

EECS 16B Designing Information Devices and Systems II

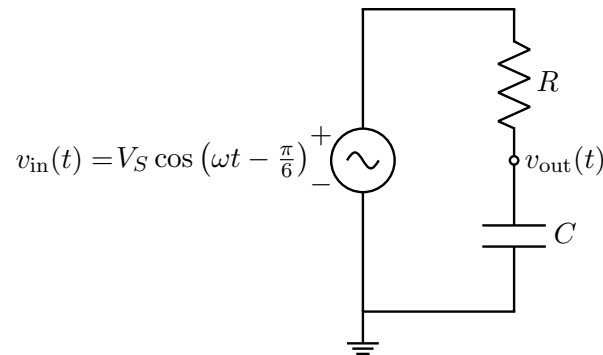
Fall 2021 Note 6: Circuit Filters

Overview

In the previous note, we developed the method of Phasor Domain Analysis to analyze the steady state behavior of circuits with sinusoidal voltage and current sources. In this note, we will consider some applications of these circuits, and begin to explore techniques for designing these circuits to fit a set of requirements. While this note may seem long, much of the space is consumed by figures, and each example provided comes with a complete set of steps.

1 Circuit Example and Transfer Functions

In the previous note, we analyzed a circuit that resembled a voltage divider, but replaced the first (top) resistor with a capacitor. Below, we consider a similar circuit where the roles of the capacitor and resistor are swapped. Our notation has changed, but only very slightly ($\tilde{U} \rightarrow \tilde{V}_{in}$, $\tilde{V}_o \rightarrow \tilde{V}_{out}$). We will also assign specific values to the input voltage and circuit elements to get a feel for a "complete" solution.



Let's say that $V_S = 5 \text{ V}$, $\omega = \sqrt{3} \cdot 10^5 \frac{\text{rad}}{\text{s}}$, $R = 1 \text{ k}\Omega$, and $C = 10 \text{ nF}$. We can calculate that the input voltage phasor \tilde{V}_{in} corresponding to $v_{in}(t) = V_S \cos(\omega t - \frac{\pi}{6})$ is:

$$\tilde{V}_{in} = \frac{V_S e^{-j\frac{\pi}{6}}}{2} = \frac{5e^{-j\frac{\pi}{6}}}{2} \quad (1)$$

Applying the equation for a voltage divider, we find the voltage phasor at $v_{out}(t)$ to be:

$$\tilde{V}_{out} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tilde{V}_{in}. \quad (2)$$

Keeping our results symbolic just a little while, we will look at the ratio of the output phasor to the input phasor. This term, $\frac{\tilde{V}_{out}}{\tilde{V}_{in}}$, is called the circuit's **transfer function** (denoted $H(\omega)$). It relates the output of the

circuit to the input while capturing the frequency dependence.

$$H(\omega) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}. \quad (3)$$

More broadly, transfer functions map **an angular frequency ω to a ratio of two phasors** — an input and an output — and are used to characterize the behavior of a system across a range of frequencies.¹

Observe that as transfer functions are the ratio of two phasors, they are complex-valued in general. Thus, we can write

$$H(\omega) = |H(\omega)| e^{j\angle H(\omega)}, \quad (4)$$

where $|H(\omega)|$ is the magnitude of the transfer function, and $\angle H(\omega)$ is the phase of the transfer function. In this example, we can rearrange $H(\omega)$ to become:

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \quad (5)$$

$$= \frac{1}{1 + j\sqrt{3}} \quad (6)$$

Now, we can compute the information we need about $H(\omega)$, symbolically and numerically. First, the magnitude²:

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} = \frac{1}{2} \quad (7)$$

Now, the phase³:

$$\text{atan2}(0, 1) \quad \text{atan}(\omega RC, 1)$$

$$\angle H(\omega) = \angle(1) - \angle(1 + j\omega RC) \quad (8)$$

$$= -\text{atan2}(\omega RC, 1) \quad (9)$$

$$= -\text{atan2}(\sqrt{3}, 1) \quad (10)$$

$$= -\frac{\pi}{3} \quad (11)$$

Now, we can compute the voltage phasor \tilde{V}_{out} using $H(\omega)$, \tilde{V}_{in} as follows:

$$\tilde{V}_{\text{out}} = H(\omega) \tilde{V}_{\text{in}} \quad (12)$$

$$= |H(\omega)| e^{j\angle H(\omega)} \tilde{V}_{\text{in}} \quad (13)$$

$$= \frac{1}{2} e^{-j\frac{\pi}{3}} \frac{5e^{-j\frac{\pi}{6}}}{2} \quad (14)$$

$$= \frac{5}{4} e^{-j\frac{\pi}{2}} \quad (15)$$

¹Here, our transfer function relates two voltage phasors, but they can also relate two current phasors, or a voltage phasor with a current phasor, in a similar manner.

