

# Calculus Notes

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Fri, 12 Feb. 2021

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**Part I**  
**Calculus II**

# 1 Vectors

## 1.1 The 3D space

When you have a constant in 3D, then the graph will have a plane parallel to the base the unit vector is perpendicular to.

e.g. for the equation  $x^2 + y^2 = 25, z = -8$  all points will lie in the horizontal plane of  $z = -8$ . Also, since  $x^2 + y^2 = 25$  then the graph will have the region with all points lying on a circle with radius  $\sqrt{25}$ .

**Common Sense Facts:** If  $A(x, y, z)$  and  $B(h, k, l)$ , then  $\vec{AB}$  will be  $AB(h - x, k - y, l - z)$ .

The length of vectors will be equal to the square root for adding their squared values. e.g. For the vector  $\vec{a} < x, y, z >$

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$$

Also, addition, subtraction, scalar multiplication of two vectors algebraically will be just the addition or subtraction or the scalar multiplication of their coordinates.

## 1.2 The sphere

For the sphere with center  $C(h, k, l)$  and radius  $r$ , the equation will be

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \quad (1)$$

e.g. For the equation  $(x + 2)^2 + y^2 + (z - 3)^2 = 16$  the center of the circle will be  $C(-2, 0, 3)$  and the radius will be  $\sqrt{16} = 4$ . Also, a sphere with center the origin will have the equation  $x^2 + y^2 + z^2 = r^2$ .

## 1.3 Vectors addition

To sum vectors  $\vec{u}, \vec{v}$  you will position the vectors as the initial point of  $\vec{v}$  is the terminal point of  $\vec{u}$ . The sum will be the vector from initial point of  $\vec{u}$  to the terminal point of  $\vec{v}$ .

On the same base, and since  $-\vec{v}$  will be the same as the vector but in the opposite direction,  $\vec{u} - \vec{v}$  will connect the tip of  $\vec{v}$  to the tip of  $\vec{u}$

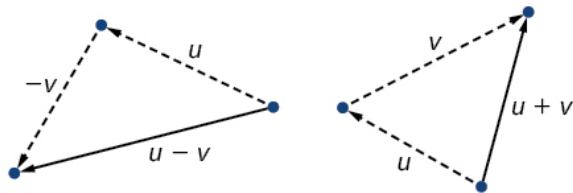


Figure 1: Representing vectors addition in two manners.

### More example

**Question** Find the unit vectors that are parallel to the tangent line to the parabola  $y = x^2$  at the point  $(2, 4)$

**Solution** The tangent line will be  $\frac{d}{dx}(x^2) = 2x = 2(2) = 4$  then the parallel lines to that vector will be the vectors  $\langle 1, 4 \rangle, \langle -1, -4 \rangle$ . To get the unit vectors, equalize the magnitude to 1.

$|\langle 1, 4 \rangle| = \sqrt{1^2 + 4^2} = \sqrt{17}$ . Thus the unit vectors will be  $\pm \frac{1}{\sqrt{17}} \langle 1, 4 \rangle$

## 1.4 Dot Product

### Formulas

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \quad (2)$$

$$\vec{a} \cdot \vec{b} = (a_1)(b_1) + (a_2)(b_2) + (a_3)(b_3) \quad (3)$$

**Direction Angles** They are  $\alpha, \beta, \gamma$  in the interval  $[0, \pi]$  that a vector  $\vec{a}$  makes with positive  $x, y, z$  axis. Cosines of these angles must equal 1.

$$\cos \alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}| |\vec{i}|} = \frac{a_1}{|\vec{a}|} \quad (4)$$

$$\cos \beta = \frac{a_2}{|\vec{a}|} \quad \cos \gamma = \frac{a_3}{|\vec{a}|}$$

Squaring the previous equations, we find that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (5)$$

Thus, we can say that direction cosines of  $\vec{a}$  are the components of unit vector in the direction of  $\vec{a}$ .

$$\frac{1}{|\vec{a}|} \vec{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \quad (6)$$

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## 1.5 Projections

Scalar Projections of  $\vec{b}$  onto  $\vec{a}$

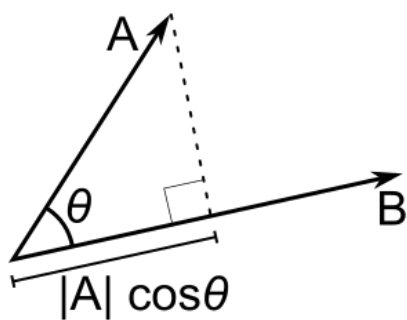
$$\text{comp}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

Vector Projections of  $\vec{b}$  onto  $\vec{a}$

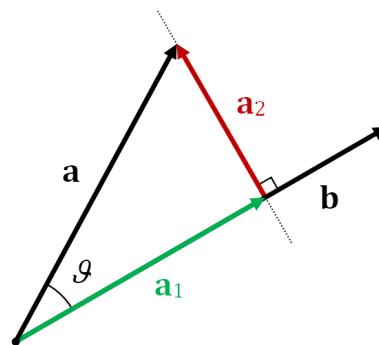
$$\text{proj}_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

The Orthogonal Vector to the projection

$$\text{orth}_a \vec{b} = \vec{b} - \text{proj}_a \vec{b}$$



(a) Scalar Projection of A onto B



(b) Vector Projection of vector  $\vec{b}$  onto  $\vec{a}$  is vector  $\vec{a_1}$



## 1.6 Cross Product

### 1.6.1 Matrix?

We will use the matrix to track where the unit vectors land after the transformation of some stuff. For instance, in the case of 2x2 Matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$a$  and  $c$  track where  $\vec{i}$  lands. and  $b$  and  $d$  track where  $\vec{j}$  lands. Thus, any transformation for the grid will be a multiplication of that unit transformation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Also, the determinant in this manner measure the factor by which the area changes. This area visualized in Figure 3 is what we'll call "cross product". However, it's really not the cross product, it's the magnitude of cross product.

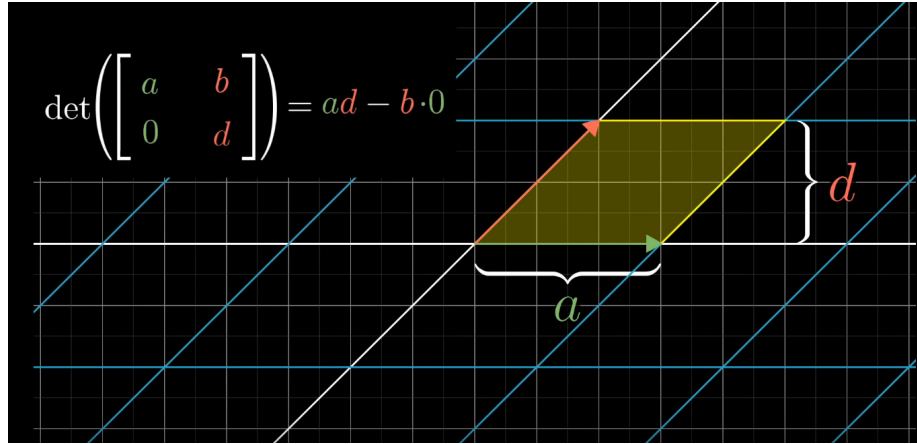


Figure 3: Visualization for the area, aka determinant by 3b1b

**Calculation** To calculate the "value" aka the determinant or magnitude of the matrix, we follow some simple rules that you can derive from Figure 3.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \vec{i} \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \vec{j} \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \vec{k} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

You will then calculate using 2D method. You can notice that we "hide" the rows and columns of the value we multiply to and select the remaining.

### 1.6.2 The Cross Product

**Intuition** In Figure 3, we said that the cross product will be the determinant of the two vectors, or the area formed by extending the parallelogram to this vector. The cross product between two vectors will produce a vector perpendicular to the plane formed by the vectors. Only the magnitude of such vector equals the area of the parallelogram formed by extending these two vectors. Direction of the cross product in 3D manner can be found using right hand rule. In 2D manner, use the unit vectors method;  $\vec{i} \times \vec{j} = +1$ .

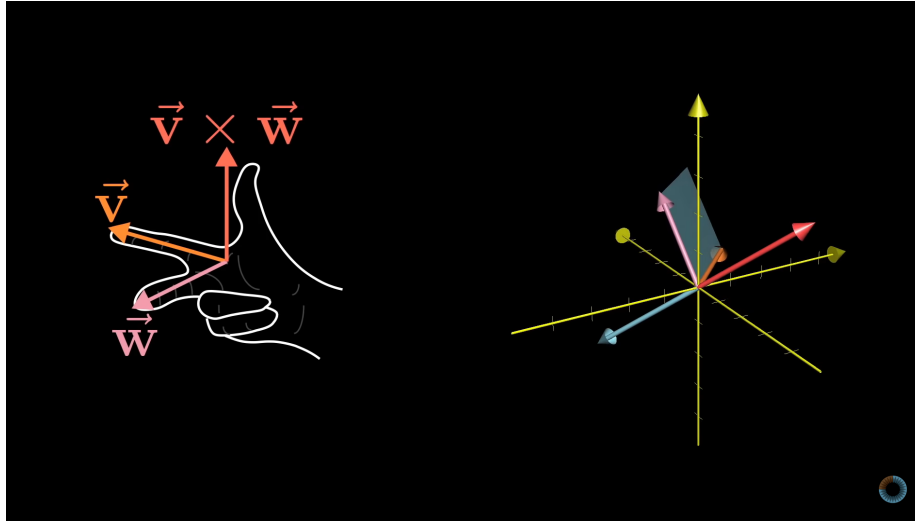


Figure 4: Cross Product Visualized

#### TL;DR Formula

$$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

$$|a \times b| = |a||b|\sin\theta$$

$$a \parallel b \implies a \times b = 0$$

Notice that the cross product is not commutative.

