Duality

Formulations

Primal

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, ..., m, \\ & x_j \ge 0, \quad j = 1, 2, ..., n. \end{aligned}$$

Dual

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad j = 1, 2, ..., n, \\ & y_i \ge 0, \quad i = 1, 2, ..., m. \end{aligned}$$

Example 1: Giapetto Problem

Primal

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 100 \\ & x_1 + x_2 \le 80 \\ & x_1 \le 40 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Dual

$$\begin{aligned} \text{min} \quad & 100y_1 + 80y_2 + 40y_3 \\ \text{s.t.} \quad & 2y_1 + y_2 + y_3 \ge 3 \\ & y_1 + y_2 \ge 2 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

Basic Variable	Eq #	z	x_1	x_2	x_3	x_4	x_5	RHS
z	0	1	0	0	1	1	0	180
x_2	1	0	0	1	-1	2	0	60
x_5	2	0	0	0	-1	1	1	20
x_1	3	0	1	0	1	-1	0	20

x_1^*	x_2^*	x_3^*	x_4^*	x_5^*
20	60	0	0	20
0	0	1	1	0
y_4^*	y_5^*	y_1^*	y_2^*	y_3^*

$z^* = w^* = 180$

Example 2

Primal

$$
\begin{aligned}
\text{Maximize} \quad & 3x_1 + 5x_2 \\
\text{subject to} \quad & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

The optimal simplex tableau is:

Eq.	B.V.	x_1	x_2	s_1	s_2	s_3	R.H.S
(0)	z	0	0	0	3/2	1	36
(1)	s_1	0	0	1	1/3	$-1/3$	2
(2)	x_2	0	1	0	1/2	0	6
(3)	x_1	1	0	0	$-1/3$	1/3	2

x_1^*	x_2^*	s_1^*	s_2^*	s_3^*
2	6	2	0	0
0	0	0	$\frac{3}{2}$	1
y_4^*	y_5^*	y_1^*	y_2^*	y_3^*

$z^* = w^* = 36$

Dual

$$
\begin{aligned}
\text{min} \quad & 4y_1 + 12y_2 + 18y_3 \\
\text{s.t.} \quad & y_1 + 3y_3 \geq 3 \\
& 2y_2 + 2y_3 \geq 5 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Dual Optimal Solution

$$(y_1^*, y_2^*, y_3^*) = \left(0, \frac{3}{2}, 1\right)$$

Example 3 (Lecture)

Primal

$$\begin{array}{rrcl} \text{Maximize} & 3x_1 + 4x_2 + 5x_3 + 4x_4 & & \\ \text{subject to} & 2x_1 + 5x_2 + 4x_3 + 3x_4 & \leq & 224 \\ & 5x_1 + 4x_2 - 5x_3 + 10x_4 & \leq & 280 \\ & 2x_1 + 4x_2 + 4x_3 - 2x_4 & \leq & 184 \\ & x_1,\ x_2,\ x_3, x_4 & \geq & 0 \end{array}$$

The optimal simplex tableau is:

Eq.	B.V.	x_1	x_2	x_3	x_4	s_1	s_2	s_3	R.H.S
(0)	z	0	14/5	0	0	6/5	1/15	2/15	312
(1)	x_4	0	1/5	0	1	1/5	0	$-1/5$	8
(2)	x_1	1	1	0	0	$-1/5$	2/15	11/30	60
(3)	x_3	0	3/5	1	0	1/5	$-1/15$	$-1/30$	20

x_1^*	x_2^*	x_3^*	x_4^*	s_1^*	s_2^*	s_3^*
60	0	20	8	0	0	0
0	$\frac{14}{5}$	0	0	$\frac{6}{5}$	$\frac{1}{15}$	$\frac{2}{15}$
y_4^*	y_5^*	y_6^*	y_7^*	y_1^*	y_2^*	y_3^*

$z^* = w^* = 312$

Dual

$$\begin{array}{rrcl} \min & 224y_1 + 280y_2 + 184y_3 & & \\ \text{s.t.} & 2y_1 + 5y_2 + 2y_3 & \geq & 3 \\ & 5y_1 + 4y_2 + 4y_3 & \geq & 4 \\ & 4y_1 - 5y_3 + 4y_3 & \geq & 5 \\ & 3y_1 + 10y_2 - 2y_3 & \geq & 4 \\ & y_1, y_2, y_3 & \geq & 0 \end{array}$$

Dual Optimal Solution

$$(y_1^*, y_2^*, y_3^*) = \left(\frac{6}{5}, \frac{1}{15}, \frac{2}{15}\right)$$