

# CS61B Lecture #20: Trees

# A Recursive Structure

- Trees naturally represent recursively defined, hierarchical objects with more than one recursive subpart for each instance.
- Common examples: expressions, sentences.
  - Expressions have definitions such as "an expression consists of a literal or two expressions separated by an operator."
- Also describe search structures in which we recursively divide a set into multiple disjoint subsets.

# Formal Definitions

- Trees come in a variety of flavors, all defined recursively:
  - **61A style:** A tree consists of a *label* value and zero or more *branches* (or *children*), each of them a tree.
  - **61A style, alternative definition:** A tree is a set of *nodes* (or *vertices*), each of which has a label value and one or more *child nodes*, such that no node descends (directly or indirectly) from itself. A node is the *parent* of its children.
  - **Positional trees:** A tree is either *empty* or consists of a node containing a label value and an indexed sequence of zero or more children, each a positional tree. If every node has two positions, we have a *binary tree* and the children are its *left and right subtrees*. Again, nodes are the parents of their non-empty children.
  - We'll see other varieties when considering graphs.

二元树

# Tree Characteristics (I)

- The **root** of a tree is a non-empty node with no parent in that tree (its parent might be in some larger tree that contains that tree as a subtree). Thus, every node is the root of a (sub)tree.
- The **order**, **arity**, or **degree** of a node (tree) is its **number** (maximum number) of children.
- The nodes of a  **$k$ -ary tree** each have at most  $k$  children.
- A **leaf** node has no children (no non-empty children in the case of positional trees).

## Tree Characteristics (II)

- The *height* of a node in a tree is the largest distance to a leaf. That is, a leaf has height 0 and a non-empty tree's height is one more than the maximum height of its children. The height of a tree is the height of its root.
- The *depth* of a node in a tree is the distance to the root of that tree. That is, in a tree whose root is  $R$ ,  $R$  itself has depth 0 in  $R$ , and if node  $S \neq R$  is in the tree with root  $R$ , then its depth is one greater than its parent's.

# A Tree Type, 61A Style

```
/** An immutable Tree whose labels are of type LABEL. */
public class Tree<Label> {

    @SuppressWarnings("unchecked") // Explained in a later lecture
    public Tree(Label label, Tree<Label>... children) {
        _label = label;
        _kids = new ArrayList<>(Arrays.asList(children));
    }

    public int arity() { return _kids.size(); }

    public Label label() { return _label; }

    public Tree<Label> child(int k) { return _kids.get(k); }

    private Label _label;
    private ArrayList<Tree<Label>> _kids;
}
```

# Fundamental Operation: Traversal

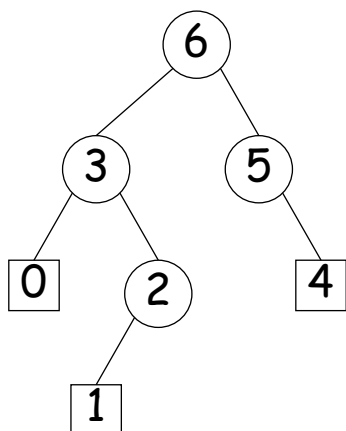
- *Traversing a tree* means enumerating (some subset of) its nodes.
- Typically done **recursively**, because that is **natural description**.
- As nodes are enumerated, we say they are *visited*.
- Three basic orders for enumeration (+ variations):

前序  
后序  
中序

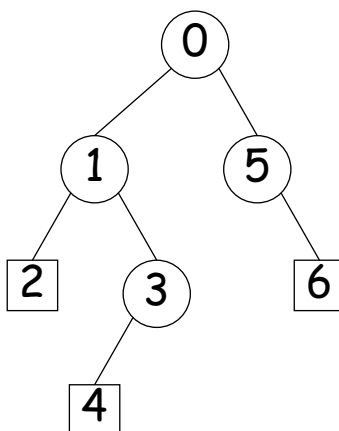
- **Preorder**: visit node, traverse its children.

- **Postorder**: traverse children, visit node.

