EECS 16B   Designing Information Devices and Systems II
Fall 2021   Note 5: Phasors and AC Circuits

1 Overview

So far in the class, we have only been dealing with circuits that have constant voltage/current sources, which are called *DC (direct current) circuits*. Additionally, we focused on how the DC circuit's behavior changed over time as the differential equation evolved, which is called *transient analysis.*

However, now we will pivot to a new class of circuits that have time-varying voltage/current sources, and so our old techniques need to be modified. Moreover, we will restrict ourselves to only sinusoidal voltage/current sources, which are called *AC (alternating current) circuits.* For this new circuit type, analyzing the differential equation won't be too helpful and so we now instead focus on the steady-state behavior of the AC circuits, which is called *frequency analysis*. The steady-state behavior is how the circuit acts after a long time, and is important to us since it shows the eventual response to our input.

It's natural to wonder: what's so special about sinusoids? For one, sinusoidal sources are a very common category of inputs, making this form of analysis very useful. Also, there's a technique known as the Fourier Transform, that can express many input signals as weighted sums of sinusoidal functions, and we can apply superposition.[1]

The most important reason, however, is that analyzing sinusoidal functions is *easy*! Whereas analyzing arbitrary input signals (like in transient analysis) requires us to solve a set of differential equations, it turns out that we can use a procedure very similar to the seven-step procedure from EECS16A in order to solve AC circuits.

2 Scalar Differential Equation with Exponential Input

We've already seen that general linear circuits with sources, resistors, capacitors, and inductors can be thought of as a system of linear, first-order, differential equations with sinusoidal input. By developing techniques to determine the steady state of such systems in general, we can hope to apply them to the special case of circuit analysis. Let's step back from circuits for a little bit and revisit the fundamentals.

First, we'll look at the scalar case, for simplicity. Consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \lambda x(t) + u(t), \tag{1}$$

where the input $u(t)$ is of the form

$$u(t) = ke^{st} \tag{2}$$

where $s \neq \lambda.$

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[1]This topic is one focus of classes like EE120.

We've previously seen (in HW 3) how to solve this equation:

$$x(t) = \left(x_0 - \frac{k}{s-\lambda}\right)e^{\lambda t} + \frac{k}{s-\lambda}e^{st}, \tag{3}$$

where $x(0) = x_0$ is the initial value of $x(t)$.

Interestingly, this is *almost* a scalar multiple of $u(t)$ - if only we could ignore the initial term involving $e^{\lambda t}$, then $x(t)$ would linearly depend on $u(t)$.

When might $e^{\lambda t} \to 0$ happen as $t \to \infty$ (i.e. steady state)? If λ were real, then the term decays to zero if and only if $\lambda < 0$. But what about for complex λ? We can try writing a complex λ in the form $\lambda = \lambda_r + j\lambda_i$, to try and reduce the problem to the real case. That is:

$$e^{\lambda t} = e^{(\lambda_r + j\lambda_i)t} \tag{4}$$
$$= e^{\lambda_r t}e^{j\lambda_i t} \tag{5}$$

The $e^{\lambda_r t}$ is exactly the real case we just saw above. But what about the $e^{j\lambda_i t}$ term? Well, the only thing we can really do here is apply Euler's formula. Expanding our expression, we find that

$$e^{\lambda t} = e^{\lambda_r t}(\cos(\lambda_i t) + j\sin(\lambda_i t)) \tag{6}$$

The first term in the product is a real exponential, which we know decays to zero exactly when $\text{Re}\{\lambda\} = \lambda_r < 0$. The second term is a sum of two sinusoids with unit amplitudes. Since the amplitude of each sinusoid is constant, their sum will not decay to zero or grow to infinity over time. Thus, the asymptotic behavior of the overall expression is governed solely by the first term ($e^{\lambda t}$ will decay to zero exactly when e^{λ_r} does).

Thus, looking back at our solution for $x(t)$, we now know that $e^{\lambda t}$ decays to 0 whenever $\text{Re}\{\lambda\} < 0$.

3 System of Differential Equations with Exponential Input

Can we apply similar techniques to what we've just seen to a system of differential equations? Specifically, consider the system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{u}(t), \tag{7}$$

where A is a fixed, real, matrix. As before, we will consider only control inputs of a special form, where each component is of the form ke^{st} for some constant k. Said differently, we will only examine cases where $\vec{u}(t)$ can be expressed in the form

$$\vec{u}(t) = \vec{u}_0 e^{st}, \tag{8}$$

where $\vec{u}_0$ does not depend on t, and s is *not* an eigenvalue of the matrix A (this will be important later).

Assume, for now [2], that A has a linearly independent set of eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with corresponding

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[2]Later in the semester, we will learn about upper triangularization that allows the following analysis to apply to all matrices, not just diagonalizable ones.

eigenvalues $\lambda_1, \ldots, \lambda_n$. Then we know that we can diagonalize A to be

$$
A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \tag{9}
$$

$$
= V \Lambda V^{-1}, \tag{10}
$$

where V and Λ are the eigenvector and eigenvalue matrices in the above diagonalization.

