Department of Industrial Engineering & Operations Research

IEOR 162: Linear Programming and Network Flows (Spring 2022)

Example of an LP with Multiple Optimal Solutions

$$\begin{array}{rlcl}
\text{Maximize} & 60x_1 + 35x_2 + 20x_3 & & \\
\text{subject to} & 8x_1 + 6x_2 - x_3 & \leq & 48 \\
& 4x_1 + 2x_2 + 1.5x_3 & \leq & 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 & \leq & 8 \\
& x_2 & \leq & 5 \\
& x_1, x_2, x_3 & \geq & 0
\end{array}$$

Tableau 1:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	−60	−35	−20	0	0	0	0	0
x_4	1	0	8	6	−1	1	0	0	0	48
x_5	2	0	4	2	1.5	0	1	0	0	20
x_6	3	0	$\boxed{2}$	1.5	0.5	0	0	1	0	8
x_7	4	0	0	1	0	0	0	0	1	5

Tableau 2:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	0	10	−5	0	0	30	0	240
x_4	1	0	0	0	−3	1	0	−4	0	16
x_5	2	0	0	−1	$\boxed{0.5}$	0	1	−2	0	4
x_1	3	0	1	0.75	0.25	0	0	0.5	0	4
x_7	4	0	0	1	0	0	0	0	1	5

Tableau 3 (Optimal):

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	0	0	0	0	10	10	0	280
x_4	1	0	0	−2	0	1	2	−8	0	40
x_3	2	0	0	−2	1	0	2	−4	0	8
x_1	3	0	1	$\boxed{1.25}$	0	0	−0.5	1.5	0	2
x_7	4	0	0	1	0	0	0	0	1	5

Tableau 4 (Optimal):

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	0	0	0	0	10	10	0	280
x_4	1	0	1.6	0	0	1	1.2	−5.6	0	43.2
x_3	2	0	1.6	0	1	0	1.2	−1.6	0	11.2
x_2	3	0	0.8	1	0	0	−0.4	1.2	0	1.6
x_7	4	0	−0.8	0	0	0	0.4	−1.2	1	3.4

Example of an Unbounded LP

$$
\begin{array}{lrcl}
\text{Maximize} & 36x_1 + 30x_2 - 3x_3 - 4x_4 & & \\
\text{subject to} & x_1 + x_2 - x_3 & \leq & 5 \\
 & 6x_1 + 5x_2 - x_4 & \leq & 10 \\
 & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}
$$

Tableau 1:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	-36	-30	3	4	0	0	0
x_5	1	0	1	1	-1	0	1	0	5
x_6	2	0	$\boxed{6}$	5	0	-1	0	1	10

Tableau 2:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	0	0	3	-2	0	6	60
x_5	1	0	0	$1/6$	-1	$\boxed{1/6}$	1	$-1/6$	$10/3$
x_1	2	0	1	$5/6$	0	$-1/6$	0	$1/6$	$5/3$

Tableau 3:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	0	2	-9	0	12	4	100
x_4	1	0	0	1	-6	1	6	-1	20
x_1	2	0	1	1	-1	0	1	0	5

Example of an LP with Degeneracy

$$\begin{array}{lrcl} \text{Maximize} & 5x_1 + 4x_2 & & \\ \text{subject to} & x_1 & \leq & 4 \\ & 2x_2 & \leq & 12 \\ & 3x_1 + 2x_2 & \leq & 18 \\ & 2x_1 + x_2 & \leq & 11 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Tableau 1:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	-5	-4	0	0	0	0	0
x_3	1	0	$\boxed{1}$	0	1	0	0	0	4
x_4	2	0	0	2	0	1	0	0	12
x_5	3	0	3	2	0	0	1	0	18
x_6	4	0	2	1	0	0	0	1	11

Tableau 2:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	0	-4	5	0	0	0	20
x_1	1	0	1	0	1	0	0	0	4
x_4	2	0	0	2	0	1	0	0	12
x_5	3	0	0	2	-3	0	1	0	6
x_6	4	0	0	$\boxed{1}$	-2	0	0	1	3

