aX+b])^2]$$

$$= E[(aX + \cancel{b} - aE[X] - \cancel{b})^2] = E[(aX - aE[X])^2] =$$

$$= E[a^2X^2 - 2a^2 X E[X] + a^2E[X]^2] = a^2 E[X^2] - 2a^2E[X\cdot E[X]] + a^2E[X]^2$$

$$= a^2E[X^2] - 2a^2E[X]\,E[\underbrace{E[X]}_{const}] + a^2E[X]^2 = a^2(E[X^2] - E[X]^2) = \boxed{a^2 Var(X)}$$

$y = aX + b \Rightarrow \boxed{E[Y]} = E[aX+b] = \boxed{aE[X]+b}$

$\Rightarrow \boxed{Var(Y) = a^2 Var(X)}$

IF $g(X)$ is linear $\Rightarrow E[g(X)] = g(E[X])$

otherwise NOT generally

$\boxed{\text{Joint probability}}$ for discrete random variab

$p_{XY}(x,y) = P(\{X=x\} \cap \{Y=y\}) \xlongequal{\text{abbreviation}} P(X=x, Y=y)$

$= P(X=x \text{ and } Y=y)$

$P((X,Y) \in A) = \sum_{(x,y)\in A} p_{XY}(x,y)$

$\boxed{p_X(x) = \sum_y p_{XY}(x,y)}$ $\boxed{p_Y(y) = \sum_x p_{XY}(x,y)}$

marginal proba

Proof

$p_X(x) = P(X=x) = \sum_y P(X=x, Y=y) = \sum_y p_{XY}(x,y)$

(25)

ThE event $\{X=x\}$ is the union of disjoint events $\{X=x, Y=y\}$ as Y ranges over all values of Y.

$$\boxed{p_Y(y) = \sum_x p_{XY}(x,y)} \qquad \boxed{p_X(x) = \sum_y p_{XY}(x,y)}$$

$$\boxed{E[g(X)] = \sum_x g(x)\, p_X(x)}$$

Usually $E[g(X)] \neq g(E[X])$ except for instance when $g(X)$ is linear

$$\boxed{E[g(X,Y)] = \sum_x \sum_y g(x,y)\, p_{XY}(x,y)}$$

Let $\boxed{Z = aX + bY + c \Rightarrow E[Z] = aE[X] + bE[Y] + c}$

Proof:

$E[aX + bY + c] = \sum_x \sum_y (ax + by + c)\, p_{XY}(x,y) =$

$\sum_x \sum_y (a \cdot x \cdot p_{XY}(x,y)) + \sum_x \sum_y b \cdot y \cdot p_{XY}(x,y) + \sum_x \sum_y c \cdot p_{XY}(x,y) =$

$= a \sum_x \sum_y x\, p_{XY}(x,y) + b \sum_x \sum_y y\, p_{XY}(x,y) + c \underbrace{\sum_x \sum_y p_{XY}(x,y)}_{1}$

$= a \sum_x \left(x \cdot \sum_y p_{XY}(x,y)\right) + b \cdot \sum_y y \cdot \sum_x p_{XY}(x,y) + c =$

$= a \sum_x x \cdot p_X(x) + b \cdot \sum_y y \cdot p_Y(y) + c = aE[X] + bE[Y] + c$

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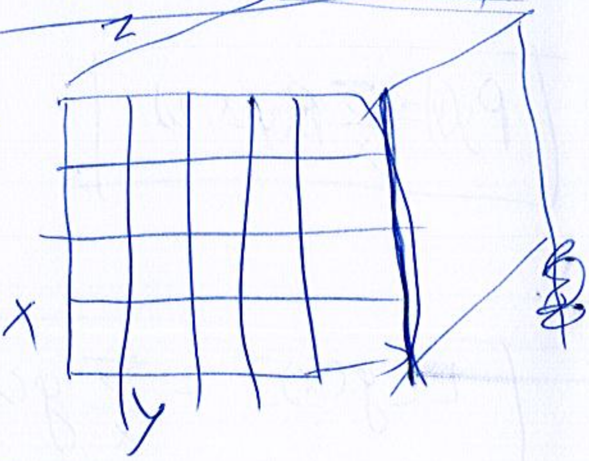
$$P_{XY}(x,y) = P(X=x, Y=y)$$