²Recall that $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$, where z_1, z_2 are both complex numbers ($a + jb$) with magnitudes as $|z| = \sqrt{a^2 + b^2}$

³ $\angle\left(\frac{z_1}{z_2}\right) = \angle(z_1) - \angle(z_2)$, where $\angle(z) = \text{atan2}(b, a)$

Now that we have \tilde{V}_{out} , we can solve for $v_{\text{out}}(t)$ using the Inverse Phasor Transform⁴:

$$v_{\text{out}}(t) = \tilde{V}_{\text{out}} e^{j\omega t} + \overline{\tilde{V}_{\text{out}}} e^{-j\omega t} \quad (16)$$

$$= \frac{5}{4} e^{-j\frac{\pi}{2}} e^{j\omega t} + \frac{5}{4} e^{j\frac{\pi}{2}} e^{-j\omega t} \quad (17)$$

$$= \frac{5}{4} \left(e^{j(\frac{\pi}{2} + \omega t)} + e^{j(\frac{\pi}{2} - \omega t)} \right) \quad (18)$$

$$= \frac{5}{2} \cos \left(\sqrt{3} \cdot 10^5 t - \frac{\pi}{2} \right), \quad (19)$$

We see here that the transfer function will scale the input signal by a factor of $H(\omega)$, and shift its phase by an angle $\angle H(\omega)$. Equipped with the notion of a transfer function that scales and delays an input sinusoid, we can follow the following circuit analysis procedure:

- Express the sinusoidal input (current or voltage) as a phasor.
- Solve for the output phasor(s), which can be voltages or currents, as functions of the input phasor, and use the output to determine the transfer function.
- Observe the magnitude and phase shift of the transfer function at the specific input frequency to determine the behavior of the circuit.

Note that we have previously solved this same example involving the output of an RC circuit with a cosine input, purely in the time domain. That process involved far more calculation and painful trigonometry to simplify the end result. The usage of phasors simplifies our calculations greatly.

2 Filters

One very common use for AC circuits is as a *filter*, which blocks certain categories of frequencies. The key idea behind filters is that of *superposition*. In EECS16A, we saw that in circuits with independent sources, the circuit could be analyzed with only 1 independent source one at a time (with the rest zeroed out).⁵ Then, the final signal was the sum of the intermediate responses from each source.

Here, rather than looking at the superposition of independent sources, we will look at the superposition of input signals with different frequencies. Specifically, let's represent our generalized input signal as follows:

$$v_{\text{in}}(t) = \sum_i \left(\tilde{U}_i e^{j\omega_i t} + \overline{\tilde{U}_i} e^{-j\omega_i t} \right) \quad (20)$$

where each ω_i is some frequency with an associated phasor \tilde{U}_i . Applying this input to a circuit with transfer function $H(\omega)$, the output voltage will be:

$$v_{\text{out}}(t) = \sum_i \left(H(\omega_i) \left(\tilde{U}_i e^{j\omega_i t} + \overline{\tilde{U}_i} e^{-j\omega_i t} \right) \right) \quad (21)$$

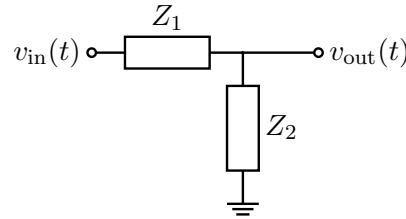
That is, we can treat the superposition of signals of two different frequencies as if the two signals were

⁴If you can arrive the final answer by inspection, that's great! The steps are here for completeness.

⁵If there are dependent sources, this doesn't hold!

separate. Each frequency sinusoid is affected by the circuit as governed by the transfer function evaluated at that frequency.

