

# Analysis of NP and QMA Problems Energy Landscape Using Mutually Unbiased Bases

M.Sc. Research Proposal

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## Abstract

In quantum mechanics, Mutually Unbiased Bases (MUB) are sets of quantum states that are maximally distinct from one another and exhibit a lack of information when a state from one basis is measured in another basis. These MUB represent a powerful tool for characterizing quantum states. Our primary objective is to demonstrate the utility of MUB in obtaining a reliable energy landscape for optimization problems. The energy landscape, as a tool for optimization, potentially plays a crucial role in efficiently identifying the global minimum solution while avoiding local minima and barren plateaus when dealing with problems (or cases) that are not NP-hard. Our secondary goal, regards the field of not NP-hard problems, is integrating the MUB approach with machine learning techniques to enhance the efficiency of the optimization process. This integration aims to reduce the number of search steps required to reach the optimal solution and represents a novel approach to the optimization of NP problems and QMA problems which can be modeled as quantum systems.

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# 1 Introduction and Background

## 1.1 Mutually Unbiased Bases

Mutually unbiased bases (MUB) are a fundamental concept in quantum information that have been studied extensively in recent years. They are a set of orthonormal bases in a Hilbert space that are pairwise unbiased, meaning that the inner product between any two states from different bases is the same [1] [?]. This property has important implications in quantum mechanics, including quantum state tomography and quantum error correction.

### 1.1.1 Mathematical Definition of MUB

The mathematical definition of mutually unbiased bases is given as follows: Let  $B1$  and  $B2$  be two orthonormal bases in a  $d$ -dimensional Hilbert space.  $B1$  and  $B2$  are said to be mutually unbiased if and only if the absolute value of the inner product between any two vectors in the two bases is equal to  $\frac{1}{\sqrt{d}}$ . Additionally, the existence of MUB in a  $d$ -dimensional Hilbert space has been proven to be linked to the mathematical structure of the unitary group and the finite Fourier transform over finite fields.

When the dimensionality  $d$  of the Hilbert space is prime  $d = p$  or power of prime  $d = p^n$ , it is possible to construct  $d+1$  MUB. [?] In addition, for certain composite dimensions, there exist subsets of MUB that satisfy certain conditions, such as the so-called "symmetric informationally complete positive operator-valued measures" (SIC-POVMs), which can be used in quantum state tomography and related applications.

### 1.1.2 Usages in Quantum Tomography

Recent research has focused on finding MUB in large-dimensional Hilbert spaces, as well as exploring their potential uses in quantum state tomography and quantum information science. Quantum state tomography is the process of learning an unknown quantum state from repeated state preparations and measurements, and yields optimal knowledge of the state's density matrix. MUB can be used to obtain this information efficiently, as they provide a way to obtain unbiased estimates of the density matrix elements. In particular, the use of symmetric informationally complete positive operator-valued measures (SIC-POVMs), which form a special subset of MUB, has been shown to provide accurate and efficient reconstructions of quantum states in a variety of contexts. For example, SIC-POVMs have been used in quantum process tomography to reconstruct the dynamics of quantum channels, as well as in experimental studies of high-dimensional entanglement [2][3].

MUB can also be used to detect and correct errors in quantum systems. By measuring the state of the system in multiple MUB, it is possible to reveal the presence of errors

and the specific basis in which the error occurred. This information can then be used to correct the error and restore the system to its original state.

### 1.1.3 Usages in Quantum Cryptography

MUB have been proposed as a tool for quantum key distribution (QKD). In QKD, a key is shared between two parties in a secure way [?], and MUB can be used to encrypt the key in such a way that any third party trying to intercept it will not be able to read it. MUB can be used to encrypt the key in a way that relative to other sets of states, it is more secure against any type of attacks [1].

## 1.2 Variational Quantum Algorithms (VQA)

Variational quantum algorithms (VQA) are a class of quantum algorithms that utilize a classical optimization loop to find the optimal parameters of a quantum circuit [4]. VQAs are designed to solve optimization problems and sample problems, including finding the ground state energy of a quantum system and solving linear systems of equations. One of the key features of VQAs is their ability to handle large quantum systems with a shallow quantum circuit.

