Projection

$P_{\vec u}\vec v$ ($\vec v$ onto $\vec u$) $= \dfrac{\langle \vec v, \vec u\rangle}{\langle \vec u, \vec u\rangle}\vec u = \dfrac{\vec u^* \vec v}{\vec u^* \vec u}\vec u$

Least Square

$A\vec x = \vec b$, $\hat{\vec x} = (A^TA)^{-1}A^T\vec b$ (real)

$\hat{\vec x} = (A^*A)^{-1}A^*\vec b$ (complex)

Upper-Triangularization

$A = UTU^* \rightarrow T = U^*AU$ (unit)

let λ and $\vec v$ be eigen value and vector of A, $\vec v^*A\vec v = \vec v^*\lambda\vec v = \lambda$

we will have basis $V = [\vec v \;\; R]$

Now, $V^*AV = \begin{bmatrix}\vec v^* \\ R^*\end{bmatrix}[A\vec v \;\; AR] = \begin{bmatrix}\vec v^*A\vec v & \vec v^*AR \\ R^*A\vec v & R^*AR\end{bmatrix} = \begin{bmatrix}\lambda & \vec v^*AR \\ \vec 0 & R^*AR\end{bmatrix}$

$R^*A\vec v = R^*\lambda\vec v = \vec 0$

$\therefore$ Name R^*AR as M_{n-1}. we will upper-triangularize M_{n-1} with U_{n-1}. thus we can regenerate a new basis:

$V = [\vec v \;\; RU_{n-1}]$ (will see that $\begin{bmatrix}1 & 0 \\ 0 & U_{n-1}\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & U_{n-1}^*\end{bmatrix} = I$.)

Spectral Theorem

If S is symmetric (Hermitian), then it has

- Orthonormal eigenvector basis
- Purely Real Eigenvalues such that
- $S = V\Lambda V^*$ where Λ is diagonal.

SVD: $\underset{m\times n}{A} = \underset{m\times m}{U}\underset{m\times n}{\Sigma}\underset{n\times n}{V^*} = \underset{m\times r}{U_r}\underset{r\times r}{\Sigma_r}\underset{r\times n}{V_r^*} = \sum_{i=1}^{r}\sigma_i\vec u_i\vec v_i^*$

$\sigma = \sqrt{\lambda}$s of ① A^*A ② AA^*.

$\vec v_i \Rightarrow \vec v$ of A^*A (also $\dfrac{A^*\vec u_i}{\sigma_i}$)

$\vec u_i \Rightarrow \vec v$ of AA^* (also $\dfrac{A\vec v_i}{\sigma_i}$)

$\Sigma \Rightarrow$ filled with σ.

- $Col(U_r) = Col(A)$
- $Col(V_{n-r}) = Null(A)$
- $Col(U_{m-r}) = Nul(A^*)$
- $Col(V_r) = Col(A^*)$

PCA.

Data point

- In columns, use $\vec u_i$
  rows, use $\vec v_i$

Linearization

$$\vec f(\vec x, \vec u) = \begin{bmatrix} f_1(\vec x,\vec u) \\ \vdots \\ f_n(\vec x,\vec u)\end{bmatrix} \approx \vec f(\vec x_*, \vec u_*) + \left.\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}\end{bmatrix}\right|_{(\vec x_*, \vec u_*)}(\vec x - \vec x_*) + \left.\begin{bmatrix}\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n}\end{bmatrix}\right|_{(\vec x_*, \vec u_*)}(\vec u - \vec u_*)$$

$$f(\vec x, \vec u) \approx f(\vec x_*, \vec u_*) + \left.\begin{bmatrix}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n}\end{bmatrix}\right|_{(\vec x_*, \vec u_*)}(\vec x - \vec x_*) + \left.\begin{bmatrix}\frac{\partial f}{\partial u_1} & \cdots & \frac{\partial f}{\partial u_n}\end{bmatrix}\right|_{(\vec x_*, \vec u_*)}(\vec u - \vec u_*)$$

$$+ (\vec x - \vec x_*)^*\left.\begin{bmatrix}\frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1\partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2}\end{bmatrix}\right|_{(\vec x_*, \vec u_*)}(\vec x - \vec x_*) + (\vec u - \vec u_*)^*\left.\begin{bmatrix}\frac{\partial^2 f}{\partial u_1^2} & \cdots & \frac{\partial^2 f}{\partial u_1\partial u_n} \\ \vdots & & \\ \frac{\partial^2 f}{\partial u_1\partial u_n} & \cdots & \frac{\partial^2 f}{\partial u_n^2}\end{bmatrix}\right|_{(\vec x_*, \vec u_*)}(\vec u - \vec u_*)$$

Loss Function

exponential : $e^{-p} \to p^+$, $e^{p} \to p^-$

log: $\ln(1 + e^{-p})$