Now, we will consider various filter configurations, and study their behavior. For now, we will look at **L-filters**, which are filters consisting of two elements placed in series, as shown:



Recall from the impedance divider equation that the transfer function of this circuit is:

$$H(\omega) = \frac{Z_2}{Z_1 + Z_2}. \quad (22)$$

We will now consider all the possible forms of impedances that we can substitute into this equation. There are nine possible configurations, and we divide them into 3 main categories: Frequency-Independent, Low-Pass, High-Pass, and LC Tank.

2.1 Frequency-Independent (RR, CC, LL)

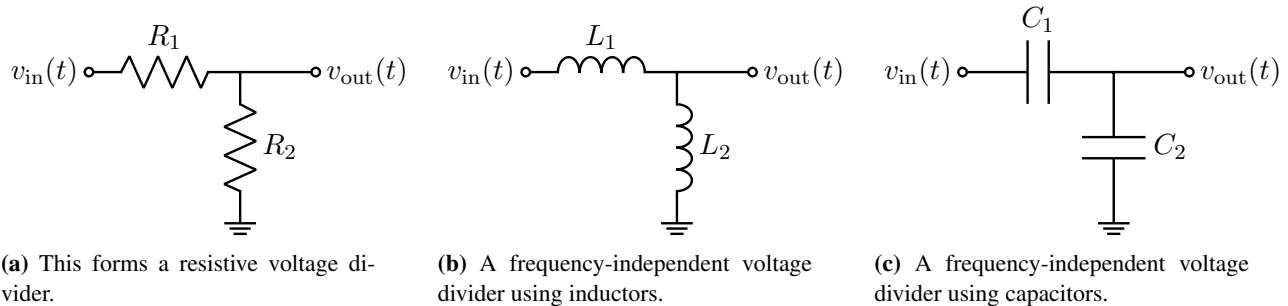


Figure 1: Frequency-Independent *L*-filters.

R-R: Both components are resistors. We can compute:

$$H(\omega) = \frac{R_2}{R_1 + R_2} \quad (23)$$

Notice there's no frequency dependence here, which makes sense since this circuit only contains resistors.

L-L: Both components are inductors. Substituting the impedance of an inductor into our equation for the transfer function, we obtain:

$$H(\omega) = \frac{j\omega L_2}{j\omega L_1 + j\omega L_2} = \frac{L_2}{L_1 + L_2} \quad (24)$$

So, this circuit exhibits behavior similar to a resistive voltage divider (once again, independently of frequency). Notice, however, that unlike resistors, inductors do not dissipate energy.⁶

⁶A natural question to ask would be: what happens to the energy if we supply a DC voltage ($\omega = 0$) to this circuit? Surely there

C-C: Both components are capacitors. Substituting in the impedance of a capacitor:

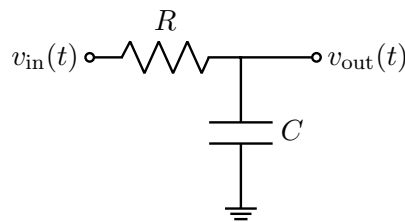
$$H(\omega) = \frac{\frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + \frac{1}{j\omega C_2}} = \frac{C_1}{C_1 + C_2}. \quad (25)$$

Again, we get something very similar to a voltage divider. Here, however, when $\omega = 0$, no current is permitted to flow since the capacitors act as open circuits. A charge-sharing argument would nevertheless demonstrate that $v_{\text{out}}(t)$ will remain as we have claimed.

Looking at these circuits' transfer functions, we see why we call them frequency-independent. Because the elements' interaction with frequency is of the same nature, it is cancelled in the transfer function expression. So we end up with transfer functions that don't contain ω terms.

2.2 Low-Pass (RC, LR)

R-C: We have a circuit with a resistor and capacitor, and we take the voltage across the capacitor.



