EECS 16B   Designing Information Devices and Systems II
Fall 2021   Note 3: Vector Differential Equations

1 Matrix Form

1.1 Introduction

Last time we saw how to solve simple scalar first order linear differential equations.

Now, we are going to expand our tool set and learn how to tackle multivariable differential equations. (You will see in homeworks that these exact same ideas will also equip us to deal with higher-order differential equations where we have second and third derivatives involved.) Let's motivate the need to understand two dimensional systems with the following circuit.
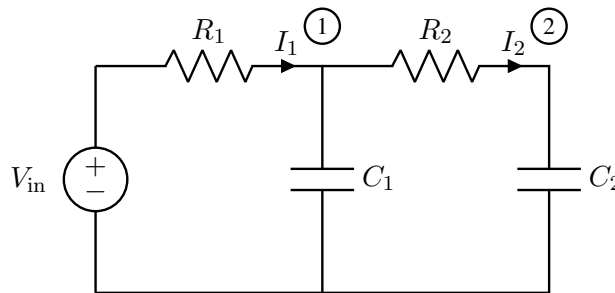
Figure 1: Two dimensional system

Let $C_1 = C_2 = 1\,\mu\text{F}$, $R_1 = \frac{1}{3}\text{M}\Omega$ and $R_2 = \frac{1}{2}\text{M}\Omega$.[1]

Concept Check: Before we begin, how would you solve for the above system if $R_2 = 0$?

$\Rightarrow C_1 + C_2 \Rightarrow$ one capa.

$C_2 = C_1 + C_2$

Answer: If $R_2 = 0$, then the C_1 and C_2 are connected in parallel and can be combined into a single effective capacitance of $C_1 + C_2$. This is like what happened earlier when we had two gate capacitances in parallel.

As in Note 1, let us first consider the discharging case. In the above system, let $V_{\text{in}} = 1\,\text{V}$ for time $t < 0$, and $V_{\text{in}} = 0\,\text{V}$ for time $t \geq 0$. With this in mind, we have two steady state conditions:

(a) Initial condition $t = 0$: The voltage has been charging the capacitors for an infinite amount of time. Hence, both capacitors have voltage $V_{C_1} = V_{C_2} = 1\text{V}$ and current $I_1 = I_2 = 0\text{A}$.

(b) As $t \to \infty$: After the capacitors have been allowed to discharge for a long period of time, they carry no charges on their plates, hence $V_{C_1} = V_{C_2} = 0\text{V}$.

Next, let us solve for the transients, i.e. how does our system go from (a) to (b)? First we need to set up the circuit equations. We will define $V_1 = V_{C_1}$ and $V_2 = V_{C_2}$ as the voltages across our capacitors.

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[1]The SI prefixes 'M' and 'μ' stand for "mega" and "micro," and correspond to the decimal multiples of 10^6 and 10^{-6} respectively.
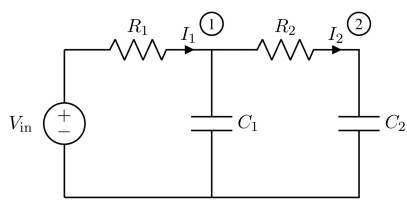
Figure 1: Two dimensional system

$$V_2 = V_1 - I_2 R_2 \tag{1}$$

$$I_2 = C_2 \frac{\mathrm{d}}{\mathrm{d}t} V_2 \tag{2}$$

$$0 - V_1 = I_1 R_1 \Rightarrow I_1 = -\frac{V_1}{R_1} \tag{3}$$

$$I_1 = I_2 + C_1 \frac{\mathrm{d}}{\mathrm{d}t} V_1 \tag{4}$$

For eq. (1) and eq. (2), we look at the current through Node ②, using Ohm's law for the former and voltage-current relation for the capacitor in the latter. Similarly, we find eq. (3) and eq. (4) by looking at Node ①. Similar to what we had seen in a single capacitor system, we have effectively introduced two new variables - $\frac{\mathrm{d}}{\mathrm{d}t} V_1$ and $\frac{\mathrm{d}}{\mathrm{d}t} V_2$. Next, to solve for the transients, we need to first define our system variables. **The standard approach that we will always take is to make anything that gets differentiated into a state variable.** Hence, we will need two state variables, V_1 and V_2.

1.2 Systems of Differential Equations

We get a system of differential equations by isolating the derivative terms and solving for them in terms of their non-differentiated selves. You can view this as doing the downward pass of Gaussian Elimination using an ordering of the variables so that the $\frac{\mathrm{d}}{\mathrm{d}t} \cdot$ variables come second to last and their non-differentiated counterparts come last. Because there will be more unknowns than equations, Gaussian elimination will stop on the last $\frac{\mathrm{d}}{\mathrm{d}t}$-ed variable. Then, we can do a limited upward-pass (back-substitution pass) of Gaussian elimination to purge any dependence of one $\frac{\mathrm{d}}{\mathrm{d}t}$-ed variable on any other $\frac{\mathrm{d}}{\mathrm{d}t}$-ed variable. This gives rise to the system of differential equations in a systematic way — it is the lower block that remains after doing this back-substitution. In principle, every other circuit quantity of interest can be found once we know how the state variables and their derivatives behave.

