

Instrumental Variables for Dynamic Spatial Models with Interactive Effects

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$$\mathbf{y}_t = \rho \mathbf{W} \mathbf{y}_{t-1} + \alpha \mathbf{y}_{t-1} + \phi \mathbf{W} \mathbf{y}_{t-1} + \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\eta}_t.$$

- Parameter of interest $\boldsymbol{\theta} := (\rho, \alpha, \phi, \boldsymbol{\beta}^\top)^\top$.

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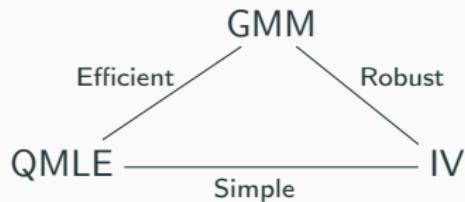
- Note Λ , f_t and ε_t are all unobserved.
- Generalisation of classic models such as individual, time or group effects:

$$\Lambda = \begin{pmatrix} \lambda_1 & 1 \\ \vdots & \vdots \\ \lambda_n & 1 \end{pmatrix}, \quad f_t = \begin{pmatrix} 1 \\ f_t \end{pmatrix}.$$

- ΛF^\top is low rank.

Related Work

- Spatial models with interactive effects:
 - (Q)MLE - [Shi and Lee \(2017\)](#); [Bai and Li \(2021\)](#). (Large n , Large T)
 - Adjusted Score - [Li and Yang \(2023\)](#). (Large n , Fixed T)
 - GMM - [Kuersteiner and Prucha \(2020\)](#). (Large n , Fixed T)
 - IV - [Cui, Sarafidis and Yamagata \(2022\)](#). (Large n , Large T)



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- Provide some additional results which are useful for applying the estimator in practice.
- Apply the method to study the relationship between economic growth, civil liberties and political rights.

Outline I

- Original model in matrices:

$$\mathbf{S}(\rho)\mathbf{Y} = \alpha\mathbf{Y}_{-1} + \phi\mathbf{W}\mathbf{Y}_{-1} + \mathbf{X} \cdot \boldsymbol{\beta} + \boldsymbol{\Lambda}\mathbf{F}^\top + \boldsymbol{\varepsilon},$$

with $\mathbf{S}(\rho) := \mathbf{I}_n - \rho\mathbf{W}$.

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- Under some conditions we can expand

$$\mathbf{Y} = \sum_{h=0}^{\infty} (\rho\mathbf{W})^h (\alpha\mathbf{Y}_{-1} + \phi\mathbf{W}\mathbf{Y}_{-1} + \mathbf{X} \cdot \boldsymbol{\beta} + \boldsymbol{\Lambda}\mathbf{F}^\top + \boldsymbol{\varepsilon}).$$

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- Generate instruments for \mathbf{Y} , i.e. $\mathbf{X}, \mathbf{W}\mathbf{X}, \mathbf{W}^2\mathbf{X}, \dots$

Outline I

- With S weights matrices

$$\mathbf{y}_t = \sum_{s=1}^S \rho_s \mathbf{W}_s \mathbf{y}_t + \alpha \mathbf{y}_{t-1} + \sum_{s=1}^S \phi_s \mathbf{W}_s \mathbf{y}_{t-1} + \mathbf{X}_t \boldsymbol{\beta} + \boldsymbol{\eta}_t.$$

- Produces a great many instruments

$$\begin{aligned} \mathbf{S}^{-1}(\boldsymbol{\rho}) &= (\mathbf{I}_n - \boldsymbol{\rho} \cdot \mathbf{W})^{-1} \\ &= \mathbf{I}_n + \sum_{s=1}^S \rho_{s_1} \mathbf{W}_{s_1} + \sum_{s_1=1}^S \sum_{s_2=1}^S \rho_{s_1} \rho_{s_2} \mathbf{W}_{s_1} \mathbf{W}_{s_2} \\ &\quad + \sum_{s_1=1}^S \sum_{s_2=1}^S \sum_{s_3=1}^S \rho_{s_1} \rho_{s_2} \rho_{s_3} \mathbf{W}_{s_1} \mathbf{W}_{s_2} \mathbf{W}_{s_3} + \dots. \end{aligned}$$

Outline II

- Re-write

$$\begin{aligned}\mathbf{Y} &= \rho \mathbf{WY} + \alpha \mathbf{Y}_{-1} + \phi \mathbf{WY}_{-1} + \mathbf{X} \cdot \boldsymbol{\beta} + \Lambda \mathbf{F}^\top + \varepsilon, \\ &= \mathbf{Z} \cdot \boldsymbol{\theta} + \Lambda \mathbf{F}^\top + \varepsilon.\end{aligned}$$

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- Let \mathcal{V} be an $n \times m$ matrix containing a set of instruments.