### 1.2.1 Variational Quantum Eigensolver (VQE)

The Variational Quantum Eigensolver (VQE) is a popular and widely used VQA, designed to find the eigenvalues of a quantum Hamiltonian, which describes the energy of a quantum system. The VQE algorithm consists of a parameterized quantum circuit that is used to generate a wavefunction that approximates the ground state of the Hamiltonian. The circuit parameters are optimized using a classical optimization algorithm to minimize the energy expectation value of the wavefunction with respect to the Hamiltonian. This process is repeated until convergence, at which point the final wavefunction is used to estimate the ground state energy [5]. VQE has been applied to a wide range of problems in quantum chemistry, including the calculation of molecular energies and properties, as well as the simulation of materials. In particular, VQE has been shown to be highly effective for simulating small and medium-sized quantum systems, where exact diagonalization and quantum Monte Carlo methods become infeasible due to the exponential scaling of the dimension of the quantum state space. Besides quantum chemistry, VQE has been used for a variety of applications, including optimization problems modeled by Ising-model Hamiltonians [6]. The Ising model is a mathematical model used to describe the behavior of interacting spins in a system. The Ising-model Hamiltonian can be mapped onto a quantum circuit, which can then be optimized using VQE[6]. By finding the ground state of the Ising-model Hamiltonian, researchers can obtain an approximate solution to the original problem. One common application of the Ising model in the context of VQE is

in solving the Max-Cut problem. In this problem, the goal is to partition a graph into two sets such that the number of edges between the sets is maximized. This problem can be transformed into an Ising-model Hamiltonian, where the spins represent the nodes of the graph and the coupling constants represent the edges. By using VQE to find the ground state of the Ising-model Hamiltonian, researchers can obtain an approximate solution to the Max-Cut problem[7]. Another application of VQE is in solving systems of linear equations, where the goal is to find the solution to a set of linear equations. The fundamental state, or ground state, of a Hamiltonian can be used to solve linear equations in a way that is similar to classical linear algebra algorithms. Specifically, the Hamiltonian can be constructed such that its ground state corresponds to the solution of the linear equations. By finding the ground state of the Hamiltonian using VQE, the solution to the linear equations can be obtained[8].

### 1.2.2 QAOA

The Quantum Approximate Optimization Algorithm (QAOA) is a VQA that was introduced in 2014 [7] to address combinatorial optimization problems. QAOA involves the construction of a parameterized quantum circuit, which is optimized using classical optimization algorithms to minimize a cost function related to the problem at hand. The cost function is formulated as a sum of Pauli products, and the QAOA circuit acts as a parameterized rotation of these products in the computational basis. The QAOA algorithm has been applied to various optimization problems, including MaxCut, Minimum Vertex Cover, and Max-SAT, among others, demonstrating its potential for solving combinatorial optimization problems on quantum computers. The QAOA algorithm has also been shown to be effective for tackling optimization problems in classical machine learning and operations research, such as portfolio optimization and logistics planning.

## 1.3 Energy Landscape

An energy landscape is a concept used in physics and chemistry to describe the potential energy of a system as a function of its configuration or of its hamiltonian parameters[9]. The configuration of a system refers to the positions and orientations of its components. For example, in a protein, the configuration refers to the positions and orientations of its amino acids [10]. The energy landscape is often represented as a surface or a plot, with energy values on the vertical axis and the coordinates of the system's configuration on the horizontal axes [9]. The landscape can have different features such as minima, maxima, and saddle points, which correspond to stable, unstable, and meta-stable states of the system, respectively. A local minimum is a point on the energy landscape where the energy is lower than that of its immediate surroundings. It corresponds to a stable state of the system, meaning that small perturbations will not cause the system to move

away from that point [9]. A global minimum is a point on the energy landscape that has the lowest energy of all possible configurations, and it represents the most stable state of the system [9]. The energy landscape is a useful concept for understanding the behavior of systems that can exist in multiple configurations, such as proteins, glasses, and phase transitions. It helps to visualize the possible configurations of the system and their relative stability. It is also used to design algorithms for finding the global minimum of the energy function [11].