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t) = -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} \right) V_1(t) + \frac{V_2(t)}{R_2 C_1} \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} V_2(t) = \frac{V_1(t)}{R_2 C_2} - \frac{V_2(t)}{R_2 C_2} \tag{6}$$

The steps outlining how we obtained eq. (5) and eq. (6) are in section 4.

The fact that every linear circuit involving capacitors (and as we will see inductors as well) can eventually be expressed into this form is effectively a consequence of the Gaussian Elimination algorithm.

Concept Check: Take a second to work out setting up the above differential equations. It is important to try to reduce all your branch equations from the circuit analysis to as few variable as possible.

As seen in EE16A, when encountering a system of equations, we try to put it into vector/matrix form to try to solve it. So, let's put eq. (5) and eq. (6) in the matrix differential form. We define $\vec{x}(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$, and:

$$\begin{bmatrix} \frac{\mathrm{d}V_1}{\mathrm{d}t} \\ \frac{\mathrm{d}V_2}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} \right) V_1 + \frac{V_2}{R_2 C_1} \\ \frac{V_1}{R_2 C_2} - \frac{V_2}{R_2 C_2} \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}(t) \tag{9}$$

In eq. (9), we have substituted in for the component values defined above so that we get a matrix with concrete numbers in it.

Quick Aside: You may come across a dot over the variable of interest (e.g. $\dot{V_1}$) as short hand to represent the $\frac{\mathrm{d}}{\mathrm{d}t}$ operator. This is alternatively known as Newton's notation and is sometimes used for conciseness, especially in fields influenced by Physics notation. Similarly, we sometimes define $\ddot{V} \equiv \frac{\mathrm{d}^2}{\mathrm{d}t^2}V$ to make it faster to write second derivatives. However, we won't be using Newton's notation in this course.

In general, we want to set up our systems in the follwing generic form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b} \tag{10}$$

So, why do we choose this form? Because it most closely resembles the first order scalar differential equation we studied in the previous note, and our goal in this note is to massage this equation to convert it to a collection of first order differential equations. In context of our above example, $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ and $\vec{b} = \vec{0}$ because there is no external voltage or current being applied for $t \geq 0$ in the above example.

2 Diagonalization by Means of an Eigenbasis

2.1 Eigenvalues and Eigenvectors

Before we delve into solving our system of differential equations, let's review a core concept from EECS16A - eigenvalues and eigenvectors. Please note this will be an abridged version of the content covered in EECS16A. If you have further questions regarding any of these concepts, please review the notes from EECS16A.

What are **eigenvalues** and **eigenvectors**? An eigenvector is a vector that is only scaled by a value (the corresponding eigenvalue) when acted upon by a matrix, i.e.

$$A\vec{v}_\lambda = \lambda\vec{v}_\lambda \tag{11}$$

Here, $\vec{v}_\lambda \neq \vec{0}$ is an eigenvector of A which corresponds to the scalar λ eigenvalue. If we look at all the eigenvectors of the matrix A corresponding to a single λ, these together form a subspace known as the λ-eigenspace. Each distinct eigenvalue is associated with its own nontrivial eigenspace[2]. We can see this

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[2]In 16A, you also saw some interesting properties of these eigenspaces. For example, any collection of nonzero vectors drawn from eigenspaces corresponding to distinct eigenvalues is linearly independent. This is proved by contradiction—the simplest case is for two distinct eigenvalues. (A vector can't simultaneously be two different multiples of the same thing!) From there, the proof continues to three vectors. If one is expressible as a combination of the others, then those two must be expressible as a multiple of each other, which we have already ruled out. And so on. If you don't remember this proof, it is worth doing for yourself. Anyway, as a result, there cannot be more than n distinct eigenvalues for an $n \times n$ matrix since each one must be associated with its own eigenspace, and there can only be n linearly independent directions in an n-dimensional space.

subspace property by rearranging the above equation into:

$$(A - \lambda I_n)\vec{v} = \vec{0}. \tag{12}$$

Here, I_n is the $n \times n$ identity matrix. Since $A\vec{v}_\lambda = \lambda\vec{v}_\lambda$, we know that the matrix $A - \lambda I_n$ has a nullspace (namely the eigenspace corresponding to the eigenvalue λ of A). Consequently, this matrix $(A - \lambda I_n)$ must be non-invertible and thus $\det(A - \lambda I_n) = 0$. Remember, the determinant of a square matrix measures the oriented volume[3] of the unit hypercube as transformed by that matrix. Non-invertible matrices destroy information by squashing the space along at least one direction (the nullspace directions), and so result in something with volume zero.

