## 25   Nearest Neighbor Algorithms: Voronoi Diagrams and k-d Trees

### NEAREST NEIGHBOR ALGORITHMS

### Exhaustive $k$-NN Alg.

Given query point $q$:
– Scan through all $n$ sample pts, computing (squared) distances to $q$.
– Maintain a max-heap with the $k$ shortest distances seen so far.
  [Whenever you encounter a sample point closer to $q$ than the point at the top of the heap, you remove the heap-top point and insert the better point. Obviously you don't need a heap if $k = 1$ or even 5, but if $k = 101$ a heap will substantially speed up keeping track of the $k$th-shortest distance.]

Time to train classifier: 0     [This is the only $O(0)$-time algorithm we'll learn this semester.]
Query time: $O(nd + n \log k)$
            expected $O(nd + k \log n \log k)$ if random pt order
[It's a cute theoretical observation that you can slightly improve the expected running time by randomizing the point order so that only expected $O(k \log n)$ heap operations occur. But in practice I can't recommend it; you'll probably lose more from cache misses than you'll gain from fewer heap operations.]

Can we preprocess training pts to obtain sublinear query time?

2–5 dimensions: Voronoi diagrams
Medium dim (up to $\sim 30$): $k$-d trees
Large dim: exhaustive $k$-NN, but can use PCA or random projection
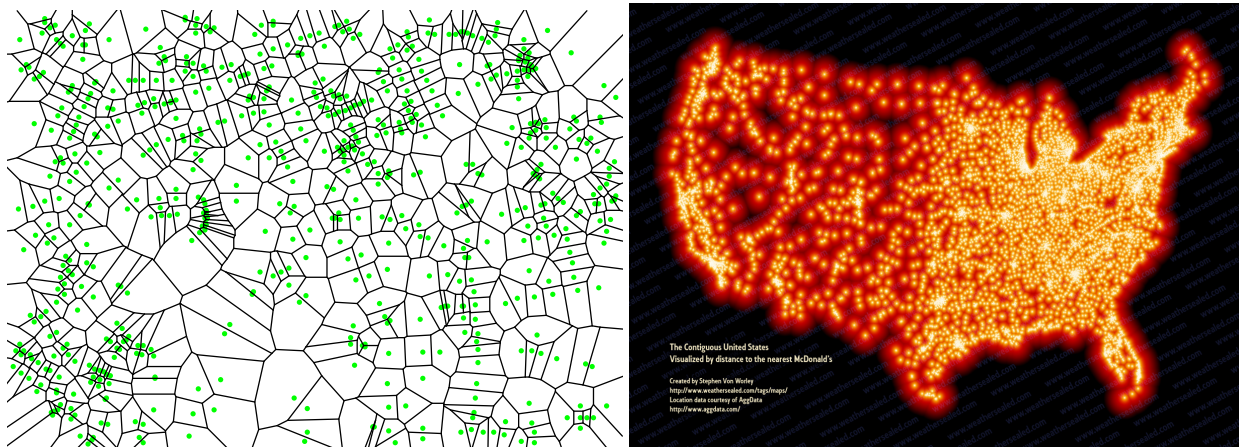            locality sensitive hashing   [still researchy, not widely adopted]
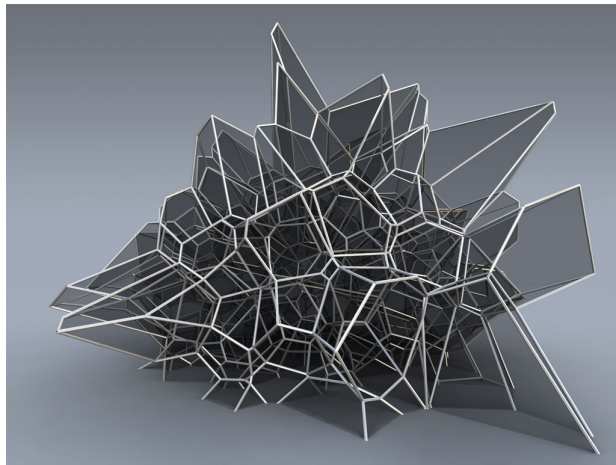
### Voronoi Diagrams

Let $X$ be a point set. The Voronoi cell of $w \in X$ is
Vor $w = \{p \in \mathbb{R}^d : \|p - w\| \le \|p - v\| \quad \forall v \in X\}$
[A Voronoi cell is always a convex polyhedron or polytope.]
The Voronoi diagram of $X$ is the set of $X$'s Voronoi cells.



