Calculate the homology without knowing its definition

Let me summarize: A "homology" of X is a "cycle" that bounds something that does not belong to X, while two cycles are in the same homology if they bound something that does belong to X. However, there is a long way to truly and deeply understand the above intuitive description of homology.

1. The object of our study: topological spaces and continuous maps between them
2. The aim of our study: to understand the homology and homotopy of topological spaces
3. The technique that we will use: combinatorics and linear algebra (of chain complexes, especially the important $\partial^2 = 0$)
4. The mathematical tool that we will use: abstract/modern algebra, such as: groups, rings, and modules.

Sets

So, what is a "topological space"? A "topological space" is a space with "topology". So, what is a "space"? By a "space" we mean a set. So, what is a "set"? By a "set" we simply mean an unordered collection of objets, called "members" or "elements" or "points".

Example

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
2. $\{$students of New UU$\}$

Elements in a set should be "definite". For example, $\{$tall students in New UU$\}$ is not a set.

Operations on sets

Warm-up:

1. subset:
2. intersection:
3. union:
4. complement:

We may also define maps between two sets, and hence injective maps, surjective maps and bijections (isomorphisms).

Product of sets

Let $\{X_i\}$, $i = 1, \cdots, k$, be a finite set of sets. Their (Cartesian) product is the set

$$\{(x_1, x_2, \cdots, x_k) : x_i \in X_i\},$$

and is denoted by $\prod_{i=1}^{k} X_i$.

Examples

1. $\mathbb{R}^n := \{(x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\}$
2. $\mathbb{C}^n :=?$
3. $\mathbb{S}^n := \left\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\right\}$
4. $\mathbb{D}^n := \left\{(x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=0}^{n} x_i^2 < 1\right\}$
5. $C(\mathbb{R}) := \{$continuous functions on $\mathbb{R}\}$
6. $C^\infty(\mathbb{R}) := \{$smooth functions on $\mathbb{R}\}$

Equivalence equation

Definition

Let S be a set and let $\sim$ be a "relation" on S , i.e., a collection of tuples (x, y) with $x, y \in S$. Given such a tuple (x, y), we write $x \sim y$. Then $\sim$ is an "equivalence relation" if the following holds.

1. For all $x, y \in S$, if $x \sim y$ then $y \sim x$.
2. For all $x \in S$ we have $x \sim x$.
3. For all $x, y, z \in S$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

In this case, we write $[x]$ to denote the subset of X whose elements are equivalent to x.

Examples

1. On $\mathbb{R}$, define an equivalence relation to be: $x \sim y$ if and only if $x - y \in \mathbb{Z}$.
2. On $\mathbb{R}^2$, define an equivalence relation like the following:

$$(x, y) \sim (x', y') \text{ if and only if } \begin{cases} x - x' = n \in \mathbb{Z} \\ y = y' \end{cases}$$

3. We can also define on $\mathbb{R}^2$ an equivalence relation like the following:

$$(x, y) \sim (x', y') \text{ if and only if } \begin{cases} x - x' = n \in \mathbb{Z} \\ y = (-1)^n y' \end{cases}$$

More examples

1. On $\mathbb{R}^2$, define an equivalence relation like the following:

$$(x, y) \sim (x', y') \text{ if and only if } \begin{cases} x - x' = n \in \mathbb{Z} \\ y - y' = m \in \mathbb{Z} \end{cases}$$

2. Let $\mathbb{S}^2$ be the (2-dimensional) sphere. Give an equivalence relation on $\mathbb{S}^2$ as follows: $x, y \in \mathbb{S}^2$ are equivalent if and only if $x = \pm y$. (We might have heard that the map $x \mapsto -x$ is called the "antipode map". We will study that later.)

Quotient space

Suppose X is a set with an equivalence relation $\sim$. Then the collection of equivalent classes forms a set, called the "quotient" space of X, and is denoted by $X/\sim$. That is, $X/\sim = \{[x]\}$.

There is a natural map $X \to X/\sim$, which maps x to $[x]$, and is usually denoted by π or p (meaning "projection").

Example

1. $S^1 = \mathbb{R}/\mathbb{Z}$
2. Infinite cylinder
3. The Möbius band
4. Torus $T^2 := S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$

More examples of quotient spaces

The quotient space $\mathbb{S}^2/\sim$ is called the "real projective plane", and is denoted by $\mathbb{RP}^2$.

Question

What does $\mathbb{RP}^2$ look like?

Homework

Try to define $\mathbb{RP}^n$ and $\mathbb{CP}^n$. They are called the "n-dimensional real and complex projective space". Show that $\mathbb{CP}^1$ is isomorphic (in topology, we usually use the word "homeomorphic") to $\mathbb{S}^2$.

Imagine what they look like. Google or Deepseek!