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- Let \mathcal{V} be an $n \times m$ matrix containing a set of instruments.
- Construct an $n \times m$ matrix $\mathbf{Q}_\mathcal{V}$ as $\mathcal{V}(\mathcal{V}^\top \mathcal{V})^{-\frac{1}{2}}$.

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- Construct an $n \times m$ matrix $\mathbf{Q}_\mathcal{V}$ as $\mathcal{V}(\mathcal{V}^\top \mathcal{V})^{-\frac{1}{2}}$.
- Premultiply by $\mathbf{Q}_\mathcal{V}^\top$ to give

$$\mathbf{Q}_\mathcal{V}^\top \mathbf{Y} = \tilde{\mathbf{Y}} = \tilde{\mathbf{Z}} \cdot \boldsymbol{\theta} + \tilde{\boldsymbol{\Lambda F}}^\top + \tilde{\boldsymbol{\varepsilon}}.$$

- Dimension reduction: $n \times T \rightarrow m \times T$.

Outline III

- Concentrated objective function:

$$\mathcal{Q}(\boldsymbol{\theta}) := \frac{1}{nT} \sum_{r=R+1}^T \mu_r \left((\tilde{\mathbf{Y}} - \tilde{\mathbf{Z}} \cdot \boldsymbol{\theta})^\top (\tilde{\mathbf{Y}} - \tilde{\mathbf{Z}} \cdot \boldsymbol{\theta}) \right),$$

where $\mu_r(\cdot)$ denotes the r -th largest eigenvalue of a matrix.

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where $\mu_r(\cdot)$ denotes the r -th largest eigenvalue of a matrix.

- IV-IFE estimator:

$$\hat{\boldsymbol{\theta}} := \arg \min_{\boldsymbol{\theta} \in \Theta} \mathcal{Q}(\boldsymbol{\theta}).$$

Assumption MD

Assumption MD (Model)

- (i) The parameter vector θ_0 is in the interior of Θ , where Θ is a compact subset of \mathbb{R}^P .
- (ii) The weights matrix \mathbf{W} has a zero diagonal, is nonstochastic and UB.
- (iii) For all $\rho \in \Theta_\rho$, $\alpha \in \Theta_\alpha$ and $\phi \in \Theta_\phi$, $|\det(\mathbf{S}(\rho))| \geq c > 0$,
 $|\det(\bar{\mathbf{B}}(\rho, \alpha, \phi))| \geq c > 0$, and $\|\mathbf{A}(\rho, \alpha, \phi)\|_2 < 1 - c$ holds for all (n, T) , and
 $\mathbf{S}^{-1}(\rho)$, $\bar{\mathbf{B}}^{-1}(\rho, \alpha, \phi)$ and $\sum_{h=1}^{\infty} |\mathbf{A}^h(\rho, \alpha, \phi)|$ are UB.
- (iv) x_{kit} , $\lambda_{0,ir}$ and $f_{0,tr}$ have uniformly bounded fourth moments.
- (v) The errors ε_{it} are independent of the factors, the loadings, and the covariates,
and are also independent over i and t with $\mathbb{E}[\varepsilon_{it}] = 0$, $\mathbb{E}[\varepsilon_{it}^2] =: \sigma_{it}^2 > 0$ and
uniformly bounded fourth moments.

Assumption CS

Assumption CS (Consistency)

- (i) $R \geq R_0 := \text{rank}(\tilde{\Lambda}_0 \mathbf{F}_0^\top).$
- (ii) $\min_{\boldsymbol{\delta} \in \mathbb{R}^P : \|\boldsymbol{\delta}\|_2 = 1} \sum_{r=R+R_0+1}^T \mu_r(\frac{1}{nT}(\boldsymbol{\delta} \cdot \tilde{\mathbf{Z}})^\top (\boldsymbol{\delta} \cdot \tilde{\mathbf{Z}})) \geq c > 0,$
w.p.a.1 as $n \rightarrow \infty.$

Consistency

Proposition 1 (Consistency)

Under Assumptions MD and CS, as $n \rightarrow \infty$,

$$\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\|_2 = \mathcal{O}_p\left(\sqrt{\frac{m}{n}}\right).$$

Assumption AD

Assumption AD (Asymptotic Distribution)

