第一問

(1)

$$\frac{\partial f}{\partial x} = 3x^2 - y \ , \ \frac{\partial f}{\partial y} = 2y - x$$

$(x, y) = (1, 2)$ 时,

$f(1,2) = 3$, $\frac{\partial f}{\partial x}(1,2) = 1$, $\frac{\partial f}{\partial y}(1,2) = 3$

曲線 $z = x^3 + y^2 - xy$ 的梯度为

$$\nabla g = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial z}{\partial z}\right) = (3x - y, 2y - x, 1)$$

$$\nabla g(1,2) = (1, 3, 1)$$

$\Rightarrow$ 点 $(1, 2, 3)$ 处法向量为 $(1, 3, 1)$

接平面方程为:

$$(x-1) + 3(y-2) + (z-3) = 0$$

$$\Rightarrow \ x + 3y + z - 10 = 0$$

(2)

$$h(x) = e^{e^{2x}-1}$$

$$h'(x) = e^{e^{2x}-1} \cdot e^{2x} \cdot 2$$

$$h(0) = 1$$

$$h'(0) = 2$$

$h(x)$在 $x=0$ 的泰勒展开为:

$$h(x) = 1 + 2x + \cdots$$

設 $h(x)$泰勒展开中 2 次以上的项为 $\varphi(x)$

$$a = \lim_{x \to 0} \frac{1-h(x)}{x^k}$$

$$= \lim_{x \to 0} \frac{-2x-\varphi(x)}{x^k} \quad \begin{cases} k<1 \text{ 时}, & a = 0 \\ k>1 \text{ 时}, & a \text{ 发散} \end{cases}$$

$\Rightarrow$ $0 < |a| < \infty$ 时, $k=1$

此时, $a = -2$

(4)

設 $h(t)=\frac{dg(t)}{dt}$，則 $\frac{dh(t)}{dt}+h(t)+\sin t=0$ …… ①

方程 $\frac{dh(t)}{dt}+h(t)=0$ 的一般解为：

$$h(t)=Ae^{-t}\quad (A为常数)$$

可将式①的解設为 $h(t)=A(t)e^{-t}$，

代入整理得：

$$A'(t)=-e^{t}\sin t$$

即：

$$\begin{aligned}A(t)&=-\int e^{t}\sin t\,dt\\&=-e^{t}\sin t+\int e^{t}\cos t\,dt\\&=-e^{t}\sin t+e^{t}\cos t-\int e^{t}\sin t\,dt\end{aligned}$$

$$\Rightarrow A(t)=\frac{1}{2}(\cos t-\sin t)e^{t}+B$$

$$h(t)=\frac{1}{2}(\cos t-\sin t)+Be^{-t}\qquad (B、C为常数)$$

$$g(t)=\frac{1}{2}(\sin t+\cos t)-Be^{-t}+C$$

代入初期值，解得：

$$B=-\frac{1}{2},\quad C=1$$

$$\Rightarrow g(t)=\frac{1}{2}(\sin t+\cos t)+\frac{1}{2}e^{-t}+1$$

(5)

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial u}{\partial y} = -1,$$
$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 3,$$

$$du\,dv = \left| \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| dx\,dy = 7\,dx\,dy$$

$$\text{原積分} = \iint_D \frac{u^3}{4+v^2}\,\frac{du\,dv}{7}$$
$$= \frac{1}{7}\int_0^1 u^3\,du \int_0^2 \frac{1}{4+v^2}\,dv$$
$$= \frac{1}{7}\left[\frac{u^4}{4}\right]_0^1 \left[\frac{1}{2}\arctan\frac{v}{2}\right]_0^2$$
$$= \frac{1}{7}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{4}$$
$$= \frac{\pi}{224}$$