Mathematics and Statistic For Data Analysis.

SECTION A

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

a) Compute the rank of A

Solution

The rank of a matrix [denoted by $\rho(A)$] is the number of linearly independent rows [or columns] in it.

* Using echelon form, the rank of the matrix A is equal to the number of non-zero rows in the resultant [row-reduced echelon].

Given $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

By reducing A;

$$R_3 \to R_3 - 2R_2 \Rightarrow A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$R_1 \to R_1 + \tfrac{1}{2} R_2 \Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$R_1 \to R_1 - \tfrac{1}{3} R_3 \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$R_2 \to \tfrac{1}{2} R_2 \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$R_3 \to \tfrac{1}{3} R_3 \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A_R [Reduced form of A] $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

i.e The rank of $A = 3$ [since we have 3 non-zero rows in A_R]

b) Provide the characteristic Polynomial for A and Compute the eigenvalues of A

Solution

Given A is an nxn matrix. The characteristic Polynomial of A is defined as function $f(\lambda)$ and the characteristic Polynomial formulae is given by:

$$f(\lambda) = \det(A - \lambda I_n)$$

* Where I represents the Identity matrix.

□ Hence, given $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

where $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The characteristic Polynomial of A $[f(\lambda)] = \det(A - \lambda I_3)$

i.e $f(\lambda) = |A - \lambda I_3|$

$$A - \lambda I_3 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I_3 = \begin{bmatrix} 1-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{bmatrix}$$

$\therefore$

$$|A - \lambda I_3| = (1-\lambda)\left[(2-\lambda)(3-\lambda)\right] - 1(0) + 1(0)$$

$$\Rightarrow |A - \lambda I_3| = (1-\lambda)[6 - 5\lambda + \lambda^2]$$

$$\Rightarrow |A - \lambda I_3| = 6 - 11\lambda + 6\lambda^2 - \lambda^3$$

i.e $|A - \lambda I_3| = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

Hence, the Characteristic Polynomial of $A: f(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$.

(ii) Recall, the roots of the Characteristic Polynomials are the eigenvalues.

i.e Given $f(\lambda) = \det(A - \lambda I_n)$ is a characteristic Polynomial, then λ_0 is an eigenvalue of A, iff $f(\lambda_0) = 0$.

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$\therefore -\lambda^3+6\lambda^2-11\lambda+6=0$

Recall, if $f(\lambda_0)=0$, then λ_0 is a root of the Polynomial $f(\lambda)=-\lambda^3+6\lambda^2-11\lambda+6$

By substituting $\lambda=2$,

$f(2)=-(2)^3+6(2)^2-11(2)+6=0$

$\therefore$ $\lambda=2$ is a root of the Polynomial $f(\lambda)$.

Then $(\lambda-2)$ is a factor. [Recall if $\lambda=a$ is a root, then $(\lambda-a)$ is a factor]

By Long division Method:

$$\begin{array}{r|l} & -\lambda^2+4\lambda-3 \\ \hline \lambda-2 & -\lambda^3+6\lambda^2-11\lambda+6 \\ & -\lambda^3+2\lambda^2 \\ \hline & \quad 4\lambda^2-11\lambda+6 \\ & \quad 4\lambda^2-8\lambda \\ \hline & \qquad -3\lambda+6 \\ & \qquad -3\lambda+6 \\ \hline & \qquad 0 \quad 0 \end{array}$$

$$\therefore \quad -\lambda^3+6\lambda^2-11\lambda+6=(\lambda-2)(-\lambda^2+4\lambda-3)$$
$$=(2-\lambda)(\lambda^2-4\lambda+3)$$

i.e $-\lambda^3+6\lambda^2-11\lambda+6=(2-\lambda)(\lambda-3)(\lambda-1)$

$\square$ $-\lambda^3+6\lambda^2-11\lambda+6=0 \Rightarrow (2-\lambda)(\lambda-3)(\lambda-1)=0$

i.e $2-\lambda=0$ or $\lambda-3=0$ or $\lambda-1=0$

$\lambda=2$ or $\lambda=3$ or $\lambda=1$.