This gives us the **characteristic equation**[4] and solving this equation will give us the eigenvalues λ. As a working example, let's compute the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$:

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \tag{13}$$

$$= \det\left(\begin{bmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{bmatrix} \right) \tag{14}$$

Taking the determinant and setting it to zero, we get the characteristic equation:

$$0 = (1 - \lambda)(3 - \lambda) - (4)(2) = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) \tag{15}$$

Setting eq. (15) to 0 and solving the quadratic equation, we get the eigenvalues as $\lambda_1 = 5$ and $\lambda_2 = -1$. Each eigenvalue will have it's own eigenspace, and it will define the nullspace of the matrix $A - \lambda I_n$. Hence to find the eigenspace, we can just find the relevant nullspace[5]

$$(A - \lambda_1 I_2)\vec{v}_1 = (A - 5I_n)\vec{v}_1 = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \vec{0} \tag{16}$$

We see that both rows provide redundant information - $4v_{11} - 2v_{12} = 0$, and hence an eigenvector is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Concept Check: Find an eigenvector $\vec{v}_2$ correspoding to λ_2.

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[3]Recall how the determinant is computed by doing Gaussian elimination. Multiplication of a row by -1 incurs a factor of -1 because it is like looking through a mirror—a mirror flips orientation. Every time we swap two rows, we also incur a factor of -1—this is because swapping two rows is like rotating and then looking through a mirror. Rotating doesn't change volume, but the mirror flips the orientation. Adding a multiple of one row to another does nothing to the oriented volume because it is like shearing a cube—just as pushing over a deck of cards doesn't change the volume of the deck, shearing doesn't change volume either. Scaling a row by a positive number has that scaling effect on the volume—and so by reversing the scalings done in Gaussian Elimination (which gets us to the Identity), we can get the volume effect of the original matrix.

[4]The characteristic equation is often also referred to as the characteristic polynomial

[5]You know from EECS 16A how you can find nullspaces systematically using Gaussian Elimination. Simply run Gaussian elimination until it terminates—you will have at least one row of zeros. Then work your way backward from the end finding free variables (variables that you did not "pivot" on — i.e. variables that you never explicitly tried to eliminate from lower equations— these are variables that got eliminated on their own!) and then expressing others in terms of them. This will give you a basis for the nullspace.

Answer: A second eigenvector is $\vec{v}_2 = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note that any $\alpha \neq 0$ would still be a valid eigenvector.

2.2 Repeated and Complex Eigenvalues

2.2.1 Repeated Eigenvalues

For a 2×2 matrix, it's possible that the two eigenvalues that you end up with have the same value, leading to a phenomenon called a **repeated eigenvalues**. This repeated eigenvalue can have one or two dimensional eigenspace (unlike a single, unrepeated eigenvalue, which will only have a one dimensional eigenspace).

For example, the following matrix has a repeated eigenvalue of λ.

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

The λ-**eigenspace** of this matrix is all of $\mathbb{R}^2$ since for any vector $\vec{v} \in \mathbb{R}^2$, $A\vec{v} = \lambda\vec{v}$.

We can also have examples like

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

that have a single eigenvalue $\lambda = 0$. (One can see this by looking at the characteristic equation, which turns out to be $\lambda^2 = 0$.) In this case, the relevant eigenspace is one-dimensional—only $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and its multiples are eigenvectors here. (If there were any other linearly independent eigenvectors corresponding to the 0 eigenvalue, then everything would have to be an eigenvector and that would mean the matrix mapped everything to the zero vector. But the matrix is definitely not the zero matrix, so that isn't what is going on.)

2.2.2 Complex Eigenvalues

It can also be the case that when we solve $\det(A - \lambda I_n) = 0$, there will be no real solutions to λ. Consider the rotation matrix (described in Note j):

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{17}$$

Concept Check: When does R_θ have real eigenvalues? Why? **Answer:** For $\theta = 0°$ or $\theta = 180°$.

However, when θ does not correspond to $0°$ or $180°$ rotation, there are no real vectors that are scaled versions of themselves after the transformation. This results in complex eigenvalues. For example, let's look at $45°$ rotation:

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\det(R - \lambda I) = \frac{1}{2}(1 - \lambda)(1 - \lambda) + \frac{1}{2}$$

Setting this determinant equal to zero and solving yields the complex eigenvalues, $\lambda = \frac{1}{\sqrt{2}}(1 + j)$ and $\lambda = $

$\frac{1}{\sqrt{2}}(1-j)$, which makes sense because there are no physical (real) eigenvectors for a rotation transformation in two dimensions.