$$P_{XYZ}(x,y) = P(X=x, Y=y, Z=z)$$

$$\boxed{P_X(x) = \sum_y \sum_z P_{X,Y,Z}(x,y,z)}$$

$$\boxed{P_Y(y) = \sum_x \sum_z P_{X,Y,Z}(x,y,z)}$$

$$\boxed{P_{X,Y}(x,y) = \sum_z P_{X,Y,Z}(x,y,z)} \longrightarrow$$



$$\boxed{E[g(X,Y,Z)] = \sum_x \sum_y \sum_z g(x,y,z) \cdot P_{X,Y,Z}(x,y,z)}$$

$$E[aX + bY + cZ] = aE[X] + bE[Y] + cE[Z]$$

$$E[X_1 + X_2 + X_3 + \ldots X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

$$E\left[\sum_{i=}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

$$\boxed{E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]}$$

$$\boxed{E[a_1X_1 + a_2X_2 + \ldots + a_nX_n] = a_1E[X_1] + a_2E[X_2] + , \ldots , + a_nE[X_n]}$$

Conditioning on an event A of a r.v. X

$$P_{X|A}(x) = \frac{P(\{X=x\}\cap A)}{P(A)}$$

$$P_{X|A}(x) = P(X=x \mid A) = \frac{P(\{X=x\}\cap A)}{P(A)}$$

$\{X=x\}$'s for all x's are disjoint and form a partition of Ω

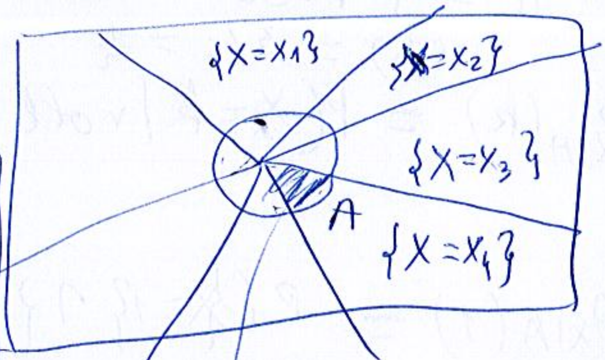
$\bigcup_x \{X=x\} = \Omega$ Hence, $\left(\bigcup_x \{X=x\}\right)\cap A = A$; thus,

or $\boxed{\bigcup_x \left(\{X=x\}\cap A\right) = A}$

$$P(A) = \sum_x P(\{X=x\}\cap A)$$

Normalization property

$$\boxed{\sum_x P_{X|A}(x) = \frac{\sum_x P(\{X=x\}\cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1}$$



$\{X=x_4\}\cap A$

$\boxed{\Omega = \bigcup_x \{X=x\}}$

$$\boxed{\sum_x P_{X|A}(x) = 1}$$

Example: $\begin{cases} \text{roll of a fair 6-sided die} \\ A = \{\text{roll is even}\} \end{cases}$

$P_{X|A}(k) = P(X=k \mid \text{roll is even})$

$= \boxed{\frac{P(X=k \text{ and } k \text{ is even})}{P(\text{roll is even})}}$

$$P_{X|A}(k) = \begin{cases} 0, \text{ for } k=1,3,5 \\ \frac{1}{3}, \text{ for } k=2,4,6 \end{cases}$$

$\{1\}\cap\{\text{is even}\} = \emptyset$

$|\{2\}\cap\{\text{is even}\}| = 1$ $\quad P(\text{roll is even}) = \frac{3}{6} = \frac{1}{2}$