Thus, we can construct a new coordinate system $z(t) = V^{-1} x(t)$ to rewrite our differential equation for $x(t)$ as

$$
\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = V \Lambda V^{-1} \vec{x}(t) + \vec{u}(t) \tag{11}
$$

$$
V^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \left(V^{-1} \vec{x}(t) \right) = \Lambda (V^{-1} \vec{x}(t)) + V^{-1} \vec{u}(t) \tag{12}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \vec{z}(t) = \Lambda \vec{z}(t) + V^{-1} \vec{u}(t) \tag{13}
$$

In equation (12), we left hand multiply both sides by V^{-1}, and in (13) we rewrite $V^{-1} \vec{x}(t)$ as $\vec{z}(t)$. We see that this diagonalized system of differential equations can be rewritten as a set of scalar differential equations of the form

$$
\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \lambda_i z_i(t) + (V^{-1} \vec{u}_0 \mathrm{e}^{st})_i, \tag{14}
$$

where the subscript $_i$ represents the ith component of the associated vector, and λ_i is the ith eigenvalue of A. Since $(V^{-1} \vec{u}_0 \mathrm{e}^{st})_i = (V^{-1} \vec{u}_0)_i \mathrm{e}^{st}$ is a multiple of e^{st} and $s \neq \lambda_i$, we know from our scalar results that the solution to $z_i(t)$ can be expressed as a linear combination of $\mathrm{e}^{\lambda_i t}$ and e^{st}, where the $\mathrm{e}^{\lambda_i t}$ decays to zero over time if and only if $\mathrm{Re}\{\lambda_i\} < 0$. In this case, this yields a steady state solution involving only a scalar multiple of e^{st}, i.e.

$$
\lim_{t \to \infty} z_i(t) = \alpha_i \mathrm{e}^{st} \tag{15}
$$

Then, we can stack these solutions into vector form and pre-multiply by V to obtain

$$
\lim_{t \to \infty} \vec{z}(t) = \vec{\alpha} \mathrm{e}^{st} \tag{16}
$$

$$
\lim_{t \to \infty} \vec{x}(t) = \lim_{t \to \infty} V \vec{z}(t) = (V \vec{\alpha}) \mathrm{e}^{st} \tag{17}
$$

So as $t \to \infty$, our steady-state solution for $\vec{x}(t)$ will also be a multiple of e^{st}.

Let us call $\vec{x}_0 = V \vec{\alpha}$ for notational simplicity. We will now try to find a formula for $\vec{x}_0$ using the fact that $\vec{x}(t) = \vec{x}_0 \mathrm{e}^{st}$ in steady-state, for some constant vector $\vec{x}_0$. Since the differential equation must still apply for the steady-state solution (by definition of what it means to be a solution),

$$
\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = A \vec{x}(t) + \vec{u}(t) \tag{18}
$$

$$
s \vec{x}_0 \mathrm{e}^{st} = A \vec{x}_0 \mathrm{e}^{st} + \vec{u}_0 \mathrm{e}^{st} \tag{19}
$$

$$
(sI - A) \vec{x}_0 \mathrm{e}^{st} = \vec{u}_0 \mathrm{e}^{st} \tag{20}
$$

$$
(sI - A) \vec{x}_0 = \vec{u}_0 \tag{21}
$$

Now, to express $\vec{x}_0$ in terms of $\vec{u}_0$, we are tempted to multiply by the inverse of $sI - A$ on both sides. Before taking the inverse, we need to ensure this operation is valid. To do this, we will use a proof by contradiction. Imagine that $sI - A$ is not invertible, so $sI - A$ has a nonempty null space, containing some vector $\vec{y}$. We could then write:

$$(sI - A)\vec{y} = \vec{0} \tag{22}$$

$$\implies A\vec{y} = s\vec{y} \tag{23}$$

so s would be an eigenvalue of A with corresponding eigenvector $\vec{y}$. However, recall our original statement that s is not an eigenvalue of A. This result contradicts our original statement and so our assumption that $sI - A$ is not invertible must be incorrect, meaning $sI - A$ is invertible!

Thus, we can rearrange equation (21) to get
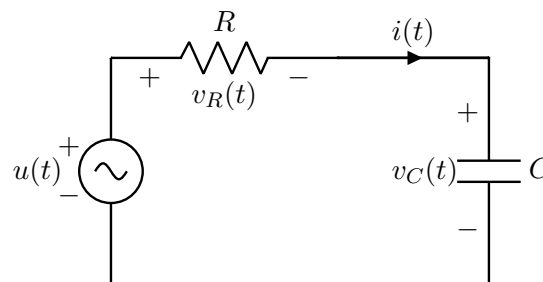
$$\vec{x}_0 = (sI - A)^{-1}\vec{u}_0 \tag{24}$$

$$\lim_{t \to \infty} \vec{x}(t) = \vec{x}_0 e^{st} = (sI - A)^{-1}\vec{u}_0 e^{st} \tag{25}$$

Thus, we have found the exact steady-state solution given our input vector [3].