We can compute the transfer function (the same as we calculated at the start of this note):

$$H(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} \quad \omega \uparrow, \text{deno} \uparrow \quad (26)$$

Now we see some more interesting behavior, since the transfer function depends on frequency. But how can we see that this is a low-pass filter? As a quick intuition check, if $\omega = 0$, $H(\omega) = 1$ and if $\omega \rightarrow \infty$, then $H(\omega) \rightarrow 0$. This makes sense, but what happens in between 0 and ∞ ? Let's take a look at a couple of values⁷ around a special frequency value, $\omega_c = \frac{1}{RC}$:

ω	$H(\omega)$	$ H(\omega) $	$\angle H(\omega)$
$0.1\omega_c$	$\frac{1}{1 + j 0.1}$	0.995	-6°
ω_c	$\frac{1}{1 + j}$	0.71	-45°
$10\omega_c$	$\frac{1}{1 + j 10}$	0.1	-84°

We see here that $\omega_c = \frac{1}{RC}$ is a very important frequency to look at since this is around the frequency where

will be some current flowing between the two potentials, which would cause energy to be dissipated, but where can this energy go? The resolution to this issue relies on the observation that our expression for $H(\omega)$ resolves to $0/0$ for $\omega = 0$, and since our simplifications assumed that $\omega \neq 0$ (allowing us to divide by $j\omega$), the pure voltage-divider formula doesn't quite hold. In the case of $\omega = 0$, we can no longer model the inductors as being ideal, and their internal resistance will dissipate energy

⁷We will soon plot such transfer functions over a continuum of frequencies.

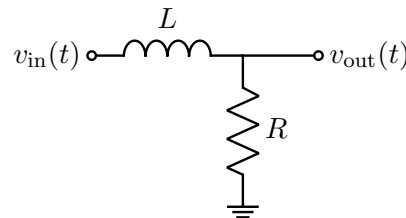
the behavior of the filter starts to qualitatively change.⁸ In fact, this is so important that we call this the **cutoff frequency**. Signals far below this frequency are allowed fully through, while signals above this frequency are largely blocked. So, this RC circuit constitutes a low-pass filter. We can write the low-pass RC transfer function in terms of ω_c : $H(\omega) = \frac{1}{1 + \frac{j\omega}{\omega_c}}$ 错

We define the cutoff frequency ω_c ,⁹ as the point where:

$$H(\omega) = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + \frac{j\omega}{\omega_c}} \quad |H(\omega_c)| = \frac{\max_{\omega} |H(\omega)|}{\sqrt{2}} \quad H(\omega_c) = \frac{1}{1+j} = \frac{1-j}{(1+j)(1-j)} = \frac{1-j}{2}$$

Here, $\max_{\omega} |H(\omega)|$ is the maximum magnitude of $H(\omega)$ over the full frequency range. For passive circuits containing passive components like R , L , C elements, this will usually be 1.

L-R: We have a circuit with a resistor and inductor, and we take the voltage across the resistor.



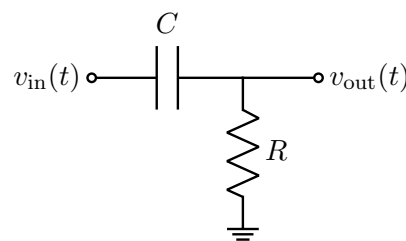
The above has the transfer function

$$H(\omega) = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega \frac{L}{R}} = \frac{1}{1 + j\omega/\omega_c} \quad (28)$$

with $\omega_c = \frac{R}{L}$. Here, $\tau = \frac{L}{R}$, as we saw in the transient analysis for an RL circuit).

2.3 High-Pass (CR, RL)

R-C: We have a circuit with a resistor and capacitor, and we take the voltage across the resistor.



We compute:

$$H(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}} \quad (29)$$

The qualitative behavior we see is different from the low-pass filters above; let's generate a table as we did before ($\omega_c = \frac{1}{RC}$):

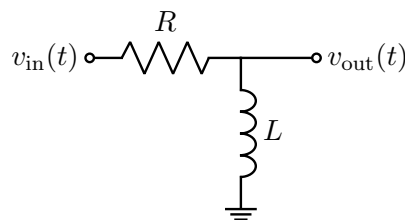
⁸Note how this is the reciprocal of the time constant from back when we studied inverter transitions. We'll explore the significance of τ in a later section.

⁹ ω_c goes by many names, including cutoff frequency, corner frequency, critical frequency, and -3 dB frequency. If you see any of these, they all mean ω_c .