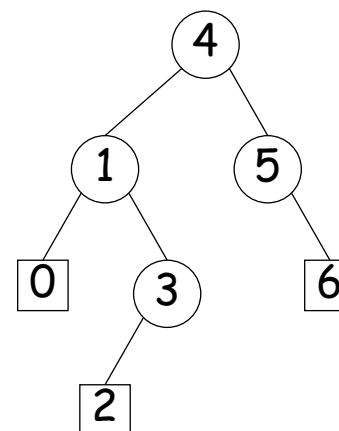
- **Inorder**: traverse first child, visit node, traverse second child (binary trees only).



Postorder



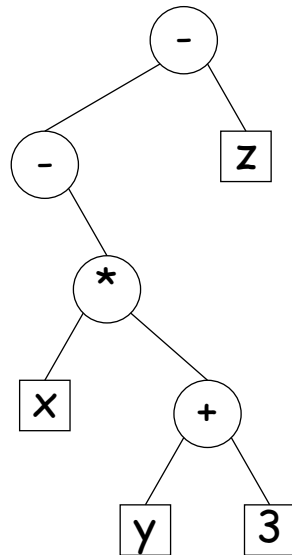
Preorder



inorder

# Preorder Traversal and Prefix Expressions

**Problem: Convert**



**into**

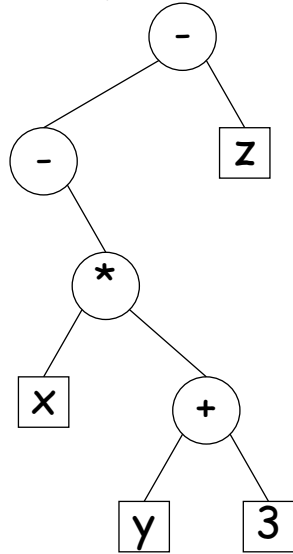
$(- (- (* x (+ y 3))) z)$

```
static String toLisp(Tree<String> T) {  
    if (T.arity() == 0) return T.label();  
    else {  
        String R; R = "(" + T.label();  
        for (int i = 0; i < T.arity(); i += 1)  
            R += " " + toLisp(T.child(i));  
        return R + ")";  
    }  
}
```



# Inorder Traversal and Infix Expressions

Problem: Convert



into

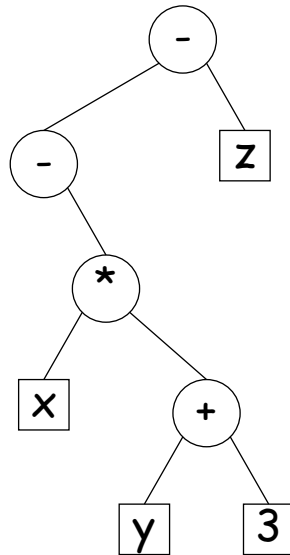
$((-(x*(y+3)))-z)$

To think about: how to get rid of all those parentheses.

```
static String toInfix(Tree<String> T) {  
    if (T.arity() == 0) {  
        return T.label();  
    } else if (T.arity() == 1) {  
        return "(" T.label() + toInfix(T.child(0)) + ")";  
    } else {  
        return "(" toInfix(T.child(0)) + T.label() + toInfix(T.child(1)) + ")";  
    }  
}
```

# Postorder Traversal and Postfix Expressions

Problem: Convert



into

x y 3 +:2 \*:2 -:1 z -:2

```
static String toPolish(Tree<String> T) {  
    String R; R = "";  
    for (int i = 0; i < T.arity(); i += 1)  
        R += toPolish(T.child(i)) + " ";  
    return R + String.format("%s:%d", T.label(), T.arity());  
}
```

# A General Traversal: The Visitor Pattern

```
void preorderTraverse(Tree<Label> T, Consumer<Tree<Label>> visit)
{
    if (T != null) {
        visit.accept(T);
        for (int i = 0; i < T.arity(); i += 1)
            preorderTraverse(T.child(i), visit);
    }
}
```