4   Circuits with Exponential Inputs

Now let's take a look at what happens if our circuit was driven by inputs of the exponential function e^{st} for some constant s.

Consider a particular capacitor C within the following circuit:

From Note 1, we know the differential equation for the capacitor voltage is

$$\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) = -\frac{v_C(t)}{RC} + \frac{u(t)}{RC} \tag{26}$$

Since $-\frac{1}{RC} < 0$, we know from our analysis of scalar differential equations in Section 2 that at steady-state, the voltage will become a multiple of e^{st}:

$$v_C(t) = \widetilde{V}_C e^{st} \tag{27}$$

for some scalar $\widetilde{V}_C$.

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[3]You can even verify yourself that the steady-state solution satisfies the differential equation, but you will need to end up using the Woodbury matrix identity.

By the known differential equation for a capacitor, we have that

$$i_C(t) = C\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) \tag{28}$$

$$= C\frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{V}_C\mathrm{e}^{st}\right) \tag{29}$$

$$= Cs\widetilde{V}_C\mathrm{e}^{st} \tag{30}$$

This means that $i_C(t)$ is also a scalar multiple of e^{st}, with this scalar factor being $\widetilde{I}_C = sC\widetilde{V}_C$. Critically, this equation resembles that of Ohm's Law! It is a purely linear equation without any time-dependence.

Similar equations can be obtained (which we do later) for an inductor $\widetilde{V}_L = \widetilde{I}_L \cdot sL$. and a resistor $\widetilde{V}_R = \widetilde{I}_R \cdot R$. Rewriting the capacitor relationship to be in the same form, we see $\widetilde{V}_C = \widetilde{I}_C \cdot (\frac{1}{sC})$. This suggests that we can view capacitors, inductors, and resistors as all being similar. In effect, capacitors and inductors just have s-dependent resistances. These are called s-impedances. The term *impedance* is a generalization of the concept of resistance to allow for different resistances at different s-values (we will soon see what governs the value of s in a circuit context.). The impedance is defined as the voltage-current ratio (similar to resistance):

$$\widetilde{Z} = \frac{\widetilde{V}}{\widetilde{I}} \tag{31}$$

A capacitor has an s-impedance of $\frac{1}{sC}$. An inductor has an s-impedance of sL. And a resistor's s-impedance is just the same as its resistance R.

This reveals an approach for circuit analysis with exponential inputs, as long as all the inputs have the same s. We can replace all the independent voltage and current sources with constant voltages and currents corresponding to only the coefficients of e^{st}. We replace all capacitors and inductors with their corresponding impedances, and then analyze the entire circuit as though it only had resistances in it to get the steady state solution. We will see examples of this in practice in the later sections.

5 Sinusoids and Phasors

So far, we stated that $\vec{u}(t)$ should be expressed as

$$\vec{u}(t) = \vec{u}_0\mathrm{e}^{st} \tag{32}$$

for some s. What kinds of s are possible in circuits? If $\mathrm{Re}\{s\} < 0$, then we know that the input approaches zero over time, so the steady-state behavior of our system won't be controlled by the input. If $\mathrm{Re}\{s\} > 0$, then our input will grow to infinity over time, so our state will blow up! This only leaves the case $\mathrm{Re}\{s\} = 0$ as neither blowing up or decaying away. If $\mathrm{Re}\{s\} = 0$, then s must be purely imaginary, so $s = \mathrm{j}\omega$ for some real ω. Then our input will be $\mathrm{e}^{st} = \mathrm{e}^{\mathrm{j}\omega t}$.

$\rightarrow \mathrm{e}^{\mathrm{j}\theta} = (\cos(\theta) + \mathrm{j}\sin(\theta))$

We now will investigate some properties of complex exponentials. We can combine Euler's formula with complex conjugates to obtain

$$\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta} = (\cos(\theta) + \mathrm{j}\sin(\theta)) + (\cos(\theta) - \mathrm{j}\sin(\theta)) = 2\cos(\theta) \tag{33}$$

inverse Euler's formulas
$$\cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2} \tag{34}$$

In other words, starting with two complex exponentials, we have pulled out a cosine function. The same

thing can be done to pull out a sine function by instead subtracting two complex exponentials:

$$e^{j\theta} - e^{-j\theta} = (\cos(\theta) + j\sin(\theta)) - (\cos(\theta) - j\sin(\theta)) = 2j\sin(\theta) \tag{35}$$

inverse Euler's formulas

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{36}$$

These 2 equations are often called the *inverse Euler's formulas*. This is a very powerful tool as it allows us to write real sinusoid functions as a sum of 2 complex exponentials, which we already know how to deal with from the previous sections.