- (i) $R = R_0$.
- (ii) $\frac{1}{n} \tilde{\Lambda}_0^\top \tilde{\Lambda}_0 \xrightarrow{p} \Sigma_{\tilde{\Lambda}_0}$ as $n \rightarrow \infty$, with $\mu_{R_0}(\Sigma_{\tilde{\Lambda}_0}) > 0$ and $\mu_1(\Sigma_{\tilde{\Lambda}_0}) < \infty$.
- (iii) $\mu_{R_0}(\frac{1}{T} \mathbf{F}_0^\top \mathbf{F}_0) > 0$ and $\mu_1(\frac{1}{T} \mathbf{F}_0^\top \mathbf{F}_0) < \infty$.
- (iv) The elements of the matrices $\bar{M}\tilde{\mathcal{H}}$ and $\bar{M}\tilde{\mathcal{R}}$ have uniformly bounded fourth moments for all (n, T) .

The matrices \mathcal{H} and \mathcal{R} are $nm \times P$ and $\bar{M} := (\mathbf{M}_{\mathbf{F}_0} \otimes \mathbf{Q}_{\mathbf{V}} \mathbf{M}_{\tilde{\Lambda}_0} \mathbf{Q}_{\mathbf{V}}^\top)$.

Asymptotic Distribution I

Theorem 1 (Asymptotic Distribution)

Under Assumptions MD, CS and AD, with $\gamma_{nm}^2 := m^2 T/n \rightarrow c \geq 0$ as $n, m \rightarrow \infty$,

$$\sqrt{nT}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(\Delta^{-1}\boldsymbol{\psi}, \Delta^{-1}\boldsymbol{\Omega}\Delta^{-1}),$$

where,

$$\boldsymbol{\psi}_n := \frac{1}{\sqrt{nT}} \begin{pmatrix} \text{tr}(\Sigma \bar{\mathbf{M}} \bar{\mathbf{W}} \bar{\mathbf{B}}^{-1}) \\ \mathbf{0}_{(K+2)} \end{pmatrix},$$

$$\boldsymbol{\psi} := \text{plim}_{n \rightarrow \infty} \boldsymbol{\psi}_n, \quad \Delta_n := (nT)^{-1}(\mathcal{H} + \mathcal{R})^\top \bar{\mathbf{M}} (\mathcal{H} + \mathcal{R}),$$

$$\Delta = \text{plim}_{n \rightarrow \infty} \Delta_n, \quad \Omega_n := (nT)^{-1}(\mathcal{H} + \mathcal{R})^\top \bar{\mathbf{M}} \Sigma \bar{\mathbf{M}} (\mathcal{H} + \mathcal{R}),$$

$\boldsymbol{\Omega} := \text{plim}_{n \rightarrow \infty} \boldsymbol{\Omega}_n$, and $\boldsymbol{\Sigma}$ is an $nT \times nT$ matrix with diagonal elements $\sigma_{11}^2, \dots, \sigma_{nT}^2$ and remaining elements equal to zero.

Asymptotic Distribution II

- Consider more closely

$$\psi_n := \frac{1}{\sqrt{nT}} \begin{pmatrix} \text{tr}(\Sigma \bar{W} \bar{B}^{-1} \bar{M}) \\ \mathbf{0}_{(K+2) \times 1} \end{pmatrix}.$$

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- The potential magnitude of this bias lies behind the restrictions on the growth of n , m and T .
- The exact magnitude of this bias will depend in part on the structure of cross-sectional dependence, represented by the weights matrix.

Asymptotic Distribution III

- Suppose that:

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_1 & 0 & \dots & 0 \\ 0 & \mathbf{W}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{W}_G \end{pmatrix},$$

with

$$\mathbf{W}_g = \begin{pmatrix} 0 & n_g^{-1} & \dots & n_g^{-1} \\ n_g^{-1} & 0 & \dots & n_g^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ n_g^{-1} & n_g^{-1} & \dots & 0 \end{pmatrix},$$

for $g = 1, \dots, G$.

Asymptotic Distribution III

- One can show (with $n \rightarrow \infty$, m/n and $T/n \rightarrow 0$)

$$\begin{aligned}\psi_{n,1} &= \frac{1}{\sqrt{nT}} \text{tr}(\Sigma \bar{\mathbf{W}} (\mathbf{I}_{nT} - \rho_0 \bar{\mathbf{W}})^{-1} \bar{\mathbf{P}}) + o(1) \\ &=: \xi + o(1),\end{aligned}$$

and establish bounds

$$-\sigma_{\min}^2 \sqrt{\frac{T}{n}} \times \frac{m}{n_{\min} + (\rho - 1)} \leq \xi \leq \sigma_{\max}^2 \sqrt{\frac{T}{n}} \left(\frac{m \wedge G}{1 - \rho} - \frac{0 \vee (m - G)}{n_{\max} + (\rho - 1)} \right),$$

where n_{\max} and n_{\min} denote maximum and minimum group sizes, respectively.