Tableau 3:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	0	0	-3	0	0	4	32
x_1	1	0	1	0	1	0	0	0	4
x_4	2	0	0	0	4	1	0	-2	6
x_5	3	0	0	0	$\boxed{1}$	0	1	-2	0
x_2	4	0	0	1	-2	0	0	1	3

Tableau 4:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	0	0	0	0	3	-2	32
x_1	1	0	1	0	0	0	-1	2	4
x_4	2	0	0	0	0	1	-4	$\boxed{6}$	6
x_3	3	0	0	0	1	0	1	-2	0
x_2	4	0	0	1	0	0	2	-3	3

Tableau 5 (Optimal):

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	0	1	0	0	0	$1/3$	$5/3$	0	34
x_1	1	0	1	0	0	$-1/3$	$1/3$	0	2
x_6	2	0	0	0	0	$1/6$	$-2/3$	1	1
x_3	3	0	0	0	1	$1/3$	$-1/3$	0	2
x_2	4	0	0	1	0	$1/2$	0	0	6

Example of an LP with Cycling

$$\begin{array}{llrl}
\text{Maximize} & 3/4x_1 - 20x_2 + 1/2x_3 - 6x_4 & & \\
\text{subject to} & 1/4x_1 - \ 8x_2 - \quad x_3 + 9x_4 & \leq & 0 \\
 & 1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 & \leq & 0 \\
 & x_3 & \leq & 1 \\
 & x_1, x_2, x_3, x_4 & \geq & 0
\end{array}$$

Tableau 1:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	$-3/4$	20	$-1/2$	6	0	0	0	0
x_5	1	0	$\boxed{1/4}$	-8	-1	9	1	0	0	0
x_6	2	0	$1/2$	-12	$-1/2$	3	0	1	0	0
x_7	3	0	0	0	1	0	0	0	1	1

Tableau 2:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	0	-4	$-5/2$	33	3	0	0	0
x_1	1	0	1	-32	-4	36	4	0	0	0
x_6	2	0	0	$\boxed{4}$	$3/2$	-15	-2	1	0	0
x_7	3	0	0	0	1	0	0	0	1	1

Tableau 3:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	0	0	-2	18	1	1	0	0
x_1	1	0	1	0	$\boxed{8}$	-84	-12	8	0	0
x_2	2	0	0	1	$3/8$	$-15/4$	$-1/2$	$1/4$	0	0
x_7	3	0	0	0	1	0	0	0	1	1

Tableau 4:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	$1/4$	0	0	-3	-2	3	0	0
x_3	1	0	$1/8$	0	1	$-21/2$	$-3/2$	1	0	0
x_2	2	0	$-3/64$	1	0	$\boxed{3/16}$	$1/6$	$-1/8$	0	0
x_7	3	0	$-1/8$	0	0	$21/2$	$3/2$	-1	1	1

Tableau 5:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	$-1/2$	16	0	0	-1	1	0	0
x_3	1	0	$-5/2$	56	1	0	$\boxed{2}$	-6	0	0
x_4	2	0	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0	0
x_7	3	0	$5/2$	-56	0	0	-2	6	1	1

Tableau 6:

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	$-7/4$	44	$1/2$	0	0	-2	0	0
x_5	1	0	$-5/4$	28	$1/2$	0	1	-3	0	0
x_4	2	0	$1/6$	-4	$-1/6$	1	0	$\boxed{1/3}$	0	0
x_7	3	0	0	0	1	0	0	0	1	1

Tableau 7 (Same as Tableau 1):

B.V.	Row	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	1	$-3/4$	20	$-1/2$	6	0	0	0	0
x_5	1	0	$1/4$	-8	-1	9	1	0	0	0
x_6	2	0	$1/2$	-12	$-1/2$	3	0	1	0	0
x_7	3	0	0	0	1	0	0	0	1	1