$$i \times j = k \quad j \times k = i \quad k \times i = j \quad j \times i = -k \quad k \times j = -i \quad i \times k = -j$$

$$a.(b \times c) = (a \times b).c \quad a \times (b \times c) = (a.c)b - (a.b)c$$

## Famous examples

**The Parallelepiped** The volume of that figure equals the product between the base and attitude. Thus, we have three vectors  $\vec{v}, \vec{u}, \vec{w}$ , the volume can be calculated by getting the cross product of the base (any two of the three vectors). Then, the volume will be the dot product of both the cross product and the third vector, as known as the triple scalar product. It can also be used to find whether some vectors are coplaner; (that triple product will equal 0)

$$(\vec{v} \times \vec{u}) \cdot \vec{w}$$

However, in the case the question gave you three vectors directly, you can just blot the three vectors on a matrix and solve, the absolute value of the determinant will be the volume.

$$Volume = \det \left( \begin{bmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right)$$

1. For any vectors  $u$  and  $v$  in  $V_3$  and any scalar  $k$ ,  $k(u \times v) = (ku) \times v$ .
2. For any vectors  $u$  and  $v$  in  $V_3$ ,  $(u + v) \times v = u \times v$ .

## Distance between a line and a point

$$\frac{|\vec{u} \times \vec{v}|}{|\vec{v}|}$$

Where  $\vec{u}$  is some line  $\vec{AB}$  For instance, and  $\vec{v}$  is the point. This equation is visualized as the area of parallelogram divided by the base, as visualized in the cross product above.

## 1.7 Lines and Planes in 3D

### 1.7.1 Equation of the line

Firstly remember that any line will be parallel to a unit vector with same direction, and each parallel vector is the scalar multiplication of the vector it is parallel to. Thus, technically any line is just some unit vector that got pushed by some scalar value. To elaborate, let  $\vec{v}$  is the unit vector parallel to  $\vec{L}$ .

$$\therefore \vec{L} = t.\vec{v} \quad (1)$$

In general, the vector equation of the line that's parallel to  $\vec{v}$  and passes through the point  $r_0$ , it's equation is

$$\vec{L} = r_0 + t\vec{v}$$

Let's let  $\vec{L} = \vec{P_0P}$  where  $P_0(x_0, y_0, z_0) \& P(x, y, z) \implies \vec{L} = \langle x - x_0, y - y_0, z - z_0 \rangle$ . Besides, let  $\vec{v} = \langle a, b, c \rangle$ . From (1):

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t. \langle a, b, c \rangle$$

$$x - x_0 = at \quad y - y_0 = bt \quad z - z_0 = ct \quad (\text{Parametric Equation})$$

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \quad (\text{Symmetric Equation})$$

**Some Important Points** If one direction is 0, then the line is a plane. If  $a = 0 \implies x = x_0$  and the line lies in  $x = x_0$  plane, parallel to the yz-plane.

For the direction vector, you do any scalar multiplication to make the thing easier. You can multiple or divide by some constants for the direction vector **except** when you're dealing with the distance.

If  $a / b / c$  equals zero, you'll write the symmetric equation with  $z$  as constant.

### 1.7.2 Planes

Remember the normal vector, which is any vector perpendicular to the plane. It can be normally got by the cross product of two vectors. Also, since all normal vectors to a plane will be all parallel, all of them are scalar multiplication of each other.

Let's play with the same line  $P_0\vec{P}$  we used in the lines. Also let the normal line to the plane  $\vec{n}$  where  $\vec{n} \perp P_0\vec{P} \implies \vec{n} \cdot P_0\vec{P} = 0$ . Actually this is the standard form of a plane. Let's play with that.

The equation of a plane can be got easily using  $\vec{n}$  which is the normal or cross product of the two vectors that form the plane. Then, the equation of the plane will be  $n_1x + n_2y + n_3z = d$

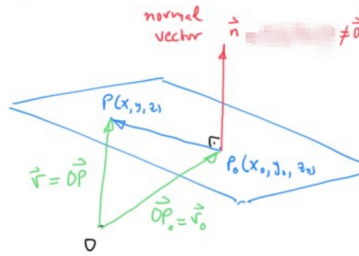


Figure 5: Equation of the plane visualization, by Dr. Lotfallah

$$\vec{n} \cdot P_0\vec{P} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Since R.H.S is all constants, they will be denoted by d

$$x + by + cz = d$$

### 1.7.3 Remember

Two planes are  $\parallel$  if their normals are  $\parallel$ .

Two planes are  $\perp$  if their normals are  $\perp$ .

Angle between planes = angle between normals.

Scalar multiplication determines parallelism, dot product determine perpendicularism (if it's equal to zero).

Two planes parallel to a line are not necessarily parallel.

### 1.7.4 Questions

1. The generic question will provide two points, P and Q for instance, where you will get  $\vec{v} = \vec{PQ}$ . Then, you'll use a point to solve. For points  $P(-1, -2, -\frac{1}{2}), Q(1, \frac{3}{2}, -3)$

$$\vec{v} = \vec{PQ} = \langle 2, \frac{7}{2}, -\frac{5}{2} \rangle \cdot 2 = \langle 4, 7, -5 \rangle$$

By using the point P

$$\therefore x = -1 + 4t \quad y = -2 + 7t \quad z = -\frac{1}{2} - 5t$$

Also

- a. You can reverse-engineer the previous equation to get the original point by plugging  $t = 0$ .
  - b. Or find the unit vector by getting the coefficients of the t, but make sure that it's only 1 x / y / z.
  - c. You can easily convert to the symmetric form by solving each of the equations to t.
2. The second generic question will be whether two lines are intersecting and their intersection point. Firstly, you will make sure they are not parallel by the normal scalar multiplication check. Then, you will equalize each of the parametric equation of the two lines, since the intersection will occur at  $x_1 = x_2, y_1 = y_2, z_1 = z_2$ . Then, solve for t.

$$L_1 : x_1 = 1 - 2t_1, y_1 = -1 - 3t_1, z_1 = -2 + t_1$$

$$L_2 : x_2 = -3 + t_2, y_2 = -2 + 2t_2, z_2 = 3 - t_2$$

$$x : 1 - 2t_1 = -3 + t_2 \quad y : -1 - 3t_1 = -2 + 2t_2 \quad z : -2 + t_1 = 3 - t_2$$

Solving two equations together.

$$1 - 2t_1 = -3 + t_2 \quad -2 + t_1 = 3 - t_2$$

$$-1 - t_1 = 0 \implies t_1 = -1, t_2 = 6$$

Then, you will equalize the values you got to the equations in x, y, and z.

$$x : 3 = 3 \quad y : 2 \neq 10 \quad z : -3 = -3$$

Thus, since the three equations did not hold with  $t_1, t_2$  values. Then, the two lines do not intersect. You can think of that as two planes flying at separate times; while they are not parallel, then do not intersect. This line is called **skew lines**.

If  $t_1 = t_2$  in the last equalization, the point where the intersection happens is simply values of  $\langle x, y, z \rangle$

3. Another similar question to the previous is asking where a line intersects a plane. Just put the equations of  $x, y, z$  in the equation of the plane given, and solve for  $t$ . Then you'll have the points.
4. The generic question will give you two vectors where you will get the normal vector, then you will use any point to get the equation for the plane. So, using a vector and a point will get you the equation of the plane or the line; normal vector in case of plane, and the unit vector parallel to the line in the latter.

5. **Find equation for line of intersection**

$$P_1 : 2x - 3y + 4z = 3 \qquad P_2 : x + 4y - 2z = 7$$

For this question you will need to get the normals of the planes first.

$$n_1 = \langle 2, -3, 4 \rangle \qquad n_2 = \langle 1, 4, -2 \rangle$$

Then we will need a point, direction vector of the line of intersection. The intuition behind the answer is that the line of intersection will be parallel to the vector which is mutually orthogonal to the two normal to the planes, as shown in 6.

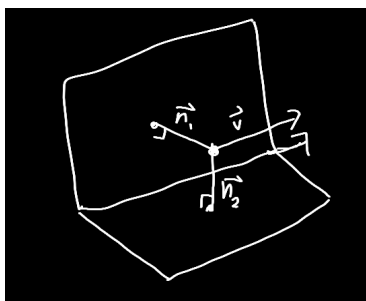


Figure 6: Intersection Line

Then, we just get the cross product to figure out  $\vec{v}$ .

$$\vec{n}_1 \times \vec{n}_2 = \langle -10, 8, 11 \rangle \qquad (Direction)$$

Now, we need a point on the intersection line since we figured the direction. Let's let any variable equal zero and solve for the rest, I'll let  $y = 0$ .

$$\begin{aligned} 2x + 4z &= 3 & (x - 2z &= 7)2 \\ \therefore x &= \frac{17}{4} & y &= 0 & z &= -\frac{11}{8} \end{aligned} \quad (Point)$$

Now, we're done! Let's just put everything in the equation of the line.

$$x = -10t + \frac{17}{4} \qquad y = 8t \qquad z = 11t - \frac{11}{8}$$

6. **Distance between a point and a plane** The distance, as viewed in Figure 7, will be just the projection of the vector  $\vec{P_0P_1}$  denoted by  $\vec{b}$  onto the normal to the plane.