- `java.util.function.Consumer<AType>` is a library interface that works as a function-like type with one void method, `accept`, which takes an argument of type `AType`.
- Now, using Java 8 lambda syntax, I can print all labels in the tree in preorder with:

```
preorderTraverse(myTree, T -> System.out.print(T.label() + " "));
```

# Iterative Depth-First Traversals

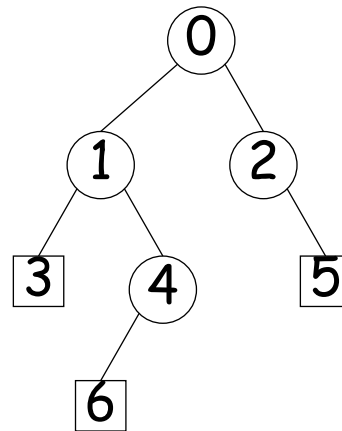
- Tree recursion conceals data: a *stack* of nodes (all the T arguments) and a little extra information. Can make the data explicit:

```
void preorderTraverse2(Tree<Label> T, Consumer<Tree<Label>> visit) {  
    Stack<Tree<Label>> work = new Stack<>();  
    work.push(T);  
    while (!work.isEmpty()) {  
        Tree<Label> node = work.pop();  
        visit.accept(node);  
        for (int i = node.arity()-1; i >= 0; i -= 1)  
            work.push(node.child(i)); // Why backward?  
    }  
}
```

- This traversal takes the same  $\Theta(\cdot)$  time as doing it recursively, and also the same  $\Theta(\cdot)$  space.
- That is, we have substituted an explicit stack data structure (*work*) for Java's built-in execution stack (which handles function calls).

# Level-Order (Breadth-First) Traversal

**Problem:** Traverse all nodes at depth 0, then depth 1, etc:

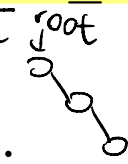


# Breadth-First Traversal Implemented

A simple modification to iterative depth-first traversal gives breadth-first traversal. Just change the (LIFO) stack to a (FIFO) queue:

```
void breadthFirstTraverse(Tree<Label> T, Consumer<Tree<Label>> visit) {  
    ArrayDeque<Tree<Label>> work = new ArrayDeque<>(); // (Changed)  
    work.push(T);  
    while (!work.isEmpty()) {  
        Tree<Label> node = work.remove(); // (Changed)  
        if (node != null) {  
            visit.accept(node);  
            for (int i = 0; i < node.arity(); i += 1) // (Changed)  
                work.push(node.child(i));  
        }  
    }  
}
```

# Times

- The traversal algorithms have roughly the form of the boom example in §1.3.3 of *Data Structures*—an exponential algorithm.
- However, the role of  $M$  in that algorithm is played by the *height* of the tree, not the number of nodes.
- In fact, easy to see that tree traversal is *linear*:  $\Theta(N)$ , where  $N$  is the # of nodes: Form of the algorithm implies that there is one visit at the root, and then one visit for every *edge* in the tree. Since every node but the root has exactly one parent, and the root has none, must be  $N - 1$  edges in any non-empty tree.
- In positional tree, is also one recursive call for each empty tree, but # of empty trees can be no greater than  $kN$ , where  $k$  is arity.
- For  $k$ -ary tree (max # children is  $k$ ),  $\underbrace{h + 1}_{\text{root}} \leq N \leq \frac{k^{h+1} - 1}{k - 1}$ , where  $h$  is height.  


each node has  $k$  children  
 $1 + k + k^2 + \dots + k^h = \text{Geometric Series.}$
- So  $h \in \Omega(\log_k N) = \Omega(\lg N)$  and  $h \in O(N)$ .
- Many tree algorithms look at one child only. For them, worst-case time is proportional to the *height* of the tree— $\Theta(\lg N)$ —assuming that tree is *bushy*—each level has about as many nodes as possible.

