1. 線形代数

(1)

(a)

$$p_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p_2 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b)

$$p_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(2)

(a)

$$X = (x_1 \ x_2) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\Rightarrow X^T X = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

(b)

$$X = (x_1 \ x_2) = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow X^T X = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$$

(3)

$$g_{i,j} = d^2_{i,n+1} + d^2_{j,n+1} - d_{i,j}$$

$$= |p_i - p_{n+1}|^2 + |p_j - p_{n+1}|^2 - |p_i - p_j|^2$$

$$= 2\left(p_i^T p_j - p_i^T p_{n+1} - p_j^T p_{n+1} + |p_{n+1}|^2\right)$$

$$X^T X = \begin{pmatrix} X_1^T \\ X_2^T \\ \vdots \\ X_n^T \end{pmatrix} (X_1 \; X_2 \; \cdots \; X_n)$$

$$= \begin{pmatrix} X_1^T X_1 & X_1^T X_2 & \cdots & X_1^T X_n \\ X_2^T X_1 & X_2^T X_2 & \cdots & X_2^T X_n \\ \vdots & & & \\ X_n^T X_1 & X_n^T X_2 & \cdots & X_n^T X_n \end{pmatrix}$$

$$X_i^T X_j = (p_i - p_{n+1})^T (p_j - p_{n+1})$$

$$= p_i^T p_j - p_i^T p_{n+1} - p_j^T p_{n+1} + |p_{n+1}|^2$$

$\Rightarrow \; G = 2X^T X$

综上，对于任意 $v \in R^n$，

$$v^T G v = 2 v^T X^T X v$$

$$= 2(Xv)^T X v$$

$$= 2|Xv|^2 \geq 0 \quad \Rightarrow \quad G\text{为半正定}$$

2. 微積分

(1)

(a)

$$\frac{df}{dt} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}\frac{dv_i}{dt}$$

(b)

$$\frac{df}{dt} = (2x_1 + x_2)\cos t + (x_1 + 4x_2)e^t$$

$(= \sin 2t + e^t \cos t + e^t \sin t + 4e^{2t}$ … 只用 t 表示 $)$

(2)

考慮 $\frac{dy}{dx} - 2xy = 0 \Rightarrow \frac{dy}{y} = 2x\,dx$

可得特殊解 $y = Ae^{x^2}$ (A为常數)

設 A 为 x 的函数，代入 $y = A(x)e^{x^2}$，得：

$$\frac{dA(x)}{dx} = 1$$

$$\Rightarrow A(x) = x + C \quad (C\text{为常數})$$

原方程一般解为：

$$y = (x + C)e^{x^2} \quad (C\text{为常數})$$

(3)

設 $w = z - \frac{\pi}{2}$，如下在 $w=0$ 处洛朗展开：

$$\frac{\cos z}{(2z-\pi)^3} = -\frac{1}{8}\frac{\sin w}{w^3}$$

$$= -\frac{1}{8w^3}\left(w - \frac{1}{6}w^3 + \cdots\right)$$

$$= -\frac{1}{8}\left(\frac{1}{w^2} - \frac{1}{6} + \cdots\right)$$

$$\Rightarrow \oint_C \frac{\cos z}{(2z-z)^2} = 0$$

3. 向量解析

(1)

S_1 上的点满足：

$$\vec{i}\cos\varphi + y\vec{j} + \vec{k}\sin\varphi \quad (0 \leq \varphi < 2\pi,\ 0 \leq y \leq 4)$$

設 S_1 指向外側的单位法線向量为 $\vec{n}$，则

$$\vec{n} = \vec{i}\cos\varphi + \vec{k}\sin\varphi$$

$$\vec{F} = \vec{i}\cos\varphi + 2y\vec{j} + 10\vec{k}\sin\varphi$$

$$\vec{F}\cdot\vec{n} = \cos^2\varphi + 10\sin^2\varphi$$

$$= \frac{11}{2} - \frac{9}{2}\cos 2\varphi$$

$$\Rightarrow \int_{S_1} dS\,\vec{F}\cdot\vec{n} = \int_0^{2\pi} d\varphi \int_0^4 dy \left(\frac{11}{2} - \frac{9}{2}\cos 2\varphi\right)$$

$$= 44\pi$$

(2)

将 S_2 分为 2 部分考虑：

S_2' : $x^2+z^2=1$ $(0 \le y \le 4,\ 0 \le z)$

S_2'' : $z=0$ $(-1 \le x \le 1,\ 0 \le y \le 4)$

S_2' 上的積分可如(1)計算：

$$\int_{S_2'} dS\,\vec{F}\cdot\vec{n} = \int_0^{\pi} d\varphi \int_0^4 dy \left(\frac{11}{2} - \frac{9}{2}\cos 2\varphi\right)$$

$$= 22\pi$$

S_2'' 上，指向外側的單位法線向量为 $\vec{n} = -\vec{k}$

且 $\vec{F} = x\vec{i} + 2y\vec{j}$, $\vec{F}\cdot\vec{n} = 0$ $\Rightarrow$ 面積分为 0

綜上，$\int_{S_2} dS\,\vec{F}\cdot\vec{n} = \int_{S_2'} dS\,\vec{F}\cdot\vec{n} = 22\pi$