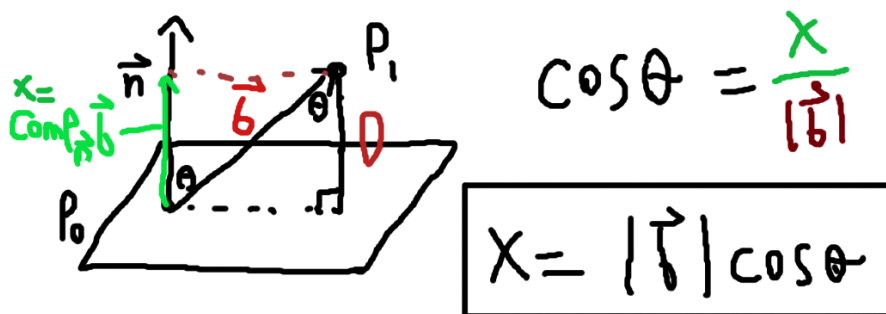


Figure 7: Scraping to visualize.

As proved in the dot product

$$D = \text{Comp}_{\vec{n}} \vec{b} = \frac{\vec{n} \cdot \vec{b}}{|\vec{n}|} = |\vec{b}| \cos \theta$$

We can play with the previous formulas. To get  $D$  from  $P_1(x_1, y_1, z_1)$  to the plane  $ax_0 + by_0 + cz_0 + d = 0$ .

$$D = \frac{\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \cdot \langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}} = \frac{|(ax_1 + by_1 + cz_1) - (ax + by + cz)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d = ax + by + cz \quad \therefore D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

## 7. Distance between two planes

$$P_1 : 5y = 2 + 2x + z \quad P_2 : 4x + 2z = 7 + 10y$$

Firstly, Check that they are parallel (If not, no fixed distance). Then, to get the distance between two planes, we will first get any point from the plane and use it to calculate using the formula  $d = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

Let  $x, y = 0$  then,  $P_{1a} = \langle 0, 0, -2 \rangle$

$$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$



Substituting numbers from the question

$$d = \frac{|2(-2) - 7|}{\sqrt{4^2 + 2^2 + (-10)^2}} = \frac{11\sqrt{30}}{60} \quad (\text{Answer})$$

**Another Situation** If components of x,y,z are equal, we can literally just use the difference between  $d_2, d_1$  as numerator in the equation, since they have common normals. This one can be used to get a function of a plane given that they are parallel and given the distance between them

$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

**Remember**, that was in case the equations are

$$ax + by + cz + d_1 = 0 \qquad ax + by + cz + d_2 = 0$$

**8. Containing skew lines between parallel planes + get the distance**

Get direction vectors of lines, denoted  $\vec{v}_1, \vec{v}_2$ . Then get the common normal easily by  $\vec{n} = \vec{v}_1 \times \vec{v}_2$ . Use points from lines, denoted  $P_1, P_2$ , and take the normal with each independent point to find the equation of planes. Then, after finding the equations you can get the distance between them using the method in the previous question.

**9. Find an equation of the plane** that passes through the line of intersection of the planes  $x - z = 2$  and  $y + 4z = 3$  and is perpendicular to the plane  $x + y - 4z = 5$

**Solution** First, we will get the line of intersection using the cross product of the normals of two lines,  $\vec{n}_1, \vec{n}_2$ .

$$\vec{n}_1 = \langle 1, 0, -1 \rangle \qquad \vec{n}_2 = \langle 0, 1, 4 \rangle \qquad \vec{n}_1 \times \vec{n}_2 = \langle 1, -4, 1 \rangle = \vec{u}$$

We then get a point in the line of intersection by setting  $z = 0$

$$x = 2 \qquad y = 3 \qquad z = 0 \qquad (1)$$

We know that the normal to the other plane is parallel to  $\vec{u}$ , so we can get the line normal to these two vectors using cross product again.

$$\vec{u} \times \langle 1, 1, -4 \rangle = \langle 15, 5, 5 \rangle \qquad (2)$$

$$\text{From (1),(2)} \implies 15x + 5y + 5z = (30 + 15) = 45 \quad \#$$

10. A cool way to get the distance  $D$  between a point  $P$  and a line  $\vec{v}$  is to firstly assume there is a vector from  $v_0$  to  $P$ , let's denote it by  $\vec{u}$ . using normal laws:

$$\begin{aligned} D &= |\vec{u}| \sin \theta \quad \times |\vec{v}| \\ D \cdot |\vec{v}| &= |\vec{u}| |\vec{v}| \sin \theta = |\vec{u} \times \vec{v}| \\ \therefore D &= \frac{|\vec{u} \times \vec{v}|}{|\vec{v}|} \end{aligned}$$

11. Find line that represents shortest distane between two skew lines

$$\begin{aligned} L_1 : x &= 1 + t & y &= 0 & z &= -t \\ L_2 : x &= -k & y &= k + 2 & z &= k \end{aligned}$$

**Intuitive Fast Solution** Let's consider the vector that links the two generic points of the lines  $\vec{t}$ . Then, we try to force perpendicularity by equalizing the dot product between the direction vector of the lines to zero  $\vec{v}_1, \vec{v}_2$ .

$$\vec{v}_1 = \langle 1, 0, -1 \rangle \quad \vec{v}_2 = \langle -1, 1, 1 \rangle$$

$$\vec{t} = \langle 1 + t - (-k), -(k + 2), -t - k \rangle$$

Then by forcing perpendicularity

$$\vec{v}_1 \cdot \vec{t} = 1 + 2t + 2k = 0 \quad \vec{v}_2 \cdot \vec{t} = -3 - 2t - 3k = 0$$

Solving those two equations will get us  $t = \frac{3}{2}$  and  $k = -2$ . Thus, we can easily derive the equation of shortest distance by putting  $t, k$  in  $\vec{t}$ .

12. **Distance between two skew lines** which passes through  $P_1P_2, P_3P_4$ . Distance will be simply the height of the parallelepiped formed by the two vectors as shown in Figure 8.

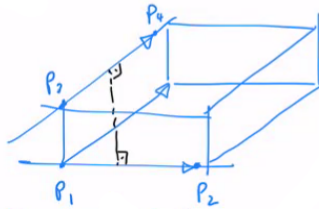


Figure 8: Parallelepiped formed by the two vectors, by Dr. Lotfallah

$$D = \frac{Vol_{parallelepiped}}{Area_{base}} = \frac{|(P_1\vec{P}_2 \times P_3\vec{P}_4) \cdot P_1\vec{P}_3|}{|P_1\vec{P}_2 \times P_3\vec{P}_4|}$$

13. If given a line perpendicular to other line, and you have only a point on the line and the equation of the other, get a vector that connects the two points and using formula known in projections  $orth_a \vec{b} = \vec{b} - proj_a \vec{b}$ . Now you have a direction and a point, voila!
14. To prove that 4 points are coplaner, proof that the volume of the parallelepiped formed by them equals 0.

## 2 Inverse Functions

### 2.1 Introduction

Firstly, a reminder that a one-to-one function;  $f(x_1) \neq f(x_2)$  whenever  $x_1 \neq x_2$ . You can usually test that using the horizontal line test. These functions are the only ones that have inverse. With domain as range, and vice versa.

$$f^{-1}(y) = x \quad \implies \quad f(x) = y$$

To find the inverse you will more likely solve the equation for  $x$  in terms of  $y$  then interchange  $x$  and  $y$ . Where the graph of  $f^{-1}$  is obtained by reflection  $f$  about the line  $y = x$ .

### 2.2 Logarithmic Function

The most famous inverse function is the logarithm, which can be obtained from the exponential function  $y = b^x$

$$b^y = x \implies \log_b x = y$$

We can get many identities off that thing as  $\log_b b^x = x$  for  $x \in \mathbb{R}$ . Also,  $b^{\log_b x} = x$  for  $x > 0$ . Also,  $\log_b xy = \log_b x + \log_b y$  and same for division but it will be subtraction. Plus,  $\log_b x = \frac{\ln x}{\ln b}$ .