$|A| = 3$ $\quad \frac{1/6}{3/6}$ or $1/3$

$$\begin{cases} \{X=x_i\}\cap\{X=x_j\} = \emptyset \\ \text{if } i \neq j \\ \begin{cases} \Omega = \bigcup_{i=1}^{n} \{X=x_i\}, \text{ if finite} \\ \Omega = \bigcup_x \{X=x\}, \text{ if infinite} \end{cases} \end{cases}$$

$$P_{X|A}(x) = P(X = x \mid A) = \frac{P(\{X=x\} \cap A)}{P(A)}$$

$$\sum_x P_{X|A}(x) = \frac{\sum P(\{X=x\} \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$$\boxed{\sum_x P_{X|A}(x) = 1}$$

Corollary: $P(\{X = x\} \cap A) = P_{X|A}(x) \cdot P(A)$

$$\boxed{P(X = x \text{ and } A) = P_{X|A}(x) \cdot P(A)}$$

Example Roll a fair 6-sided die

$A = \{\text{roll is even}\} = \{2, 4, 6\} \Rightarrow |A| = 3$

$P(A) = 3/6 = \frac{1}{2}$

$$P_{X|A}(k) = P(X = k \mid \text{roll is even}) = \frac{P(\{X=k\} \cap \{\text{roll is even}\})}{P(\{\text{roll is even}\})}$$

$$P_{X|A}(1) = \frac{P(\{X=1\} \cap \{\text{roll is even}\})}{P(\text{roll is even})} = \frac{0}{3/6}$$

$$P_{X|A}(2) = \frac{P(\{X=2\} \cap \{X \text{ is even}\})}{P(\text{roll is even})} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$P_{X|A}(k) \begin{cases} 1/3, & \text{for } k = 2, 4, 6 \\ 0, & \text{otherwise} \end{cases}$$

(29)

Conditioning a random v. on another

$$P_{X|Y}(x|y) = \cancel{P(\{X=x\}|\{Y=y\})} = P(X=x \mid Y=y)$$

$$= \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$\boxed{P_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} \Rightarrow P_{X,Y}(x,y) = P_{X|Y}(x|y) \cdot P_Y(y)$$

$$P_{Y|X}(y,x) = \frac{P_{X,Y}(x,y)}{P_X(x)} \Rightarrow P_{X,Y}(x,y) = P_{Y|X}(y|x) \cdot P_X(x)$$

Let's fix y and $P_Y(y) > 0 \Rightarrow P_{X|Y}(x,\overset{\text{fixed}}{y})$ is a function of x (a valid PMF of X)

$$\sum_x P_{X|Y}(x|y) = 1 \quad \Leftarrow \quad P_{X|Y}(x|y) = P_{X|A}(x|A)$$

where $A = \{Y=y\}$

or $P_{X|Y}(x|y) = \overset{= P(X=x|Y=y)}{\dfrac{P(\{X=x\} \cap \{Y=y\})}{P(\{Y=y\})}} = \dfrac{P(\{X=x\} \cap A)}{P(A)} = P_{X|A}(x)$

as we have shown $\boxed{\sum_x P_{X|A}(x) = 1}$

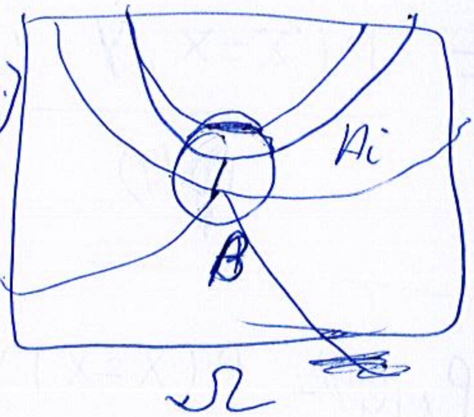
(30) Prove that $\boxed{P_X(x) = \sum_{i=1}^{m} P_{X|A_i}(x) \cdot P(A_i)}$

We have shown that if:

$\{A_i\}_i^m$ is a partition of Ω $\iff \begin{cases} A_i \cap A_j = \emptyset, & i \neq j \\ \bigcup_{i=1}^{m} A_i = \Omega \end{cases}$

event $B \subset \Omega$, $P(A_i) \geq 0 \ \forall i$

then $P(B) = \sum_i P(A_i \cap B) = \sum_i P(B \cap A_i)$

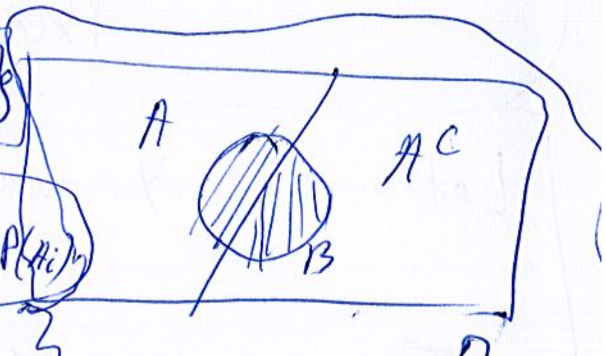


$$\boxed{P(B) = \sum_i^m P(B|A_i) \cdot P(A_i)}$$

$$= \sum_i^m P(B \cap A_i)$$

Let event B be such that ~~if all outcomes give~~ it includes all outcomes that give rise ~~to event consisting~~ to the value of X equal x

$\boxed{B = \{X = x\}}$ $\boxed{B = \{\omega \in \Omega : X(\omega) = x\}}$



~~$P_X(x)$~~ $P(B) = P(\{X=x\}) = \sum_i P(\{X=x\}|A_i) \cdot P(A_i)$

$P(B) = P(\{X=x\}) = \sum_{i=1}^{m} P(\{X=x\}|A_i) \cdot P(A_i)$

$P(X=x) = P_X(x) = \sum_{i=1}^{m} P(X=x|A_i) \cdot P(A_i)$

$P(B) = P(B \cap A) + P(B \cap A^C)$

$P(B) = P(B|A) \cdot P(A) + P(B|A^C) \cdot P(A^C)$

$= \sum_{i=1}^{m} P_{X|A_i}(x) \cdot P(A_i)$

Different start: Let $\{A_i\}_1^m$ be a partition of Ω; we know that $\Omega = \bigcup \{X=x\}$

$P_X(x) = P(X=x) = P(\{X=x\}) = \sum_{i=1}^{m} P(\underbrace{\{X=x\}}_{B \text{ if we want}} \cap A_i) = \sum_{i=1}^{m} P(X=x|A_i) \cdot P(A_i)$

$= \sum_{i=1}^{m} P_{X|A_i}(x) \cdot P(A_i)$

(31)

Just Notes :

$$\boxed{P(A\cap B) = P(A|B)\,P(B)}$$

A, B independent $\boxed{P(A\cap B) = P(A)\cdot P(B)}$

and if $P(B)>0$ $\boxed{P(A|B) = P(A)}$

A, B ind given C $\boxed{P(A\cap B|C) = P(A|C)\cdot P(B|C)}$

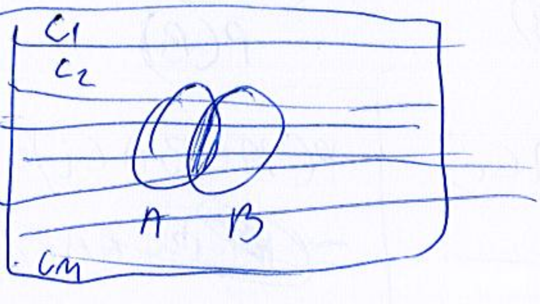
and if $P(B\cap C) > 0$

$$\boxed{P(A|B\cup C) = P(A|C)}$$

Conditional version of the Total Prob Theorem

$$\boxed{P(A|B) = \sum_{i=1}^{n} P(C_i|B)\cdot P(A|B\cap C_i)}$$

$\{C_i\}_{i=1}^{n}$ a partition of Ω



$$A\cap B = \bigcup_{i=1}^{n} (A\cap B)\cap C_i$$

$$= [(A\cap B)\cap C_1]\cup[(A\cap B)\cap C_2]\ldots$$

$$P(A\cap B) = \sum_{i=1}^{n} P((A\cap B)\cap C_i)$$

$$P(X_3\cap X_2\cap X_1) = \cancel{P(X_3\cap X_2|X_1)\cdot P(X_1)} = P(X_3|X_2\cap X_1)\cdot P(X_2\cap X_1)$$