The Big-M Method

$$\begin{aligned}
\text{Minimize} \quad & 2x_1 + 3x_2 \\
\text{subject to} \quad & \frac{1}{2}x_1 + \tfrac{1}{4}x_2 \leq 4 \\
& x_1 + 3x_2 \geq 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \geq 0
\end{aligned}$$

$$\begin{aligned}
\text{Minimize} \quad & 2x_1 + 3x_2 + Ma_1 + Ma_2 \\
\text{subject to} \quad & \frac{1}{2}x_1 + \tfrac{1}{4}x_2 + x_3 = 4 \\
& x_1 + 3x_2 - x_4 + a_1 = 20 \\
& x_1 + x_2 + a_2 = 10 \\
& x_1, x_2, x_3, x_4, a_1, a_2 \geq 0
\end{aligned}$$

Tableau 1 (set for minimization):

B.V.	Row	z	x_1	x_2	x_3	x_4	a_1	a_2	RHS
w	0	1	$2M-2$	$4M-3$	0	$-M$	0	0	30M
x_3	1	0	1/2	1/4	1	0	0	0	4
a_1	2	0	1	$\boxed{3}$	0	-1	1	0	20
a_2	3	0	1	1	0	0	0	1	10

Tableau 2:

B.V.	Row	z	x_1	x_2	x_3	x_4	a_1	a_2	RHS
w	0	1	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{3-4M}{3}$	0	$\frac{60+10M}{3}$
x_3	1	0	5/12	0	1	1/12	$-1/12$	0	7/3
x_2	2	0	1/3	1	0	$-1/3$	1/3	0	20/3
a_2	3	0	$\boxed{2/3}$	0	0	1/3	$-1/3$	1	10/3

Tableau 3 (Optimal):

B.V.	Row	z	x_1	x_2	x_3	x_4	a_1	a_2	RHS
w	0	1	0	0	0	$-\frac{1}{2}$	$\frac{1-2M}{2}$	$\frac{3-2M}{2}$	25
x_3	1	0	0	0	1	$-1/8$	1/8	$-5/8$	1/4
x_2	2	0	0	1	0	$-1/2$	1/2	$-1/2$	5
x_1	3	0	1	0	0	1/2	$-1/2$	3/2	5

Spotting an infeasible LP: Suppose we modify the LP by changing the right hand side of the second constraint from 20 to 36. The following tableau is the optimal tableau obtained at the end of the Big M method for this modified problem. The presence of an artificial variable with a positive value in the optimal basis indicates that the modified problem is infeasible.

B.V.	Row	z	x_1	x_2	x_3	x_4	a_1	a_2	RHS
w	0	1	$1-2M$	0	0	$-M$	0	$3-4M$	$30+6M$
x_3	1	0	1/4	0	1	0	0	$-1/4$	3/2
a_1	2	0	-2	0	0	-1	1	-3	6
x_2	3	0	1	1	0	0	0	1	10

The Phase I - Phase II Method

Original LP:

$$
\begin{array}{lrrrrrrrrrl}
\text{Maximize} & 40x_1 + 10x_2 & & & & +7x_5 & +14x_6 & & & & \\
\text{subject to} & x_1 - & x_2 & & & +2x_5 & & +a_1 & & & = 0 \\
 & -2x_1 & +x_2 & & & -2x_5 & & & +a_2 & & = 0 \\
 & x_1 & & +x_3 & & +x_5 & -x_6 & & & +a_3 & = 3 \\
 & & 2x_2 & +x_3 & +x_4 & +2x_5 & +x_6 & & & & = 4 \\
 & & & & & & & & & & x_i, a_i \geq 0,\ \forall i
\end{array}
$$

Phase I LP:

$$
\begin{array}{lrrrrrrrrrl}
\text{Minimize} & & & & & & & a_1 + a_2 + a_3 & & & \\
\text{subject to} & x_1 - & x_2 & & & +2x_5 & & +a_1 & & & = 0 \\
 & -2x_1 & +x_2 & & & -2x_5 & & & +a_2 & & = 0 \\
 & x_1 & & +x_3 & & +x_5 & -x_6 & & & +a_3 & = 3 \\
 & & 2x_2 & +x_3 & +x_4 & +2x_5 & +x_6 & & & & = 4 \\
 & & & & & & & & & & x_i, a_i \geq 0,\ \forall i
\end{array}
$$