### 2.3 Inverse trigonometry

#### 2.3.1 Introduction

Since they're not one-to-one, we will just restrict the domains of functions to be one-to-one.

$$\sin^{-1} x = y, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \quad -1 \leq x \leq 1$$

$$\cos^{-1} x = y, \quad 0 \leq y \leq \pi \quad -1 \leq x \leq 1$$

$$\tan^{-1} x = y, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

The arccsc and arcsec aren't used a lot. To get the arcsec for example go for  $\cos^{-1} \frac{a}{b} = \sec^{-1} \frac{b}{a}$ . Or use the identities of the cheat sheet.

$$\cot^{-1} x = y \quad y \in (0, \pi)$$

### 2.3.2 Derivatives

Firstly there is a fact that the inverse of function of any one-to-one differentiable function is differentiable as well. To find the derivative of arcsin for example, let's differentiate  $\sin(y) = x$  implicitly.

$$\cos y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos y} \implies \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

With some other proofs, you'll find that

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}} \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

### 2.3.3 Questions

1. **Simplify**  $\sin(\tan^{-1}(x))$ .  $\tan(x)$  is the  $\frac{opp}{adj}$ , so in  $\tan^{-1} x$ , it can be imagined as a triangle with opposite side  $x$  and adjacent side 1. Thus, the sine of this angle will equal the opposite over hypotenuse, which can be got by Pythagoras.

$$\frac{x}{\sqrt{x^2+1^2}}$$

## 2.4 Hyperbolic Functions

### 2.4.1 Identities

1.  $\sinh$  is an odd function.
2.  $\cosh$  is an even function.
3.  $\sinh x = \frac{e^x - e^{-x}}{2}$
4.  $\cosh x = \frac{e^x + e^{-x}}{2}$
5.  $\operatorname{csch} x = \frac{1}{\sinh x}$
6.  $\operatorname{sech} x = \frac{1}{\cosh x}$
7.  $\cot h x = \frac{\cosh x}{\sinh x}$
8.  $\cosh^2 x - \sinh^2 x = 1$
9.  $\operatorname{sech}^2 x = 1 - \tanh x$
10.  $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
11.  $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
12.  $\sinh(2x) = 2 \sinh x \cosh x$
13.  $\cosh(2x) = \cosh^2 x + \sinh^2 x$
14.  $\cosh^2 x = \frac{1}{2}(1 + \cosh(2x))$
15.  $\sinh^2 x = \frac{1}{2}(-1 + \cosh(2x))$
16.  $\frac{d}{dx} \sinh x = \cosh x$
17.  $\frac{d}{dx} \cosh x = \sinh x$
18.  $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x} - \operatorname{sech}^2 x$
19.  $\frac{d}{dx} \csc hx = -\csc hx \cot hx$
20.  $\frac{d}{dx} \sec hx = -\sec hx \tanh x$
21.  $\frac{d}{dx} \cot h = -\csc h^2 x$
22. Integrals of  $\sinh$  and  $\cosh$  are the opposite + C. That's since there is no negative as the normal trig functions.
23.  $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$   $(-\infty, \infty)$
24.  $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$   $[1, \infty]$
25.  $\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$   $(-1, 1)$

$$26. \frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$

$$27. \frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$$

$$28. \frac{d}{dx} \tanh^{-1} x = \frac{1}{1-x^2}$$

$$29. \frac{d}{dx} \sec h^{-1} x = -\frac{1}{x\sqrt{1-x^2}}$$

$$30. \frac{d}{dx} \csc h^{-1} x = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$31. \frac{d}{dx} \cot h^{-1} = \frac{1}{1-x^2}$$

## 3 More Integration

### 3.1 Integration by Substitution

Let's say we have some complex equation that's consists of two terms and it looks like an "inverse chain rule". Here you will substitute one of the parts with  $u$  where a manipulation of  $u$  or  $du$  come up with a simpler expression. Note that after you get  $du$ , solve for  $dx$  so you can replace in the original integral equation. You now understand nothing, so let's see an example.

$$\int 4x(x^2 + 5)^3 dx$$

Let  $u = x^2 + 5$ .

$$u = x^2 + 5 \implies \frac{du}{dx} = 2x \implies dx = \frac{du}{2x}$$

Let's plug both  $du$  and  $\frac{du}{2x}$  in the main equation.

$$\int \frac{4x}{2x} (u)^3 du = 2 \int u^3 du = 2 \frac{u^4}{4} + C = \frac{u^4}{2} + C$$

Then, replace  $u$  with  $x^2 + 5$

$$\implies \frac{(x^2 + 5)^4}{2} + C$$

Note that if you couldn't put  $du$  directly to cancel the other term in the integral, you can always solve for the other term.

$$\int x\sqrt{3x+2} dx$$

$$u = 3x + 2 \implies \frac{du}{3} = dx \quad x = \frac{u-2}{3}$$

Plugging in the question.

$$\int \frac{u-2}{3} \sqrt{u} \frac{du}{3} = \frac{1}{9} \int (u-2)\sqrt{u} du$$

And you can figure out the rest.



### 3.1.1 Questions

1.

$$\int \frac{\sin(2x)}{37 + \cos^2 x} dx$$

**Solution** Here you will use the identity  $\sin(2x) = 2 \sin x \cos x$ . Then, let  $u = 37 + \cos^2 x \implies du = -2 \cos x \sin x$ . Now the problem is solved! Remember  $\int \frac{1}{x} = \ln x$ .

2.

$$\int \frac{x^5}{1 + x^{12}} dx$$

**Solution** Here, the trick is to set  $u = x^6$  where the derivative will cancel with the numerator.

$$\begin{aligned} u = x^6 &\implies du = 6x^5 dx \implies \int \frac{x^5}{(6x^5)(1 + u^2)} du \\ &\implies \int \frac{1}{5(1 + u^2)} du = \frac{1}{5} \int \frac{1}{1 + u^2} du \end{aligned}$$

And as you don't remember,  $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$ . So, now you can solve it.

3.

$$\int_0^1 5 \cos\left(\frac{\pi x}{2}\right) dx$$

**Solution** In definite integrals, remember to change the boundaries as you change the equation. So by letting  $u = \frac{\pi x}{2} \implies du = \frac{\pi}{2} dx$ . Now you will get the  $u$  values for the boundaries 0,1:  $x = 0 \implies u = 0$ ,  $x = 1 \implies u = \frac{\pi}{2}$ . Therefore the equation will be

$$\int_0^{\frac{\pi}{2}} 5 \cos u \frac{2}{\pi} du = \frac{10}{\pi} [\sin u]_0^{\frac{\pi}{2}} = \frac{10}{\pi}$$

4.

$$\int_{-4}^4 (x+4)\sqrt{16-x^2} dx$$

**Solution** This will be solved by writing the integral as the sum between two integrals. Then, you will find that one of them is an odd function and since  $\int_{-a}^a f(x) = 0$  if  $f$  is odd, so the problem will be much easier.

$$\int_{-4}^4 (x+4)\sqrt{16-x^2} dx = \int_{-4}^4 x\sqrt{16-x^2} + \int_{-4}^4 4\sqrt{16-x^2} = 0 + \int_{-4}^4 4\sqrt{16-x^2}$$

Now, solution will be intuitive a bit since we know that this is the equation of a semicircle with radius  $\sqrt{16}$  multiplies by 4  $\implies \int = 4 \frac{(\pi 4^2)}{2} = 32\pi$ .

### 3.2 Integration by parts

It is derived from the product rule by integrating its normal formula. Proof is left as exercise.

$$\int f(x).g'(x) dx = f(x).g(x) - \int g(x).f'(x) dx$$

$$\int u dv = u.v - \int v du$$

Choose  $u$  so that  $\int v$  is possible and  $\frac{d}{dx}u$  is simple. See the example:

$$\int x e^x dx \quad u = x \quad du = dx \quad v = \int e^x$$

$$x e^x - \int e^x dx = x e^x - e^x + C$$

#### 3.2.1 Questions

$$\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos \theta^2 d\theta$$

**Solution** First, you will apply integration by u-substitution using  $t = \theta^2 \implies dt = 2\theta d\theta$

$$\frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} t \cos t du$$

Then, you will use integration by parts. Let  $u = t \quad du = dt \quad v = \int \cos(t) dt$   
 $\frac{1}{2} t \sin t + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \sin t dt = \frac{1}{2} (t \sin t - [\cos t]_{\frac{\pi}{2}}^{\pi})$

### 3.3 Integration by partial fractions

When you have some rational function as  $\int \frac{1}{(x-3)(x+4)} dx$ , you will separate the fraction into two fractions. If the denominator is not factored, factor it first. Let's elaborate.

$$\frac{1}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4} \implies 1 = A(x+4) + B(x-3)$$

We will then substitute  $x$  as what will make the other function zero. For instance, in the previous example, if we substituted  $x = 3$  the part  $B(x-3)$  will equal zero, and we will have the value of A, and the same with the value of B by substituting  $x$  with -4. Now the question is a fairly simple one.

$$x = 3 \implies A = \frac{1}{7} \quad x = -4 \implies B = -\frac{1}{7}$$

$$\int \frac{1}{(x-3)(x+4)} dx = \int \frac{\frac{1}{7}}{x-3} - \frac{\frac{1}{7}}{x+4} dx = [\ln|x-3| - \ln|x+4|] \frac{1}{7}$$

Also, a quicker way is to replace the function, ignoring the "B" part replacing  $x$  with the number that will make the ignored fraction zero. Let's explain more.

$$\text{In } \int \frac{3x}{(x-4)(x+5)}$$

$$A = \left[ \frac{3x}{x+5} \right]_{x=4} = \frac{4}{3} \quad B = \left[ \frac{3x}{x-4} \right]_{x=-5} = \frac{5}{3}$$

Another example with 3 fractions in the bottom. In  $\int \frac{2x+5}{x(x-1)(x+5)}$ , We will do the same. A is left as a 5-second-quick mind exercise.

$$B = \left[ \frac{2x+5}{x(x-5)} \right]_{x=1} = \frac{7}{6} \quad C = \left[ \frac{2x+5}{x(x-1)} \right]_{x=-5} = -\frac{1}{6}$$

In some situations with equal powers and different limits like  $\int \frac{2x^2+5}{x^2+2x}$  it won't be equal  $\frac{A}{x} + \frac{B}{x+2}$  as when  $x$  approaches  $\infty$ , the main function approach 2 while the new A and B functions approach 0, which is wrong. Now, you will need to perform 'long division' till the power of denominator be more than that of numerator and deal with the remainder. For the mentioned question:

$$\begin{array}{r} \phantom{2} \text{Quotient} \\ x^2 + 2x \overline{) 2x^2 + 5} \\ \underline{-(2x^2 + 4x)} \phantom{5} \\ -4x + 5 \end{array}$$

*Remainder*

Note that now the power of the numerator is less than that of the denominator.

$$\therefore \frac{2x^2+5}{x^2+2x} = 2 + \frac{-4x+5}{x(x+2)} = 2 + \frac{A}{x} + \frac{B}{x+2}$$

In other cases where calculating B will yield complex solution, as  $\int \frac{1}{x(x^2+4x+5)}$ . This one will need solving  $x^2 + 4x + 5$  which does not have real solutions. Here you will add a new variable to be added in the equation in a way so you can use it to calculate coefficients of  $x, x^2, \dots$

$$\frac{1}{x(x^2 + 4x + 5)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4x + 5} \implies 1 = A(x^2 + 4x + 5) + (Bx + C)x$$

You will then compare the R.H.S to the L.H.S. To find A easily substitute  $x = 0$  and get  $A = \frac{1}{5}$

$$\begin{aligned} \text{Coefficient of } x^2 : A + B &= 0 \implies B = -\frac{1}{5} \\ \text{Coefficient of } x^1 : 4A + C &= 0 \implies C = -\frac{4}{5} \end{aligned}$$

Now we're done and you can finish the problem smoothly.

