Primal-Dual Relationships

Primal

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, ..., m, \\ & x_j \ge 0, \quad j = 1, 2, ..., n. \end{aligned}$$

Dual

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad j = 1, 2, ..., n, \\ & y_i \ge 0, \quad i = 1, 2, ..., m. \end{aligned}$$

The Dual Problem in Equality Form

$$\begin{array}{rl} \text{minimize} & b_1y_1 + b_2y_2 + ... + b_my_m \\ \text{subject to} & a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m - y_{m+1} \qquad\qquad = c_1 \\ & a_{12}y_1 + a_{22}y_2 + ... + a_{m2}y_m \qquad - y_{m+2} \qquad = c_2 \\ & . \\ & . \\ & . \\ & a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m \qquad\qquad - y_{m+n} = c_n \\ & y_1, y_2, ..., y_{m+n} \ge 0. \end{array}$$

Recall that y_{m+1}, y_{m+2},..., y_{m+n} are called surplus variables because they measure how much the left hand sides of the constraints exceed the right hand sides.

Note that the primal and dual problems in equality form both have $m+n$ variables. You should become comfortable with the following notation:

Primal

$$(\underbrace{x_1, x_2, ..., x_n}_{\text{original variables}}, \underbrace{x_{n+1}, x_{n+2}, ..., x_{n+m}}_{\text{slack variables}})$$

Dual

$$(\underbrace{y_1, y_2, ..., y_m}_{\text{original variables}}, \underbrace{y_{m+1}, y_{m+2}, ..., y_{m+n}}_{\text{surplus variables}})$$

Primal-Dual Relationships

1. **Weak duality theorem**:

   If $(x_1, x_2, ..., x_{n+m})$ and $(y_1, y_2, ..., y_{n+m})$ are feasible solutions to the primal and dual problems, respectively, then:

$$\begin{aligned} c_1x_1 + c_2x_2 + ... + c_nx_n &\le x_1 \sum_{i=1}^{m} a_{i1}y_i + x_2 \sum_{i=1}^{m} a_{i2}y_i + ... + x_n \sum_{i=1}^{m} a_{in}y_i \\ &= y_1 \sum_{j=1}^{n} a_{1j}x_j + y_2 \sum_{j=1}^{n} a_{2j}x_j + ... + y_m \sum_{j=1}^{n} a_{mj}x_j \le y_1b_1 + y_2b_2 + ... + y_mb_m \end{aligned}$$

The first inequality is obtained by multiplying both sides of each dual constraint ($\sum_{i=1}^{m} a_{ij}y_i \geq c_j$) by the corresponding primal variable x_j and summing the constraints from $j = 1 \ldots n$. The second equality by switching the order of summation. The third inequality is obtained by multiplying both sides of each primal constraint ($\sum_{j=1}^{n} a_{ij}x_j \leq b_i$) by the corresponding dual variable y_i and summing the constraints from $i = 1 \ldots m$. The final result shows that for a feasible pair of solutions to the primal and dual problems, the dual objective is always greater than or equal to the primal objective (when the primal is a maximization problem).

2. **Strong duality**: The dual solution vector $(y_1, y_2, .., y_m)$ has the following interpretation: y_i is the quantity by which we multiply row i prior to adding it to the objective function row.

   Consider the coefficients in the objective function row in the optimal tableau:

$\bar{c}_1$	$\bar{c}_2$	.	.	.	$\bar{c}_n$	$\bar{c}_{n+1}$	.	.	.	$\bar{c}_{n+m}$	$\bar{z}$

$$\begin{cases} \bar{c}_j = -c_j + \sum_{i=1}^{m} a_{ij}y_i & j = 1, .., n+m \\ \bar{z} = 0 + \sum_{i=1}^{m} b_i y_i \end{cases}$$

   where $\bar{z}$ is the optimal objective value of the maximization problem. The reduced cost for the $(n+i)^{th}$ variable (the slack variable for the i^{th} constraint) is given by:

$$\bar{c}_{n+i} = 0 + \sum_{k=1}^{m} a_{k,n+i}\ y_k = y_i \geq 0$$

   which shows that the reduced cost of the i^{th} slack variable is equal to the i^{th} dual variable. For the other reduced costs, we have from before:

$$\bar{c}_j = -c_j + \sum_{i=1}^{m} a_{ij}y_i \geq 0$$

$$\Longrightarrow \sum_{i=1}^{m} a_{ij}y_i \geq c_j$$

   which shows that the vector $(y_1, y_2, ..., y_m)$ is feasible for the dual problem. The associated objective value of the dual problem is thus given by:

$$\bar{w} = \sum_{i=1}^{m} b_i y_i$$

   which is equal to the optimal value of the primal problem. This leads to the *strong duality theorem.*

   Conclusion
   Strong Duality Theorem: If the primal (dual) problem has a finite optimal solution, then so does the dual (primal) problem, and these two values are equal.