2.3 Change of Basis and Diagonalization

Let's start with a vector $\vec{v} \in \mathbb{R}^n$. This vector represents a point in space. When you think about this vector written out using the coordinates, you are scaling the vectors in the standard basis (i.e. the columns of the identity matrix, I_n) by the components of $\vec{v}$ and then adding them up:

$$\vec{v} = I_n \vec{v} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

But, suppose that I think about this vector in terms of a different set of directions. We can define a new coordinate system using a basis, i.e. n linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$. Now, let's say I have a vector $\vec{u}$ which I am representing as $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ measuring with respect to my coordinate system. How can I translate this to the coordinates you are familiar with? Well, instead of scaling the vectors of the standard basis, we are now scaling the vectors defining my new basis. Assuming that both of us were thinking of the same physical point in space, then the vector $\vec{v}$ in the standard basis is:

$$\vec{v} = u_1 \vec{v}_1 + u_2 \vec{v}_2 + \cdots + u_n \vec{v}_n \tag{18}$$

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \tag{19}$$

$$= V\vec{u} \tag{20}$$

Hence, to tranform a vector from my basis to your standard basis involves just a matrix multiplication. This can be expressed in a diagram given in Figure 2 using the up and down arrows.
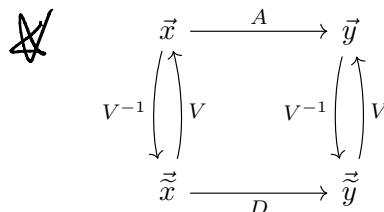
Figure 2: Change of Basis Mapping. The top row has everything in the standard basis. The bottom row is in V-basis. The matrix D is supposed to represent the same linear transformation as A, except that it does so for vectors expressed in V-basis. It turns out $D = V^{-1}AV$ since vectors are columns. We use tilde-ed coordinates for things in the lower basis.

Next, let's see what happens if we have a matrix A that transforms vectors. Consider the 2×2 matrix

example, $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. We can imagine this matrix is performing a linear tranformation, $\vec{y} = A\vec{x}$. The question is whether there is another basis within which this transformation is much simpler to understand.

What can we wish for? The transformation is doing something nontrivial, and so it is not possible to find a basis in which the transformation is just the identity matrix. Nor is one going to be found where the matrix is just the zero matrix. So not all wishes can be fulfilled. What about wishing for a diagonal transformation? That would certainly be r to understand. But is that a reasonable wish? Let's see whether it is in this case.

Let's suppose we had our new basis V so that $\vec{x} = V\vec{\tilde{x}}$ and correspondingly, $\vec{\tilde{x}} = V^{-1}\vec{x}$. Similarly, $\vec{y} = B\vec{\tilde{y}}$ and correspondingly, $\vec{\tilde{y}} = B^{-1}\vec{y}$. Then:

$$A\vec{x} = AB\vec{\tilde{x}} \tag{21}$$

$$= A(\tilde{x}_1\vec{b}_1 + \tilde{x}_2\vec{b}_2) \tag{22}$$

$$= \tilde{x}_1 A\vec{b}_1 + \tilde{x}_2 A\vec{b}_2 \tag{23}$$

Now, if we chose our new basis to be the ones defined by the eigenvectors of A, then we can simplify:

$$= \tilde{x}_1 \lambda_1 \vec{b}_1 + \tilde{x}_2 \lambda_2 \vec{b}_2 \tag{24}$$

$$= \begin{bmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{25}$$

$$= BD\vec{\tilde{x}} \tag{26}$$

$$= BDB^{-1}\vec{x}, \tag{27}$$

where D is the diagonal matrix of eigenvalues and B is a matrix with the corresponding eigenvectors as its columns. Thus we have proved that $A = BDB^{-1}$. Furthermore, this also means that $D = B^{-1}AB$.

So, our wish can indeed be fulfilled—at least for this specific matrix. In general, the pattern we have used above will hold whenever we can find an eigenbasis—a full basis consisting of eigenvectors. Why? From the eigenvalue-eigenvector relationship we have $AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{b}_1 & \lambda_2\vec{b}_2 & \cdots & \lambda_n\vec{b}_n \end{bmatrix} = BD$. If B is full rank and thus invertible, we can left multiply both sides by B^{-1} to get:

$$B^{-1}AB = B^{-1}BD = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

But we know that there also exist matrices for which an eigenbasis does not exist. For those, this trick won't work.

3 How to Solve?

Now that we understand how to convert our systems of differential equations into vector/matrix form, let's understand how to solve them. The first thing we should try to do is to use the tools we developed for the scalar case. But, how can we do this, when our equations seem to be fundamentally dependent on two independent variables? We wouldn't have this problem of 'coupling' if our A matrix were diagonal. So, let's try to change our basis to one where the A matrix does become diagonal.