ω	$H(\omega)$	$ H(\omega) $	$\angle H(\omega)$
$0.1\omega_c$	$\frac{j 0.1}{1 + j 0.1}$	0.1	84°
ω_c	$\frac{j}{1 + j}$	0.71	45°
$10\omega_c$	$\frac{j 10}{1 + j 10}$	0.995	6°

Here, instead of passing low frequencies and blocking high frequencies, this circuit does the opposite!

R-L: We have a circuit with a resistor and inductor, and we take the voltage across the inductor.

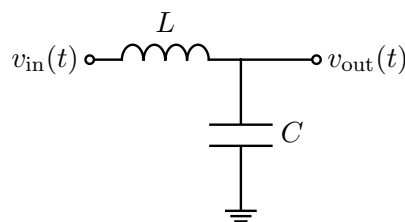


We compute ($\omega_c = \frac{R}{L}$):

$$H(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega \frac{L}{R}}{1 + j\omega \frac{L}{R}} = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}} \quad (30)$$

2.4 LC Tank (LC, CL)

We introduced LC circuits like the one below in **Note 4**, and observed how energy in such a circuit alternates between being stored in the inductor's magnetic field and the capacitor's electric field.



We can compute its transfer function in the standard way and arrive at the expression below, but we will study them **more in the context of resonance in RLC circuits.**

$$H(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + j\omega L} = \frac{1}{1 - \omega^2 LC} \quad (31)$$

2.5 Summary of Filters

Well, that was a lot of filters! But looking closely at their transfer functions, we primarily see two distinct forms:

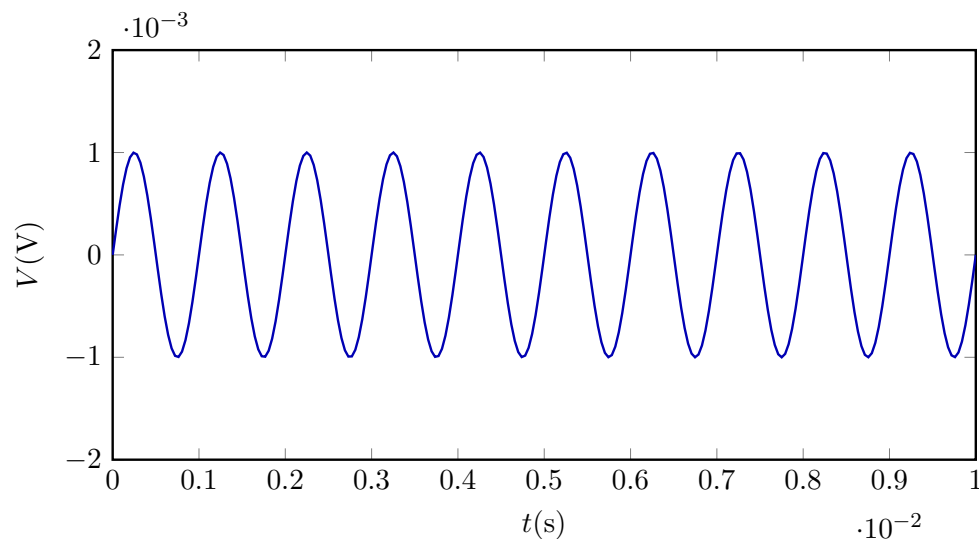
$$H_{LP}(\omega) = \frac{1}{1 + j\omega/\omega_c} \quad H_{HP}(\omega) = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c} \quad (32)$$

The first of these two transfer functions describes a *low-pass filter*, since for values of $\omega \rightarrow 0$, $H(\omega) \rightarrow 1$, while for values of $\omega \rightarrow \infty$, $H(\omega) \rightarrow 0$. In essence, low frequencies can pass through, while high frequencies cannot. In a symmetric manner, the second of these transfer functions describes a *high-pass filter*, which has the opposite behavior.