Asymptotic Distribution III

- If $n, m, G \rightarrow \infty$ and n_{\max}, n_{\min} remain fixed

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- A spectrum of cases lie in between.
- Direct relationship between the properties of the estimator and the structure of the weights matrix.

Asymptotic Distribution IV

- Construct a bias correct estimator

$$\tilde{\boldsymbol{\theta}} := \hat{\boldsymbol{\theta}} - \frac{1}{\sqrt{nT}} \hat{\boldsymbol{\Delta}}^{-1} \hat{\boldsymbol{\psi}}.$$

- Under assumptions MD, CS and AD, as $n, m \rightarrow \infty$ with $m^2 T/n \rightarrow c \geq 0$

$$\sqrt{nT}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(\mathbf{0}_{P \times 1}, \boldsymbol{\Delta}^{-1} \boldsymbol{\Omega} \boldsymbol{\Delta}^{-1}).$$

Illustration I

Design:

$$\mathbf{Y} = \rho \mathbf{WY} + \alpha \mathbf{Y}_{-1} + \beta_1 \mathbf{X}_1 + \beta_2 \mathbf{X}_2 + \boldsymbol{\Lambda} \mathbf{F}^\top + \boldsymbol{\varepsilon},$$

with

- $R_0 = 2, R = 2, f_{tr}, \lambda_{ir} \sim \mathcal{N}(0, 1)$,
- $\mathbf{X}_1 = \boldsymbol{\Lambda} \mathbf{F}^\top + \mathbf{e}$, where $e_{it} \sim \mathcal{N}(0, 1)$,
- $X_{2,it} \sim \mathcal{N}(0, 1)$,
- $\text{vec}(\boldsymbol{\varepsilon}) := \boldsymbol{\Sigma}^{\frac{1}{2}} \text{vec}(\mathbf{u})$, $u_{it} \sim \mathcal{N}(0, 1)$, $\boldsymbol{\Sigma}$ diagonal with elements $\sigma_{it}^2 \in (0, 2)$,
- $\mathcal{V} := (\mathbf{X}_1, \mathbf{X}_2, \mathbf{W}\mathbf{X}_1, \mathbf{W}\mathbf{X}_2)$.
- Notice that $m = \mathcal{O}(T)$.

Illustration II

- Partition the cross-section into G disjoint groups.
- Within each group all units are connected only to a single central unit who reciprocates the link.
- Produces a block diagonal weights matrix representing multiple stars.
- E.g. with $G = 1$

$$\mathbf{W} = \begin{pmatrix} 0 & \boldsymbol{\iota}_{n-1}^\top \\ \boldsymbol{\iota}_{n-1} & \mathbf{0}_{n-1 \times n-1} \end{pmatrix},$$

before being row-normalised.

- Easy to verify that $\text{rank}(\mathbf{W}) = 2G$ whereby

$$\psi_{n,1} = \mathcal{O}\left((m \wedge G) \times \sqrt{\frac{T}{n}}\right) = \mathcal{O}\left((T \wedge G) \times \sqrt{\frac{T}{n}}\right).$$

Illustration III

Table 1a: Coverage 95% Confidence Intervals - $G = 5$

$n \setminus T$	$\hat{\rho}$			$\tilde{\rho}$		
	6	9	12	6	9	12
100	0.902	0.843	0.853	0.932	0.951	0.964
300	0.928	0.921	0.916	0.953	0.944	0.960
500	0.939	0.939	0.929	0.954	0.949	0.947

Table 1b: Coverage 95% Confidence Intervals - $G = 25$

$n \setminus T$	$\hat{\rho}$			$\tilde{\rho}$		
	6	9	12	6	9	12
100	0.815	0.798	0.353	0.956	0.951	0.962
300	0.918	0.892	0.698	0.956	0.961	0.961
500	0.935	0.913	0.861	0.948	0.965	0.960

Specification Testing I

- IV-IFE estimator is based the moment condition

$$\mathbb{E} \left[(\mathbf{M}_{\mathbf{F}_0} \otimes \mathbf{Q}_{\mathcal{V}} \mathbf{M}_{\tilde{\Lambda}_0} \mathbf{Q}_{\mathcal{V}}^\top) \text{vec}(\boldsymbol{\eta}) \right] = \mathbf{0}_{nT}.$$

- A total of $(T - R_0)(m - R_0)$ linearly independent restrictions.
- Motivates the following statistic:

$$\mathcal{J} := \text{vec}(\hat{\boldsymbol{\eta}})^\top (\mathbf{M}_{\hat{\mathbf{F}}} \otimes \mathbf{Q}_{\mathcal{V}} \mathbf{M}_{\hat{\Lambda}} \mathbf{Q}_{\mathcal{V}}^\top) \text{vec}(\hat{\boldsymbol{\eta}}),$$

with $\hat{\boldsymbol{\eta}} := \mathbf{y} - \mathbf{Z} \cdot \hat{\boldsymbol{\theta}}$.