Hence, the eigen values are: 2, 3 and 1.

$-8+24+\ldots$

Projection of u on v $= \text{Proj}_v u = \left(\frac{u \cdot v}{\|v\|^2}\right) v$

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* Let u_i be the i-th Column of A for $i = 1, 2, 3$. Use the Gram-Schmidt Process to generate an orthonormal basis for $\{u_1, u_2, u_3\}$.

Solution

Recall, assume a basis $\{b_1, \ldots, b_n\}$ is given. Then it generates inductively:

$$b_1^* = b_1,$$
$$b_2^* = b_2 - \text{Proj}_{b_1^*}(b_2)$$
$$b_3^* = b_3 - \text{Proj}_{b_1^*}(b_3) - \text{Proj}_{b_2^*}(b_3)$$
$$b_4^* = b_4 - \text{Proj}_{b_1^*}(b_4) - \text{Proj}_{b_2^*}(b_4) - \text{Proj}_{b_3^*}(b_4)$$
$$\vdots$$
$$b_n^* = b_n - \sum_{i=1}^{n-1} \text{Proj}_{b_i^*}(b_n).$$

The generated $\{b_1^*, \ldots, b_n^*\}$ is a basis that all the distinct elements are orthogonal to each other.

So given u_i [the ith column of A] for $i = 1, 2, 3$.

Then $\{u_1, u_2, u_3\} \Rightarrow u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

$\Rightarrow u_1^* = u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow u_2^* = u_2 - \text{Proj}_{u_1^*}(u_2) = u_2 - \left(\frac{u_2 \cdot u_1^*}{\|u_1^*\|^2}\right) \cdot u_1^*$

Where $u_2 \cdot u_1^*$ is the dot Product of u_2 and u_1^*

$$= \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} - \left(\frac{\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\left(\sqrt{1^2+0^2+0^2}\right)^2}\right) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} - \left(\frac{-1}{1}\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore u_2^* = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$

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Also,

$$u_3^* = u_3 - \text{Proj}_{u_1^*}(u_3) - \text{Proj}_{u_2^*}(u_3)$$

$$= u_3 - \left(\frac{u_3 \cdot u_1^*}{\|u_1^*\|^2}\right) \cdot u_1^* - \left(\frac{u_3 \cdot u_2^*}{\|u_2^*\|^2}\right) \cdot u_2^*$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{(\sqrt{1})^2}\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}}{(\sqrt{0^2+2^2+4^2})^2}\right) \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left[\frac{12}{(\sqrt{20})^2}\right] \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} - \frac{3}{5} \cdot \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\frac{6}{5} \\ \frac{3}{5} \end{bmatrix}$$

$$\therefore \quad u_3^* = \begin{bmatrix} 0 \\ -6/5 \\ 3/5 \end{bmatrix}$$

Hence, the generated $u_1^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [1, 0, 0]$

$u_2^* = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = [0, 2, 4]$

and $u_3^* = \begin{bmatrix} 0 \\ -6/5 \\ 3/5 \end{bmatrix} = [0, -6/5, 3/5]$

forms an Orthogonal basis for R^3. Normalizing the vectors in the Orthogonal basis, we obtain the Orthonormal basis:

$$\left\{\frac{u_1^*}{\|u_1^*\|}, \frac{u_2^*}{\|u_2^*\|}, \frac{u_3^*}{\|u_3^*\|}\right\} = \left\{\frac{[1,0,0]}{\sqrt{1}}, \frac{(0,2,4)}{\sqrt{20}}, \frac{(0, -6/5, 3/5)}{\sqrt{45/25}}\right\}$$

$$= \left\{(1,0,1), \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(0, \frac{-6}{\sqrt{45}}, 3/\sqrt{45}\right)\right\}$$

Orthonormal basis $= \left\{(1,0,1), \left(0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right), \left(0, \frac{-2\sqrt{45}}{15}, \frac{\sqrt{45}}{15}\right)\right\}$