2022 . H1

(1) a)

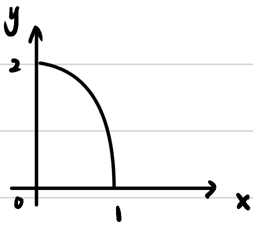
$V_1 = xyz$

b)

yx 平面上，$z=0$.

$16x^2+4y^2=16$

$x^2+\frac{y^2}{4}=1$



c)

在点 P 处，$g(x,y,z)=0$ 与 $f(x,y,z)=f(a,b,c)$ 相切

d)

$$\begin{cases} bc = 32a\lambda \\ ac = 8b\lambda \\ ab = 2c\lambda \\ 16a^2+4b^2+c^2=16 \end{cases}$$

e)

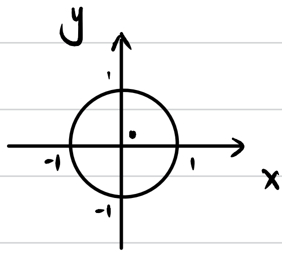
解得：$a=\frac{1}{\sqrt{3}}$，$b=\frac{2}{\sqrt{3}}$，$c=\frac{4}{\sqrt{3}}$，$\lambda=\frac{1}{4\sqrt{3}}$

(2)

(a)

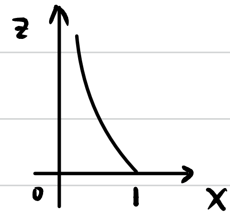
与 xy 平面的交线上 $z=0$，则有

$$x^2+y^2=1$$



与 xz 平面的交线上 $y=0$，则有

$$z=-2\log x$$



b)

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \Rightarrow z=-2\log r \qquad \begin{cases} d\le r\le e^{-\frac{z}{2}} \\ 0\le\theta\le 2\pi \end{cases}$$

$$V_2' = \iiint dx\,dy\,dz$$
$$= \int_0^{-2\ln d} dz \int_0^{2\pi} d\theta \int_d^{e^{-\frac{z}{2}}} dr\cdot r$$

c)

$$V_2' = \pi(2d^2\ln d+1-d^2)$$

H2

(1). a)

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 & 5 \\ 0 & 2 & -6 & 1 & -1 \\ -1 & 1 & 0 & 2 & -1 \\ 2 & 3 & -15 & 0 & -4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -2 & 3 & -4 & 5 \\ 0 & 2 & -6 & 1 & -1 \\ 0 & 1 & -3 & 2 & -4 \\ 0 & 7 & -21 & 8 & -14 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & -4 & 5 \\ 0 & 1 & -3 & 2 & -4 \\ 0 & 0 & 0 & -3 & 7 \\ 0 & 0 & 0 & -6 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & -4 & 5 \\ 0 & 1 & -3 & 2 & -4 \\ 0 & 0 & 0 & 1 & -7/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \operatorname{rank} A = 3$

b)

$$A \to \begin{pmatrix} 1 & 2 & 3 & -4 & 5 \\ 0 & 1 & -3 & 2 & -4 \\ 0 & 0 & 0 & 1 & -9/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -3 & 0 & -3 \\ 0 & 1 & -3 & 0 & 2/3 \\ 0 & 0 & 0 & 1 & -9/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 3x_3 + 3x_5 \\ x_2 = 3x_3 - 2/3\, x_5 \\ x_3 = x_3 \\ x_4 = 9/3\, x_5 \end{cases}$$

$$x = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} 3 \\ -2/3 \\ 0 \\ 9/3 \\ 1 \end{pmatrix} t \qquad (s, t \neq 0)$$

基底：$\frac{1}{\sqrt{143}} \begin{pmatrix} 3 \\ -2/3 \\ 0 \\ 9/3 \\ 1 \end{pmatrix}$

(2) a)

$$\det B = \begin{vmatrix} -2 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & -2 \end{vmatrix} = -2$$

b)

$$\det(\lambda E - B) = \begin{vmatrix} \lambda+2 & -1 & 1 \\ -1 & \lambda & -1 \\ 1 & -1 & \lambda+2 \end{vmatrix} = (\lambda+1)^2(\lambda+4)$$

固有値：-1，-4

c)

$\lambda_1 = -1$ 時：

$$-E-B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}，固有向量 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}，\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -4$ 時：

$$-4E-B = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}，固有向量 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

d)

$$D = P^{-1} B P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$$\begin{aligned} B^n &= (P D P^{-1})^n \\ &= P D^n P^{-1} \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-4)^n \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2(-1)^n + (-4)^n & (-1)^n - (-4)^n & -(-1)^n + (-4)^n \\ (-1)^n - (-4)^n & 2(-1)^n + (-4)^n & (-1)^n - (-4)^n \\ -(-1)^n + (-4)^n & (-1)^n - (-4)^n & 2(-1)^n + (-4)^n \end{pmatrix} \end{aligned}$$

e)

实对称行列可通过直交行列 R 对角化. 該对角行列为 Λ

$$\Lambda = R^{-1} C R$$

則有 $C^k = R \Lambda^k R = 0$

$$R^{-1} R \Lambda^k R R^{-1} = R^{-1} 0 R$$

$$\Lambda^k = 0$$

即：C 只有值为 0 的固有值 $\Rightarrow C = O$