Specification Testing II

- Define

$$\mathbf{M}_{\mathcal{J}} \coloneqq \bar{\mathbf{M}} - \mathbf{P}_{\mathcal{J}},$$

$$\mathbf{P}_{\mathcal{J}} \coloneqq \bar{\mathbf{M}}(\mathbf{H} + \mathbf{R})((\mathbf{H} + \mathbf{R})^\top \bar{\mathbf{M}}(\mathbf{H} + \mathbf{R}))^{-1}(\mathbf{H} + \mathbf{R})^\top \bar{\mathbf{M}},$$

$$\ell \coloneqq (T - R_0)(m - R_0) - P,$$

$$\begin{aligned}\sigma_{\mathcal{J}}^2 &\coloneqq \text{tr}((\mathbf{M}^{(4)} - 3\boldsymbol{\Sigma}^2)(\mathbf{M}_{\mathcal{J}} \odot \mathbf{M}_{\mathcal{J}})) \\ &+ 2\boldsymbol{\iota}_{nT}^\top (\boldsymbol{\Sigma}(\mathbf{M}_{\mathcal{J}} \odot \mathbf{M}_{\mathcal{J}})\boldsymbol{\Sigma})\boldsymbol{\iota}_{nT},\end{aligned}$$

where $\mathbf{M}^{(4)}$ is an $nT \times nT$ matrix with diagonal elements $\mathbb{E}[\varepsilon_{11}^4], \dots, \mathbb{E}[\varepsilon_{nT}^4]$ and all remaining elements equal to zero.

Assumption JS (J-Test)

- (i) The errors ε_{it} have uniformly bound eighth moments.
- (ii) $\ell^{-1}\sigma_{\mathcal{J}}^2 \geq c > 0$ w.p.a.1.

Specification Testing III

Theorem 2 (J -Test)

Under Assumptions MD, CS, AD and JS, with $\gamma_{nm}^2 \rightarrow c \geq 0$ as $n, \ell \rightarrow \infty$,

$$\frac{\mathcal{J} - \varphi_{\mathcal{J}}}{\sigma_{\mathcal{J}}} \xrightarrow{d} \mathcal{N}(0, 1),$$

where

$$\varphi_{\mathcal{J}} := \text{tr}(\Sigma M_{\mathcal{J}}) - \psi_n^\top \Delta_n^{-1} (\mathcal{H} + \mathcal{R})^\top \bar{M} (\mathcal{H} + \mathcal{R}) \Delta_n^{-1} \psi_n.$$

- E.g. if $\varepsilon_{it} \sim \mathcal{N}(0, 1)$, $\text{tr}(\Sigma M_{\mathcal{J}}) = \ell$ and $\sigma_{\mathcal{J}}^2 = 2\ell$.

Specification Testing IV

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- For simplicity assume that $\varepsilon_{it} \sim \mathcal{N}(0, 1)$ and $\rho_0 = 0$ such that $\varphi_{\mathcal{J}} = \ell$ and $\sigma_{\mathcal{J}}^2 = 2\ell$.

Specification Testing IV

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- For simplicity assume that $\varepsilon_{it} \sim \mathcal{N}(0, 1)$ and $\rho_0 = 0$ such that $\varphi_{\mathcal{J}} = \ell$ and $\sigma_{\mathcal{J}}^2 = 2\ell$.
- Then if $R < R_0$ it can be shown that under the assumptions of Proposition 2 (except AD(i))

$$\xi_R := \frac{\mathcal{J}_R - \ell_R}{\sqrt{2\ell_R}} \rightarrow \infty.$$

Specification Testing IV

- This result can also be used to test for correct number of factors.
- For simplicity assume that $\varepsilon_{it} \sim \mathcal{N}(0, 1)$ and $\rho_0 = 0$ such that $\varphi_{\mathcal{J}} = \ell$ and $\sigma_{\mathcal{J}}^2 = 2\ell$.
- Then if $R < R_0$ it can be shown that under the assumptions of Proposition 2 (except AD(i))