$$= P(X_3|X_2\cap X_1)\cdot P(X_2|X_1)\cdot P(X_1)$$

$$P(X_4\cap X_3\cap X_2\cap X_1) = P(X_4|X_3\cap X_2\cap X_1)\,P(X_3\cap X_2\cap X_1) =$$

$$= P(X_4|X_3\cap X_2\cap X_1)\,P(X_3|X_2\cap X_1)\cdot P(X_2|X_1)\cdot P(X_1)$$

$$= \prod_{k=1}^{n} P\left(X_k \,\middle|\, \bigcap_{i=1}^{k-1} X_i\right)$$

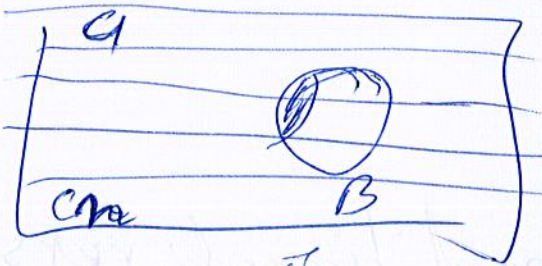
| $n=2$ $k=1$ $P(X_1)$; $k=2$ $P(X_2|X_1)$

$n=3$ $k=1$ $P(X_1)$; $k=2$ $P(X_2|X_1)$; $k=3$ $P(X_3|X_2\cap X_1)$

Theorem] $\{C_i\}_{i=1}^{m}$ → partition of Ω

$$\boxed{P(A|B) = \sum_{i=1}^{m} P(A|B\cap C_i)\cdot P(C_i|B)}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$



$$A\cap B = \bigcup_{i=1}^{m} (A\cap B)\cap C_i$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\sum_{i=1}^{m} P((A\cap B)\cap C_i)}{P(B)} =$$

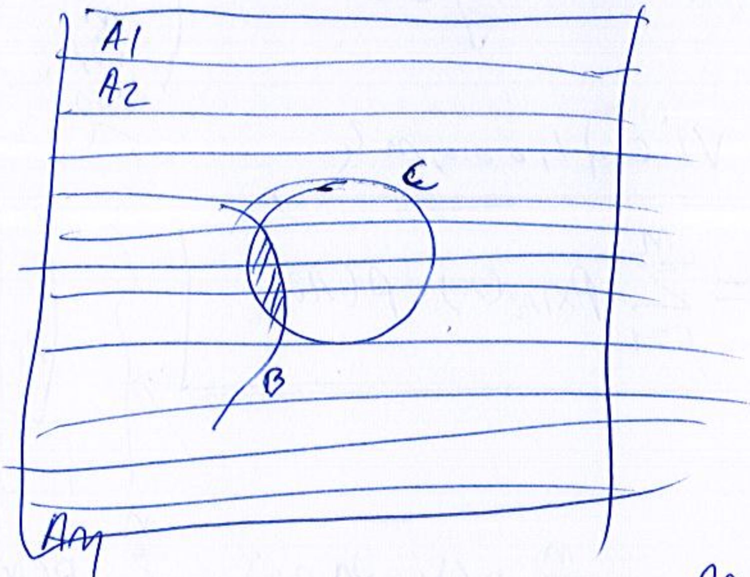
$$= \frac{\sum_{i=1}^{m} P(A\cap B\cap C_i)}{P(B)}$$

$$P(A\cap B\cap C_i) = P(A\cap C_i\cap B) = P(A|(C_i\cap B))\cdot P(C_i\cap B)$$
$$= P(A|C_i\cap B)P(C_i|B)\cdot P(B)$$

$$= \frac{\sum_{i=1}^{m} P(A|C_i\cap B)\cdot P(C_i|B)\cdot P(B)}{P(B)} = \frac{P(B)\sum P(A|C_i\cap B)\cdot P(C_i|B)\cancel{P(B)}}{P(B)}$$