In another case of some squared part in the denominator of the function, you will split it in this way

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Now, solve the normal way for A and C. B is harder to get, but solving for it using A and C values will easily yield its value.

In case of higher power, you can have multiple variables where two of them will be easily got using the normal method. The rest of the variables can be got by equating  $x$  to some values and then you can have infinite number of equations with the number of variables desires and you can solve them easily.

A fast formula to work with

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

**TL;DR** Different Limits/ higher power in numerator  $\implies$  Long division

Repeated Power  $\implies$  multiple graduated variables

Unsolvable/ Complex solutions (where  $b^2 - 4ac < 0$ )  $\implies$  use  $Bx+C$  and coefficients, and for the unsolvable part use the identity  $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$  after completing the square if necessary (and then using u-substitution)

**Note** Always check whether the numerator is the derivative of the denominator. If so, it will be an easy u-substitution problem.

### 3.3.1 Questions

1.

$$\int \frac{4}{3 \sin x - 4 \cos x} dx$$

**Solution** Here, it is normal not to get its intuition fast. Weierstrass noticed that substituting  $t = \tan \frac{x}{2}$  will convert any rational function of  $\sin$  and  $\cos$  to a normal rational function.  $t = \tan \frac{x}{2}, \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}, dx = \frac{2}{1+t^2} dt$ . Now you can normally solve it using partial fractions. Solution left as a practice.

$$\int \frac{\sqrt{x+4}}{x} dx$$

**Solution** Here, you will do a substitution trick that will convert it to a rational function,  $u = \sqrt{x+4} \implies u^2 = x+4, dx = 2u du$  now the new equation will be

2.

$$\int \frac{u}{u^2-4} 2u du$$

Now, solving as a normal rational function, we'll long divide since the power of top is higher than the bottom

$$2 \int 1 + \frac{4}{u^2-4} du = 2u + 8 \int \frac{1}{(u-2)(u+2)} du$$

The left part can be solved easily now using our normal rules. The final answer is

$$\int \frac{\sqrt{x+4}}{x} dx = 2\sqrt{x+4} + 2 \ln \left| \frac{\sqrt{x+4}-2}{\sqrt{x+4}+2} \right| + C$$

### 3.4 Trigonometric Integrals

In general, we try to write an integrand involving powers of sine and cosine in a form where we have only one sine factor (and the remainder of the expression in terms of cosine) or only one cosine factor (and the remainder of the expression in terms of sine). The identity  $\sin^2 x + \cos^2 x = 1$  enables us to convert back and forth between even powers of sine and cosine.

1.

$$\int \sin^m x \cos^n x \, dx$$

If  $m$  is odd, factor the sines to cosines using the identity, then substitute using  $u = \cos x$

If  $n$  is odd, factor out the cosines except one and use  $u = \sin x$ .

If both of them are even, use the identities  $\sin^2 x = \frac{1 - \cos(2x)}{2}$  and  $\cos^2 x = \frac{1 + \cos(2x)}{2}$ .

If both are odd, factor out the one with less power for a quicker answer.

2.

$$\int \tan^m x \sec^n x$$

If  $n$ , the power of secant, is even, save one factor of  $\sec^2 x$  and use the identity  $\sec^2 x = 1 + \tan^2 x$ . Then, substitute using  $u = \tan x$ .

If  $m$ , the power of tangent is odd, save a factor of  $\sec x \tan x$  and then use the identity  $\tan^2 x = \sec^2 x - 1$  to represent the remaining factors in terms of  $\sec x$ , and substitute then  $u = \sec x$

#### Remember

$$\int \tan x \, dx = \ln |\sec x| + C \qquad \int \sec x \, dx = \ln |\sec x + \tan x|$$

Go check the proofs or the exam will choke you, unless you like that.

3. Repetitive angles are easy, just use these identities whenever you see them.

$$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$$

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

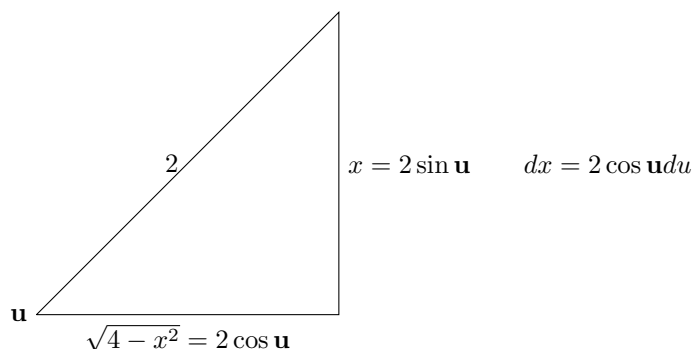
$$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$$

### 3.5 Trigonometric Substitution

Let's start by an example to get the intuition.

$$\int \frac{1}{\sqrt{4-x^2}} dx$$

Normal substitution will be pathetic, let's think of another intuitive approach. If you focused on the bottom term, it kind of looks like Pythagoras theorem for a triangle with sides  $x$ ,  $\sqrt{4-x^2}$  and hypotenuse 2.



You can notice that after some normal trigonometric manipulations you can change the form of the original equation.

$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{2 \cos \mathbf{u}} 2 \cos \mathbf{u} du = \int d\mathbf{u} = \mathbf{u} + C$$

Now, we can substitute from  $x = 2 \sin \mathbf{u}$  to get  $x$ .

$$\implies \int \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1}\left(\frac{x}{2}\right) + C$$

Let's verify the domain and our bounds.

$$-2 \leq x \leq 2 \implies -2 \leq 2 \sin \mathbf{u} \leq 2 \implies -1 \leq \sin \mathbf{u} \leq 1$$

$$\mathbf{u} \leq \frac{\pi}{2} \quad (\text{In domain})$$

Thus, your solution was right. You may need to validate normally directly after substitution.

Now, since you won't have the time to visualize in each question, **Remember**

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &\implies x = a \sin \theta & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} & \quad 1 - \sin^2 \theta = \cos^2 \theta \\ \int \sqrt{a^2 + x^2} dx &\implies x = a \tan \theta & -\frac{\pi}{2} < \theta < \frac{\pi}{2} & \quad 1 + \tan^2 \theta = \sec^2 \theta \\ \int \sqrt{x^2 - a^2} dx &\implies x = a \sec \theta & 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2} & \quad \sec^2 \theta - 1 = \tan^2 \theta \end{aligned}$$

### 3.5.1 Questions

1. Find the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

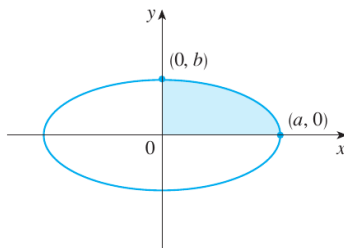


Figure 9:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

**Solution** First, Let's start by solving for y.

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

You can see from the graph that the area of circle is four times the area of each quadrant. We will solve for the shaded part then multiply by four. The area of the shaded part is

$$A = \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

Now solve using normal trigonometric substitution using  $x = a \sin \theta$ . After solving that you will get the total area by multiplying  $A$  by four, which will equal  $\pi ab$ .

2.

$$\int \frac{x}{\sqrt{3 - 2x - x^2}} dx$$

**Solution** The trick in this question is completing the square.

$$3 - 2x - x^2 = 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1) = 4 - (x + 1)^2$$

Now, substituting  $u = x + 1$

$$\int \frac{u - 1}{\sqrt{4 - u^2}} du$$

Now, it's a normal trig-sub question. Put  $u = 2 \sin \theta$  and voila.



### 3.6 Improper Integrals

To summarize, if the integral was on an interval  $[0, \pm\infty]$  or  $[\pm\infty, 0]$ , you can denote the infinity with  $t$  and solve normally, then evaluate the limit. If the limit exists, the integral is convergent, if not, the integral diverges.

Remember is that L'Hopital rule will come in handy in multiple situations. For instance, in calculating  $\lim_{t \rightarrow 0^+} t \ln t$  It will be easier to think of it as  $\frac{\ln t}{\frac{1}{t}}$  so we can use l'hopital to calculate the limit.

I will list the main set of equations you will normally use for reference.

a.

$$\int_a^\infty f(x)dx = \lim_{t \rightarrow \infty} \int_a^t f(x)dx$$

b.

$$\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx$$

c.

$$\int_{-\infty}^\infty f(x)dx = \int_a^\infty f(x)dx + \int_{-\infty}^a f(x)dx$$

d.

$$\int_1^\infty \frac{1}{x^p}, p > 1 \implies \text{converge}, p \leq 1 \implies \text{diverges}$$

This was the normal, or easy type of questions you will be able to momentarily solve and notice. In the other one, suppose you are getting the integral of a function on  $[a, b]$  that has a vertical asymptotic at  $b$  ( $f$  is on  $[a, b)$ ).

e.