$$\xi_R := \frac{\mathcal{J}_R - \ell_R}{\sqrt{2\ell_R}} \rightarrow \infty.$$

- Then,

$$\hat{R} := \min_{R=1, \dots, R_{\max}} \{R : \xi_r \geq c_{1-\delta_n} \text{ for } r = 1, \dots, R-1, \xi_r < c_{1-\delta_n}\},$$

is a consistent estimator of R_0 , with $c_{1-\delta_n}$ being the $100(1 - \delta_n)$ percentile of $\Phi(x)$ and $\delta_n \rightarrow 0$, $\ln(\delta) = o(m/nT)$ as $n, m \rightarrow \infty$ and $m^2T/n \rightarrow c \geq 0$.

Spatially/Seriously Correlated Errors I

- Can extend the model of cross-sectional and serial dependence to the error term.
- Suppose the errors are generated according to

$$\varepsilon_t = \rho_\varepsilon \mathbf{W} \varepsilon_t + \alpha_\varepsilon \varepsilon_{t-1} + \phi_\varepsilon \mathbf{W} \varepsilon_{t-1} + u_t.$$

Spatially/Serially Correlated Errors II

Assumption ER (Error)

1. The vector $\boldsymbol{\theta}_{\varepsilon,0} := (\rho_{\varepsilon}, \alpha_{\varepsilon}, \phi_{\varepsilon})^\top$ lies in the interior of Θ_{ε} , where Θ_{ε} is a compact subset of \mathbb{R}^3 in which $\inf_{\boldsymbol{\theta}_{\varepsilon} \in \Theta_{\varepsilon}} \det(\mathbf{S}_{\varepsilon}(\boldsymbol{\theta}_{\varepsilon})) \neq 0$ and $\inf_{\boldsymbol{\theta}_{\varepsilon} \in \Theta_{\varepsilon}} \det(\mathbf{B}_{\varepsilon}(\boldsymbol{\theta}_{\varepsilon})) \neq 0$, $\mathbf{S}_{\varepsilon}^{-1}(\boldsymbol{\theta}_{\varepsilon})$, and $\bar{\mathbf{B}}_{\varepsilon}^{-1}(\boldsymbol{\theta}_{\varepsilon})$ are UB, and $\|\mathbf{A}_{\varepsilon}^h(\boldsymbol{\theta}_{\varepsilon})\|_2 < 1 - c$ for some $c > 0$.
2. The errors u_{it} are independent of the factors, the loadings, and the covariates, and are also independent over i and t with $\mathbb{E}[u_{it}] = 0$, $\mathbb{E}[u_{it}^2] =: \sigma_{u,it}^2 > 0$ and uniformly bounded fourth moments.

Spatially/Serially Correlated Errors III

Assumption IC (Initial Condition)

Assume the initial conditions are generated as

$$\mathbf{y}_0 = \boldsymbol{\Sigma}_{y_0}^{\frac{1}{2}} \boldsymbol{\nu}_1$$

$$\boldsymbol{\varepsilon}_0 = \boldsymbol{\Sigma}_{\varepsilon_0}^{\frac{1}{2}} \boldsymbol{\nu}_2$$

where $\{\nu_{1,j}, \nu_{2,j}\}$ are independent of the independent of the factors, the loadings, the covariates, and the errors \mathbf{u} , are also independent over j with $\mathbb{E}[\nu_{i,1}] = \mathbb{E}[\nu_{i,2}] = 0$, $\mathbb{E}[\nu_{i,1}^2] = \mathbb{E}[\nu_{i,2}^2] = 1$ and uniformly bounded fourth moments. Moreover,

$$\mathbb{E} \left[\begin{pmatrix} \mathbf{y}_0 \\ \boldsymbol{\varepsilon}_0 \end{pmatrix} \begin{pmatrix} \mathbf{y}_0 \\ \boldsymbol{\varepsilon}_0 \end{pmatrix}^\top \right] := \begin{pmatrix} \boldsymbol{\Sigma}_{\varepsilon_0} & \boldsymbol{\Sigma}_{\varepsilon_0 y_0} \\ \boldsymbol{\Sigma}_{\varepsilon_0 y_0} & \boldsymbol{\Sigma}_{y_0} \end{pmatrix} =: \boldsymbol{\Sigma}_0,$$

where $\boldsymbol{\Sigma}_0$ is UB with $\mu_{\min}(\boldsymbol{\Sigma}_0) \geq c > 0$.