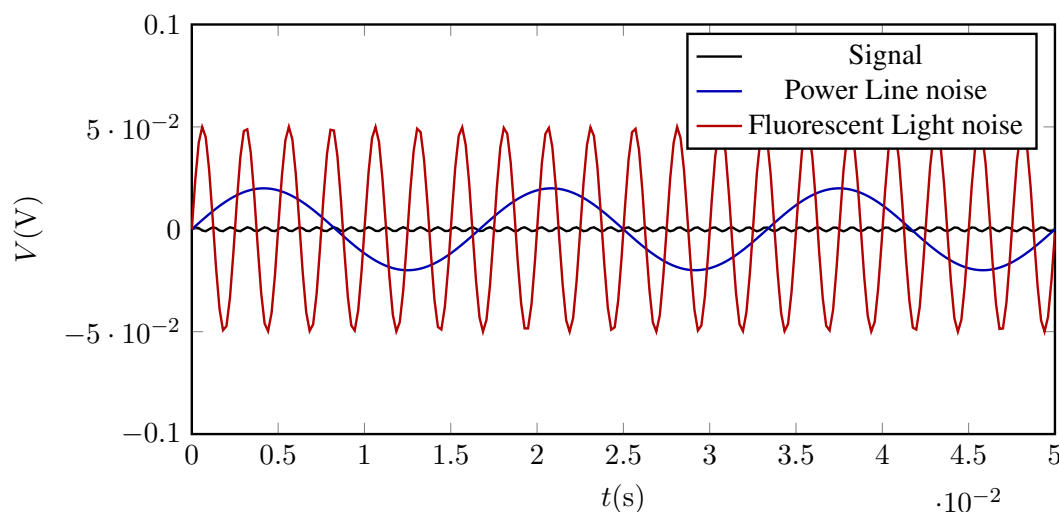
3 Design

Now that we have a better understanding of low- and high-pass filters, it's time to look at a toy example to understand why filters are important in practical applications. We also will recognize how useful it will be to have more powerful visualization tools than the tables in the previous section, and we will cover such tools in Note 6 on Bode Plots.

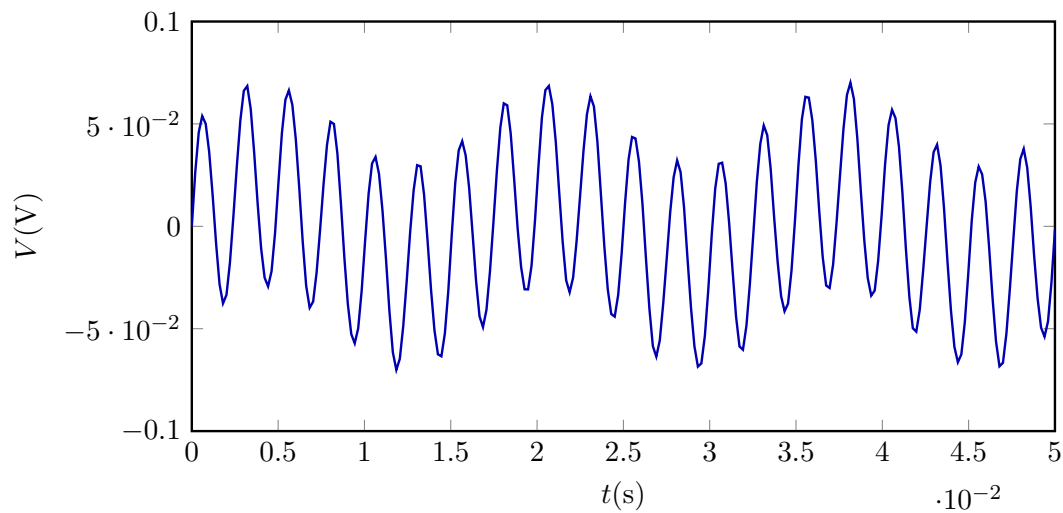
Imagine that we have a signal with amplitude 1 mV and frequency approximately 1 kHz, as shown:



Now, we introduce some 60 Hz noise (a common contamination from power lines) at a stronger 50 mV, as well as some 100 kHz noise (such as from a malfunctioning fluorescent light ballast) at 20 mV. These 3 waveforms (1 desired signal, and 2 undesired sources of noise) are shown below:



Adding these noise sources to our signal, we obtain the result:



At this point, our original signal is not even visible. But, there's still hope! Someone might have told you about the magic of the **Fourier transform, which can break up a signal into its constituent sinusoids**. All we need to do is record this signal precisely in a digital format, send it to a computer, and let a software library figure the rest out.