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx, \text{ If the limit exists as a finite number.}$$

Also, the same occurs if  $a$  was the one non existent ( $f$  is on  $(a, b]$ ). Besides, if  $f$  has a discontinuity at  $c$  where  $a < c < b$  and both  $\int_a^c f(x)dx$  and  $\int_c^b f(x)dx$  are convergent, then

f.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

In any situation of definite integral, make sure that the whole interval is continuous. Other than that, you may solve wrong if you did not consider the case of improper integral as (f).

g. **Comparison Theorem** Suppose that  $f$  and  $g$  are continuous functions with  $f(x) \geq g(x) \geq 0$  for  $x \geq a$ .

1. If  $\int_a^\infty f(x)dx$  is convergent, then  $\int_a^\infty g(x)dx$  is convergent.
2. If  $\int_a^\infty g(x)dx$  is divergent, then  $\int_a^\infty f(x)dx$  is divergent.

### 3.6.1 Questions

Show whether it is convergent or not

1.

$$\int_0^{\infty} e^{-x^2} dx$$

**Solution** This one is a direct example for the comparison theorem. Since we can not evaluate the integral, we will start by simplifying the question by separating it.

$$\int_0^{\infty} e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x^2} dx$$

The first part of the R.H.S is a normal definite integral. While the second we can use the fact that  $x^2 \geq x \implies -x^2 \leq -x \implies e^{-x^2} \leq e^{-x}$ . This fact reminds us of the comparison theorem where we will check whether  $\int_1^{\infty} e^{-x} dx$  is convergent or not, yielding the solution.

$$\int_1^{\infty} e^{-x} = \lim_{t \rightarrow \infty} [-e^{-x}]_1^t = \lim_{t \rightarrow \infty} -e^{-t} + e^{-1} = e^{-1}$$

## 4 Area and Volume

### 4.1 Area between curves

The area bounded by  $f(x)$  and  $g(x)$  where both are continuous and  $f(x) \geq g(x)$  for  $[a, b]$  is

$$A = \int_a^b [f(x) - g(x)] dx$$

Some questions require finding the area enclosed by two graphs, here the points  $a, b$  are the points of intersection. You can get them by solving the two equations simultaneously, either using calculator or by your working brain cells.

In case you have two curves that  $f(x) \geq g(x)$  only in some values, use the absolute in the formula as shown below.

$$A = \int_a^b |f(x) - g(x)| dx$$

However, in some curves it will make sense to treat it as a function of  $y$   $x = f(y)$ . With  $f(y)$  as the right boundary, and  $g(y)$  the left one

$$A = \int_a^b [f(y) - g(y)] dy$$

### 4.2 Volumes

Volume of a solid that lies between  $x = a$  and  $x = b$  with cross sectional area  $A(x)$  and  $A$  is a continuous function is

$$V = \int_a^b A(x) dx$$

The main example I will include is deriving the volume of a sphere of radius  $r$ . Let's start looking at a graph.

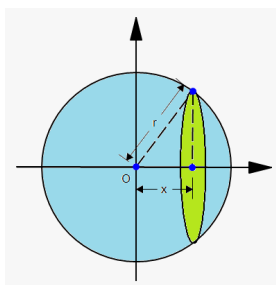


Figure 10: Sphere graph

Here, using Pythagoras we will find that  $y = \sqrt{r^2 - x^2}$ , and so, the cross sectional area will be  $A(x) = \pi(r^2 - x^2)$ , and using our main formula, the volume will be

$$V = \int_{-r}^r \pi(r^2 - x^2)dx = 2\pi \int_0^r r^2 - x^2 dx = \frac{4}{3}\pi r^3$$

This method of obtaining volume using disks is known as the disk method. Usually, try to find what will be the radius of each disk using variables of  $x$  ( $R(x)$ ) then use this formula where the bounding is normally given or intuitively got

$$V = \pi \int_a^b R(x)^2 dx$$

Graphing and visualizing the given in the question prompt will make everything easier. Also, you can do the same with vertical disks instead of horizontal ones, and it will make sense to do so if you are revolving around the  $y$  axis.

Another case that includes the area bounded by two graphs, here the area we want will be the subtraction of the area of the circle formed by the smaller graph from the area of the circle formed by the bigger one as viewed in the graph below.  $V = \pi \int_a^b R(x)^2 - r(x)^2 dx$ , where  $R$  is the bigger radius ( $x$ ) and  $r$  is the smaller one.

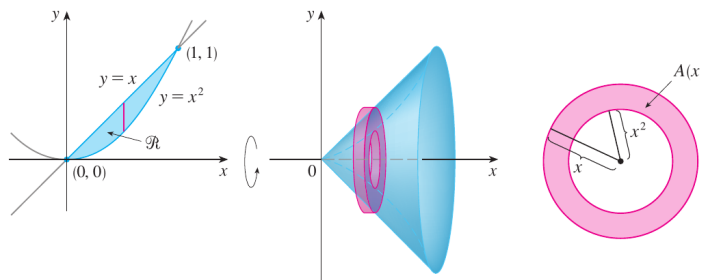


Figure 11: Example of the washer volume bounded by curves  $x$  and  $x^2$  around  $x$ -axis

### TL;DR of solids of revolution

$$V = \pi \int_a^b R(x)^2 dx \quad \text{or} \quad V = \pi \int_c^d R(y)^2 dy \quad (\text{Disk})$$

$$V = \pi \int_a^b R(x)^2 - r(x)^2 dx \quad \text{or} \quad V = \pi \int_c^d R(y)^2 - r(y)^2 dy \quad (\text{washer})$$

**Note**, when you are rotating around a line parallel to the  $x$ -axis, you will get  $R(x)$  and use horizontal slices, while if the line is parallel to the  $y$ -axis, you will get  $R(y)$  and use vertical slices.

### 4.3 Volumes by shells

Rotating around **y-axis**

$$V = \int_a^b (\text{circumference})(\text{height})(\text{thickness})$$
$$\Rightarrow V = 2\pi \int_a^b x f(x) dx \quad \text{where } 0 \leq a < b$$

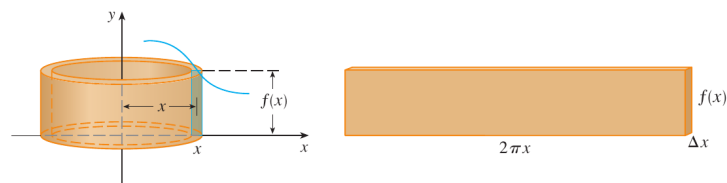


Figure 12: Volume using shells visualization

Where you are rotating around a line parallel to x-axis, use  $dy$ , and vice versa. You get the distance between a point on the curve and the axis of rotation ( $x \pm a$ ) or ( $y \pm a$ ), and  $f(x)$  or  $f(y)$ .

## 5 Sequences

Here, questions are really very straightforward. You will denote the sequence  $a_n$  as a function of  $n$ ,  $f(n)$ , then calculate the limit as  $n$  approaches  $\infty$ .

### Some Limit Reminders

1.

$$\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0, \text{ if } r > 0$$

2.

$$\lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

3.

$$\lim_{n \rightarrow \infty} a_n^p = \left[ \lim_{n \rightarrow \infty} a_n \right]^p \quad \text{as } p, a_n > 0$$

4. **SQUEEZE** If you have a function that came from an oscillating function as sine and cosine, manipulate the original function's boundaries to be the same as  $a_n$ . You will more probably get an evaluation for the limit.

"The squeeze theorem states that if  $f(x) \leq g(x) \leq h(x)$  for all numbers, and at some point  $x = k$  we have  $f(k) = h(k)$ , then  $g(k)$  must also be equal to them. We can use the theorem to find tricky limits like  $\sin(x)/x$  at  $x = 0$ , by "squeezing"  $\sin(x)/x$  between two nicer functions and using them to find the limit at  $x = 0$ ."

5.

$$\text{If } \lim_{n \rightarrow \infty} |a_n| = 0 \implies \lim_{n \rightarrow \infty} a_n = 0$$

6.

If the limit of  $a_n$   $L$  exists, and  $f$  is continuous at  $L$ , then  $\lim_{n \rightarrow \infty} f(a_n) = L$

7.

$$\lim_{n \rightarrow \infty} r^n = 0 \text{ for } -1 < r < 1 \qquad \lim_{n \rightarrow \infty} r^n = 1 \text{ for } r = 1$$

**Monotonic Sequences** They are the sequences that are either increasing ( $a_n < a_{n+1}$ ) or decreasing ( $a_n > a_{n+1}$ ) for all  $n \geq 1$

**Bounded Sequences** They are sequences which are bounded from above and below. Note that if the function is positive it is bounded below from zero. Besides, if the function is convergent, it is bounded from above.

**Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

## 5.1 Questions

1.

$$a_n = \frac{(8n-1)!}{(8n+1)!}$$

**Solution** Here, the trick is to notice that by the definition of the factorial, and since the numerator is less than the denominator by two, we can rewrite the above equation as below to cancel the two terms and simplify the question

$$a_n = \frac{(8n-1)!}{((8n+1)(8n))(8n-1)!}$$

Now, we can simply evaluate the limit

$$\lim_{n \rightarrow \infty} \frac{1}{(8n+1)(8n)} = 0 \quad (\text{Answer})$$

2.

$$a_n = \frac{(-1)^n}{n}$$

Here, we can use the fifth limit reminder and get the limit of the absolute value.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

3.

$$\lim_{n \rightarrow \infty} \sin \frac{\pi}{n}$$

According the sixth limit reminder, we can rewrite the question and simplify it since sine function is continuous at 0

$$\lim_{n \rightarrow \infty} \sin \frac{\pi}{n} = \sin \left( \lim_{n \rightarrow \infty} \left( \frac{\pi}{n} \right) \right) = \sin 0 = 0$$

## 6 Series

A series  $s_n$  is  $\sum_{n=1}^{\infty} a_n$ , where  $a_n$  is an infinite sequence. If  $\lim_{n \rightarrow \infty} s_n = s$ , then  $s_n$  converges.

**Geometric series** It is the series that is obtained by increasing a common ratio  $r$ , For instance

$$s_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} = \sum ar^{n-1} \quad a \neq 0$$

$$s_n = \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r}$$

$$\sum ar^{n-1} = \frac{a}{1 - r} \quad |r| < 1$$

If  $|r| \geq 1$ , the geometric series diverges.

Always notice that it is  $n-1$  not  $n$ , so if it is to the power of  $n$ ,  $a = \text{excess} \times r$

### Examples

1. Calculate  $\sum 2^{2n} 3^{1-n}$

**Solution** Let's do some algebraic manipulations to make it looks like this

$$\sum (2^2)^n 3^{-(n-1)} = \sum \frac{4^n}{3^{n-1}} = \sum 4 \left(\frac{4}{3}\right)^{n-1}$$

Now, it is normal geometric series with  $r = \frac{4}{3}$ , since  $r > 1$ , it diverges.

- 2.

$$\sum \frac{1}{n(n+1)}$$

**Solution** Since this one is not geometric, let's write the first few terms to spot a pattern

$$s_n = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \dots + \frac{1}{n(n+1)}$$

Now, let's try to use partial fraction decomposition to make it simpler

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Thus, we have

$$\sum \frac{1}{i} - \frac{1}{i+1} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots + (\frac{1}{n} - \frac{1}{n+1})$$



We notice that the second part of each term cancels with the first part of the term after it, thus the whole thing will be only

$$1 - \frac{1}{n+1}$$

And by taking the limit

$$\lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1$$

**Divergence test** If  $\lim_{n \rightarrow \infty} a_n \neq 0$  then the series  $\sum a_n$  diverges.

**Note**

$$a_n = s_n - s_{n-1}$$


---

**Integral Test** If  $f$  is continuous, positive, and decreasing function on  $[1, \infty)$  and  $a_n = f(n)$ . Then series  $\sum_{n=1}^{\infty} a_n$  converge if the improper integral is convergent.

**TL;DR** If  $\int_1^{\infty} f(x)dx$  converges, then  $\sum a_n$  converges. And vice versa.

**Note** You don't have to start  $n$  from 1 here, you can start it from any digit and start the integral from the same digit.

**p-series**  $\sum \frac{1}{n^p}$  converge if  $p > 1$  and diverges if  $p \leq 1$

**Note**

$$\sum a_n \neq \int_1^{\infty} f(x)dx$$

**Remainder/ Error estimate** If a function  $f(x) = a_k$  and  $f$  is continuous, positive, and decreasing for  $x \geq n$  and  $\sum a_n$  converge. If  $R_n = s - s_n$ , then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_n^{\infty} f(x)dx$$


---

**Comparison Tests** Suppose  $\sum a_n, \sum b_n$  are positive series  
 If  $\sum b_n$  converge and  $a_n \leq b_n$  for all  $n$ , then  $\sum a_n$  converge.  
 If  $\sum b_n$  diverge and  $a_n \geq b_n$  for all  $n$ , then  $\sum a_n$  diverge.

**Limit Comparison Test** Suppose  $\sum a_n, \sum b_n$  are positive series

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

And  $c$  is finite  $c > 0$ , then either both series diverge or both converge.

**Remainder/ Error** Let  $T_n = t - t_n$  the error of  $b_n$ , since  $a_n \leq b_n$  for all  $n$ , we have  $R_n \leq T_n$ . If  $\sum b_n$  is a  $p$ -series, we can estimate the remainder as in the previous subsection. If it was a geometric series, then  $T_n$  is the summation exactly.

**Alternating Series** If  $\sum (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 \dots b_n > 0$  ( $b_n = |a_n|$ ) satisfies:

$$b_{n+1} \leq b_n \quad \text{for all } n \text{ (Function is decreasing)}$$

$$\lim_{n \rightarrow \infty} b_n = 0$$

Then, the series is convergent

**Estimation** If  $s = \sum (-1)^{n-1} b_n$ , where  $b_n > 0$  is a sum of alternating series that satisfy the conditions mentioned above, then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

**Absolute Convergence** A series  $\sum a_n$  is absolutely convergent (and convergent) if the series  $\sum |a_n|$  converges. A series is conditionally convergent if it is convergent but not absolutely convergent.

**Ratio Test** Let's consider a limit  $L$

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

If  $L < 1$  the series is absolutely convergent. If  $L > 1$ , the series is divergent. If  $L = 1$ , the test is inconclusive, aka pathetic.

**Question** Test convergence of

$$\sum \frac{n^n}{n!}$$

**Answer** Using the ratio test, you will find that you are calculating limit as  $n$  goes to  $\infty$  of  $\left[\frac{n+1}{n}\right]^n$  which equals  $e$ , meaning that the series diverges. However, use the normal divergence test as sequence doesn't approach 0.

**Root Test**

$$\text{Let } L = \sqrt[n]{|a_n|}$$

If  $L < 1$  the series is absolutely convergent. If  $L > 1$ , the series is divergent. If  $L = 1$ , the test is inconclusive, aka pathetic.

**Note**

$$\cos(n\pi) = (-1)^n$$

## Power Series

For a given power series  $\sum c_n(x-a)^n$  there are only three possibilities:

- i. The series converges only when  $x = a$
- ii. The series converges for all  $x$
- iii. There is a positive number  $R$  (radius of convergence) that series converges if  $|x-a| < R$  and diverge if  $|x-a| > R$

Radius is found using Ratio test or sometimes Root test. Note that they will always fail when  $x$  is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

---

## Representations of functions as power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

More formally, if a power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$  has  $R > 0$ , then  $f$  is defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum c_n(x-a)^n$$

and it is differential and continuous on the interval  $(a-R, a+R)$

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x) dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

$$\frac{d}{dx} \left[ \sum_{n=0}^{\infty} c_n (x-a)^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} [c_n (x-a)^n]$$

$$\int \left[ \sum_{n=0}^{\infty} c_n (x-a)^n \right] dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx$$

Note, the radii of convergence of the derivative and integral are both  $R$ , but note that does not mean that the interval of convergence stays the same. (endpoints may change)

$$\arctan(x) = \int \frac{1}{1+x^2} dx = \int (1-x^2+x^4-x^6+\dots) dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad R=1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C \quad \ln(1-x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} \quad |x| < 1$$

Derivatives

$\frac{d}{dx} e^x = e^x$   
 $\frac{d}{dx} \sin(x) = \cos(x)$   
 $\frac{d}{dx} \cos(x) = -\sin(x)$   
 $\frac{d}{dx} \tan(x) = \sec^2(x)$   
 $\frac{d}{dx} \cot(x) = -\csc^2(x)$   
 $\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$   
 $\frac{d}{dx} \csc(x) = -\csc(x) \cot(x)$   
 $\frac{d}{dx} \sin^{-1} = \frac{1}{\sqrt{1-x^2}}, x \in [-1, 1]$   
 $\frac{d}{dx} \cos^{-1} = \frac{-1}{\sqrt{1-x^2}}, x \in [-1, 1]$   
 $\frac{d}{dx} \tan^{-1} = \frac{1}{1+x^2}, \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$   
 $\frac{d}{dx} \sec^{-1} = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$   
 $\frac{d}{dx} \sinh(x) = \cosh(x)$   
 $\frac{d}{dx} \cosh(x) = \sinh(x)$   
 $\frac{d}{dx} \tanh(x) = \operatorname{sech}^2(x)$   
 $\frac{d}{dx} \coth(x) = -\operatorname{csch}^2(x)$   
 $\frac{d}{dx} \operatorname{sech}(x) = -\operatorname{sech}(x) \tanh(x)$   
 $\frac{d}{dx} \operatorname{csch}(x) = -\operatorname{csch}(x) \coth(x)$   
 $\frac{d}{dx} \sinh^{-1} = \frac{1}{\sqrt{x^2+1}}$   
 $\frac{d}{dx} \cosh^{-1} = \frac{-1}{\sqrt{x^2-1}}, x > 1$   
 $\frac{d}{dx} \tanh^{-1} = \frac{1}{1-x^2}, -1 < x < 1$   
 $\frac{d}{dx} \operatorname{sech}^{-1} = \frac{1}{x\sqrt{1-x^2}}, 0 < x < 1$   
 $\frac{d}{dx} \ln(x) = \frac{1}{x}$   
 $\frac{d}{dx} b^x = b^x \ln b$