Before we go on, we will discuss some common properties of sinusoid functions. Consider the example function $x(t) = A\cos(\omega t)$, where $x(t)$ can be thought of as representing an input in our circuit, like an alternating current or voltage.
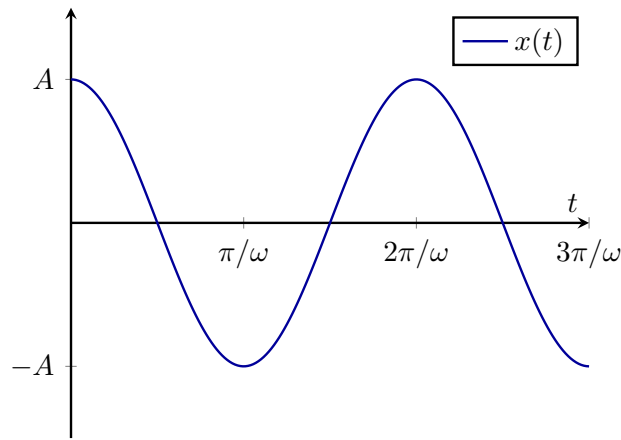
There are a couple properties of $x(t)$ that are apparent from the figure: we call the maximum value of $x(t)$ above the mean (in this case, the x-axis) the *amplitude* (A), and the spacing between repetitions of the function the *period* $(T = 2\pi/\omega)$.

However, there's one other important property of sinusoids: their *phase*. Consider a similar function $y(t) = A\cos(\omega t + \phi)$.
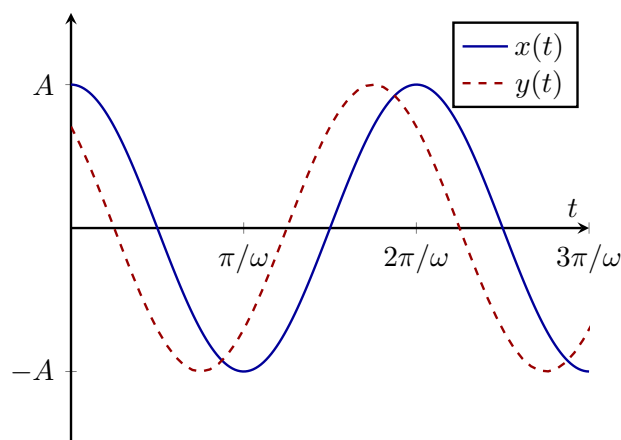
Here, ϕ represents the *phase shift* of $y(t)$ with respect to $x(t)$. As can be seen, a positive phase shift moves the function to the left by that amount. In particular, notice that the sine and cosine functions are really the same sinusoid, but just with a $\pi/2$ radian phase shift between them!

Now that we know a little about sinusoids, let's see how we can express a sinusoidal voltage input $v(t) = V_0 \cos(\omega t + \phi)$ in terms of exponential functions.

From the above inverse Euler formulas,

$$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta} \tag{37}$$

$$\implies \cos(\omega t + \phi) = \frac{1}{2}e^{j\omega t + j\phi} + \frac{1}{2}e^{-j\omega t - j\phi} \tag{38}$$

$$= \frac{e^{j\phi}}{2}e^{j\omega t} + \frac{e^{-j\phi}}{2}e^{-j\omega t} \tag{39}$$

$$\implies v(t) = V_0 \cos(\omega t + \phi) \tag{40}$$

$$= \frac{V_0 e^{j\phi}}{2}e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2}e^{-j\omega t}. \tag{41}$$

Notice that the coefficients of the two exponential functions are complex conjugates of one another. Then since $e^{-j\theta} = \overline{e^{j\theta}}$, we can rewrite the above as

$$v(t) = \frac{V_0 e^{j\phi}}{2}e^{j\omega t} + \frac{\overline{V_0 e^{j\phi}}}{2}e^{-j\omega t}. \tag{42}$$

Thus, the coefficient of the $e^{j\omega t}$ can be used to represent the entire sinusoid $v(t)$ (assuming the frequency ω is known). We call this coefficient of $e^{j\omega t}$ the **phasor** representing $v(t)$, and denote it with a capital and tilde as

$$\widetilde{V} = \frac{V_0 e^{j\phi}}{2} \tag{43}$$

which gives us $v(t) = \widetilde{V}e^{j\omega t} + \overline{\widetilde{V}}e^{-j\omega t}$.