Phase I:

Tableau 1:

B.V.	Row	w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS
w	0	1	0	0	1	0	1	-1	0	0	0	3
a_1	1	0	1	-1	0	0	2	0	1	0	0	0
a_2	2	0	-2	1	0	0	-2	0	0	1	0	0
a_3	3	0	1	0	$\boxed{1}$	0	1	-1	0	0	1	3
x_4	4	0	0	2	1	1	2	1	0	0	0	4

Tableau 2 (Phase I Optimal):

B.V.	Row	w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS
w	0	1	-1	0	0	0	0	0	0	0	-1	0
a_1	1	0	1	-1	0	0	2	0	1	0	0	0
a_2	2	0	-2	1	0	0	-2	0	0	1	0	0
x_3	3	0	1	0	1	0	1	-1	0	0	1	3
x_4	4	0	-1	2	0	1	1	2	0	0	-1	1

Phase II: The variables x_1 and a_3 can never be basic (positive) so we can omit their columns from the tableau in Phase II.

Tableau 1:

B.V.	Row	z	x_2	x_3	x_4	x_5	x_6	a_1	a_2	RHS
z	0	1	-10	0	0	-7	-14	0	0	0
a_1	1	0	-1	0	0	2	0	1	0	0
a_2	2	0	1	0	0	-2	0	0	1	0
x_3	3	0	0	1	0	1	-1	0	0	3
x_4	4	0	2	0	1	1	$\boxed{2}$	0	0	1

Tableau 2 (Phase II Optimal):

B.V.	Row	z	x_2	x_3	x_4	x_5	x_6	a_1	a_2	RHS
z	0	1	4	0	7	0	0	0	0	7
a_1	1	0	0	0	0	2	0	1	0	0
a_2	2	0	1	0	0	0	0	0	1	0
x_3	3	0	1	1	1/2	3/2	0	0	0	7/2
x_6	4	0	0	0	1/2	1/2	1	0	0	1/2

Complications and their Resolutions

Formulation:

Complication	**Resolution**
1) Minimize instead of maximize	Minimize z is equivalent to Maximize (-z), or choose incoming basic variable with positive coefficient
2) Variable x_j unconstrained in sign	Replace x_j by $x'_j - x''_j$ with $x'_j \geq 0$, $x''_j \geq 0$
3) Negative RHS for some constraint	Multiply both sides by (-1)
4) Equality constraint: $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_i$	Replace by $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + a_i = b_i$, $a_i \geq 0$ (artificial variable), then apply Big M Method or phase I, phase II
5) Constraint $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \geq b_i$	Replace by $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n - e_i = b_i$, $e_i \geq 0$, and treat as equality constraint

Computation:

Complication	**How to recognize it**	**Resolution**
7) Tie for entering variable	More than one-non basic variable has the most negative (positive) coefficient in row (0) for maximization (minimization)	Arbitrarily choose one
8) Degeneracy or tie for leaving basic variable	Two or more basic variables reach zero simultaneously as entering basic variable is increased; that is, minimum ratio test ends in tie	Arbitrarily choose one. Degeneracy is theoretically important because it requires an algorithm to avoid cycling of simplex method but is usually ignored in practice.
9) Unbounded z or no leaving basic variable	There is a column in the tableau corresponding to a nonbasic variable x_j where all entries are non-positive.	STOP! As x_j increases, all basic variables increase. Thus, x_j can be increased as much as desired. Since z increases as x_j increases, z is unbounded.
10) Multiple optimal solutions	At optimality, there is a non-basic variable in the objective function with a coefficient of zero.	Let that variable be an entering basic variable and pivot to obtain another optimal solution.
11) Infeasibility or no feasible solutions	At optimality of Big M Method, at least one artificial variable has a positive value.	STOP!