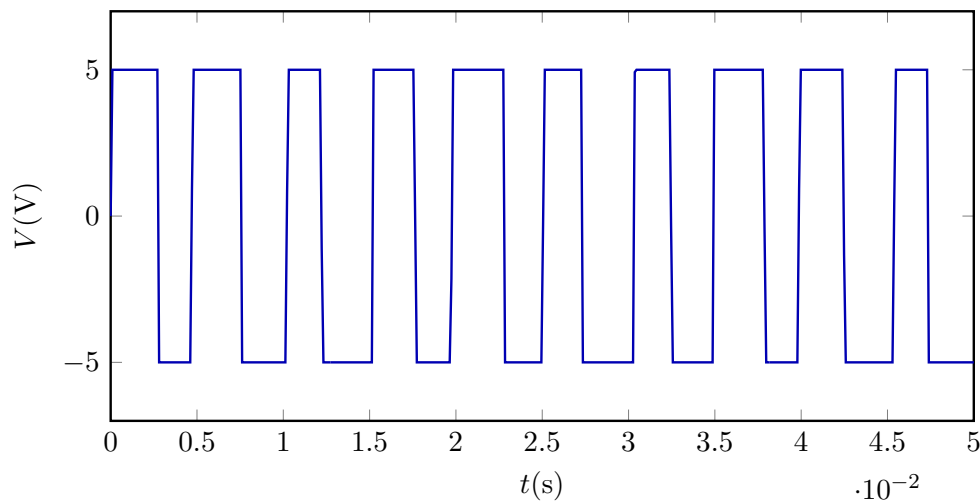
Unfortunately, we are likely encoding this signal using something akin to a 4-bit ADC, much like the one from lab. If our ADC has a maximum voltage of 3.3 V, then the **least significant bit indicates a precision of:**

$$\frac{3.3 \text{ V}}{2^4} \approx 0.2 \text{ V}. \quad (33)$$

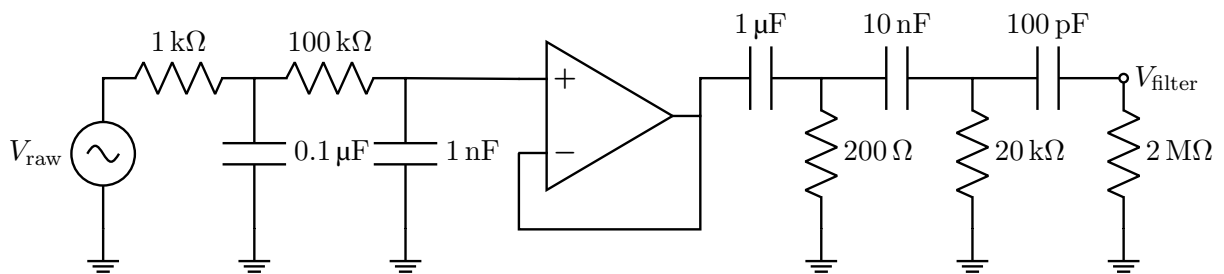
But our signal only has an amplitude of 1 mV! What can we do? One natural option is to **amplify our signal**, such as by using an op-amp circuit, until the amplitude of our signal is on the same order as our measurement ability. Suppose we amplify our signal by a factor of $\frac{3.3 \text{ V}}{1 \text{ mV}} = 3300$. Unfortunately, **when we do this, our noise signals will shoot up, reaching amplitudes of $50 \text{ mV} \cdot 3300 = 165 \text{ V}$!**

Of course, our op-amps can only produce voltages between their rails (perhaps around $\pm 5 \text{ V}$). So **our amplified signal will rail much of the time. Plotting the end result of this amplification, we obtain¹⁰:**

¹⁰The diagram is actually fairly inaccurate due to limitations of the plotting software - it should really look more like a solid block of color.



Notice that our signal is almost always railing, meaning that information has been irretrievably lost before even arriving at a computer. Therefore, analog processing is necessary to remove the noise. We want to remove both the high and low frequency noise while maintaining the amplitude of our signal frequency. We could then amplify the signal, without worrying about our op-amp railing. To do so, consider the following circuit:



The final amplification of V_{signal} is omitted for reasons of space, but can be constructed using a non-inverting amplifier.

Recall that our analysis of L -filters relied heavily on the voltage divider equation. That equation, in turn, assumed that all the current flowing into the top impedance flowed out of the bottom one. But here, some current flows out of the top impedance of each filter and into the next one in series (that is, later filtering stages load the previous ones). There are two ways to handle this issue. This first is to use a unity gain buffer, as done here between the low-pass and high-pass filter chains, but is not done between all pairs of adjacent individual filters.