With the phasor in hand, we can now find the steady state of systems of differential equations with sinusoidal inputs. First, use the above inverse Euler transform to write the real input as a linear combination of complex conjugate exponential functions

$$\vec{u}(t) = \vec{u}_0 e^{j\omega t} + \overline{\vec{u}_0}e^{-j\omega t} \tag{44}$$

where $\vec{u}_0$ is the vector of phasors. Then we can use our work from Section 3 to find the steady-state solution $\vec{x}_i e^{st}$ for each exponential term in the input, and then sum them using superposition.

For the first term $\vec{u}_0 e^{j\omega t}$, we just need to solve

$$(j\omega I - A)\vec{x}_1 = \vec{u}_0 \tag{45}$$

for $\vec{x}_1$, where we substituted $s = j\omega$. For the second term $\overline{\vec{u}_0}e^{-j\omega t}$, we need to solve

$$(-j\omega I - A)\vec{x}_2 = \overline{\vec{u}_0} \tag{46}$$

for $\vec{x}_2$, where we substituted $s = -j\omega$. However, note that by taking the complex conjugate of both sides of (45),

$$\overline{(j\omega I - A)\vec{x}_1} = \overline{\vec{u}_0} \tag{47}$$

$$\overline{(j\omega I - A)}\ \overline{\vec{x}_1} = \overline{\vec{u}_0} \tag{48}$$

$$(-j\omega I - A)\overline{\vec{x}_1} = \overline{\vec{u}_0} \tag{49}$$

This is exactly what we wanted to solve for in (46) so $\vec{x}_2 = \overline{\vec{x}_1}$. We can then apply superposition and sum the steady state solutions from each exponential input, finally getting

$$\vec{x}(t) = \vec{x}_1 e^{j\omega t} + \overline{\vec{x}_1} e^{-j\omega t} \tag{50}$$

$$= (j\omega I - A)^{-1} \vec{u}_0 e^{j\omega t} + \overline{(j\omega I - A)^{-1} \vec{u}_0} e^{-j\omega t} \tag{51}$$

6   Impedances

In principle, at this point we already know how to find the steady-state response when given a circuit with sinusoidal inputs all at the same frequency. But we can simplify it even more with some key findings about $j\omega$-*impedances*, which are a specialization of the s-impedances from Section 4 when letting $s = j\omega$. The analysis will be almost the exact same as what was done in Section 4, except now the voltage/current is a pure sinusoid and thus a combination of two exponential terms instead of just one. In addition, we will derive the $j\omega$-impedances in depth for the inductor and resistor along with the capacitor like before.

6.1   Impedance of a Capacitor

We examine a capacitor provided with the sinusoidal voltage $v_C(t) = V_0 \cos(\omega t + \phi)$, as shown:
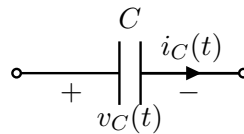
Note that we aren't assuming anything about the origin of the $v_C(t)$ – it could come from a voltage supply directly, or from some other complicated circuit. From before, we know that $v_C(t)$ has a phasor representation. Since:

$$v_C(t) = \frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2} e^{-j\omega t}, \tag{52}$$

it can be represented by the phasor

$$\widetilde{V}_C = \frac{V_0 e^{j\phi}}{2}. \tag{53}$$

Now, by the capacitor equation, we know that:

$$i_C(t) = C \frac{d}{dt} v_C(t) \tag{54}$$

$$= C \frac{d}{dt} \left(\frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2} e^{-j\omega t} \right) \tag{55}$$

$$= C \frac{d}{dt} \left(\widetilde{V}_C e^{j\omega t} + \overline{\widetilde{V}}_C e^{-j\omega t} \right) \tag{56}$$

$$= C \left(\widetilde{V}_C (j\omega) e^{j\omega t} + \overline{\widetilde{V}}_C (-j\omega) e^{-j\omega t} \right) \tag{57}$$

$$= (j\omega C) \widetilde{V}_C e^{j\omega t} + (-j\omega C) \overline{\widetilde{V}}_C e^{-j\omega t} \tag{58}$$

Noting that we find phasors by taking the coefficient of the $e^{j\omega t}$ term (and recognizing that the coefficient of

the $\mathrm{e}^{-\mathrm{j}\omega t}$ term is guaranteed to be the complex conjugate.). So we can represent the current as the phasor

$$\widetilde{I}_C = (\mathrm{j}\omega C)\widetilde{V}_C. \tag{59}$$

In other words, having already shown that all steady state circuit quantities will be sinusoids with frequency ω in response to the input voltage, we can relate the phasors of the voltage across and the current through a capacitor by a ratio that depends only on the frequency and the capacitance.

