Dot Product

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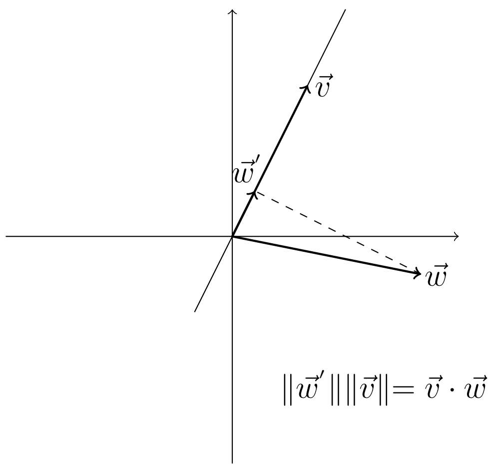
There are a few ways to multiply the vectors. One of them is called **dot product** and it is defined as follows: for any vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{bmatrix}$,

$$\vec{v} \cdot \vec{w} = \sum_{k=1}^{n} v_k w_k$$

Let's see what this means geometrically:



We assumed $\vec{w} - \vec{w}' \perp \vec{v}$ above. We will proove the equation $\|\vec{w}'\| \|\vec{v}\| = \vec{v} \cdot \vec{w}$. We will use another vector multiplication to proove this, if $\vec{v}$ and $\vec{w}$ are two vectors, then their **cross product** defined as

$$\vec{v} \times \vec{w} = \|\vec{v}\|\|\vec{w}\|\text{sin}(\theta)\vec{\mathbf{n}}$$

where $\vec{\mathbf{n}}$ is the normal vector and θ is the angle between these two vectors. By using this definition, we can see that

$$\begin{aligned}
\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}&,\\
\mathbf{i} \times \mathbf{j} = \mathbf{k}&,\\
\mathbf{k} \times \mathbf{i} = \mathbf{j}&,\\
\mathbf{j} \times \mathbf{k} = \mathbf{i}&,\\
\mathbf{j} \times \mathbf{i} = -\mathbf{k}&,\\
\mathbf{i} \times \mathbf{k} = -\mathbf{j}&,\\
\mathbf{k} \times \mathbf{j} = -\mathbf{i}&
\end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are basis vectors in three-dimensional space. So we can define the cross product in two-dimensional space by

$$\begin{aligned}
\vec{v} \times \vec{w} &= (v_1\mathbf{i} + v_2\mathbf{j}) \times (w_1\mathbf{i} + w_2\mathbf{j})\\
&= v_1w_1(\mathbf{i} \times \mathbf{i}) + v_1w_2(\mathbf{i} \times \mathbf{j})\\
&v_2w_1(\mathbf{j} \times \mathbf{j}) + v_21w_2(\mathbf{j} \times \mathbf{j})\\
&= v_1w_2\mathbf{k} - v_2w_1\mathbf{k}.
\end{aligned}$$

Although $\mathbf{k}$ is not two dimensional, we can say that

$$\|\vec{v} \times \vec{w}\| = v_1w_2 - v_2w_1$$

and it means

$$\|\vec{v}\|\|\vec{w}\|\sin(\theta) = v_1w_2 - v_2w_1.$$

According the figure, the distance $\|\vec{w}\|\sin(\theta)$ must be equal to the distance $\|\vec{w} - \vec{w}^{'}\|$ so

$$\|\vec{w} - \vec{w}^{'}\| = \frac{v_1w_2 - v_2w_1}{\|\vec{v}\|}.$$

By the Pythagorean theorem, it must be true that

$$\begin{aligned}
\|\vec{w}^{'}\| &= \sqrt{\|\vec{w}\|^2 - \frac{(v_1w_2 - v_2w_1)^2}{\|\vec{v}\|^2}}\\
&\implies \|\vec{v}\|\|\vec{w}^{'}\| = \sqrt{\|\vec{v}\|^2\|\vec{w}\|^2 - (v_1w_2 - v_2w_1)^2}\\
&= \sqrt{v_1^2w_1^2 + v_2^2w_1^2 + v_1^2w_2^2 + v_2^2w_2^2 - v_1^2w_2^2 + 2v_1v_2w_1w_2 - v_2^2w_1^2}\\
&= \sqrt{v_1^2w_1^2 + v_2^2w_2^2 + 2v_1v_1w_1w_2}\\
&= v_1w_1 + v_2w_2 = \vec{v} \cdot \vec{w}
\end{aligned}$$

so the geometric interpretation for the dot product is true for the two-dimensional space. By this way it can be seen that it is true for other dimensions as well.