Integrals

$\int \sin(x)dx = -\cos(x)$   
 $\int \cos(x)dx = \sin(x)$   
 $\int \tan(x)dx = -\ln|\cos(x)|$   
 $\int \sec x \tan x dx = \sec x$   
 $\int \sec x dx = \ln|\sec x + \tan x|$   
 $\int \sec^2(x)dx = \tan(x) + C$   
 $\int \sec^3(x)dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x \tan x|)$   
 $\int \csc^2(x)dx = -\cot(x)$   
 $\int \csc(x) \cot(x)dx = -\csc(x)$   
 $\int \csc(x)dx = \ln|\csc(x) - \cot(x)|$   
 $\int \cot(x)dx = \ln|\sin(x)|$   
 $\int \frac{1}{x} dx = \ln|x|$   
 $\int e^x dx = e^x$   
 $\int a^x dx = \frac{1}{\ln a} a^x$   
 $\int e^{ax} dx = \frac{1}{a} e^{ax}$   
 $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(\frac{x}{a})$   
 $\int \frac{1}{a^2+ x^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a})$   
 $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln|\frac{x-a}{x+a}|$   
 $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln|x + \sqrt{x^2 \pm a^2}|$   
 $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}(x)$   
 $\int \sinh(x)dx = \cosh(x)$   
 $\int \cosh(x)dx = \sinh(x)$   
 $\int \tanh(x)dx = \ln|\cosh(x)|$   
 $\int \tanh(x) \operatorname{sech}(x)dx = -\operatorname{sech}(x)$   
 $\int \operatorname{sech}^2(x)dx = \tanh(x)$   
 $\int \operatorname{csch}(x) \coth(x)dx = -\operatorname{csch}(x)$   
 $\int \ln(x)dx = (x \ln(x)) - x$   
 $\int_{-r}^r \sqrt{r^2-x^2} dx = \frac{\pi r^2}{2}$   
 $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$

Integrals of inverse trig functions are solved using integration by parts.

Weierstrass

$t = \tan \frac{x}{2} \quad \cos x = \frac{1-t^2}{1+t^2}$   
 $\sin x = \frac{2t}{1+t^2} \quad dx = \frac{2}{1+t^2} dt$

$\int \sin^m x \cos^n x$   
 $m$  is odd factor sines, then sub  $u = \cos x$   
 $n$  is odd factor cosines, then sub  $u = \sin x$ .

If both even, use  $\frac{1-\cos(2x)}{2}, \frac{1+\cos(2x)}{2}$ . If both odd, factor one with less power.

$\int \tan^m x \sec^n x$   
 $n$  even save one  $\sec^2 x$  and sub  $u = \tan x$ .  
 $m$  odd save one  $\sec x \tan x$  and sub  $u = \sec x$

$\int \sqrt{a^2-x^2} dx \implies x = a \sin \theta$   
 $\int \sqrt{a^2+x^2} dx \implies x = a \tan \theta$   
 $\int \sqrt{x^2-a^2} dx \implies x = a \sec \theta$   
 $\int_1^\infty \frac{1}{x^p}, p > 1$  : converge,  $p \leq 1$  : diverges  
e.o=o   e.e=e   o.o=e   e+e=e   o+o=o

**Integration by Parts**  
 $\int u dv = uv - \int v du$

Functions/ Identities

$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$   
 $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$   
 $\sec(\tan^{-1}(x)) = \sqrt{1+x^2}$   
 $\tan(\sec^{-1}(x)) = (\sqrt{x^2-1} \text{ if } x \geq 1)$   
 $= (-\sqrt{x^2-1} \text{ if } x \leq -1)$   
 $\sin^2(x) + \cos^2(x) = 1$   
 $1 + \tan^2(x) = \sec^2(x)$   
 $1 + \cot^2(x) = \csc^2(x)$   
 $\sin(x \pm y) = \sin(x) \cos(y) \pm \cos(x) \sin(y)$   
 $\cos(x \pm y) = \cos(x) \cos(y) \pm \sin(x) \sin(y)$   
 $\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x) \tan(y)}$   
 $\sin(2x) = 2 \sin(x) \cos(x)$   
 $\cos(2x) = \cos^2(x) - \sin^2(x)$   
 $1 + \tan^2(x) = \sec^2(x)$   
 $1 + \cot^2(x) = \csc^2(x)$   
 $\sin^2(x) = \frac{1-\cos(2x)}{2}$   
 $\cos^2(x) = \frac{1+\cos(2x)}{2}$   
 $\tan^2(x) = \frac{1-\cos(2x)}{1+\cos(2x)}$   
 $\sin(-x) = -\sin(x)$   
 $\cos(-x) = \cos(x)$   
 $\tan(-x) = -\tan(x)$   
 $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$   
 $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$   
 $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$   
 $\cosh(x) = \frac{e^x + e^{-x}}{2}$   
 $\cosh^2 x - \sinh^2 x = 1$   
 $1 - \tanh^2(x) = \operatorname{sech}^2(x)$   
 $\sinh^{-1}(x) = \ln x + \sqrt{x^2+1}$   
 $\sinh^{-1}(x) = \ln x + \sqrt{x^2-1}, x \geq -1$   
 $\tanh^{-1}(x) = \frac{1}{2} \ln x + \frac{1+x}{1-x}, 1 < x < -1$   
 $\operatorname{sech}^{-1}(x) = \ln[\frac{1+\sqrt{1-x^2}}{x}], 0 < x \leq -1$   
 $\cosh^2(x) - \sinh^2(x) = 1$   
 $\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$   
 $\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$

3D

given two points:  
 $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ ,  
Distance between them:  
 $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$   
Midpoint:  
 $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$   
Sphere with center (h,k,l) and radius r:  
 $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Vectors

Vector:  $\vec{u}$   
Unit Vector:  $\hat{u}$   
Magnitude:  $||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$   
Unit Vector:  $\hat{u} = \frac{\vec{u}}{||\vec{u}||}$

**Dot Product**  
 $\vec{u} \cdot \vec{v}$   
Produces a Scalar  
(Geometrically, the dot product is a vector projection)  
 $\vec{u} = \langle u_1, u_2, u_3 \rangle$   
 $\vec{v} = \langle v_1, v_2, v_3 \rangle$   
 $\vec{u} \cdot \vec{v} = 0$  means the two vectors are Perpendicular  
 $\theta$  is the angle between them.  
 $\vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos(\theta)$   
 $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$   
NOTE:  
 $\hat{u} \cdot \hat{v} = \cos(\theta)$   
 $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$   
 $\vec{u} \cdot \vec{v} = 0$  when  $\perp$   
Angle Between  $\vec{u}$  and  $\vec{v}$ :  
 $\theta = \cos^{-1}(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||})$   
Projection of  $\vec{u}$  onto  $\vec{v}$ :  
 $pr_{\vec{v}} \vec{u} = (\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}) \vec{v}$

**Cross Product**  
 $\vec{u} \times \vec{v}$   
Produces a Vector  
(Geometrically, the cross product is the area of a parallelogram with sides  $||\vec{u}||$  and  $||\vec{v}||$ )  
 $\vec{u} = \langle u_1, u_2, u_3 \rangle$   
 $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\vec{u} \times \vec{v} = \vec{0}$  means the vectors are parallel  
 $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$

Volume of Parallelepiped

$(\vec{v} \times \vec{u}) \cdot \vec{w}$

However, in the case the question gave you three vectors directly

$$Volume = \det \left( \begin{bmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right)$$

Lines and Planes

**Equation of a Plane**  
 $(x_0, y_0, z_0)$  is a point on the plane and  $\langle A, B, C \rangle$  is a normal vector  
 $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$   
 $\langle A, B, C \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$   
 $Ax + By + Cz = D$  where  
 $D = Ax_0 + By_0 + Cz_0$

Equation of a line

A line requires a Direction Vector  
 $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and a point  $(x_1, y_1, z_1)$

Parametric equation:  
 $x = u_1 t + x_1 \quad y = u_2 t + y_1 \quad z = u_3 t + z_1$   
Symmetric equations:  
 $t = \frac{x-x_1}{u_1} = \frac{y-y_1}{u_2} = \frac{z-z_1}{u_3}$   
**Distance from a Point to a Plane**  
The distance from a point  $(x_0, y_0, z_0)$  to a plane  $Ax+By+Cz=D$  can be expressed by the formula:  
 $d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}$

**A line intersects a plane** Just put the equations of  $x, y, z$  in the equation of the plane given, and solve for  $t$ . Then you'll have the points.

Distance between two skew lines through  $P_1 P_2, P_3 P_4$

$$D = \frac{Vol_{parallelepiped}}{Area_{base}} = \frac{|(P_1 \vec{P}_2 \times P_3 \vec{P}_4) \cdot P_1 \vec{P}_3|}{|P_1 \vec{P}_2 \times P_3 \vec{P}_4|}$$

Or go for the intuitive approach of forcing perpendicularity.

**If two vectors are parallel**  $|\vec{v}_1 \times \vec{v}_2| = 0$

**Distance between a point and a line  $\vec{v}$**   
 $D = \frac{|P \vec{v}_0 \times \vec{v}|}{|\vec{v}|}$

Other Information

$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$   
Law of Cosines:  
 $a^2 = b^2 + c^2 - 2bc(\cos(\theta))$   
Quadratic Formula:  
 $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ , when  $ax^2 + bx + c = 0$

Logarithmic

$\log(xy) = \log(x) + \log(y)$   
 $\log(\frac{x}{y}) = \log(x) - \log(y)$   $\log({}^y \sqrt{x}) = \frac{\log(x)}{y}$   
 $\log_b(1) = 0$   $\log_a(x) = \frac{1}{\log_x(a)}$