## 1.4 Vehicle Routing Problem

The Vehicle Routing Problem (VRP) is a central problem in the field of transportation and logistics. It aims to find the most efficient routes for a fleet of vehicles to visit a set of customers, typically referred to as "stops" or "delivery points", while minimizing the total distance traveled. VRP is known for being a NP-hard problem [12], meaning that it is computationally difficult to find an exact solution in a reasonable amount of time for large instances of the problem. Some variants of the VRP are known to be NP-complete, meaning that finding the optimal solution is computationally intractable as the problem size grows. One of the most common NP-complete versions of the VRP is the Capacitated Vehicle Routing Problem (CVRP) [13]. In the CVRP, a fleet of vehicles with limited capacity must be used to serve a set of customers, each with a known demand. The objective is to minimize the number of vehicles and the total distance traveled while satisfying the capacity constraints. Another common NP-complete version of the VRP is the Vehicle Routing Problem with Time Windows (VRPTW) [14]. In the VRPTW, each customer has a time window during which they can be serviced, and the objective is to minimize the total travel time while ensuring that all time windows are satisfied. Both the CVRP and the VRPTW are NP-complete, meaning that finding the optimal solution for large problem instances is extremely difficult.

### 1.4.1 Classical Computing Approaches for the Solution

The Vehicle Routing Problem (VRP) has been extensively studied by researchers from various fields, including operations research, computer science, and mathematics. Researchers have proposed a wide range of solution methods for the VRP, including mathematical programming, heuristics, and meta-heuristics. There are several approaches that researchers have used to try and solve the VRP:

- **Mathematical Programming:** Researchers have tried to solve the VRP by formulating it as a mathematical programming problem and using various optimization algorithms to find the optimal solution. Due to the complexity of the problem, finding an exact optimal solution for large instances of the VRP is computationally infeasible. Therefore, researchers have developed various approximations and

optimization algorithms to find near-optimal solutions. Some of the most common methods used to solve the VRP include linear programming, integer programming, and constraint programming [15]. e.g. constraint programming involves formulating the VRP as a set of constraints and using constraint satisfaction algorithms to find a feasible solution.

- **Heuristics:** Researchers have also developed heuristics and meta-heuristics, such as genetic algorithms, simulated annealing, tabu search, and ant colony optimization, to quickly find near-optimal solutions to the VRP.
- **Artificial Intelligence:** Researchers have also applied various artificial intelligence techniques, such as neural networks and decision trees, to find good solutions to the VRP.
- **Exact Algorithms:** Researchers have also developed exact algorithms, such as branch-and-cut and branch-and-price, to find the optimal solution to the VRP. These algorithms are typically very slow but guarantee an optimal solution [16].
- **Hybrid Approaches:** Researchers have also combined multiple approaches, such as combining heuristics and exact algorithms, to find better solutions to the VRP [17].

All of these approaches have been successful to some extent in solving the VRP, but the VRP remains a challenging problem and there is still ongoing research in this area. The VRP is a complex problem that has many variations and applications, and researchers continue to explore new approaches to solve it.

#### **1.4.2 Quantum Computing Approaches for the Solution**

One of the quantum computing approaches for solving the VRP is the quantum annealing algorithm. This algorithm uses a quantum system to find the global minimum of an objective function by gradually reducing the energy of the system. In the case of the VRP, the objective function represents the total distance or cost of all the routes, and the quantum system can be used to find the optimal solution in a much faster time compared to classical optimization algorithms.

Another approach is the quantum-inspired optimization algorithms, which mimic the behavior of quantum systems to solve classical optimization problems. These algorithms have been shown to be effective in finding near-optimal solutions to the VRP in a much faster time compared to classical optimization algorithms.

A recent article applied QAOA to the VRP [18]. In this work, the authors propose a heuristic based on QAOA for the VRP and compare its performance with classical heuristics. The results show that the proposed heuristic outperforms the classical heuristics in terms of solution quality and computational time. There were also VQE solution approaches for VRP [19]. Analysis of the effects of noise in a gate-based simulation of an algorithm to

solve the vehicle routing problem. It was found that amplitude damping noise causes the least impact on the results of an optimized variational circuit. In contrast bit-phase-flip, depolarising and phase-flip noise channels had the maximum negative impact.

## 2 Related Work

There are few papers about using quantum annealing to solve those kind of optimization problems, that are intractable for classical computers. Quantum Annealing (QA) is a method for solving optimization problems that is inspired by the process of annealing in metallurgy. Like classical simulated annealing, it is a method that uses a random search to find the global minimum of an energy function, but it uses the principles of quantum mechanics to guide the search. In QA, the optimization problem is mapped to a physical system, called the quantum annealer, which consists of a large number of interacting quantum bits (qubits). The energy function of the optimization problem is encoded into the Hamiltonian of the quantum annealer, and the ground state of the Hamiltonian corresponds to the optimal solution of the problem.

