Homework assignment 2

Hand in on DTU Learn before 5 November 10pm. **The overall page limit exclude Appendix is 6.**

1 Report on the exercises for week 9 (60%, the page limit is 3)

2 Review: Rosenbrock problem (20%)

The Rosenbrock problem that we studied in 4-hour exercise in Week 5 can be formulated as a system of nonlinear equations:

$$\boldsymbol{r}(\boldsymbol{x}) = \sqrt{2}\begin{bmatrix} 10(x_2 - x_1^2) \\ 1 - x_1 \end{bmatrix} = \mathbf{0}.$$

It is easy to see that this system has the unique solution $\boldsymbol{x}^* = [1, 1]^T$, and this is the unique minimizer for $f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{r}(\boldsymbol{x})\|_2^2$.

1. (by hand, 5%) Calculate the Jacobian J of $\boldsymbol{r}$.

2. (5%) Implement a Matlab function to calculate $\boldsymbol{r}$ and J with a given $\boldsymbol{x}$. You can start your function with

   ```
   function [r,J]=fun_rJ_Rosen(x)
   ```

   Please include this Matlab function in your answers.

3. (10%) Call the Matlab function `Levenberg_Marquardt_yq.m`, which can be found in DTU Learn, to apply the Levenberg-Marquardt method to solve the minimization problem

   $$\min_{\boldsymbol{x}} f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{r}(\boldsymbol{x})\|_2^2.$$

   Set the starting point as $\boldsymbol{x}_0 = [-1.2, 1]^T$ and the initial λ as $10^{-3}\|J(\boldsymbol{x}_0)^T J(\boldsymbol{x}_0)\|_2$. Plot $\boldsymbol{e}_k = \|\boldsymbol{x}_k - \boldsymbol{x}^*\|_2$, $f(\boldsymbol{x}_k)$ and $\|\nabla f(\boldsymbol{x}_k)\|_2$ as functions of the iteration number. How many iterations did the method need until meeting the default stopping criteria? Which convergence rate did you get? Please explain how you found out the convergence rate?

3 Linear least squares with weights (12%)

In this exercise, we determine the parameters in the fit function

$$\phi(t) = x_1 e^{-27t} + x_2 e^{-8t} + x_3$$

to fit a NMR signal. We know that the exact parameters $\boldsymbol{x}^* = [1.27, 2.04, 0.3]^T$. In addition, we know that the first 10 data points was added larger Gaussian noise with mean 0 and standard deviation 0.5, and the rest data points was added Gaussian noise with mean 0 and standard deviation 0.1.

1. (4%) Load the data in `data_exe3_2023.mat`. Compute the least squares fit without taking the difference in noise levels into account. What is the solutions of all three parameters? What is the 2-norm of the absolute error $\|\boldsymbol{e}\|_2 = \|\boldsymbol{x} - \boldsymbol{x}^*\|_2$?
2. (6%) Now, we take the difference of the noise levels into account and apply the linear weighted least squares fit. According to the standard deviations of the noise, how should we add weights to the problem, i.e, what is the weight matrix? Compute the weighted least squares solution and the 2-norm of the absolute error.
3. (2%) Compare the solutions without the weights and with the weights. Which one is more accurate?

4 Solutions of linear least squares (by hand, 8%)

Consider the linear least squares problem

$$\min_{\boldsymbol{x}} \|\boldsymbol{y} - A\boldsymbol{x}\|_2^2.$$

We have the singular value decomposition (SVD) of $A \in \mathbb{R}^{m \times n} (m \geq n)$ as

$$A = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T,$$

where $r > 0$ is the rank of A. Prove that

$$\boldsymbol{x}^* = \sum_{i=1}^{r} \frac{\boldsymbol{u}_i^T \boldsymbol{y}}{\sigma_i} \boldsymbol{v}_i + \sum_{i=r+1}^{n} \tau_i \boldsymbol{v}_i$$

for arbitrary coefficients $\{\tau_i\}$ is a minimizer of this least squares problem.