This ratio is called **impedance** and as mentioned before in Section 4, can be thought of as the "AC resistance" of a capacitor, since it relates the phasor representations of an element's voltage and current by a constant ratio. For a capacitor, the impedance is

$$Z_C = \frac{\widetilde{V}_C}{\widetilde{I}_C} = \frac{1}{\mathrm{j}\omega C}. \tag{60}$$

6.2 Impedance of a Resistor

Imagine some resistor R labeled as follows:
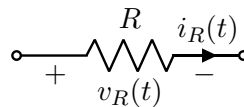
Let $v_R(t)$ be represented by some phasor $\widetilde{V}_R$. Thus, by Ohm's Law,

$$v_R(t) = \widetilde{V}_R \mathrm{e}^{\mathrm{j}\omega t} + \overline{\widetilde{V}}_R \mathrm{e}^{-\mathrm{j}\omega t} \tag{61}$$

$$i_R(t) = \frac{1}{R} v_R(t) \tag{62}$$

$$= \frac{\widetilde{V}_R}{R} \mathrm{e}^{\mathrm{j}\omega t} + \frac{\overline{\widetilde{V}_R}}{R} \mathrm{e}^{-\mathrm{j}\omega t}, \tag{63}$$

so we may represent the output current with the phasor

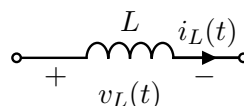$$\widetilde{I}_R = \frac{\widetilde{V}_R}{R}, \tag{64}$$

so the impedance is

$$Z_R = \frac{\widetilde{V}_R}{\widetilde{I}_R} = R. \tag{65}$$

From this, we see that the impedance behaves very much like resistance does, except that it generalizes to non-resistor components.

6.3 Impedance of an Inductor

Consider an inductor with voltage and current across it as follows:

Let the current $i_L(t) = I_0 \cos(\omega t + \phi)$ be represented by some phasor $\widetilde{I}_L$. Thus, by the equation of an inductor,

$$i_L(t) = \widetilde{I}_L e^{j\omega t} + \overline{\widetilde{I}_L} e^{-j\omega t} \tag{66}$$

$$v_L(t) = L \frac{di_L(t)}{dt} \tag{67}$$

$$= L \left((j\omega) \widetilde{I}_L e^{j\omega t} - (j\omega) \overline{\widetilde{I}_L} e^{-j\omega t} \right) \tag{68}$$

$$= (j\omega L) \widetilde{I}_L e^{j\omega t} + \overline{(j\omega L) \widetilde{I}_L} e^{-j\omega t}, \tag{69}$$

so the voltage can be represented by the phasor

$$\widetilde{V}_L = j\omega L \widetilde{I}_L. \tag{70}$$

Thus, the impedance of an inductor is

$$Z_L = \frac{\widetilde{V}_L}{\widetilde{I}_L} = j\omega L. \tag{71}$$

6.4 A Remark on Conjugation

When we have an expression like $v(t) = \frac{V_0 e^{j\phi}}{2} e^{j\omega t} + \frac{V_0 e^{-j\phi}}{2} e^{-j\omega t}$, we have consistently been defining the phasor as $\widetilde{V} = \frac{V_0 e^{j\phi}}{2}$, the coefficient of the $e^{j\omega t}$ term. But what about the conjugate phasor $\overline{\widetilde{V}} = \frac{V_0 e^{-j\phi}}{2}$ associated with the complementary exponential, $e^{-j\omega t}$? Why does it not appear in our time-to-phasor-domain transformation?

Well, since the two phasor terms form a complex-conjugate pair, changing one automatically impacts the other. This balance happens in such a way that the ultimate time-domain signal stays fully real-valued. In a sense, having the conjugate phasor is necessary to "cancel out" the imaginary parts from the original phasor. Notice that $e^{j\omega t}$ has a real and imaginary component (think of it as circling around the unit circle in the complex plane, as t or ω varies). Its imaginary part can only be cancelled by adding its complex conjugate (for any complex a, $a + \bar{a} = 2\operatorname{Re}\{a\}$, which is fully real.)

7 Circuit Analysis

We now discuss how to apply the phasor and impedance techniques we just derived to an actual circuit. The major insight allowing us to do this is that with a phasor, we are representing the *entire sinusoid* function of time. We will find this very convenient mathematically, and could do this only because linear circuits (resistors, capacitors, and inductors) will never alter the frequency of a sinusoid, as our exponential input's (e^{st}) frequency is unchanged by differentiation.

7.1 Summarizing the Connection Between Time and Phasor Domains

Given a general sinusoidal time-domain input signal $u(t) = A \cos(\omega t + \phi)$, we derived a way to conveniently represent $u(t)$ as a weighted sum of a complex exponential $e^{j\omega t}$ and its complex conjugate $e^{-j\omega t}$. This led us to naturally define the phasor, which is a term $\widetilde{U} = \frac{A e^{j\phi}}{2}$. We showed how the phasor captures all the critical information about $u(t)$, but without the time-dependent terms. This transformation $u(t) \to \widetilde{U}$

is often called a *Phasor-Transform*.