Note that if our system of differential equations were 'uncoupled' to begin with, our A matrix would have been diagonal. Hence, we could proceed to solve the first order differential equations independently of each other, as seen in Note 1. Consider the following circuit:
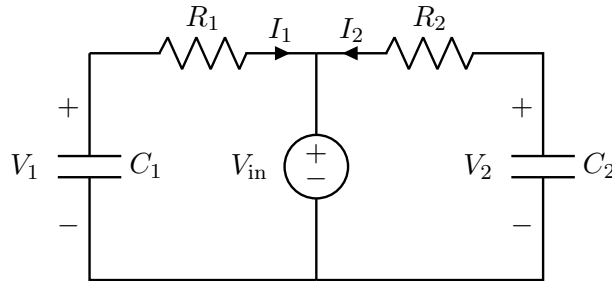
Figure 3: Diagonal System

Both the capacitors have been charged to V_{in} and at $t = 0$, we set $V_{\text{in}} = 0V$, and allow the capacitors to discharge. Hence our initial conditions are $V_1(0) = V_2(0) = V_{\text{in}}$. We get the following branch equations:

$$I_1 = -C_1 \frac{\mathrm{d}}{\mathrm{d}t} V_1 = \frac{V_1}{R_1} \tag{28}$$

$$I_2 = -C_2 \frac{\mathrm{d}}{\mathrm{d}t} V_2 = \frac{V_2}{R_2} \tag{29}$$

Hence from eq. (28) and eq. (29), we get the following uncoupled differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} \tag{30}$$

Concept Check: We can easily solve the above system of equations by separately solving for V_1 and V_2. Review Note 1 if you are unsure about how to solve for the voltages.

Coming back to our original system, we define $\vec{x}(t) = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$. Then, in eq. (9), we set up the differential equations as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \vec{x}(t) \tag{31}$$

As discussed, let's begin by finding a coordinate system in which the transformation represented by this matrix is simply diagonal.

First, we must find its eigenvalues, i.e. the roots of its characteristic equation:

$$\det\left(\lambda I - A\right) = \det\left(\begin{bmatrix} \lambda + 5 & -2 \\ -2 & \lambda + 2 \end{bmatrix}\right) \tag{32}$$

$$= (\lambda + 5)(\lambda + 2) - 4 \tag{33}$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda + 6)(\lambda + 1) \tag{34}$$

Solving the above characteristic equation, we get the eigenvalues as $\lambda_1 = -6$ and $\lambda_2 = -1$ and hence we can find a set of eigenvectors as $\vec{v}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$. Defining the new basis as $V = [\vec{v}_1, \vec{v}_2]$, we get the diagonal matrix $\Lambda = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}$.

Hence, we can write $A = V \Lambda V^{-1}$ and rewrite eq. (9) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = V \Lambda V^{-1}\vec{x}(t) \tag{35}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \vec{x}(t) \tag{36}$$

Let's perform a change of basis by setting $\vec{\tilde{x}}(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = V^{-1}\vec{x}(t)$. By left multiplying V^{-1} on both sides of eq. (35), we get the following:

$$V^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = V^{-1}A\vec{x}(t) = V^{-1}V \Lambda V^{-1}\vec{x}(t) \tag{37}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}V^{-1}\vec{x}(t) = \Lambda\vec{\tilde{x}}(t) \tag{38}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}} = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \vec{\tilde{x}}(t) \tag{39}$$

Because V^{-1} is a constant matrix, we can pull it inside the derivative to go from eq. (37) to eq. (38). In eq. (39), we have successfully uncoupled our equations and we can proceed to solve them independenly as mentioned earlier:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1(t) = -6\tilde{x}_1 t) \tag{40}$$

$$\implies \tilde{x}_1 = k_1 \mathrm{e}^{-6t} \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_2(t) = -\tilde{x}_2(t) \tag{42}$$

$$\implies \tilde{x}_2 = k_2 \mathrm{e}^{-t} \tag{43}$$

Next, we need to solve for our constants k_1 and k_2. Recall our initial conditions, $V_1(0) = V_2(0) = 1V$.

Hence, $\widetilde{x}_1(0)$ and $\widetilde{x}_2(0)$ are given by[6]:

$$\vec{\widetilde{x}}(0) = V^{-1} \begin{bmatrix} V_1(0) \\ V_2(0) \end{bmatrix} \tag{44}$$