The Contiguous United States
Visualized by distance to the nearest McDonald's

Created by Stephen Von Worley
http://www.weathersealed.com/tags/maps/
Location data courtesy of AggData
http://www.aggdata.com/

voro.pdf, vormcdonalds.jpg, voronoiGregorEichinger.jpg, saltflat-1.jpg
[Voronoi diagrams sometimes arise in nature (salt flats, giraffe, crystallography).]
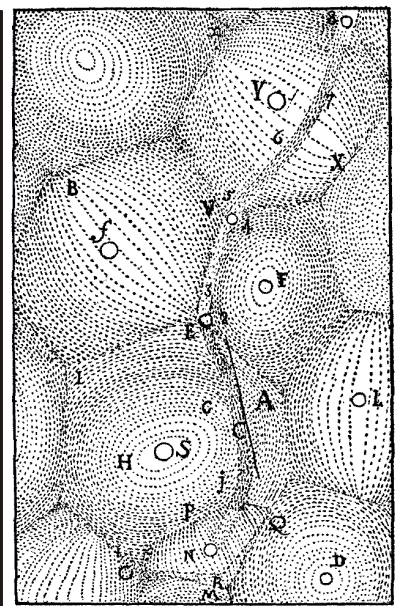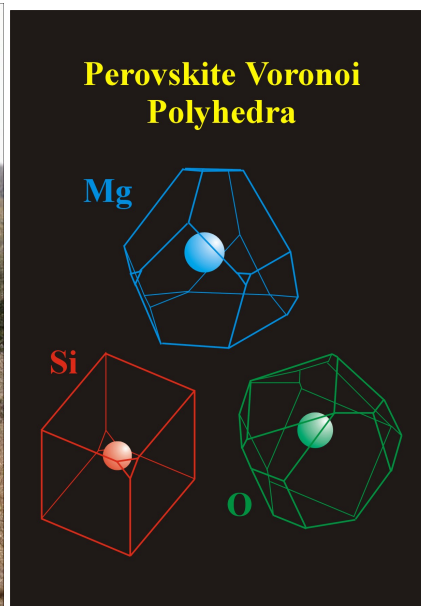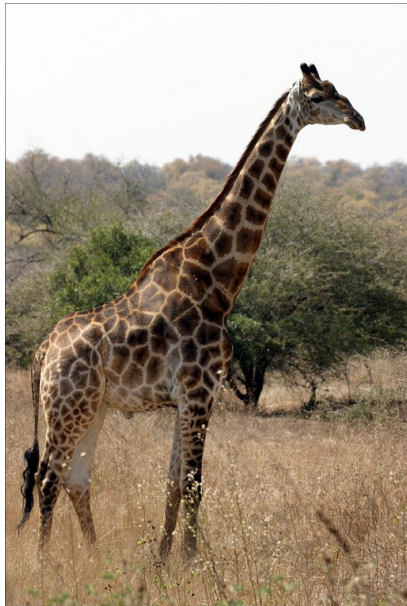


giraffe-1.jpg, perovskite.jpg, vortex.pdf

[Believe it or not, the first published Voronoi diagram dates back to 1644, in the book "Principia Philosophiae" by the famous mathematician and philosopher René Descartes. He claimed that the solar system consists of vortices. In each region, matter is revolving around one of the fixed stars (vortex.pdf). His physics was wrong, but his idea of dividing space into polyhedral regions has survived.]

Size (e.g., # of vertices) $\in O(n^{\lceil d/2 \rceil})$
[This upper bound is tight when $d$ is a small constant. As $d$ grows, the tightest asymptotic upper bound is somewhat smaller than this, but the complexity still grows exponentially with $d$.]
...but often in practice it is $O(n)$.
[Here I'm leaving out a constant that may grow exponentially with $d$.]

Point location: Given query point $q \in \mathbb{R}^d$, find the point $w \in X$ for which $q \in$ Vor $w$.
[We need a second data structure that can perform this search on a Voronoi diagram efficiently.]

2D:    $O(n \log n)$ time to compute V.d. and a trapezoidal map for pt location

    $O(\log n)$ query time    [because of the trapezoidal map]
[That's a pretty great running time compared to the linear query time of exhaustive search.]

$d$D: Use binary space partition tree (BSP tree) for pt location
[Unfortunately, it's difficult to characterize the running time of this strategy, although it is likely to be reasonably fast in 3–5 dimensions.]

1-NN only!
[A standard Voronoi diagram supports only 1-nearest neighbor queries. If you want the $k$ nearest neighbors, there is something called an order-$k$ Voronoi diagram that has a cell for each possible $k$ nearest neighbors. But nobody uses those, for two reasons. First, the size of an order-$k$ Voronoi diagram is $O(k^2 n)$ in 2D, and worse in higher dimensions. Second, there's no software available to compute one.]

[There are also Voronoi diagrams for other distance metrics, like the $L_1$ and $L_\infty$ norms.]

