1 DeBoor definition

$$\{\Omega\}_{jk} = \int B_j''(t) B_k''(t) dt \tag{1}$$

$$[\tau_i, ..., \tau_{i+r}]g = \frac{\tau_{i+1}, ..., \tau_{i+r}]g - \tau_i, ..., \tau_{i+r-1}]g}{\tau_{i+r} - \tau_i} \tag{2}$$

$$B_{j,k,t}(x) := (t_{j+k} - t_j)[t_j, ..., t_{j+k}](\cdot - x)_+^{k-1} \tag{3}$$

2 Hastie definition

An order-M spline with knots $\xi_j,\ j = 1, ..., K$ is a piece-wise polynomial of order M, and has continuous derivatives up to order $M - 2$. A cubic spline has order $M = 4$.

$$\begin{array}{l} \tau_1 \leq \tau_2 \leq \cdots \leq \tau_M \leq \xi_0 \\ \tau_{j+M} = \xi_j, j = 1, \cdots, K \\ \xi_{K+1} \leq \tau_{K+M+1} \leq \tau_{K+M+2} \leq \cdots \leq \tau_{K+2M} \end{array} \tag{4}$$

$$B_{i,1}(x) = \begin{cases} 1 & \text{if } \tau_i \leq x < \tau_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

for $i = 1, ..., K + 2M - 1$. (Haar functions.)

$$B_{i,m}(x) = \frac{x - \tau_i}{\tau_{i+m-1} - \tau_i} B_{i,m-1}(x) + \frac{\tau_{i+m} - x}{\tau_{i+m} - \tau_{i+1}} B_{i+1,m-1}(x) \tag{6}$$

for $i = 1, ..., K + 2M - m$.

3 Hastie smoothing

$$RSS(f, \lambda) = \sum_{i=1}^{N} \{y_i - f(x_i)\}^2 + \lambda \int \{f''(t)\}^2 dt \tag{7}$$

$$f(x) = \sum_{j=1}^{N} N_j(x) \theta_j \tag{8}$$

$$RSS(\theta, \lambda) = (\mathbf{y} - \mathbf{N}\theta)^T (\mathbf{y} - \mathbf{N}\theta) + \lambda \theta^T \mathbf{\Omega}_N \theta \tag{9}$$

...where $\{\mathbf{N}\}_{ij} = N_j(x_i)$ and $\{\mathbf{\Omega}_N\}_{jk} = \int N_j''(t) N_k''(t) dt$.

$$\hat{\theta} = (\mathbf{N}^T \mathbf{N} + \lambda \mathbf{\Omega}_N)^{-1} \mathbf{N}^T \mathbf{y} \tag{10}$$

$$\hat{f}(x) = \sum_{j=1}^{N} N_j(x) \hat{\theta}_j \tag{11}$$

$$\hat{f}'(x) = \sum_{j=1}^{N} N_j'(x) \hat{\theta}_j \tag{12}$$

From here on we need to deal with multiple spline fits to different probes' time series.

$$f_p(x) = \sum_{j=1}^{N} N_j(x) \theta_{pj} \tag{13}$$

Notice that the basis need not change. It is determined solely by the knots (time points).

$$
\begin{aligned}
D(p,q) &= \int |\hat{f}_p'(x) - \hat{f}_q'(x)| dx \\
&= \int |\sum_{j=1}^{N} N_j'(x)\theta_{pj} - \sum_{j=1}^{N} N_j'(x)\theta_{qj}| dx \\
&= \sum_{i=0}^{4} \int_{\xi_i}^{\xi_{i+1}} |\sum_{j=i}^{i+3} N_j'(x)\theta_{pj} - \sum_{j=i}^{i+3} N_j'(x)\theta_{qj}| dx \\
&= \sum_{i=0}^{4} \int_{\xi_i}^{\xi_{i+1}} |\sum_{j=i}^{i+3} (\theta_{pj} - \theta_{qj}) N_j'(x)| dx \\
&= \sum_{i=0}^{4} \left\{ \int_{\xi_i}^{\alpha_i} \sum_{j=i}^{i+3} (\theta_{pj} - \theta_{qj}) N_j'(x) dx + \int_{\alpha_i}^{\beta_i} \sum_{j=i}^{i+3} (\theta_{pj} - \theta_{qj}) N_j'(x) dx + \int_{\beta_i}^{\xi_{i+1}} \sum_{j=i}^{i+3} (\theta_{pj} - \theta_{qj}) N_j'(x) dx \right\}
\end{aligned}
\tag{14}
$$

Because the $N_j'(x)$ are quadratics, in each interval there can be no more than 2 curve crossings, so it is possible to break up the full integral into a small number of intervals (at most $3k$) on each of which D is exactly solvable.

4 Implementation guides

	Interval				
	0	1	2	3	4
Spline	[0,1)	[1,2)	[2,4)	[4,6)	[6,12]
0	x	.	.	.	.
1	x	x	.	.	.
2	x	x	x	.	.
3	x	x	x	x	.
4	.	x	x	x	x
5	.	.	x	x	x
6	.	.	.	x	x
7	.	.	.	.	x

Following shows how the various polynomial errors cover intervals. Notice that the left and right 3 intervals are not real integers.