$$\Rightarrow \begin{bmatrix} \widetilde{x}_1(0) \\ \widetilde{x}_2(0) \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{45}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix} \tag{46}$$
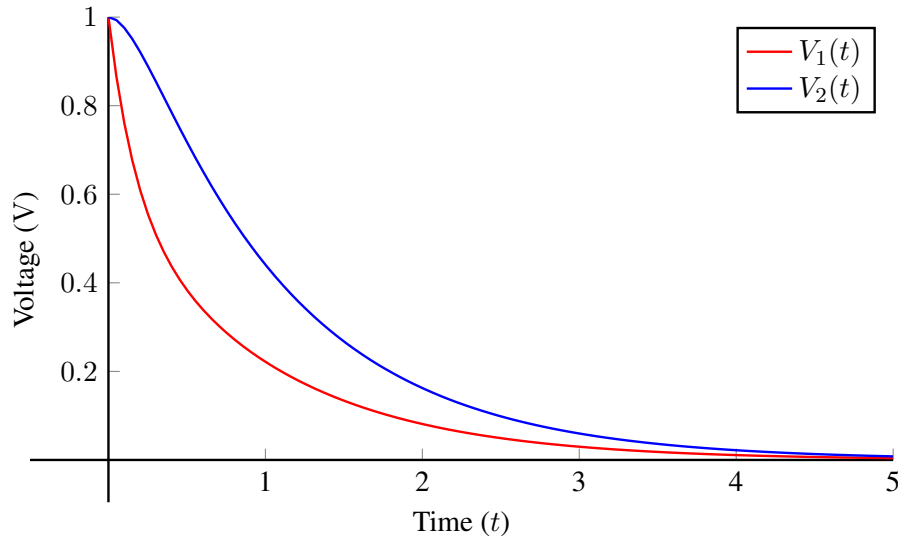
Hence, $k_1 = -\frac{1}{\sqrt{5}}$ and $k_2 = \frac{3}{\sqrt{5}}$. Now, we can transform back into our original basis as follows to find $V_1(t)$ and $V_2(t)$:

$$\vec{x} = V\vec{\widetilde{x}} \tag{47}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{5}}e^{-6t} \\ \frac{3}{\sqrt{5}}e^{-t} \end{bmatrix} \tag{48}$$

$$= \begin{bmatrix} \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t} \\ -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t} \end{bmatrix} \tag{49}$$
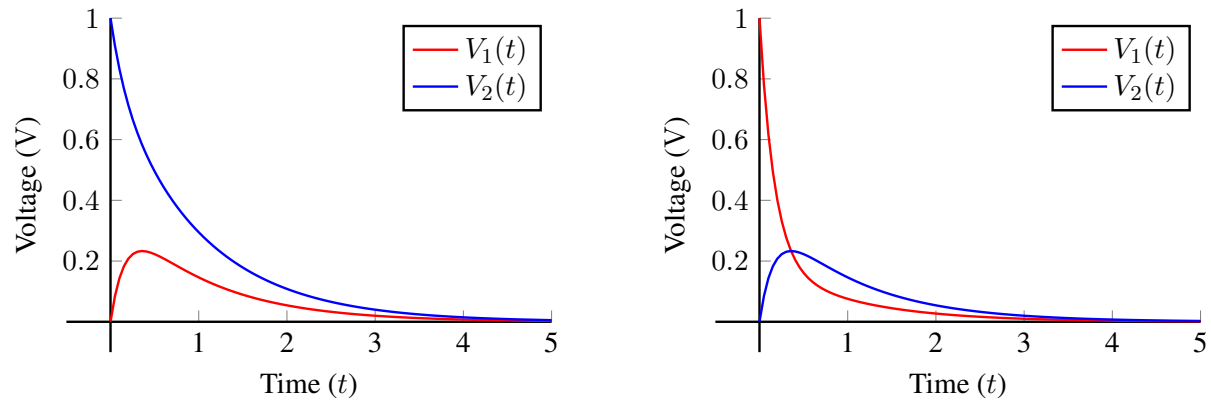
For $t \geq 0$, we find that $V_1 = \frac{2}{5}e^{-6t} + \frac{3}{5}e^{-t}$ and $V_2 = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$. Figure 4 is a plot of our solutions:

Figure 4: Initial Conditions: $V_1(0) = 1V$ and $V_2(0) = 1V$. Homogeneous Case Solution.

Using the same argument, we can see how the voltage will vary with different initial conditions (as shown in Figure 5):

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[6]Note that in this case, $V^{-1} = V^{\top} = V$ so when we plug in V^{-1} below, it's the same as plugging in V.

(a) Initial Conditions: $V_1(0) = 0V$ and $V_2(0) = 1V$. Here, $V_1(t) = -\frac{2}{5}e^{-6t} + \frac{2}{5}e^{-t}$, and $V_2(t) = \frac{1}{5}e^{-6t} + \frac{4}{5}e^{-t}$.

(b) Initial Conditions: $V_1(0) = 1V$ and $V_2(0) = 0V$. Here, $V_1(t) = \frac{4}{5}e^{-6t} + \frac{1}{5}e^{-t}$, and $V_2(t) = -\frac{2}{5}e^{-6t} + \frac{2}{5}e^{-t}$.

Figure 5: Voltage transients for different initial conditions on the capacitors. The specific solutions are in the subcaptions for your reference.

Concept Check: Take a minute to qualitatively reason about the initial increase in voltages in Figure 5.