Another option (that we selected here) is to try to make that current draw (the load of a filter) as low as possible, by increasing the impedance of subsequent filters. On the low-pass side, notice that the resistance of the resistor jumps by a factor of 100 from the first filter to the second, as does the impedance of the capacitors.¹¹ On the high-pass side, we again increase the impedance of each filter by a factor of 100 as well (for both the resistor and capacitor). Thus, each consecutive filter imposes only a very small load on the previous filters, and we approximate the filters as being independent.

Of course, it may not be clear why chaining multiple low-pass or high-pass filters is necessary. Why not just use a single filter of each type? The logic is that we want to build higher-order filters, described more (and with plots) in the next Note. Notice that the transfer function $H(\omega)$ of two filters chained together is the

¹¹A capacitor's impedance is inversely proportional to its capacitance, so reducing the capacitance actually increases the impedance.

product of their two transfer functions $H_1(\omega)$ and $H_2(\omega)$.¹² Consider the case of two low-pass filters with time constant $\tau = RC$ driven by a frequency significantly greater than ω_p . Making linear approximations:

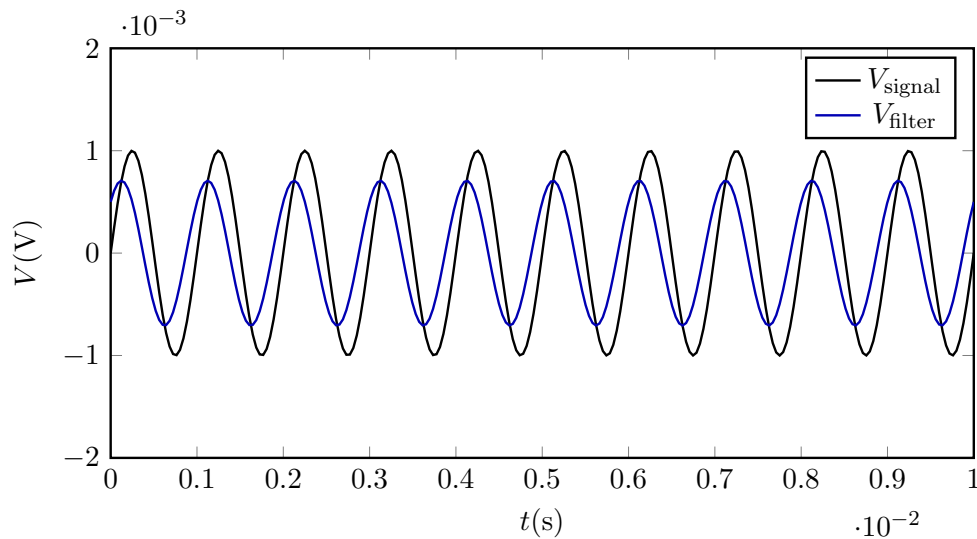
$$|H_1(\omega)| \approx \frac{1}{\omega RC} \quad (34)$$

$$|H_2(\omega)| \approx \frac{1}{\omega RC} \quad (35)$$

$$|H(\omega)| \approx \frac{1}{(\omega RC)^2}. \quad (36)$$

Notice that $|H(\omega)|$ drops with ω^2 , rather than ω . If we were to plot it on a log-log plot, its slope would be -2 , whereas $H_1(\omega)$ and $H_2(\omega)$ would only have a slope of -1 . The magnitude of this slope is known as a filter's *roll-off*. Chaining filters enables a greater roll-off, so it attenuates undesired-frequency signals much more.

Looking at the ultimate end-result V_{filter} versus V_{signal} , we observe:



Notice that we have successfully used our low- and high-pass filters to reject undesired frequencies! However, we have attenuated our signal a bit, and its phase has also shifted. We may have to adjust our amplification factor a bit to compensate for this attenuation, but we can now actually measure our signal with just a 4-bit DAC! This issue of noise sources of different frequencies contaminating the desired signal is ubiquitous in circuit design, so it's quite nice that we can develop a solution to this problem using our knowledge about filters!

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¹²Assuming that the second filter draws very little current from the first, as is true here.

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