[Voronoi diagrams are good for 1-nearest neighbor queries in 2 or 3 dimensions, maybe 4 or 5, but $k$-d trees are much simpler and probably faster in 6 or more dimensions.]

*choose direction with*
*greatest width.*
*Best aspect ratio.*



### $k$-d Trees

"Decision trees" for NN search. Differences:    [compared to decision trees]
  – Choose splitting feature w/greatest width: feature $i$ in $\max_{i,j,k}(X_{ji} - X_{ki})$.
    [With nearest neighbor search, we don't care about the entropy. Instead, what we want is that if we draw a sphere around the query point, it won't intersect very many boxes of the decision tree. So it helps if the boxes are nearly cubical, rather than long and thin.]

*saves*
*build time* ←
  – Cheap alternative: rotate through the features. [We split on the first feature at depth 1, the second feature at depth 2, and so on. This builds the tree faster, by a factor of $O(d)$.]
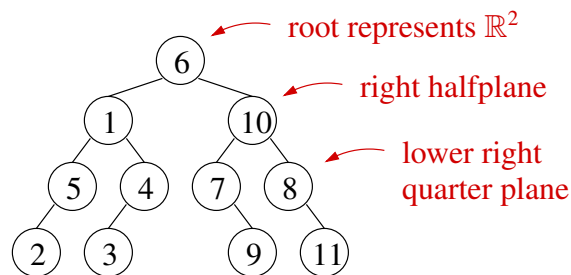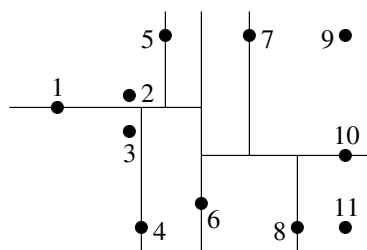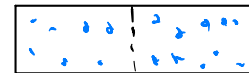  – Choose splitting value: median point for feature $i$, or $\frac{X_{ji}+X_{ki}}{2}$.
    Median guarantees $\lfloor \log_2 n \rfloor$ tree depth; $O(nd \log n)$ tree-building time.
    [...or just $O(n \log n)$ time if you rotate through the features. An alternative to the median is splitting at the box center, which improves the aspect ratios of the boxes, but it could unbalance your tree. A compromise strategy is to alternate between medians at odd depths and centers at even depths, which also guarantees an $O(\log n)$ depth.]
  – Each internal node stores a sample point.    [...that lies in the node's box. Usually the splitting point.]
    [Some $k$-d tree implementations have points only at the leaves, but it's better to have points in internal nodes too, so when we search the tree, we often stop searching earlier.]



root represents $\mathbb{R}^2$

right halfplane

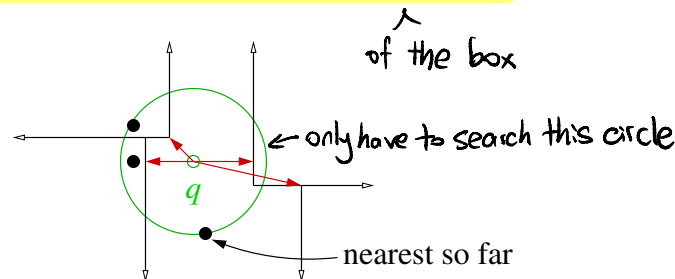lower right quarter plane

[Draw this by hand.  kdtreestructure.pdf ]

Goal:    given <mark>query pt $q$</mark>, <mark>find a sample pt $w$ such that $\|q - w\| \le (1 + \epsilon)\|q - s\|$,</mark>
         where $s$ is the closest sample pt.
         $\epsilon = 0 \implies$ exact NN;    <mark>$\epsilon > 0 \implies$ approximate NN.</mark>

Query alg. maintains:
– Nearest neighbor found so far (or $k$ nearest).                          goes down ↓
– <mark>Binary min-heap of unexplored subtrees, keyed by distance from $q$.</mark>    goes up    ↑ (eliminating boxes as we
                                                                                          go)

of the box

← only have to search this circle

$q$

nearest so far

[Draw this by hand. kdtreequery.pdf] [A query in progress.]

[Each subtree represents an axis-aligned box. The query tries to avoid searching most of the boxes/subtrees by <mark>searching the boxes close to $q$ first</mark>. We measure the distance from $q$ to a box and use it as a key for the subtree in the heap. The search <mark>stops when the distance from $q$ to the $k$th-nearest neighbor found so far $\le$ the distance from $q$ to the nearest unexplored box (times $1 + \epsilon$)</mark>. For example, in the figure above, the query never visits the box at far lower right, because it doesn't intersect the circle.]

