Experimental Validation and Adjustment of a Theoretical Model for Mid-Air Collision of Projectiles

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October 24, 2024

1 Abstract

This experiment explored the collision dynamics of two projectiles launched simultaneously from distinct starting points to apply principles of projectile motion in predicting precise collision coordinates. Positioned two meters apart, the launch apparatuses propelled projectiles in parallel, necessitating accurate calculation of initial velocities and launch angles. Measured velocities obtained from speed traps and calculated launch angles were used to establish initial conditions for predicting and observing the collision of both projectiles. A comparison of theoretical predictions with observed results demonstrated the reliability of projectile motion equations and highlighted their relevance to practical applications, reinforcing our understanding of kinematic principles in physics.

2 Introduction

2.1 Theory

The setup consisted of two projectile launchers, A and B, positioned at the same height and angled so that their projectile paths would be co-planar. The following variables are defined:

- Coordinate plane: A co-planar reference for both projectile paths, with unit vectors $\hat{i}$ and $\hat{j}$ equal to 1 mm.

- Time t (in seconds): $t = 0$ at the moment of launch.

- $x^A(t)$, $y^A(t)$, $x^B(t)$, and $y^B(t)$: The x- and y-positions for projectiles from launchers A and B, respectively.

- Displacement vectors $\vec{s^A}(t)$ and $\vec{s^B}(t)$ as functions of t:

$$\vec{s^A}(t) = \begin{bmatrix} s_x^A(t) \\ s_y^A(t) \end{bmatrix} = \begin{bmatrix} x^A(t) - x^A(0) \\ y^A(t) - y^A(0) \end{bmatrix}$$

$$\vec{s^B}(t) = \begin{bmatrix} s_x^B(t) \\ s_y^B(t) \end{bmatrix} = \begin{bmatrix} x^B(t) - x^B(0) \\ y^B(t) - y^B(0) \end{bmatrix}$$

- Velocities $v_x^A(t)$, $v_y^A(t)$, $v_x^B(t)$, and $v_y^B(t)$: The horizontal and vertical velocities for each projectile as functions of t.

  - These velocity functions are the derivatives of their respective positions, e.g.,

$$v_x^A(t) = \frac{dx^A(t)}{dt}.$$

The combined velocities can be expressed as vectors:

$$\vec{v^A}(t) = \begin{bmatrix} v_x^A(t) \\ v_y^A(t) \end{bmatrix} \quad \text{and} \quad \vec{v^B}(t) = \begin{bmatrix} v_x^B(t) \\ v_y^B(t) \end{bmatrix}$$

- Displacement vector $\vec{D} = \begin{bmatrix} D_x \\ D_y \end{bmatrix}$: The vector from launcher A to launcher B, with initial positions:

$$\vec{s^A}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{s^B}(0) = \vec{s^A}(0) + \vec{D}$$

- $g = 9.81$ m/s^2: Acceleration due to gravity.

- t_c: Collision time, defined as the smallest $t_c \geq 0$ for which:

$$\vec{s^B}(t_c) = \vec{s^A}(t_c).$$

2.2 Initial Speed Measurements

To measure initial projectile velocities, launchers were positioned horizontally on a table, loaded to various levels, and fired. Projectile speeds were recorded with a speed trap and an iPad application, and average velocities were used in further calculations. The tables below lists measured exit velocities for both launchers.

Speed Mode	Trial 1(m/s)	Trial 2(m/s)	Trial 3(m/s)	Average(m/s)
Slow	3.22	3.17	3.18	**3.19**
Medium	4.74	4.84	4.80	**4.79**
High	6.64	6.27	6.44	**6.45**

Table 1: Launcher B Exit Speeds by Speed Mode

Speed Mode	Trial 1(m/s)	Trial 2(m/s)	Trial 3(m/s)	Average(m/s)
Slow	2.49	2.55	2.55	**2.53**
Medium	3.55	3.55	3.55	**3.55**
High	4.69	4.73	4.75	**4.72**

