Question 2

The multi-dimensional random variables $\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}$ follows

$N_4(\mu, \Sigma)$ where $\mu = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 7 & 3 & \Upsilon & 2 \\ 3 & 6 & 0 & 4 \\ \Upsilon & 0 & 5 & -2 \\ 2 & 4 & -2 & 4 \end{bmatrix}$

Where for Σ you need to substitute the last digit of your student ID for Υ. Answer the followings after the substitution of Υ, as mentioned in Σ:

a) Determine all the independent univariate random variables

Solution

Where $\Upsilon = 0$ [Last digit of Student ID]

$$\Sigma = \begin{bmatrix} 7 & 3 & \Upsilon & 2 \\ 3 & 6 & 0 & 4 \\ \Upsilon & 0 & 5 & -2 \\ 2 & 4 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 0 & 2 \\ 3 & 6 & 0 & 4 \\ 0 & 0 & 5 & -2 \\ 2 & 4 & -2 & 4 \end{bmatrix}$$

Recall, suppose X has a $N_n(\mu, \Sigma)$ distribution, Partitioned as in the expression $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$

Then X_1 and X_2 are independent if and only if the Co-variance between their components are all 0; i.e $\Sigma_{12} = 0$ and $\Sigma_{21} = 0$.

Here, The independent univariate random variables are x_1 and x_2, with the distribution: $X_1 \sim N(\mu_1, \Sigma_{11})$ and $X_2 \sim N(\mu_2, \Sigma_{22})$.

Similarly for a 4-dimensional random variable, with $X = \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} \\ \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44} \end{bmatrix}$

The Independent univariate random variables are: $X_1 \sim N(\mu_1, \Sigma_{11})$

$X_2 \sim N(\mu_2, \Sigma_{22})$

$Y_1 \sim N(\mu_3, \Sigma_{33})$

and $Y_2 \sim N(\mu_4, \Sigma_{44})$

Where $\mu = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 7 & 3 & 0 & 2 \\ 3 & 6 & 0 & 4 \\ 0 & 0 & 5 & -2 \\ 2 & 4 & -2 & 4 \end{bmatrix}$

$\therefore \quad X_1 \sim N(\mu_1, \Sigma_{11}) \Rightarrow x_1 \sim N(2, 7)$

$x_2 \sim N(\mu_2, \Sigma_{22}) \Rightarrow x_2 \sim N(-1, 6)$

$x_3 \sim N(\mu_3, \Sigma_{33}) \Rightarrow x_3 \sim N(3, 5)$

$x_4 \sim N(\mu_4, \Sigma_{44}) \Rightarrow x_4 \sim N(1, 4)$.

Question 1b

What is the Probability density function for $x_1 + 2x_2$?

Solution

Recall, the univariate distribution function is:

$$\boxed{P(x; \mu, \sigma^2) = f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}} \quad \cdots (A)$$

For a multivariate distribution [where $n=2$ i.e Bivariate]

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

Then $$P(x; \mu, \Sigma) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{\left(\frac{-1}{2\sigma_1^2}(x_1-\mu_1)^2\right)} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{\left(\frac{-1}{2\sigma_2^2}(x_2-\mu_2)^2\right)}$$

Recall, if x and y are independent, then: $X + Y \sim N_d(\mu_x + \mu_y, \Sigma_{xx} + \Sigma_{yy})$

Where $x = x_1$ and $y = 2x_1$

$$\Longrightarrow x_1 + 2x_1 \sim N_d(\mu_{x_1} + \mu_{2x_1}, \Sigma_{x_1x_1} + \Sigma_{2x_12x_1})$$

Let $z = x_1 + 2x_2$

$\mu_z = E(z) = E(x_1 + 2x_1) = E(x_1) + 2E(x_2) = \mu_1 + 2\mu_2$

$\therefore$ Recall $\mu = [2 \quad -1 \quad 3 \quad 1]$

i.e $\mu_1 = 2$, and $\mu_2 = -1$

$\Longrightarrow \mu_z = 2 + 2(-1) = 0$

Also $\Sigma_z = Var(z) = Var(x_1 + 2x_2) = Var(x_1) + Var(2x_2)$

$= \Sigma_{11} + 2^2 Var(x_2)$

$= \Sigma_{11} + 4\Sigma_{22} = 7 + 4(6)$

$\therefore \Longrightarrow \Sigma_z = 31$.

i.e.