$$= \boxed{\sum_{i=1}^{m} P(A|C_i\cap B)\cdot P(C_i|B)} \quad \therefore$$

just repetition



$$C \cap B = \bigcup_{i=1}^{m} \left[(C \cap B) \cap A_i \right]$$

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{\sum_{i=1}^{m} P(C \cap B \cap A_i)}{P(B)} = \frac{\sum_{i=1}^{m} P(C|B \cap A_i) \cdot P(B \cap A_i)}{P(B)}$$

$$= \sum_{i=1}^{m} \frac{P(C|B \cap A_i) \cdot P(A_i|B) \cdot \cancel{P(B)}}{\cancel{P(B)}} = \sum_{i=1}^{m} P(C|B \cap A_i) P(A_i|B)$$

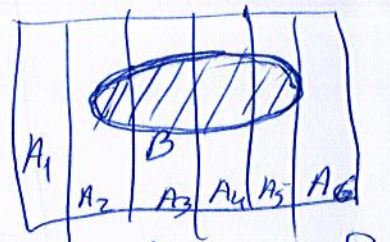
(34)

Again

$\{A_i\}_{i=1}^{n}$ is a partition of Ω i.e $\begin{cases} A_i \cap A_j = \emptyset, & i \neq j \quad i,j \in \{1,\dots,n\} \\ \bigcup_{i=1}^{n} A_i = \Omega \end{cases}$

$P(A_i) > 0 \quad \forall i \in \{1, \dots, n\}$

Then $\boxed{p_X(x) = \sum_{i=1}^{n} p_{X|A_i}(x) \cdot P(A_i)}$



$\Omega = \bigcup_{i=1}^{n} A_i$ $\quad\Omega$

Proof:

$\boxed{p_X(x)} = P(X=x) = \sum_{i=1}^{n} P(\{X=x\} \cap A_i) = \sum_{i=1}^{n} P(\{X=x\} | A_i) \cdot P(A_i)$

$B = \{X=x\}$

$= \sum_{i=1}^{n} P(X=x | A_i) \cdot P(A_i) = \boxed{\sum_{i=1}^{n} p_{X|A_i}(x) \cdot P(A_i)}$ $\therefore$

Different way.

$B = \{X=x\}$ $\quad B = \bigcup_{i=1}^{n} (B \cap A_i)$ $\quad \begin{cases} \{X=x\} = \bigcup_{i=1}^{n} (B \cap A_i) \\ (B \cap A_i) \cap (B \cap A_j) = \emptyset \\ \quad i \neq j \end{cases}$

$\{X=x\} = \{w \in \Omega : X(\Omega) = x\}$

$\Rightarrow P(\{X=x\}) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i) \cdot P(A_i) =$

$= \sum_{i=1}^{n} P(\{X=x\}|A_i) P(A_i) = \sum_{i=1}^{n} P(X=x|A_i) P(A_i) =$

$= \boxed{\sum_{i=1}^{n} p_{X|A_i}(x) \cdot P(A_i)}$

Proove that:

$$\boxed{P_{X|B}(x) = \sum_{i=1}^{n} P_{X|A_i \cap B}(x) \cdot P(A_i|B)} \quad (1)$$

$\{A_i\}_{i=1}^{n}$ is partition of Ω

Proof:

We have proven that (pg 32) Conditional Version of the Total Probability Theorem

$$P(A|B) = \sum_{i=1}^{n} P(A|B \cap C_i) \cdot P(C_i|B)$$

$\{C_i\}_{i=1}^{n}$ a partition of Ω (2)

Let's rewrite it

$X' \leftrightarrow A$, $B \leftrightarrow B$

$\begin{cases} A \to X'; \; B \to B \\ C_i \to A_i \end{cases}$

$$\boxed{P(X'|B) = \sum_{i=1}^{n} P(X'|B \cap A_i) \cdot P(A_i|B)} \quad (3)$$

To get from (3)

$$\boxed{P_{X|B}(x) = \sum_{i=1}^{n} P_{X|A_i \cap B}(x) \cdot P(A_i|B)} \quad (4)$$