Table 2: Launcher A Exit Speeds by Speed Mode

2.3 Calculations

Let the launch angles of launchers A and B be denoted as θ_A and θ_B, respectively. The initial velocities of the projectiles can be written as:

$$\vec{v^A}(0) = |\vec{v^A}(0)| \begin{bmatrix} \cos(\theta_A) \\ \sin(\theta_A) \end{bmatrix}, \quad \vec{v^B}(0) = |\vec{v^B}(0)| \begin{bmatrix} \cos(\theta_B) \\ \sin(\theta_B) \end{bmatrix}$$

Since $\vec{v_x}$ is constant, while $\vec{v_y}$ varies due to gravity, the displacement functions based on initial positions are:

$$\vec{s^A}(t) = \begin{bmatrix} v_x^A(0) \cdot t \\ v_y^A(0) \cdot t - \frac{1}{2}gt^2 \end{bmatrix}, \quad \vec{s^B}(t) = \begin{bmatrix} D_x + v_x^B(0) \cdot t \\ D_y + v_y^B(0) \cdot t - \frac{1}{2}gt^2 \end{bmatrix}$$

Setting $\vec{s^A}(t_c) = \vec{s^B}(t_c)$ for collision at time t_c leads to:

$$\begin{cases} |\vec{v^A}(0)| \cos(\theta_A) \cdot t_c = D_x + |\vec{v^B}(0)| \cos(\theta_B) \cdot t_c \\ |\vec{v^A}(0)| \sin(\theta_A) \cdot t_c - \frac{1}{2}gt_c^2 = D_y + |\vec{v^B}(0)| \sin(\theta_B) \cdot t_c - \frac{1}{2}gt_c^2 \end{cases}$$

Dividing each by t_c (if $t_c \neq 0$), we derive:

$$\vec{v^A}(0) - \vec{v^B}(0) = \frac{\vec{D}}{t_c}$$

This equation asserts that for collision, the difference between initial velocities must match the displacement scaled with t_c.

The **collision point** can be expressed as $\vec{s^A}(t_c)$ or $\vec{s^B}(t_c)$

2.4 Predicted Calculations

Calculations:

1. **Equating Vertical Positions:**

   Set $y_A(t_c) = y_B(t_c)$:

   $$v_A \sin(\theta_A)t_c - \frac{1}{2}gt_c^2 = v_B \sin(\theta_B)t_c - \frac{1}{2}gt_c^2$$

   Simplify:

   $$v_A \sin(\theta_A)t_c = v_B \sin(\theta_B)t_c$$

   Since t_c is common on both sides of the equation, it would cancel out.

   Equation (1):

   $$v_A \sin(\theta_A) = v_B \sin(\theta_B)$$

2. **Equate Horizontal Positions:**

Set $x_A(t_c) = x_B(t_c)$:

$$v_A \cos(\theta_A)t_c = D + v_B \cos(\theta_B)t_c$$

Taking all terms that have t_c to one side:

$$(v_A \cos(\theta_A) - v_B \cos(\theta_B))t_c = D$$

Equation (2):

$$t_c = \frac{D}{v_A \cos(\theta_A) - v_B \cos(\theta_B)}$$

Solve for θ_A and θ_B: We have two equations with three unknowns $(\theta_A, \theta_B, t_c)$. To solve this, we would choose a value for one of the angles and solve for the others.

For this, we would take the angle of our launcher B, i.e. $\theta_B = 74°$.

Calculate $\sin(\theta_B)$ and $\cos(\theta_B)$:

- $\sin(74°) = 0.9613$

- $\cos(74°) = 0.2756$

Using Equation (1) that we found above to solve for θ_A:

$$v_A \sin(\theta_A) = v_B \sin(\theta_B)$$

$$6.45 \sin(\theta_A) = 4.735 \times 0.9613$$

$$6.45 \sin(\theta_A) = 4.552 \,\text{m/s}$$

Solving for $\sin(\theta_A)$:

$$\sin(\theta_A) = \frac{4.552}{6.45} \Rightarrow 0.7066$$

Calculate θ_A:
$$\theta_A = \arcsin(0.7066) = 44.9°$$

Therefore, our predicted value for our large launcher is 44.9 degrees, making $\theta_A = 44.9°$.