# Recursive Breadth-First Traversal: Iterative Deepening

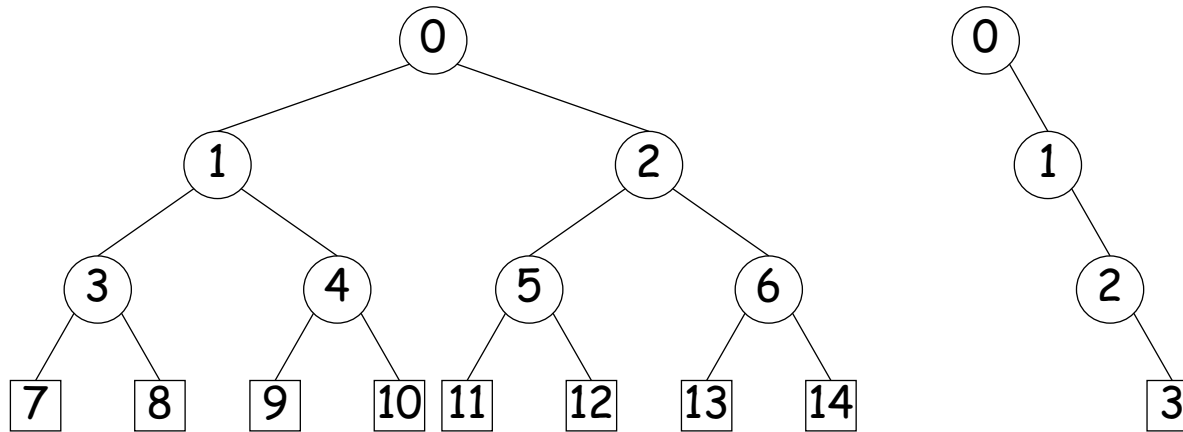
- Previous breadth-first traversal used space proportional to the *width* of the tree, which is  $\Theta(N)$  for bushy trees, whereas depth-first traversal takes  $\lg N$  space on bushy trees.
- Can we get breadth-first traversal in  $\lg N$  space and  $\Theta(N)$  time on bushy trees?
- For each level,  $k$ , of the tree from 0 to *lev*, call `doLevel(T, k)`:

```
void doLevel(Tree T, int lev) {  
    if (lev == 0)  
        visit T  
    else  
        for each non-null child, C, of T {  
            doLevel(C, lev-1);  
        }  
}
```

- So we do breadth-first traversal by repeated (truncated) depth-first traversals: *iterative deepening*.
- In `doLevel(T, k)`, we skip (i.e., traverse but don't visit) the nodes before level  $k$ , and then visit at level  $k$ , but not their children.



# Iterative Deepening Time?



- Let  $h$  be height,  $N$  be # of nodes.
- Count # edges traversed (i.e, # of calls, not counting null nodes).
- First (full) tree: 1 for level 0, 3 for level 1, 7 for level 2, 15 for level 3.
- Or in general  $(2^1 - 1) + (2^2 - 1) + \dots + (2^{h+1} - 1) = 2^{h+2} - h \in \Theta(N)$ , since  $N = 2^{h+1} - 1$  for this tree.
- Second (*right leaning*) tree: 1 for level 0, 2 for level 2, 3 for level 3.
- Or in general  $(h + 1)(h + 2)/2 = N(N + 1)/2 \in \Theta(N^2)$ , since  $N = h + 1$  for this kind of tree.

# Iterators for Trees

- Frankly, iterators are not terribly convenient on trees.
- But can use ideas from iterative methods.

```
class PreorderTreeIterator<Label> implements Iterator<Label> {  
    private Stack<Tree<Label>> s = new Stack<Tree<Label>>();  
  
    public PreorderTreeIterator(Tree<Label> T) { s.push(T); }  
  
    public boolean hasNext() { return !s.isEmpty(); }  
    public T next() {  
        Tree<Label> result = s.pop();  
        for (int i = result.arity()-1; i >= 0; i -= 1)  
            s.push(result.child(i));  
        return result.label();  
    }  
}
```

**Example:** (what do I have to add to class Tree first?)

```
for (String label : aTree) System.out.print(label + " ");
```

```

graph TD
    0((0)) --- 1((1))
    0 --- 2((2))
    0 --- 3((3))
  
```