Now that we have a good understanding of the homogenous case, let's look at the voltage transients of charging our two capacitor system. In this case, we have two uncharged capacitors, i.e. $V_1(0) = V_2(0) = 0V$, and we apply a voltage $V_{in} = 1V$ for time $t > 0$. We get the following equations:
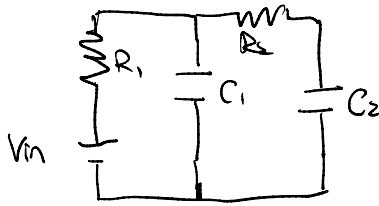
$$V_2 = V_1 - I_2 R_2 \tag{50}$$

$$I_2 = C_2 \frac{d}{dt} V_2 \tag{51}$$

$$V_{in} - V_1 = I_1 R_1 \Rightarrow I_1 = \frac{V_{in} - V_1}{R_1} \tag{52}$$

$$I_1 = I_2 + C_1 \frac{d}{dt} V_1 \tag{53}$$

Hence, our matrix differential equation is:

$$\frac{d}{dt}\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{V_{in}}{R_1 C_1} \\ 0 \end{bmatrix} \tag{54}$$

$$= \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{55}$$

$$= A\vec{x}(t) + \vec{b}. \tag{56}$$

Now the system has a non-homogeneous term which we haven't seen before, but let's see if diagonalization can help us simplify the problem. Letting $\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$ and using $A = V\Lambda V^{-1}$, we plug in:

$$\frac{d}{dt}\vec{\tilde{x}}(t) = \frac{d}{dt}V^{-1}\vec{x}(t) = V^{-1}\left(\frac{d}{dt}\vec{x}(t)\right) \tag{57}$$

$$= V^{-1}\left(A\vec{x}(t) + b\right) \tag{58}$$

$$= V^{-1}A\vec{x}(t) + V^{-1}b \tag{59}$$

$$V^{-1}x(t)$$

$$= V^{-1}AV\vec{\widetilde{x}}(t) + V^{-1}b \tag{60}$$

$$= \Lambda\vec{\widetilde{x}}(t) + V^{-1}b. \tag{61}$$

In this case, recall that from earlier in the notes

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \tag{62}$$

so then

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\widetilde{x}}(t) = \Lambda\vec{\widetilde{x}}(t) + V^{-1}b \tag{63}$$

$$= \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}\vec{\widetilde{x}}(t) + \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \tag{64}$$

$$= \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}\vec{\widetilde{x}}(t) + \begin{bmatrix} -\frac{6}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} \end{bmatrix}. \tag{65}$$

We can thus write the following uncoupled system of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_1(t) = -6\widetilde{x}_1(t) - \frac{6}{\sqrt{5}} \tag{66}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_2(t) = -\widetilde{x}_2(t) + \frac{3}{\sqrt{5}} \tag{67}$$

We already know how to solve these non-homogeneous differential equations! (see Section 4.4 in Note 1):

$$\widetilde{x}_1(t) = \mathrm{e}^{-6t}\left(\widetilde{x}_1(0) + \frac{1}{\sqrt{5}}\right) - \frac{1}{\sqrt{5}} \tag{68}$$

$$\widetilde{x}_2(t) = \mathrm{e}^{-t}\left(\widetilde{x}_2(0) - \frac{3}{\sqrt{5}}\right) + \frac{3}{\sqrt{5}} \tag{69}$$

To get a full function for $\vec{\widetilde{x}}(t)$, we need to find $\vec{\widetilde{x}}(0)$. By the definition of $\vec{\widetilde{x}}(t)$,

$$\vec{\widetilde{x}}(t) = V^{-1}\vec{x}(t) \implies \vec{\widetilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{70}$$

Thus plugging these values in for $\vec{\widetilde{x}}(t)$,

$$\widetilde{x}_1(t) = \frac{1}{\sqrt{5}}\mathrm{e}^{-6t} - \frac{1}{\sqrt{5}} \tag{71}$$

$$\widetilde{x}_2(t) = -\frac{3}{\sqrt{5}}\mathrm{e}^{-t} + \frac{3}{\sqrt{5}} \tag{72}$$

or, putting it together,

$$\vec{\widetilde{x}}(t) = \begin{bmatrix} \frac{1}{\sqrt{5}}\mathrm{e}^{-6t} - \frac{1}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}}\mathrm{e}^{-t} + \frac{3}{\sqrt{5}} \end{bmatrix} \tag{73}$$

Remember, we require $\vec{x}(t)$, not $\vec{\tilde{x}}(t)$. By definition of $\vec{\tilde{x}}(t)$,

$$\vec{x}(t) = V\vec{\tilde{x}}(t) \tag{74}$$

$$= \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}}e^{-6t} - \frac{1}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}}e^{-t} + \frac{3}{\sqrt{5}} \end{bmatrix} \tag{75}$$