From (A): $P(z; \mu_z, \Sigma_z) = f(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z} \cdot e^{-\frac{1}{2}\left(\frac{z-\mu_z}{\sigma_z}\right)^2}$

Recall $\Sigma = \sigma^2$ [i.e $\sigma_z = \sqrt{\Sigma}$]

i.e $\mu_z = 0$, $\sigma_z = \sqrt{31}$

$\therefore$

$$P(z; \mu_z, \Sigma_z) = f(z) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{31}} \cdot e^{-\frac{1}{2}\left(\frac{z-0}{\sqrt{31}}\right)^2}$$

$$\therefore f(z) = \frac{1}{\sqrt{62\pi}} e^{-\frac{1}{2}\cdot z^2/31}$$

$$f(x_1+2x_2) = \frac{1}{\sqrt{62\pi}} e^{-\frac{1}{62}(x_1+2x_2)^2}$$

1c

Let $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y := [y_1]$. Determine the Conditional distribution density $P(y|x)$ and $P(x|y=1)$.

Solution

* Given $x \in \mathbb{R}^d$ as an input and $y \in \mathbb{R}$ (dependent variable)

we consider the Likelihood function: $P(y|x) = N(y|f(x), \sigma^2)$

where $f: \mathbb{R}^d \to \mathbb{R}$ is the regression function

* Recall, $P(y|x) = \frac{f(x,y)}{f_1(x)}$ [where $f_1(x) > 0$]

where $f_1(x)$ is the marginal Probability density of X.

$$f(x,y) = N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}\right), \quad f(x) = \int f(x,y)\, dy = N(x|\mu_x, \Sigma_{xx})$$

$$P(y|x) = N(\mu_{y|x}, \Sigma_{y|x})$$

$$\mu_{y|x} = \mu_y + \Sigma_{yy}\Sigma_{xx}^{-1}(x-\mu_x)$$

$$\Sigma_{y|x} = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy},$$

Given: $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y := [y_1]$, The joint normal distribution where $\mu = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 7 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

is given by $X \sim N_d(\mu, \Sigma)$, where

$y_1 \quad \ldots \quad \Sigma_{22} \quad \Sigma_{23}$

$\Sigma_{31} \quad \Sigma_{32} \quad \Sigma_{33}$

Hence we have $X \sim N_d\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right) \quad \ldots \text{(B)}$

i.e $P(y|x) = P(y_1|x) = N(\mu_{y_1|x}, \Sigma_{y_1|x})$

where $\mu_{y_1|x} = \mu_{y_1} + \Sigma_{y_1 x} \Sigma_{xx}^{-1}(x - \mu_x)$

from (B): $\mu_{y_1} = 3$, Since y_1 is independent of x_1, x_2

we have $\Sigma_{y_1 x} = 0$.

$\Rightarrow \mu_{y_1|x} = 3$

$\Sigma_{y_1|x} = \Sigma_{y_1 y_1} - \Sigma_{y_1 x}\Sigma_{xx}^{-1}\Sigma_{x y_1}$

~~from (B)~~: $\Sigma_{y_1 y_1} = 5$, [Recall $\Sigma_{y_1 x} = 0$]

$\Rightarrow \Sigma_{y_1|x} = 5 - 0 = 5$.

$\therefore P(y_1|x) = N(\mu_{y_1|x}, \Sigma_{y_1|x}) = N(3, 5)$.

* $P(x|y=1) = N(\mu_{x|y=1}, \Sigma_{x|y_1=1})$

$(\mu_{x|y_1=1}) = \mu_x + \Sigma_{xy_1}\Sigma_{y_1 y_1}^{-1}(y - \mu_y)$

From (B): $\mu_x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\Sigma_{xy_1} = 0$

$\Rightarrow \mu_{x|y_1=1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Sigma_{x|y_1=1} = \Sigma_{xx} - \Sigma_{xy_1}\Sigma_{y_1 y_1}^{-1}\Sigma_{y_1 x}$

from (B): $\Sigma_{xx} = \begin{bmatrix} 7 & 3 \\ 3 & 6 \end{bmatrix}$, Recall $\Sigma_{xy_1} = 0$

$\therefore \Sigma_{x|y_1=1} = \begin{bmatrix} 7 & 3 \\ 3 & 6 \end{bmatrix}$

Hence $P(x|y=1) = N\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 & 3 \\ 3 & 6 \end{bmatrix}\right)$