$X' = \{X = x\}$

$$\boxed{P_{X|B}(x) = P(\{X = x\}|B) = P(X'|B)}$$

$$\boxed{P_{X|A_i \cap B}(x) = P(\{X = x\}|A_i \cap B) = P(X'|A_i \cap B)}$$

$$\boxed{P(X'|B) = P_{X|B}(x)} \quad (5)$$

$$\boxed{P(X'|A_i \cap B) = P_{X|A_i \cap B}(x)} \quad (6)$$

we plug (5) and (6) onto (3) and we get (4)

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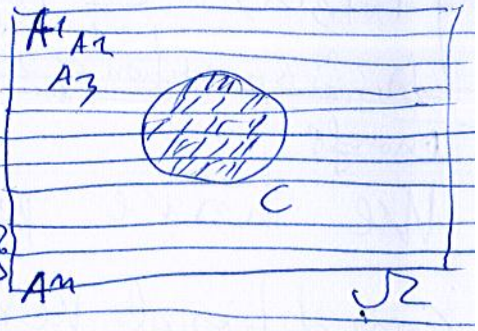
$\{A_i\}_{i=1}^{m}$ → portition of Ω i.e. $\begin{cases} A_i \cap A_j = \emptyset & i \neq j \\ \bigcup_{i=1}^{m} A_i = \Omega \end{cases}$

$$\boxed{P(C) = \sum_{i=1}^{m} P(C|A_i)\cdot P(A_i)} \quad (1)$$

$$\boxed{p_X(x) = \sum_{i=1}^{m} p_{X|A_i}(x)\cdot P(A_i)} \quad (2)$$

Theorem of Total Probabi

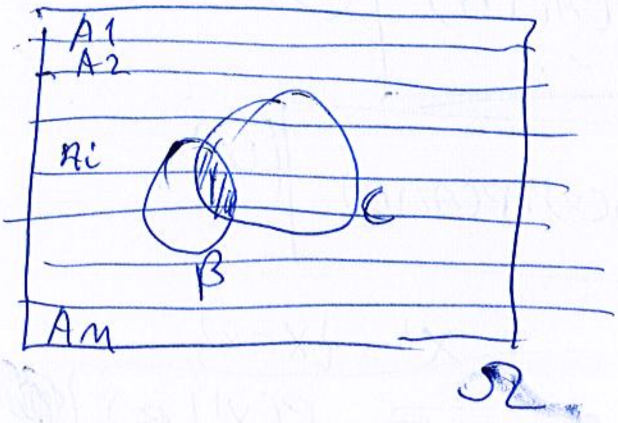


$C = \{X = x\} = \{\omega \in \Omega : X(\omega) = x\}$

$$\boxed{P(C|B) = \sum_{i=1}^{m} P(C|A_i \cap B)\cdot P(A_i|B)} \quad (3)$$

$$\boxed{p_{X|B}(x) = \sum_{i=1}^{m} p_{X|A_i \cap B}(x)\cdot P(A_i|B)} \quad (4)$$

Conditioning Version of Theorem of Total Probability



$$\boxed{C = \{X = x\} = \{\omega \in \Omega : X(\omega) = x\}}$$

(5) plug onto (1) to get (2)

Again proof of Cond ve (3)

$\to (C\cap B\cap A_i) \cap (C\cap B\cap A_j) = \emptyset$

$$P(C|B) = \frac{P(C\cap B)}{P(B)} \overset{C\cap B = \bigcup_{i=1}^{m}[(C\cap B)\cap A_i]}{=\!=} \frac{\sum_{i=1}^{m} P(C\cap B\cap A_i)}{P(B)} =$$

$P(B\cap A_i) = P(A_i \cap B) = P(A_i|B)\cdot P(B)$

$$= \frac{\sum_{i=1}^{m} P(C|B\cap A_i)\cdot P(B\cap A_i)}{P(B)} = \frac{\sum_{i=1}^{m} P(C|B\cap A_i)\cdot P(A_i|B)\cdot \cancel{P(B)}}{\cancel{P(B)}}$$

$$= \boxed{\sum_{i=1}^{m} P(C|B\cap A_i)\, P(A_i|B)} \;\therefore$$

To proove (4) use 5 onto (3).