The quantum annealer is initialized in a high-energy state, and then it is slowly cooled down to its ground state by applying a time-dependent Hamiltonian. The cooling process is guided by a schedule that controls the rate of change of the Hamiltonian. The schedule is designed to avoid getting stuck in local minima and to increase the chances of finding the global minimum.

## 3 Objectives of the Research

The purpose of the M.Sc. research is to configure a way to draw the energy landscape of a quantum system using MUB. Through that, a great number of realizations can be reached, for example avoid local minima and find the absolute minimum, avoid barren plateaus. In our research, we are going to focus on a problem which preoccupies the entire optimization world, the Vehicle Routing Problem. Although our research will focus on this problem as our quantum system, one can refer to other NP-hard problem, after the usage of MUB as a potential tool for drawing the energy landscape will be demonstrated. In the pursuit of our research objective, the utilization of quantum algorithms and the implementation of a hybrid quantum-classical approach may provide significant benefits. In particular, the modeling of our problem into VQE or QAOA has the potential to offer improved approximations and the discovery of larger subclasses of NP-hard problems that are not NP-hard themselves.

Despite the limitations imposed by shallow quantum circuits, such a hybrid solution may still outperform traditional classical methods, providing a promising avenue for further

exploration and advancement in the field. VRP can be modeled as a search for the global minimum of an energy function, when the energy function represents the cost of a particular solution, such as the total distance traveled by vehicles or the total time taken to visit all customers. The configuration space is defined by the possible routes and the assignments of customers to vehicles. The global minimum of the energy function corresponds to the optimal solution of the problem.

The search in the energy space can be done by using machine learning model or neural networks, for the correct differential fluctuations needed in the angles of each quantum eigenstate of the mutually unbiased bases in use. The use of machine learning in the context of MUB is motivated by two main considerations. Firstly, the exponential growth of the number of MUB states for a large number of qubits necessitates the development of techniques for efficiently selecting a portion of these states or selecting them only for a subset of qubits. Secondly, even for small numbers of qubits, there is a need for innovative methods to explore the space beyond the vicinity of MUB states. This approach can significantly reduce the computational resources required for the VQE algorithm, making it more scalable for larger problems. We think about using MUB to construct energy landscapes of quantum systems using machine learning techniques. This can provide new insights into the behavior of the system that can be hard to gain by using other methods. The application of machine learning algorithms can help to overcome challenges by allowing for the characterization of smaller-scale problems and facilitating the transfer of insights to more complex systems with a larger number of qubits. In order to gain a comprehensive understanding of the landscape, it is necessary to not only examine the exact MUB-states points but also their surrounding regions. This can be achieved through two methods: By initiating the quantum circuit at each MUB state while incorporating a parametrized circuit that combines the MUB state with a standard ansatz. By starting at each MUB state and constructing a small-angle ansatz through the implementation of small-angle rotations of each qubit and small-angle rotations of the controlled-phase operation.

## 4 Other Research Directions

In addition to our main research topic, Prof. Mor and I are collaborating on other research problems. We plan to expand Qandies [20] [21] [22] terminology usage for explaining other quantum key distribution protocols and quantum phenomenas such as entanglement-swapping and delay-choice entanglement-swapping [23].