Calculating Collision Time (t_c) Calculate $\cos(\theta_A)$:

$$\cos(44.9) = \sqrt{1 - \sin^2(44.9)} = \sqrt{1 - (0.7066)^2} = 0.7076$$

Calculate $v_A \cos(\theta_A)$ and $v_B \cos(\theta_B)$:

$$v_A \cos(\theta_A) = 6.45 \times 0.7076 = 4.563 \, \text{m/s}$$

$$v_B \cos(\theta_B) = 4.735 \times 0.2756 = 1.304 \, \text{m/s}$$

From here, we use Equation (2) to calculate the time (t_c):

$$t_c = \frac{D}{v_A \cos(\theta_A) - v_B \cos(\theta_B)}$$

$$t_c = \frac{D}{4.563 - 1.304} \Rightarrow \frac{2.0}{3.259} = 0.6136 \, \text{s}$$

Calculate Collision Point Horizontal position at t_c

- **Projectile A:**

$$x_A(t_c) = v_A \cos(\theta_A)t_c = 4.563 \times 0.6136 = 2.798 \, \text{m}$$

- **Projectile B:**

$$x_B(t) = D + v_B \cos(\theta_B)t$$

$$x_B(t) = 2.0 + 1.304 \times 0.6136 = 2.0 + 0.8 = 2.798 \, \text{m}$$

These calculations show that our horizontal values are equal, which means that our projectile would be at the same position on the x-axis.

Vertical position at t_c

- **Projectile A:**

$$y_A(t_c) = v_A \sin(\theta_A)t - \frac{1}{2}gt^2$$

$$y_A(t_c) = 4.552 \times 0.6136 - 4.905 \times (0.6136)^2$$

$$y_A(t_c) = 2.793 - 1.847 = 0.946\,\text{m}$$

- **Projectile B:**

$$y_B(t_c) = v_B \sin(\theta_B)t - \frac{1}{2}gt^2$$

$$y_B(t_c) = 4.552 \times 0.6136 - 4.905 \times (0.6136)^2$$

$$y_B(t_c) = 2.793 - 1.847 = 0.946\,\text{m}$$

Since vertical and horizontal velocities are equal, we can predict that the projectiles would collide at these coordinates, given these values.

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3 Materials and Methods

3.1 Materials

- **Launcher A (Big Launcher):**
    - Plastic ball (25.3 mm in diameter, 9.73 g mass)
    - Positioned at $x = 0$ m
    - Velocity is measured using a speed trap

- **Launcher B (Small Launcher):**
    - Metal ball (15.7 mm diameter, 16.36 g mass)
    - Positioned at $x = D = 2$ m
    - Velocity is measured using a speed trap

- Sparkview Speed Trap

- Measuring Tape to measure the distance between launchers

3.2 Methods

1. Position Launcher A (big launcher) at $x = 0$ m and Launcher B (small launcher) at $x = 2$ m, both set to the same vertical height.

2. Record the exit velocities of both projectiles by conducting several trials.

3. Calculate the average velocity for both projectiles.

4. Use kinematic equations to predict the required launch angles for both projectiles to collide in mid-air, considering that Launcher A is two meters behind Launcher B.