Metric spaces

We turn to study "topological spaces", but let us start with some simple examples.
Suppose NewUU has 500 students and each student is assigned with an ID number between 1 and 500. We may say that two numbers, 250 and 251, are close to each other, but we cannot say two students are close to each other, although there is an isomorphism between the two sets.
Why? Here we have implicitly used the fact that the set of numbers has a "metric", and with such a metric, we may measure the distance of two numbers. The set of NewUU students does not have a metric a priori.

Definition (Metric space)

A metric space is specified by a pair (X, d_X) where X is a set and d_X is a function $d_X : X \times X \to R_{\geq 0}$ (called the distance function) such that the following holds:

1. $d_X(x, y) = 0$ if and only if $x = y$;
2. $d_X(x, y) = d_X(y, x)$;
3. For any $x, y, z \in X$, the "triangle inequality" holds:

$$d_X(x, z) \leq d_X(x, y) + d_X(y, z).$$

Examples of metric space

1. $\mathbb{R}^n$
2. $\mathbb{C}^n$
3. $M_n(\mathbb{R})$

Hamming distance

Definition

Fix an alphabet Σ, i.e., a set of symbols we will call letters (we can just take $\Sigma = \{A, B, C, D, \cdots, Z\}$). Let x and y be two words of length n with letters in Σ. Then the "Hamming distance" between x and y is defined to be the number of positions at which the letters of x and y differ:

$$d_H(x, y) = \#\{i : x_i \neq y_i\}.$$

Recall that $\#$ of a set X is the "cardinality" of X, i.e., the number of elements in X.

Example

The Hamming distance between

1. "karolin" and "kathrin" is 3.
2. "karolin" and "kerstin" is 3.
3. "kathrin" and "kerstin" is 4.
4. 0000 and 1111 is 4.
5. 2173896 and 2233796 is 3.

Homework

Write, in Python or in C (or any language you like), a program to find the distance of two words.

With the definition of a metric, we may talk about the distance of two points in a set. Remember that our goal is to define the topological spaces and continuous maps. However, with only a metric, this is not enough. Recall that in Calculus I, a function is continuous if $\lim_{n\to\infty} f(x_i) = f(\lim_{n\to\infty} x_i)$, but sometimes even $\lim_{n\to\infty} x_i$ does not exist (although they may go closer and closer to each other)!

For example, in $\mathbb{Q}$, let $\{x_i\}$ be the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \cdots$$

We know the sequence converges to π, but π is not in $\mathbb{Q}$.

Cauchy sequence

Definition

For a metric space (X, d_X), a "Cauchy sequence" is a sequence of points $\{x_i\}$ in X such that for all $\varepsilon > 0$, there exists a natural number N such that $d_X(x_j, x_k) < \varepsilon$, for all $j, k > N$.

A metric space (X, d_X) is "complete" if every Cauchy sequence converges to a point $x \in X$.

Continuous functions revisited:

Definition

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f\colon X \to Y$ is "continuous" if for every sequence $\{x_i\}$ in X converging to x, the sequence $\{f(x_i)\}$ converges in Y to $f(x)$; that is, $\lim_{n\to\infty} f(x_i) = f(\lim_{n\to\infty} x_i)$.

Topology

Suppose (X, d_X) is a metric space. For any $x \in X$ and and $r \geq 0$, the set

$$U_x(r) := \{y \in X : d(x, y) < r\}$$

is called an open neighborhood of x (with radius r). More generally, a subset U of X is called an "open" subset (neighborhood) if for any $x \in U$, there is an $r > 0$, such that $U_x(r) \subset U$. All such open neighborhoods together form the "topology" of X.

Remark

From above, we see that the union of (possibly infinite number of) open neighborhoods like above is open, and so is the intersection of finite number of neighborhoods like above. In particular, X and $\emptyset$ are both open.

Topological space

For a metric space (X, d), we have defined a topology on X. X with a topology is called a "topological space". In the following we are going to define topological space without specifying a metric on it.

Definition

A topological space is a pair $(X, \mathcal{U})$, where X is a set and $\mathcal{U}$ is a topology of X, that is, a collection of subsets of X, which we refer to as open sets. The open sets satisfy the following conditions.

1. Both the empty subset $\emptyset$ and X are elements of $\mathcal{U}$.
2. Any union of elements of $\mathcal{U}$ is an element of $\mathcal{U}$.
3. The intersection of a finite collection of elements of $\mathcal{U}$ is an element of $\mathcal{U}$.

A subset $Z \subset X$ is "closed" if the complement of Z in X is open.

Examples of topological space

1. discrete topology
2. indiscrete topology
3. Zariski topology on $\mathbb{R}$

Continuous functions revisited again:

Definition

Let $(X,\mathcal{U})$ and $(Y,\mathcal{W})$ be two topological spaces. A map $f\colon X\to Y$ is "continuous" if the inverse of any open set is open.

Homework

Check that the above definition, when both X and Y are $\mathbb{R}$ with the usual (Euclidean) topology, is equivalent to the definition of continuous functions given before.

Induced topology