## References

- [1] William K Wootters and Brian D Fields. Optimal state-determination by mutually unbiased measurements. *Annals of Physics*, 191(2):363–381, 1989.
- [2] Thomas Durt, Berthold-Georg Englert, Ingemar Bengtsson, and Karol Życzkowski. On mutually unbiased bases. *International journal of quantum information*, 8(04):535–640, 2010.
- [3] Huangjun Zhu and Masahito Hayashi. Symmetric informationally complete positive operator-valued measures and mutually unbiased bases in dimension six. *Physical Review Research*, 2(1):013338, 2020.
- [4] David Gross, Yi-Kai Liu, Steven T Flammia, Simon Becker, and Jens Eisert. Quantum state tomography via compressed sensing. *Physical Review Letters*, 105(15):150401, 2010.
- [5] Charles H. Bennett and Gilles Brassard. Quantum cryptography: Public key distribution and coin tossing. In *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing*, pages 175–179, 1984.
- [6] Marco Cerezo, Andrew Arrasmith, Ryan Babbush, Simon C Benjamin, Suguru Endo, Keisuke Fujii, Jarrod R McClean, Kosuke Mitarai, Xiao Yuan, Lukasz Cincio, et al. Variational quantum algorithms. *Nature Reviews Physics*, 3(9):625–644, 2021.
- [7] Alberto Peruzzo, Jarrod McClean, Peter Shadbolt, Man-Hong Yung, Xiao-Qi Zhou, Peter J Love, Alán Aspuru-Guzik, and Jeremy L O’Brien. A variational eigenvalue solver on a photonic quantum processor. *Nature communications*, 5(1):4213, 2014.
- [8] Nikolaj Moll, Paul D Nation, and Alán Aspuru-Guzik. *Introduction to Quantum Algorithms for Physics and Chemistry*. CRC Press, Boca Raton, FL, 2018.
- [9] Edward Farhi, Jeffrey Goldstone, and Sam Gutmann. A quantum approximate optimization algorithm. *arXiv preprint arXiv:1411.4028*, 2014.
- [10] Aram W Harrow, Avinatan Hassidim, and Seth Lloyd. Quantum algorithm for linear systems of equations. *Physical Review Letters*, 103(15):150502, 2009.
- [11] David J Wales and Harold A Scheraga. *Energy Landscapes: With Applications to Clusters, Biomolecules and Glasses*. Cambridge University Press, 2003.
- [12] Ken A Dill and Justin L MacCallum. The protein-folding problem, 50 years on. *Science*, 338(6110):1042–1046, 2012.

- [13] David J Wales and Jonathan PK Doye. Global optimization by basin-hopping and the lowest energy structures of lennard-jones clusters containing up to 110 atoms. *The Journal of Physical Chemistry A*, 101(28):5111–5116, 1997.
- [14] Paolo Toth and Daniele Vigo. The vehicle routing problem. *Society for Industrial and Applied Mathematics*, 46(3):412–432, 2002.
- [15] Paolo Toth and Daniele Vigo. *Vehicle Routing: Methods and Studies*. Society for Industrial and Applied Mathematics, 2 edition, 2014.
- [16] Martin Desrochers, Jan Karel Lenstra, and Martin W. P. Savelsbergh. Vehicle routing and scheduling with time windows and multiple routes. *Operations Research*, 40(1):110–125, 1992.
- [17] Paolo Toth and Daniele Vigo. An overview of vehicle routing problems. *The vehicle routing problem*, pages 1–26, 2002.
- [18] Gilbert Laporte and Yves Nobert. Exact algorithms for the vehicle routing problem. In *North-Holland mathematics studies*, volume 132, pages 147–184. Elsevier, 1987.
- [19] Mhand Hifi and Lei Wu. A hybrid metaheuristic for the vehicle routing problem with time windows. In *2014 International Conference on Control, Decision and Information Technologies (CoDIT)*, pages 188–194. IEEE, 2014.
- [20] Whei Yeap Suen, Chun Yat Lee, and Hoong Chuin Lau. Quantum-inspired algorithm for vehicle sharing problem. In *2021 IEEE International Conference on Quantum Computing and Engineering (QCE)*, pages 17–23, 2021.
- [21] Nishikanta Mohanty, Bikash K Behera, and Christopher Ferrie. Analysis of the vehicle routing problem solved via hybrid quantum algorithms in presence of noisy channels. *arXiv preprint arXiv:2205.07630*, 2022.
- [22] Junan Lin and Tal Mor. Quantum candies and quantum cryptography. In *Theory and Practice of Natural Computing: 9th International Conference, TPNC 2020, Taoyuan, Taiwan, December 7–9, 2020, Proceedings 9*, pages 69–81. Springer, 2020.
- [23] Junan Lin, Tal Mor, and Roman Shapira. Quantum information and beyond—with quantum candies. *arXiv preprint arXiv:2110.01402*, 2021.
- [24] Tal Mor, Roman Shapira, and Guy Shemesh. Digital signatures with quantum candies. *Entropy*, 24(2):207, 2022.
- [25] Xiao-song Ma, Johannes Kofler, and Anton Zeilinger. Delayed-choice gedanken experiments and their realizations. *Reviews of Modern Physics*, 88(1):015005, 2016.