Spatially/Serially Correlated Errors IV

Theorem 3 (Asymptotic Distribution - CE)

Under Assumptions MD, CS, AD and ER, with $m^2T/n \rightarrow c \geq 0$ as $n, m \rightarrow \infty$,

$$\sqrt{nT}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}\left(\Delta_*^{-1}(\psi_{*,1} + \psi_{*,2} + \psi_{*,3}), \Delta_*^{-1}\boldsymbol{\Omega}_*\Delta_*^{-1}\right).$$

- $\psi_{*,1}$ generalises ψ :

$$\psi_{*,1} := \frac{1}{\sqrt{nT}} \begin{pmatrix} \text{tr}\left(\Sigma_u \bar{B}_\varepsilon^{-\top} \bar{M} W \bar{B}_\varepsilon^{-1} \bar{B}_\varepsilon^{-1}\right) \\ \text{tr}\left(\Sigma_u \bar{B}_\varepsilon^{-\top} \bar{M} \bar{\Pi} \bar{B}_\varepsilon^{-1} \bar{B}_\varepsilon^{-1}\right) \\ \text{tr}\left(\Sigma_u \bar{B}_\varepsilon^{-\top} \bar{M} \bar{W} \bar{\Pi} \bar{B}_\varepsilon^{-1} \bar{B}_\varepsilon^{-1}\right) \\ 0_{K \times 1} \end{pmatrix}.$$

- $\psi_{*,2}$ and $\psi_{*,3}$ arise due to correlation with the initial condition ε_0 and are $\mathcal{O}(T^{-\frac{1}{2}})$.

Spatially/Serially Correlated Errors V

- Let $\hat{\xi}(\theta_\varepsilon) := \text{vec}(\hat{\varepsilon}) - \rho_\varepsilon \bar{W} \text{vec}(\hat{\varepsilon}) - \alpha_\varepsilon \text{vec}(\hat{\varepsilon}_{-1}) - \phi_\varepsilon \bar{W} \text{vec}(\hat{\varepsilon}_{-1})$ where $\hat{\varepsilon} := (\mathbf{Y} - \mathbf{Z} \cdot \hat{\theta}) \mathbf{M}_{\hat{F}}$, and

$$\varphi(\theta_\varepsilon) := \frac{1}{nT} \begin{pmatrix} \hat{\xi}(\theta_\varepsilon)^\top \Psi_1 \hat{\xi}(\theta_\varepsilon) \\ \vdots \\ \hat{\xi}(\theta_\varepsilon)^\top \Psi_L \hat{\xi}(\theta_\varepsilon) \end{pmatrix},$$

where Ψ_1, \dots, Ψ_L are a series of $n \times n$ matrices with zero diagonals.

- Estimate of $\theta_{0,\varepsilon}$ can be obtained as

$$\hat{\theta}_\varepsilon := \arg \min_{\hat{\theta}_\varepsilon \in \Theta_\varepsilon} \|\varphi(\theta)\|_2^2.$$

- Can be used to construct a test for $\rho_\varepsilon = \alpha_\varepsilon = \phi_\varepsilon = 0$.

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- These biases do not occur through not knowing the factors and the loadings.
- The ‘usual’ LS-IFE biases are also present, though of a lower stochastic order.
- It is also possible to correct for these.

Application I

- Study the relationship economic growth, civil liberties and political rights in the 21st century.
- Similar in spirit to Acemoglu et al. (2019). (ANRR for short)
- Data covers a panel of 180 countries observed between 2001 and 2020.
- Outcome y_{it} log of GDP per capita taken from the World Bank.
- Binary regressor d_{it} derived from Freedom House index

$$d_{it} = \begin{cases} 0 & \text{if classified as } \textit{not free}, \\ 1 & \text{if classified as } \textit{partially free or free}. \end{cases}$$

Application II

- The World Bank provides high resolution latitude and longitude coordinates of international boundaries.
- These are rounded to generate a lower resolution projection which describes the shape of countries using a fewer data points.
- Great-circle distance is calculated between every pair of coordinates.
- For each country pair ij , let δ_{ij} denote the shortest distance between two countries, and let e denote half the distance of the equator.
- The $n \times n$ weights matrix \mathbf{W} is generated by setting element w_{ij} equal to

$$w_{ij} = \begin{cases} 1 - \delta_{ij}/e & \text{if } \delta_{ij}/e < \tau, \\ 0 & \text{otherwise,} \end{cases}$$

where τ is a cut-off point set to 0.1.

Application III

- Outcome equation: $y_t = \alpha y_{t-1} + \beta d_t + \eta_t$.
- Instruments: $\mathcal{V} = (d_1, \dots, d_T)$.
- Long term effect: $\gamma := (1 - \alpha)^{-1} \beta$.