Alg. for 1-NN query:                   ↗ $k$-d Tree    we have to search in this box
         $Q \leftarrow$ heap containing <mark>root node</mark> <mark>with key zero</mark> ←
         $r \leftarrow \infty$ → closest neighbour dist
         while $Q$ not empty and $(1 + \epsilon) \cdot \text{minkey}(Q) < r$
             $B \leftarrow \text{removemin}(Q)$
             $w \leftarrow B$'s sample point
             $r \leftarrow \min\{r, \text{dist}(q, w)\}$                         [For speed, store square of $r$ instead.]
             $B', B'' \leftarrow$ child boxes of $B$
             if $(1 + \epsilon) \cdot \text{dist}(q, B') < r$ then insert($Q, B', \text{dist}(q, B')$)    [The key for $B'$ is dist($q, B'$)]
             if $(1 + \epsilon) \cdot \text{dist}(q, B'') < r$ then insert($Q, B'', \text{dist}(q, B'')$)
         return point that determined $r$

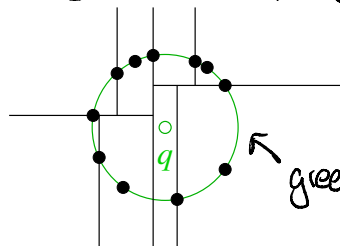<mark>For $k$-NN, replace "$r$" with a max-heap holding the $k$ nearest neighbors</mark>
[... just like in the exhaustive search algorithm.]

Works with any $L_p$ norm for $p \in [1, \infty]$.
[$k$-d trees are not limited to the Euclidean ($L_2$) norm.]

Why $\epsilon$-approximate NN?    All training points  equally far away.

$q$

green circle intersects every k-d tree

[Draw this by hand. kdtreeproblem.pdf] [A worst-case exact NN query.]

[In the worst case, we may have to visit every node in the *k*-d tree to find the exact nearest neighbor. In that case, the *k*-d tree is slower than simple exhaustive search. This is an example where an *approximate* nearest neighbor search can be much faster. In practice, settling for an approximate nearest neighbor sometimes improves the speed by a factor of 10 or even 100, because you don't need to look at most of the tree to do a query. This is especially true in high dimensions—remember that in high-dimensional space, the nearest point often isn't much closer than a *lot* of other points.]

Software: ANN (U. Maryland), FLANN (U. British Columbia), GeRaF (U. Athens)   [random forests!]

## Example: im2gps

[I want to emphasize the fact that exhaustive nearest neighbor search really is one of the first classifiers you should try in practice, even if it seems too simple. So here's an example of a modern research paper that uses 1-NN and 120-NN search to solve a problem.]

Paper by James Hays and [our own] Prof. Alexei Efros.
[Goal: given a query photograph, determine where on the planet the photo was taken. Called geolocalization. They evaluated both 1-NN and 120-NN. What they did not do, however, is treat each photograph as one long vector. That's okay for tiny digits, but too expensive for millions of travel photographs. Instead, they reduced each photo to a small descriptor made up of a variety of features that extract the essence of each photo.]
[Show slides (im2gps.pdf). Sorry, images not included here. http://graphics.cs.cmu.edu/projects/im2gps/]

[Bottom line: With 120-NN, their most sophisticated implementation came within 64 km of the correct location about 50% of the time.]

## RELATED CLASSES [if you like machine learning, consider taking these courses]

CS 182 (fall, probably spring): Deep Neural Networks
CS C281A (spring): Statistical Learning Theory [C281A is the most direct continuation of CS 189/289A.]
EECS 127/227AT (both), EECS 227BT (fall): Numerical Optimization [a core part of ML]
[It's hard to overemphasize the importance of numerical optimization to machine learning, as well as other CS fields like graphics, theory, and scientific computing.]
EECS 126 (both): Random Processes [Markov chains, expectation maximization, PageRank]
EE C106A/B: Intro to Robotics [dynamics, control, sensing]
Math 110: Linear Algebra [but the real gold is in Math 221]
Math 221: Matrix Computations [how to solve linear systems, compute SVDs, eigenvectors, etc.]
CS 194-26/294-26 (fall): Computational Photography (Efros, Kanazawa)
CS C267 (spring): Scientific Computing [parallelization, practical matrix algebra, some graph partitioning]
CS C280 (spring): Computer Vision
CS 285 (fall): Deep Reinforcement Learning
CS 287H (spring): Algorithmic Human-Robot Interaction
CS 288 (spring): Natural Language Processing
CS 294-43 (fall): Visual Object/Activity Recognition (Darrell)
CS 294-150 (spring): ML & Biology (Listgarten)
CS 294-162 (spring): ML Systems (Gonzalez)
CS 294-182 (spring): ML & Econ (Haghtalab, Jordan)
CS 294-186 (spring): Equity in Algorithms (Abebe, Britton)
VS 265: Neural Computation
  ↓
Vision Science