Phasor Transform: $u(t) = A\cos(\omega t + \phi) \rightarrow u(t) = \dfrac{Ae^{j\phi}}{2}e^{j\omega t} + \dfrac{Ae^{-j\phi}}{2}e^{-j\omega t} \implies \widetilde{U} = \dfrac{Ae^{j\phi}}{2}$ (72)

Using $\widetilde{U}$, we have seen how to analyze the behavior of R, L, C circuit elements to find their voltages and currents. That's great, but we ultimately want to know what the output voltage or current is as a function of time. This motivates the *Inverse Phasor Transform.* The inverse transformation takes some $\widetilde{W} = Be^{j\psi}$ that we've solved for, and converts it into $w(t) = B\cos(\omega t + \psi)$. Notice that the frequency term stayed the same throughout! This was one of our critical assumptions, related to how the RLC circuits we study only contain elements which preserve the frequency of the sinusoids they act on. They only impact the amplitude and phase.

Inverse Phasor Transform: $\quad \widetilde{W} = Be^{j\psi} \implies w(t) = 2B\cos(\omega t + \psi)$ (73)

7.2 KCL and KVL with Phasors

We will now show that a sum of sinusoidal functions is zero if and only if the sum of the phasors of each of those functions equals zero as well. Let the sinusoid functions be $x_1(t), x_2(t), \ldots, x_n(t)$. Let the phasors representing the sinusoids be $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$. Then,

$$\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n = 0 \tag{74}$$

$$\Longleftrightarrow \qquad (\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n)e^{j\omega t} = 0 \tag{75}$$

$$\Longleftrightarrow \qquad (\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n)e^{j\omega t} + (\overline{\widetilde{X}_1 + \widetilde{X}_2 + \ldots + \widetilde{X}_n})e^{-j\omega t} = 0 \tag{76}$$

$$\Longleftrightarrow \qquad \sum_{k=1}^{n}\left(\widetilde{X}_k e^{j\omega t} + \overline{\widetilde{X}_k}e^{-j\omega t}\right) = 0 \tag{77}$$

$$\Longleftrightarrow \qquad x_1(t) + x_2(t) + \ldots + x_n(t) = 0, \tag{78}$$

so we have proved that a sum of sinusoids is zero if and only if the sum of their corresponding phasors is zero as well. This proof also generalizes both KCL and KVL to phasors:

1. This proof shows that the sum of currents leaving a node is 0 if and only if the sum of current phasors leaving a node is 0, which is exactly KCL.

2. This proof shows that the sum of voltage differences in a loop is 0 if and only if the sum of voltage phasors in a loop is 0, which is exactly KVL.

7.3 Problem-Solving Process

In previous sections, we essentially obtained "equivalents" to Ohm's Law for inductors and capacitors, using the impedance to relate their voltage and current phasors. With KCL and KVL established to work in the phasor domain as well, we have now successfully generalized all of our techniques of DC analysis to AC/frequency analysis, and thus makes working in the phasor domain the same as analyzing resistor circuits from 16A.

We now outline the key steps to analyze and solve a general circuit using phasors:

(a) Confirm that the circuit can actually be usefully analyzed using phasors. *This requires the voltages and currents to be sinusoidal!*

(b) Convert all $v(t), i(t)$ information to the phasor-domain $(\widetilde{V}, \widetilde{I})$.

(c) Solve for the voltages and currents using EECS16A techniques (KCL, KVL, NVA, etc.)

(d) Convert all phasor results back to the time-domain.

We can now consider some basic circuits, to exercise this technique, and verify that it works correctly. Consider a voltage divider, where we introduce a capacitor in place of one of the resistors, as follows:
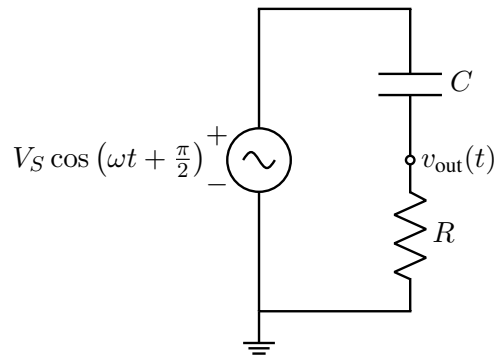
We are interested in knowing how the voltage $v_{\text{out}}(t)$ varies over time in response to the input supply, $u(t) = V_S \cos\left(\omega t + \frac{\pi}{2}\right)$. Recall that we proved the voltage divider equation in the context of DC circuit analysis. However, that proof carries over to the phasor domain in a straightforward manner. Thus, the phasor $\widetilde{V}_o$ representing the voltage $v_{\text{out}}(t)$ can be represented in terms of the phasor $\widetilde{U}$ representing the supply voltage as follows:

$$\widetilde{V}_o = \frac{Z_R}{Z_C + Z_R}\widetilde{U}, \tag{79}$$

where Z_C and Z_R are the impedances of the capacitor and resistor, respectively. Note also that, since the supply is at frequency ω, all other voltages and currents in the system will also be at the same frequency ω. The circuit elements do not change this frequency. Thus, using our results from earlier, we know that:

$$Z_C = \frac{1}{\mathrm{j}\omega C} \qquad Z_R = R \tag{80}$$

Note also that $\widetilde{U} = \frac{V_S e^{\mathrm{j}\frac{\pi}{2}}}{2} = \frac{\mathrm{j}V_S}{2}$, using our equation for $\cos\theta$ from earlier.