3. **The simplex method solves the primal and dual problems simultaneously because**: If $(x_1^*, x_2^*, ..., x_{n+m}^*)$ and $(y_1^*, y_2^*, ..., y_{n+m}^*)$ are the optimal solutions to the primal and dual problems, respectively, then:

   a) y_i^*, the optimal value of the i^{th} original dual variable, is given by the row 0 coefficient of x_{n+i} in the optimal simplex tableau (the reduced cost of the i^{th} slack variable)

   b) y_{m+j}^*, the optimal value of the j^{th} surplus dual variable, is given by the row 0 coefficient of x_j (j^{th} primal variable) in the optimal simplex tableau

   c) The optimal value z^* of the primal objective function equals the optimal value w^* of the dual objective function, that is,

$$\begin{aligned}
c_1x_1^* + c_2x_2^* + ... + c_nx_n^* &= x_1^* \sum_{i=1}^{m} a_{i1}y_i^* + x_2^* \sum_{i=1}^{m} a_{i2}y_i^* + ... + x_n^* \sum_{i=1}^{m} a_{in}y_i^* \\
&= y_1^* \sum_{j=1}^{n} a_{1j}x_j^* + y_2^* \sum_{j=1}^{n} a_{2j}x_j^* + ... + y_m^* \sum_{j=1}^{n} a_{mj}x_j^* \\
&= y_1^*b_1 + y_2^*b_2 + ... + y_m^*b_m
\end{aligned}$$

   Note that the first equality gives us that:

$$\sum_{j=1}^{n} x_j^* \left(\sum_{i=1}^{m} a_{ij}y_i^* - c_j \right) = 0$$

   And the last equality gives us that:

$$\sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij}x_j^* \right) = 0$$

   Together, this set of equations imply the complementary slackness conditions stated next.

4. **Complementary slackness conditions:**

   a) $x_j^* y_{m+j}^* = 0 \quad j = 1, 2, ..., n$ (j^{th} primal original)$\cdot$(j^{th} dual surplus)

   b) $x_{n+i}^* y_i^* = 0 \quad i = 1, 2, ..., m$ (i^{th} primal slack)$\cdot$(j^{th} dual original)

   Property a) states that if the j^{th} primal original variable is positive, then the j^{th} dual constraint is "tight", or equivalently, if the j^{th} dual constraint is "loose", then the j^{th} primal variable must be equal to zero.

   Property b) states that if the i^{th} primal constraint is "loose", then the i^{th} dual variable must equal zero, or equivalently, if the i^{th} dual variable is positive, then the i^{th} primal constraint is "tight".

5. **The dual of the dual is the primal**.

6. **One of the following four possibilities must occur**:

Case	**Primal**	**Dual**
1	Feasible with finite optimal objective	Feasible with finite optimal objective
2	Feasible with unbounded optimal objective	Infeasible
3	Infeasible	Feasible with unbounded optimal objective
4	Infeasible	Infeasible

Case 1 happens since if the primal is feasible with finite optimum, then the dual solution is available from the simplex optimal tableau. The same goes if the dual is feasible with finite optimum.
Case 2 is a consequence of the weak duality theorem.
For cases 3 and 4, when the primal is infeasible, either possibility of unboundedness or infeasibility hold for the dual. See examples below.

Example for Case 3:

$$\begin{aligned} \text{maximize } & x_1 - x_2 \\ \text{subject to } & x_1 - x_2 \leq 5 \\ & -x_1 + x_2 \leq -6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Example for Case 4:

$$\begin{aligned} \text{maximize } & x_1 + x_2 \\ \text{subject to } & x_1 - x_2 \leq 5 \\ & -x_1 + x_2 \leq -6 \\ & x_1, x_2 \geq 0 \end{aligned}$$