	FE	IV-IFE	ANRR
β	0.0161	0.0141	0.0078 - 0.0097
t-stat	3.9000	2.5343	
α	0.9228	0.8435	0.938 - 0.973
t-stat	146.3841	24.7520	
γ	0.2086	0.0901	0.1264 - 0.3558
t-stat	3.7379	2.7803	
J -stat	-	0.2096	-

Application III

- Outcome equation: $y_t = \alpha y_{t-1} + \beta d_t + \eta_t$.
- Instruments: $\mathcal{V} = (d_1, \dots, d_T, Wd_1, \dots, Wd_T)$.
- Long term effect: $\gamma := (1 - \alpha)^{-1}\beta$.

	FE	IV-IFE	ANRR
β	0.0161	0.0124	0.0078 - 0.0097
t-stat	3.9000	2.286	
α	0.9228	0.9335	0.938 - 0.973
t-stat	146.3841	76.8929	
γ	0.2086	0.1865	0.1264 - 0.3558
t-stat	3.7379	2.4492	
J -stat	-	0.4632	-

Application IV

- Outcome equation:

$$\mathbf{y}_t = \rho \mathbf{W} \mathbf{y}_{t-1} + \alpha \mathbf{y}_{t-1} + \phi \mathbf{W} \mathbf{y}_{t-1} + \beta \mathbf{d}_t + \boldsymbol{\eta}_t.$$

- Instruments: $\mathcal{V} = (\mathbf{d}_1, \dots, \mathbf{d}_T, \mathbf{W}\mathbf{d}_1, \dots, \mathbf{W}\mathbf{d}_T)$.
- Long term effects:
 - Direct effect: $\gamma_D := \text{tr}(((1 - \alpha)\mathbf{I}_n - (\rho + \phi)\mathbf{W})^{-1}\beta)/n$.
 - Indirect effect: $\gamma_I := \boldsymbol{\iota}_n^\top ((1 - \alpha)\mathbf{I}_n - (\rho + \phi)\mathbf{W})^{-1}\beta\boldsymbol{\iota}_n/n - \gamma_D$.

Application IV

	FE	IV-IFE	IV-IFE-BC
β	0.0158	0.0111	0.0112
t-stat	3.8585	2.0618	2.0781
α	0.9224	0.9347	0.9374
t-stat	145.5072	78.0726	78.2993
ρ	0.0125	0.0164	0.0082
t-stat	6.9085	8.0257	4.0043
ϕ	-0.0118	-0.0156	-0.0084
t-stat	-7.3287	-8.0175	-4.3520
γ_D	0.2042	0.1716	0.1793
t-stat	3.7029	2.1954	2.2006
γ_I	0.0645	0.1142	-0.0184
t-stat	0.9342	0.8839	-0.5809
J -stat	-	0.1592	-

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 - Misspecification.
 - Unit roots.
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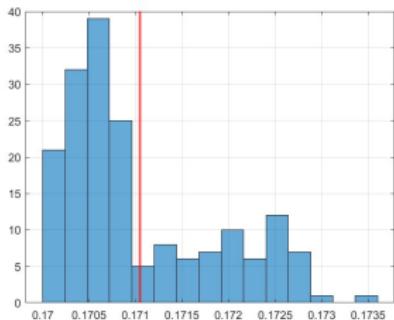
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Application V

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- Explore more complex specification and utilise the additional results.
- Any comments/suggestions most welcome.

Application VI

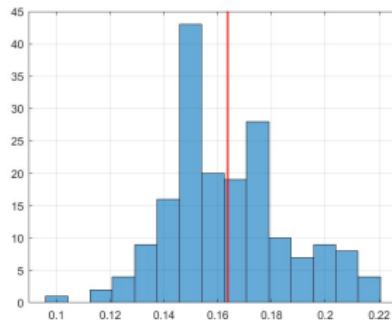


$$\beta = 0.0111$$

$$\alpha = 0.9347$$

$$\rho = 0.0164$$

$$\phi = -0.0156$$



$$\beta = 0.0111$$

$$\alpha = 0.9347$$

$$\rho = 0.0164$$

$$\phi = -0.0100$$

Closing Remarks

- Introduce a simple IV estimator which is consistent and asymptotically normally distributed as long as the number of cross-sectional units n grows sufficiently fast relative to the number of instruments m and the number of time periods T .
- Circumstances exist where, depending on the weights matrix, the estimator can exhibit considerable bias.
- Constructing a bias corrected estimator significantly ameliorates this issue.
- Application applies the method to study the relationship between economic growth, political rights and civil liberties.
- Multiple extensions.