Substituting these values into our equation for $\widetilde{V}_o$, we find that

$$\widetilde{V}_o = \frac{R}{\frac{1}{\mathrm{j}\omega C} + R}\frac{\mathrm{j}V_S}{2} = \frac{\frac{\mathrm{j}R}{2}}{\frac{1}{\mathrm{j}\omega C} + R}V_S. \tag{81}$$

It'll be convenient to have a magnitude-phase representation (and to multiply both top and bottom by $\mathrm{j}\omega C$ to rationalize the denominator):

$$\widetilde{V}_o = \frac{V_S\frac{\mathrm{j}R}{2}\mathrm{j}\omega C}{1 + \mathrm{j}\omega RC} \tag{82}$$

$$= \frac{-V_S\,\omega RC}{2\left(1 + \mathrm{j}\omega RC\right)} \tag{83}$$

$$= \frac{V_S\,\omega RC}{2\sqrt{1 + (\omega RC)^2}}e^{\mathrm{j}\left(\pi - \mathrm{atan2}(\omega RC,1)\right)} \tag{84}$$

Handwritten annotations:

$a + bj$

$z = |z|\,e^{\mathrm{j}\angle z}$

$= |z|\,(\cos(\theta) + \mathrm{j}\sin(\theta))$

$\cong -\dfrac{V_S\frac{\omega RC}{2}}{1 + \mathrm{j}\omega RC} \cong -\dfrac{2 \times V_S\frac{\omega RC}{2}}{2(1 + \mathrm{j}\omega RC)}$

Now, we can use the Inverse Phasor Transform formula:

$$v_{\text{out}}(t) = \widetilde{V_{\text{o}}}e^{j\omega t} + \overline{\widetilde{V_{\text{o}}}}e^{-j\omega t} \tag{85}$$

$$= 2\left|\widetilde{V_{\text{o}}}\right|\cos\left(\omega t + \measuredangle\widetilde{V_{\text{o}}}\right) \tag{86}$$

$$= \frac{V_S\,\omega RC}{\sqrt{1 + (\omega RC)^2}}\cos\left(\omega t + \pi - \text{atan2}(\omega RC, 1)\right). \tag{87}$$

This formula might look complicated, but it'll become significantly simpler when we have actual values of R, C, ω to plug in. An example of this will be at the start of the next note!

Appendix A Warning

Be aware that in this course phasors are defined slightly differently from how it is often done elsewhere. Essentially, there is a factor of 2 difference.

In this course, we define the phasor representation $\widetilde{X}$ of a sinusoid $x(t)$ to be such that

$$x(t) = \widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}. \tag{88}$$

However, elsewhere, the phasor representation may be defined such that

$$x(t) = \frac{1}{2}(\widetilde{X}e^{j\omega t} + \overline{\widetilde{X}}e^{-j\omega t}). \tag{89}$$

Our definition is more natural and aligns to what you will see in later courses when you learn about Laplace and Fourier transforms. This is because our definition arises from the mathematics, and the same spirit of definition works even when working with inputs of the form e^{st} where s is not a purely imaginary number.

But then why would anyone ever use the alternative, more common definition? Its main advantage is that the magnitude of the phasor equals the amplitude of the signal. For instance, if we have the signal $A\cos(\omega t + \phi)$, then the alternative definition yields the phasor $Ae^{j\phi}$, with magnitude A. In contrast, our definition yields the phasor $(A/2)e^{j\phi}$. The former definition is convenient when conducting physical observations - when using an oscilloscope, one can easily see[4] the amplitude A of a signal, not the half-amplitude $A/2$.

Furthermore, it turns out that there are some slight calculation advantages (i.e. it makes some formulas simpler) to the more common definition when working with power systems and power electronics, which you may see if you take the relevant upper-division EE courses. However, for the purposes of the scope of this course, our definition is simpler and easier to understand, so we will stick with it throughout.

Of course, if the mathematics is done correctly, there is no real difference between the two definitions, in that both describe the same physical behaviors.

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[4]Actually in practice, if there is a DC component to the circuit — i.e. there are some inputs that are constants too — then the easiest thing to see is the peak-to-peak swing of the voltage which corresponds to twice the amplitude. So even the more common definition often forces the person using it to have to divide by two.

Note 5: Phasors and AC Circuits, © UCB EECS 16B, Fall 2021. 13

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