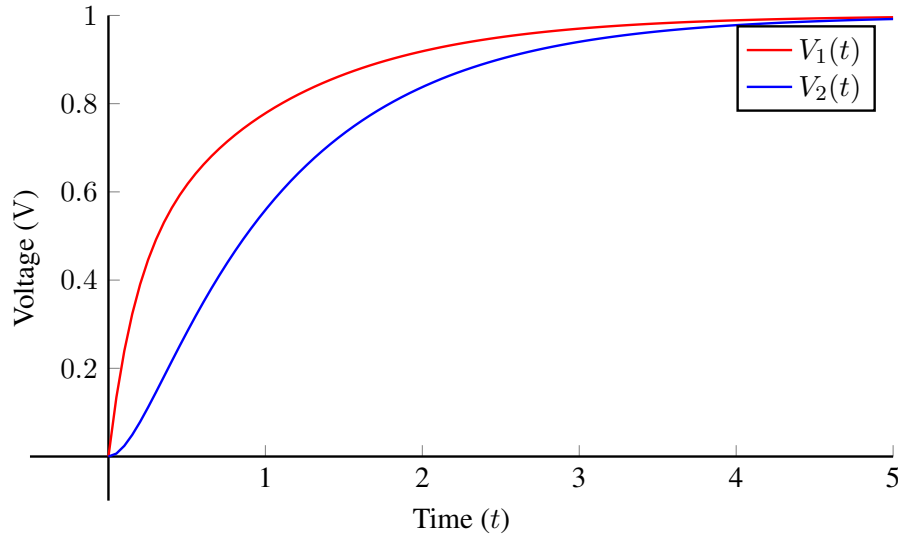$$= \begin{bmatrix} -\frac{2}{5}e^{-6t} - \frac{3}{5}e^{-t} + 1 \\ \frac{1}{5}e^{-6t} - \frac{6}{5}e^{-t} + 1 \end{bmatrix}. \tag{76}$$

Finally, we have $V_1(t) = 1 - \frac{2}{5}e^{-6t} - \frac{3}{5}e^{-t}$ and $V_2(t) = 1 + \frac{1}{5}e^{-6t} - \frac{6}{5}e^{-t}$.

Figure 6: Initial Conditions: $V_1(0) = 0V$ and $V_2(0) = 0V$. Nonhomogeneous Case Solution.

4   Appendix: Algebra for Formation of Matrix Equation

We start with the following equations:

$$V_2 = V_1 - I_2 R_2 \tag{77}$$

$$I_2 = C_2 \frac{\mathrm{d}}{\mathrm{d}t} V_2 \tag{78}$$

$$0 - V_1 = I_1 R_1 \Rightarrow I_1 = -\frac{V_1}{R_1} \tag{79}$$

$$I_1 = I_2 + C_1 \frac{\mathrm{d}}{\mathrm{d}t} V_1 \tag{80}$$

Our goal is to rearrange this into a form where the derivatives of our state quantities (voltages in this case) are expressed in terms of the voltages themselves and the input voltage.[7] Keeping this in mind, we start with

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[7]There are multiple ways to arrive at eq. (5) and eq. (6); this is just one valid sequence of steps.

the simpler equation to derive:

$$\text{eq. (78)} \implies \qquad \frac{\mathrm{d}}{\mathrm{d}t}V_2(t) = \frac{I_2}{C_2} \tag{81}$$

$$\text{eq. (77)} \implies \qquad I_2 = \frac{V_1(t)}{R_2} - \frac{V_2(t)}{R_2} \tag{82}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_2(t) = \frac{V_1(t)}{R_2 C_2} - \frac{V_2(t)}{R_2 C_2} \tag{83}$$

And now:

$$\text{eq. (80)} \implies \qquad \frac{\mathrm{d}}{\mathrm{d}t}V_1(t) = \frac{I_1}{C_1} - \frac{I_2}{C_1} \tag{84}$$

$$\text{eq. (82), eq. (79)} \implies \qquad \frac{\mathrm{d}}{\mathrm{d}t}V_1(t) = \frac{I_1}{C_1} - \frac{\frac{V_1(t)}{R_2} - \frac{V_2(t)}{R_2}}{C_1} \tag{85}$$

$$= \frac{-\frac{V_1(t)}{R_1}}{C_1} - \frac{\frac{V_1(t)}{R_2} - \frac{V_2(t)}{R_2}}{C_1} \tag{86}$$

Simplifying:

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) = -\frac{V_1(t)}{R_1 C_1} - \frac{V_1(t)}{R_2 C_1} + \frac{V_2(t)}{R_2 C_1} \tag{87}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1(t) = \left(-\frac{1}{R_1 C_1} - \frac{1}{R_2 C_1} \right) V_1(t) + \frac{1}{R_2 C_1} V_2(t) \tag{88}$$

And these two equations (eq. (88), eq. (83)) exactly match eq. (5), eq. (6).

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