5. Launch both projectiles simultaneously and record the actual launch angles and collision points.

6. Compare the calculated values with the actual results and analyze any differences.

4 Results

4.1 Launcher A (Big Launcher)

	Slowest mode	Medium mode	High Speed
Trial 1	3.22 m/s	4.74 m/s	6.64 m/s
Trial 2	3.17 m/s	4.84 m/s	6.27 m/s
Trial 3	3.18 m/s	4.80 m/s	6.44 m/s
Average	3.19 m/s	4.79 m/s	6.45 m/s

Table 3: Velocity data for Launcher A (Big Launcher)

4.2 Launcher B (Small Launcher)

	Slowest mode	Medium mode	High Speed
Trial 1	2.49 m/s	3.55 m/s	4.69 m/s
Trial 2	2.55 m/s	3.55 m/s	4.73 m/s
Trial 3	2.55 m/s	3.55 m/s	4.75 m/s
Average	2.53 m/s	3.55 m/s	4.72 m/s

Table 4: Velocity data for Launcher B (Small Launcher)

4.3   Additional Information

- Distance between launchers: $D = 2.0$ m

- Gravity: $g = 9.81$ m/s^2

5   Discussion

5.1   Summary of Experimental Results

In this experiment, we aimed to determine the required launch angles and timing for two projectiles, launched in the same direction from different starting points, to collide mid-air. After conducting the experiment and analyzing the slow-motion footage, we obtained the following values:

- **Launcher A (Large Projectile)**: Experimental launch angle of 40°.

- **Launcher B (Small Projectile)**: Experimental launch angle of 74°.

- **Collision Time**: 0.62 seconds, calculated by multiplying the time measured in slow motion by the frames per second (FPS) of the recording.

These values closely align with the pre-lab calculations, which predicted angles close to 44.9° for Launcher A and exactly 74° for Launcher B, as well as a collision time of approximately 0.6136 seconds.

5.2   Comparison with Pre-Lab Calculations

The experimentally obtained angle for Launcher A is 40°, while the pre-lab calculation predicted an angle of 44.9°. This corresponds to a difference of approximately 4.9°, or a 12.25% discrepancy from the predicted angle. For Launcher B, the experimental and calculated angles match exactly at 74°. The collision time calculated from the experiment (0.62 seconds) is also remarkably close to the predicted time of 0.6136 seconds, differing by less than 1%.

These results suggest that the pre-lab calculations provided the predicted values necessary for a collision. The slight discrepancy in Launcher A's angle could be due to minor errors in angle measurement, subtle inconsistencies in initial velocity, or the effects of air resistance, which were neglected in the theoretical model.

5.3   Evaluation of Measurement Techniques

The collision time was determined using a slow-motion recording. To obtain the time, I multiplied the recorded slow-motion time by the video's FPS. This technique allowed for accurate time measurement, minimizing human error in timing but introducing potential error due to slight frame rate variations or imprecisions in playback speed.

5.4   Potential Sources of Error

1. **Angle Measurement**: Small errors in measuring the launch angles could contribute to the observed discrepancies. Even a minor deviation in the launcher setup could slightly alter the launch angle.

2. **Initial Velocity Variations**: The experimental setup may not perfectly reproduce the exact initial velocities used in calculations. Variations in launcher tension or minor friction effects could impact the speed. The reason for this is spring overuse; as we keep loading and releasing a spring, its base length tends to grow shorter because of how often it is compressed, which can lead to the spring being more compressed and releasing less force on the ball.

3. **Air Resistance**: Although the theoretical calculations assumed no air resistance, real-world conditions introduce a slight drag on each projectile, which could affect both angle and collision timing.

4. **Data Collection**: While collecting our initial velocities, we had the launchers shoot the projectile horizontally and vertically to find the velocity of the ball now because we couldn't predict at what angle our ball would be launching just to get our velocity values, we shot the ball horizontally along the table. Although this method did work, there are chances that the ball might have rubbed against the barrel of the turret, which could've slowed down its initial release trajectory.

5. Laser Discrepencies:

6   Conclusion

The experiment successfully validated the pre-lab calculations, demonstrating that the theoretical model for projectile motion can accurately predict collision conditions within an acceptable margin of error. The small discrepancies observed emphasize the impact of real-world factors like measurement precision and air resistance, which should be accounted for in further studies. This experiment underscores the reliability of kinematic equations for predicting projectile motion, even in complex scenarios with multiple variables.